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Black Holes and Calabi-Yau Threefolds

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Abstract

We compute the microscopic entropy of certain 4 and 5 dimensional extermal black holes which arise for compactification of M-theory and type IIA on Calabi-Yau 3-folds. The results agree with macroscopic predictions, including some subleading terms. The macroscopic entropy in the 5 dimensional case predicts a surprising growth in the cohomology of moduli space of holomorphic curves in Calabi-Yau threefolds which we verify in the case of elliptic threefolds.

1 Introduction

Microscopic degeneracy counts of BPS states in 4 and 5 dimensional compactifications and agreement with the corresponding Bekenstein-Hawking formula for black hole entropy has been tested in many cases. In this paper we extend these results by considering a wide class of supersymmetric compactification of M-theory to 4 and 5 dimensions. In particular we consider compactifications of M-theory on elliptic Calabi-Yau threefolds to 5 dimensions, and type IIA compactifications on Calabi-Yau threefolds to 4 dimensions and verify the match with the macroscopic prediction of Bekenstein-Hawking Entropy formula. The method we use is unaffected whether the Calabi-Yau threefold has $SU(3)$ holonomy, $SU(2)$ holonomy (such as $K3 \times T^2$) or trivial holonomy (T^6). When we refer to Calabi-Yau threefolds in this paper, we have any of these three possible cases in mind (in particular our work includes as a special case the results obtained in [1]).

In the case of 5-dimensional BPS states we perform the counting in two different ways, one using properties of F-theory and its relation to M-theory, and the other using a direct count of M2 branes wrapped around 2-cycles of CY 3-folds. The latter is not only consistent with computations of holomorphic curves in CY 3-folds based on mirror symmetry but actually sheds light on some aspects of it. For the case of 4-dimensional compactifications of type IIA strings, in a recent paper [2] it was shown how properties of M-theory 5-brane and its relation to type IIA branes leads to a prediction of BPS states in agreement with macroscopic results. Here we show that the same results follow from a direct type IIA description as well in the spirit of BPS count of type IIA strings on $K3$ and T^4 [3] [4]. Thus the situation in 4 dimensions is completely parallel with the 5 dimensional case which can also be computed in two different ways.

2 F-theory, M-theory, Type IIA Chain

Let us review some aspects of F-theory, M-theory and Type IIA theories which are relevant for us. In particular the relation between F-theory and M-theory will prove useful for understanding the entropy of 5d black holes and the relation between M-theory and type IIA will prove useful for the computation of entropy of 4d black holes.

2.1 F-theory, M-theory Duality

Consider F-theory on an elliptic manifold K with a section B, which is the base of the elliptic fibration. Moreover consider a three brane of type IIB wrapped around a holomorphic two cycle $C \subset B$ which will correspond to a string in the uncompactified space time. Upon further compactification of F-theory on circle, we obtain M-theory on the same manifold K [5]. Moreover one can map what corresponds to the string obtained by wrapping F-theory 3-brane around C in the M-theory setup. There are two possibilities to consider upon further compactification on a circle: A string which is wrapped around the circle, carrying some momentum $p = n/R$ along the circle, or a string which is unwrapped. In terms of M-theory compactification on K, the wrapped strings with momentum n correspond to $M2$ branes wrapped around a 2-cycle in K in the homology class $[C] + n[E]$ where $[E]$ denotes the class of the elliptic fiber. The strings which are unwrapped correspond to $M5$ branes wrapped around the four manifold $\widehat{C}$ which is the total space of the curve C together with the elliptic fiber on top of it. This is quite an interesting link: If we wish to count the number of M2 BPS states wrapped in the class $[C] + n[E]$, it suffices to compute the number of BPS states of a wrapped string of momentum n. The easiest way to find the degrees of freedom on this string is to consider a further compactification on S^1 and consider not wrapping the string around S^1 and use the M-theory description of the string. The reason for doing this, instead of a direct count of the low energy modes on the string in the 6-dimensional setup is that the coupling constant on the 3-brane worldvolume is undergoing $SL(2, \mathbf{Z})$ monodromy and it is thus difficult to formulate a Lagrangian description of the low energy degrees of freedom on the left-over string. The degrees of freedom on this string, apart from compactification of one bosonic direction of the F-theory string, is the same degrees of freedom as those of the string obtained by wrapping $M5$ brane on $\widehat{C}$. Let us count these degrees of freedom. The string we obtain by wrapping M5 brane on $\widehat{C}$ has $(0,4)$ supersymmetry. To compute the number of BPS states it suffices to compute the left-moving degrees of freedom on the string. Apart form the center of mass degree of freedom, the left-moving bosonic modes arise from the modes corresponding to moving the $\widehat{C}$ in the Calabi-Yau, which has complex dimension $h^{2,0}(\widehat{C})$ as well as anti-self dual harmonic two forms on $\widehat{C}$ which correspond to left-moving degrees of freedom on the string due to self-duality of the anti-symmetric field strength on $M5$ brane. The number of anti-self dual 2-forms is equal to $h^{1,1}(\widehat{C})-1$. Similarly the left-moving fermions on the string come from the odd cohomologies of $\widehat{C}$, which is equal to $4h^{1,0}(\widehat{C})$ real fermions. In addition there are three scalar modes corresponding to the center of mass of the string. So all told the number of real left-moving bosons is

