

# Spherical Membranes in M(atrix) Theory

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## Abstract

We consider membranes of spherical topology in uncompactified Matrix theory. In general for large membranes Matrix theory reproduces the classical membrane dynamics up to  $1/N$  corrections; for certain simple membrane configurations, the equations of motion agree exactly at finite  $N$ . We derive a general formula for the one-loop Matrix potential between two finite-sized objects at large separations. Applied to a graviton interacting with a round spherical membrane, we show that the Matrix potential agrees with the naive supergravity potential for large  $N$ , but differs at subleading orders in  $N$ . The result is quite general: we prove a pair of theorems showing that for large  $N$ , after removing the effects of gravitational radiation, the one-loop potential between classical Matrix configurations agrees with the long-distance potential expected from supergravity. As a spherical membrane shrinks, it eventually becomes a black hole. This provides a natural framework to study Schwarzschild black holes in Matrix theory.

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## 1 Introduction

It was conjectured by Banks, Fischler, Shenker, and Susskind (BFSS) that M-theory in the infinite momentum frame is exactly described by a supersymmetric matrix quantum mechanics [1]. This seminal conjecture has passed many consistency checks. Gravitons [1], membranes [2, 3], and 5-branes [4–6] have been constructed as states in Matrix theory corresponding to block matrices with particular commutation properties. Numerous calculations have shown that at one-loop order Matrix theory correctly reproduces long-range forces expected from supergravity (see for example [1, 7–17]). Some progress has been made in understanding two-loop corrections [16, 18–21] as well as processes involving M-momentum transfer [22]. The conjecture has been put on a more systematic footing by the work of Sen and Seiberg [23, 24]. For a recent review see [25].

To date, most discussions of extended objects in Matrix theory have focused on static branes, which are either infinite in extent or have been wrapped around toroidally compactified dimensions. Such membranes preserve some supersymmetry. In this paper, we focus on a different class of states, namely membranes of spherical topology. Such membranes break all of the supersymmetries of the theory, and as solutions to the equations of motion are time-dependent. Nonetheless, the means for studying such objects was developed long ago [2, 3]. In light-front gauge there is an explicit map between embeddings of a spherical membrane and Hermitian matrices. In this paper, we use this map to discuss spherical membranes in uncompactified Matrix theory. Although we focus here on membranes with spherical topology, analogous results could be derived for higher genus membranes [26, 27].

Our calculation of the Matrix theory potential will be performed for finite-sized matrices, and thus should be understood in the context of discrete light-front quantization [28]. We will see that the Matrix potential only agrees with the naive discrete light-front quantization of supergravity at leading order in large  $N$ , while the subleading terms in  $N$  differ. This supports the conjecture that large- $N$  Matrix theory is equivalent to 11-dimensional M-theory. But it also supports the idea that finite- $N$  Matrix theory, although equivalent to the DLCQ of M-theory [24], simply does not reduce to the DLCQ of supergravity at low energies [25, 29]. Other finite- $N$  discrepancies have been found, involving curved backgrounds in Matrix theory [30–32].

Our primary motivation for this paper is simply to understand the behavior of large, semiclassical membranes in Matrix theory. But ultimately a more interesting set of questions involves the behavior of a membrane in the quantum domain. A spherical membrane will shrink with time until it

is smaller than its Schwarzschild radius. Eventually quantum effects will become important, and cause the individual partons composing the membrane to fly off in different directions. Thus spherical membranes provide a natural context for studying black holes in Matrix theory.

The structure of this paper is as follows. In Section 2, we review the prescription developed in [3] for describing spherical membranes in Matrix theory. We show that even at finite  $N$  one has the expected interpretation in terms of a spherical geometry. We also show that for large membranes, Matrix theory reproduces the semiclassical membrane equations, up to  $1/N$  corrections. In Section 3, we derive a general formula for the one-loop Matrix theory potential between two finite-sized objects. In Section 4, we compare the Matrix calculation to the results expected from supergravity. We show that the potential between a spherical membrane and a graviton is correctly reproduced at leading order in  $N$  but that there are discrepancies at sub-leading order. We also prove a general pair of theorems which show that, after removing the effects of gravitational radiation, the leading term in the supergravity potential is always reproduced by large  $N$  Matrix theory. Section 5 contains conclusions and a discussion of further directions for research, including some comments about Matrix black holes.