$$2h^{2,0}+(h^{1,1}-1)+3=h^{0,0}+(h^{2,0}+h^{1,1}+h^{0,2})+h^{2,2}=b^{even}(\widehat{C})$$

where b^{even} denotes the total number of even cohomology elements and we have used $h^{0,0}=h^{2,2}=1$ and $h^{2,0}=h^{0,2}$. Moreover the number of left-moving fermions is equal to

$$h^{1,0}+h^{0,1}+h^{2,1}+h^{1,2}=b^{odd}(\widehat{C})$$

where b^{odd} denotes the total number of odd cohomology elements. The number of BPS states of this string with left-momentum n and no right-moving momentum would correspond to the coefficient of $d(n)$ in the expansion

$$\sum d(n)q^n=\frac{[\prod_n(1+q^n)]^{b^{odd}}}{[\prod_n(1-q^n)]^{b^{even}}}$$

Using Hardy-Ramanujan formula it is easy to see that for large enough n we have

$$d(n)\sim \exp(2\pi\sqrt{\frac{nc_L}{6}})=\exp(2\pi\sqrt{\frac{n(b^{even}+\frac{1}{2}b^{odd})}{6}} \tag{2.1}$$

Now let us compare this to the direct count of $M2$ branes wrapped in the class $[C]+n[E]$. The mathematical problem involved turns out to be simpler if we consider one compact spatial direction in which case we obtain type IIA on the same manifold, and instead of wrapped M2 branes we consider wrapped D2 branes[1]. Clearly the physics of the counting will not change as we are not considering any excitations along the compact direction, except for a possible quantized momentum along the compact direction. We will show that the counting of the states in the Type IIA setup will give the same result for all values of discrete momentum along the extra compactified circle, and thus in the large radius limit (strong coupling limit of type IIA) will give the BPS spectrum for a particle in one higher dimension. The number of BPS states for type IIA in this setup case can be computed by using similar techniques as was done in [4](see also the review article [7]). The relevant D2 brane configuration consists of bound states of n D2 branes wrapped around $[E]$ with one single $D2$ brane $[C]$. The moduli space of such bound states is the same as the choice of n points on $[C]$ to which the $D2$ branes wrapped around E are attached. In other words, the total configuration of the $D2$ brane looks like a singular Riemann surface, consisting of C with n "elliptic ears". If we include the choice of Wilson line on the elliptic D2 brane, which is equivalent for each torus to the choice of a point on the dual torus, we see that the full moduli space involves the choice of n points on the $\widehat{C}$ manifold, where the elliptic fiber is the dual to the original one. Of course the order of the points is irrelevant and thus the number of BPS states is the same as

$$H^*(Sym^n(\widehat{C})) \tag{2.2}$$

where H^* denotes the cohomologies and Sym^n denotes the $n-$fold symmetric product. Note that if we turn on m-units of $U(1)$ flux on the D2 brane, which would correspond to considering momentum m/L along the extra compact dimension, the moduli space we obtained above would not change as that would still give the Jacobian as the moduli space. We thus see the count of BPS states will give the same result for arbitrary momentum m/L and so in the large L limit (strong coupling limit of type IIA) we obtain an M2 prediction.

We are thus left to compute the cohomologies (2.2). This question was encountered before in the course of studying $N = 4$ gauge theories in 4-dimensions [8] and also in the math literature [9] where one uses orbifold techniques to compute it. One finds that the cohomology can be computed by considering the Fock space of b^{even} bosonic oscillators and b^{odd} fermionic oscillators, exactly in accordance with what we found in the con-

[1]The same comment applies to counting the BPS degeneracies of wrapped M2 branes of M-theory on K3. The direct count for M2 branes in the $K3$ case and the present case is also possible [6].

text of wrapped string of F-theory. Thus they agree, as anticipated by duality between F-theory and M-theory.[2] This also explains part of the results in [12] [13] where it was observed that the E_8 string carries 12 bosonic oscillators on the worldsheet–12 is the number of even cohomologies of P^2 blown up at 9 points, which gives rise to E_8 string by wrapping the M5 brane around it.

2.2 M-theory, Type IIA Duality

If we consider M-theory on an arbitrary CY 3-fold $\mathcal{K}$ (not necessarily elliptic), further compactification on S^1 gives a type IIA theory on manifold $\mathcal{K}$ [14] [15]. Now consider a holomorphic four-cycle $\mathcal{C}$ in $\mathcal{K}$. Then if we consider the M5 brane wrapped around $\mathcal{C}$ with some left-moving momentum n/R along the extra circle, the computation is identical to what we did above and was recently considered in [2]. From the type IIA perspective this has the interpretation of a $D4$ brane wrapping around $\mathcal{C}$ bound to n 0-branes (modulo the shift induced by wrapped $D4$ brane which is $-c_2(\mathcal{C}/24)$ [4] [16]), as is standard from comparing the branes of M-theory with those of type IIA. However this count of BPS states can also be done quite directly in the type IIA setup as was done in [7]: Namely each 0-brane binds to the 4-brane $\mathcal{C}$ [17]. Thus the moduli space of this bound state involves the choice of n points stuck on $\mathcal{C}$ and we thus obtain again the moduli space of symmetric product of n points on $\mathcal{C}$ as the moduli space of bound states, whose cohomology again gives the same answer as mentioned before. In fact the agreement between the 5-brane count of BPS states [18] and the type IIA count [7] [4] in the special cases of $K3$ and T^4 was already known. Here we have shown this to be true for more general four manifolds.

3 Black Holes in 5 Dimensions

Consider M-theory compactified on a Calabi-Yau 3-fold K down to 5 dimension. Let Q denote an element of $H_2(K, Z)$ (corresponding to the M2 charge). Let k denote the Kahler class of Calabi-Yau. Then the BPS mass of this state is given by

$$M = \int_{[Q]} k$$

The entropy predicted for this configuration according to the results [19] is

$$S = a\mathrm{Min}_k[M^{3/2}], \qquad \text{subject to} \quad \int_K k^3 = 1 \tag{3.1}$$

[2]The similarity of this result with recent results [10] [11] showing the existence of a Virasoro action on partition function of topological strings which count holomorphic curves is very suggestive.

where a is a universal constant. In other words we vary the kahler class k, subject to the total volume of the CY being 1 and then compute the mass M for this particular k, in terms of which the entropy formula has the universal form given above. For the application we have in mind we will assume that K is elliptic CY 3-fold with a section B. Let C denote a curve (i.e. a Riemann surface) in B. Let e_i denote a basis for $H_2(B,Z)$. Let us define

$$k_E = \int_{T^2} k \qquad k_i = \int_{e_i} k \qquad d_{ij} = e_i \cdot e_j \qquad [C] = C^i e_i$$

Let us consider the charge state

$$Q = n[T^2] + [C]$$

Then

$$M = nk_E + C^i k_i$$

which is to be minimized as we vary k_E and k_i subject to $V = k_E k_i d^{ij} k_j = 1$. This can be easily done and one finds that at the minimum

$$M = \frac{3}{4}(nC \cdot C)^{1/3}$$

Moreover one finds that at the minimum $k_E = 4^{-1/3} n^{-2/3} (C \cdot C)^{1/3}$ and $k_i = 4^{1/6} C_i n^{1/3} (C \cdot C)^{-2/3})$. Using the universal entropy formula (including the universal numerical factor) this gives

$$S = 2\pi \sqrt{\frac{1}{2} C \cdot Cn}$$

Now we come to the microscopic computation. From the discussion in the previous section it is clear that all we need to do is to compute the cohomology of $\widehat{C}$, which is the four manifold consisting of the elliptic manifold over the Riemann surface C. It is straight forward to do this. Let c_1 denote the first Chern class of the base B. Then one can show using index theory and the vanishing of the first chern class of K that

$$h^{2,0}(\widehat{C}) = \frac{1}{2}(C \cdot C + c_1(C))$$

$$h^{1,0}(\widehat{C}) = \frac{1}{2}(C \cdot C - c_1(C)) + 1$$

$$h^{1,1}(\widehat{C}) = C \cdot C + 9c_1(C) + 2$$

We thus see that

$$b^{even} + \frac{1}{2} b^{odd} = (3C \cdot C + 9c_1(C) + 6)$$

Thus according to (2.1) the Hardy-Ramanujan formula gives that for large n

$$S = 2\pi\sqrt{\frac{1}{6}n(3C\cdot C + 9c_1(C) + 6)}$$

To leading order in large n and C, this is in agreement with the black hole formula above. The microscopic prediction has some subleading terms which do not vanish for large charges. The leading correction goes as

$$3\pi\sqrt{\frac{nc_1(C)^2}{2C\cdot C}}$$

This is similar to what has been found in certain 4d cases using M-theory recently [2] as we will rederive below in the context of type IIA. In that case the subleading terms were matched with corrections to the effective action in 4 dimensions involving R^2 type terms which are proportional to Euler characteristic in 4 dimensions. In fact there are such corrections already in the 5-dimensional theory [20] which go as

$$[\int_K c_2(K)\wedge k]R^2$$

Here the R^2 is not the Euler characteristic anymore (as that vanishes in 5 dimensions) but rather it corresponds to a specific contraction of two R's which in 4d would give the Euler character density. If we consider a Euclidean black hole, this term corrects the entropy by $\exp(-\delta S)$. We will now verify that the leading correction this induces has the same charge dependence as was found by the microscopic derivation above.

For large n the dominant term comes from the expansion involving k_i of the base as can be seen from the fixed point values of moduli at the horizon given above. Let $\widehat{e}_i$ denote the elliptic four manifold over e_i which form a basis for $H_2(B)$. Then $c_2(\widehat{e}_i) = 12c_1(e_i)$, where c_1 is the first chern class of the base B. Expanding the effective action the leading term will thus come from

$$R^2 c_1(e_i)k_i$$

Integrating $\int R^2$ over the 5-dimensional extremal black hole will give a term which scales as (length) (as R^2 has dimension 4). Since the mass in the extremal black hole appears in the combination M/r^2, this means that the integral will go as $M^{1/2}$. Using the fact that $M\sim [n(C\cdot C)]^{1/3}$ and that $k_i\sim C_i n^{1/3}(C\cdot C)^{-2/3}$ we learn that the correction to the effective action in the presence of the 5d black hole scales as

$$[n(C\cdot C)]^{1/6}(c_1(e_i)C_i n^{1/3}(C\cdot C)^{-2/3}) = \sqrt{\frac{nc_1(C)^2}{C\cdot C}}$$

which has precisely the same charge dependence as predicted by the subleading corrections from the microscopic computation. It would be interesting also to check that the numerical coefficients in front also matches–this would require computing the integral of R^2 over the 5d extremal euclidean black hole, which we have not performed.

In this section we have mainly considered an elliptic Calabi-Yau in order to do the explicit count of holomorphic curves. The question arises as to whether we can do this count more generally for M-theory compactifications on arbitrary Calabi-Yau manifolds. In this case the following puzzle has been pointed out by Strominger [21]: The black hole entropy formula predicts that the entropy of holomorphic curves should scale as the $N^{3/2}$ as we rescale the $M2$ charges $Q \in H_2(\mathcal{K})$ by $Q \to NQ$. However, in some examples, such as quintic it is known that the entropy of genus zero curves scales as N (i.e. the number of rational curves of degree N grows as $\exp(aN)$ for some a) [22]. One could hope to overcome this by assuming that the higher genus holomorphic curves will grow at a different rate. However this hope is dashed by the results in [23] where it was shown that at least for the quintic 3-fold the growth in entropy is again linear in N. Moreover it was conjectured in [23] based on universality arguments (and relations with non-critical $c = 1$ strings) that the entropy being linear in N should be a generic behavior for generic Calabi-Yau threefolds. We seem to have arrived at a serious contradiction! This is not so and we will now explain the resolution of the puzzle.

The computation done by mirror symmetry is not strictly speaking 'counting' holomorphic curves, but rather computing the Euler characteristic of the moduli space [3]. For example if we consider T^6 the mirror symmetry answer for the 'count' of holomorphic curves gives zero, which should be interpreted as the statement about the Euler characteristic of the moduli space of such curves. In fact clearly in the case of T^6 the construction above indicates that there is a huge growth in the number of cohomologies of the moduli of holomorphic curves. Also there are examples known where the 'number' of curves is negative [24] [25] implying that there is a moduli space of curves with more odd classes than even classes. Thus the only resolution of the puzzle for the discrepancy between the microscopic and macroscopic prediction of black hole entropy will be if for large enough N we *always* get a non-trivial moduli space of holomorphic curves[4], and moreover the number of even and odd cohomologies each go to leading order as $\exp[cN^{3/2}]$ and that there is a near perfect cancellation between them in computing Euler characteristic, and that there is a subleading term in the computation of

[3]More precisely the Euler characteristic of a specific bundle over the moduli space.

[4]This is for cases of directions in H_2 for which there is a finite horizon black hole, which exists if there exists a kahler class which squares to the particular element in H_2.

Euler characteristic that survives and that would go as $\exp[aN]$. This sounds on the face of it rather like wishful thinking, but we will in fact prove that in the elliptic case considered above this is exactly what happens!

We showed that the cohomologies of moduli space of holomorphic curves are given by n-th level states in a fock space involving bosons and fermions. Note that if the number of bosons and fermions were exactly equal we would get equal number of even and odd cohomologies and thus zero Euler character. In fact if we wish to compute Euler character for the moduli space wrapping the elliptic fiber n times and the curve C once, instead of considering the coefficient of q^n in

$$\frac{\prod_n (1+q^n)^{b^{odd}}}{\prod_n (1-q^n)^{b^{even}}}$$

we should weight the odd classes by a minus sign and thus consider the coefficient of q^n in

$$\frac{\prod_n (1-q^n)^{b^{odd}}}{\prod_n (1-q^n)^{b^{even}}} = \frac{1}{\prod_n (1-q^n)^{b^{even}-b^{odd}}}$$

In our case $b^{even} - b^{odd} = 12c_1(C)$, which thus implies that the coefficient of q^n for the Euler character of moduli spaces grows as

$$\exp 2\pi\sqrt{2nc_1(C)}$$

Note that under uniform rescaling of charges by a factor of N this grows as $\exp(aN)$ as expected on general grounds. Having confirmed the general idea in this particular case we can now be more confident and make the following prediction (based on black hole analysis): For large enough degrees of curves, there will generically be a moduli space of curves in CY 3-folds, with nearly perfect equality of the number of even and odd cohomologies where each goes as $\exp(bN^{3/2})$ (where the coefficient of b can be fixed using the black hole prediction as explained above). The subleading difference in the growth of even and odd cohomologies should then lead to a growth in the *Euler characteristic* of moduli space as $\exp(aN)$. It would be extremely interesting to verify this prediction; the case of quintic 3-fold seems a good test case.

4 Black Holes in 4 Dimensions

We now consider type IIA strings compactified on a CY 3-fold $\mathcal{K}$ (not necessarily elliptic). The black hole entropy in this case has been recently computed in [2] using M-theory. Our derivation here uses type IIA to arrive at the same conclusions, in the spirit of [3] [4]. Consider a 4-cycle realized through a holomorphic manifold $\mathcal{C}$. Let us assume $h^{1,0}(\mathcal{C}) = 0$. We are interested in a BPS black hole which is a bound state of $D4$ charge $[\mathcal{C}]$ and n of $D0$ charge. According to [26] [27] the entropy in this case is predicted to be

$$S = 2\pi\sqrt{(n - \frac{c_2(\mathcal{C})}{24})(\frac{1}{6}\mathcal{C}^3)} \tag{4.1}$$

where $c_2(\mathcal{C})$ denotes the second chern class of $\mathcal{K}$ evaluated on $\mathcal{C}$ and $\mathcal{C}^3$ denotes the triple self intersection of $\mathcal{C}$.

As was discussed in the previous section there are two ways to count the microscopic entropy in this situation. One using M-theory, by considering M5 branes wrapped on $\mathcal{C} \times S^1$ with momentum n/R along S^1. This was the approach followed in [2]. The other is by considering the bound states of 0-brane and 4-brane, which for n large enough gives the answer (using the fact that since $b^{odd} = 0, b^{even} = \chi(\mathcal{C})$)

$$S = 2\pi\sqrt{\frac{1}{6}n(\chi(\mathcal{C}))} \tag{4.2}$$

$\chi(\mathcal{C})$ can be computed topologically in a standard way [25], using the fact that the normal bundle of $\mathcal{C}$ is the canonical bundle of $\mathcal{C}$, with the result

$$\chi(\mathcal{C}) = \mathcal{C}^3 + c_2(\mathcal{C})$$

where $c_2(\mathcal{C})$ denotes the second chern class of the CY 3-fold evaluated on $\mathcal{C}$. This is in agreement with the macroscopic entropy prediction (4.1) for $n >> C$, which is required for the use of Hardy-Ramanujan formula. The microscopic term also includes a further linear correction in $\mathcal{C}$ which was interpreted in [2] as a one loop correction for effective action in type IIA string of the form $[\int_{\mathcal{K}} c_2 \wedge k]R^2$ [23] which gives a subleading correction to the black hole entropy. As mentioned before this is similar to what we have found in the 5 dimensional case as well.

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