## 2 Matrix Description of Spherical Membranes

### 2.1 Membrane-Matrix Correspondence

In the original work of de Wit, Hoppe and Nicolai [3], the supermembrane was quantized in light front gauge. An essential observation of these authors was that the Lie algebra of area-preserving diffeomorphisms of the sphere can be approximated arbitrarily well by the algebra of  $N \times N$  Hermitian matrices, when  $N$  is taken to be sufficiently large. The Poisson bracket of two functions on the sphere is then naturally associated with a commutator in the  $U(N)$  algebra. This correspondence can be made explicit by considering the three Cartesian coordinate functions  $x_1, x_2, x_3$  on the unit sphere. With a rotationally invariant area form (which we identify with a symplectic form), these Cartesian coordinates are generators whose Hamiltonian flows correspond to rotations around the three axes of the sphere. The Poisson brackets of these functions are given by

$$\{x_A, x_B\} = \epsilon_{ABC} x_C.$$

It is therefore natural to associate these coordinate functions on  $S^2$  with the generators of  $SU(2)$ . The correct correspondence turns out to be

$$x_A \leftrightarrow \frac{2}{N} J_A$$

where  $J_1, J_2, J_3$  are generators of the  $N$ -dimensional representation of  $SU(2)$ , satisfying the commutation relations

$$-i[J_A, J_B] = \epsilon_{ABC} J_C.$$

In general, the transverse coordinates of a membrane of spherical topology can be expanded as a sum of polynomials of the coordinate functions:

$$X^i(x_A) = \sum_k t^i_{A_1 \dots A_k} x_{A_1} \dots x_{A_k}.$$

The coefficients  $t^i_{A_1 \dots A_k}$  are symmetric and traceless (because  $x_A x_A = 1$ ). Using the above correspondence, a matrix approximation to this membrane can be constructed, which we denote by  $\mathbf{X}^i$ . It is necessary to truncate the expansion at terms of order  $N - 1$ , because higher order terms in the power series do not generate linearly independent matrices

$$\mathbf{X}^i = \sum_{k < N} \left(\frac{2}{N}\right)^k t^i_{A_1 \dots A_k} J_{A_1} \dots J_{A_k}. \quad (2.1)$$

This is the map between membrane configurations and states in Matrix theory developed in [3].

To motivate the correspondence, we begin by recording the membrane equations of motion in light-front gauge. The membrane action is<sup>3</sup>

$$S = -\frac{T_2}{2} \int d^3\xi \sqrt{-g} \left( g^{\alpha\beta} \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu} - 1 \right).$$

Here  $\xi^\alpha = (t, \sigma^a)$  are coordinates on the membrane worldvolume,  $g_{\alpha\beta}$  is an auxiliary worldvolume metric, and  $X^\mu(\xi)$  are embedding coordinates. The light-front gauge fixing of this action is carried out in [3, 34]. In terms of equal-area coordinates  $\sigma^a$  on a round unit two-sphere, the equations of motion can be written either in terms of an induced metric  $\gamma_{ab} \equiv \partial_a X^i \partial_b X^i$ , or in terms of a Poisson bracket  $\{f, g\} \equiv \epsilon^{ab} \partial_a f \partial_b g$  where  $\epsilon^{12} = 1$ . In

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<sup>3</sup>The spacetime metric is  $\eta_{\mu\nu} = (- + \dots +)$ . Light-front coordinates are  $X^\pm \equiv \frac{1}{\sqrt{2}}(X^0 \pm X^{10})$ .  $X^+$  is light-front time; the conjugate energy is  $p^- = \frac{1}{\sqrt{2}}(E - p^{10})$ . The coordinate  $X^-$  is compactified,  $X^- \approx X^- + 2\pi R$ , and the conjugate momentum  $p^+ = \frac{1}{\sqrt{2}}(E + p^{10})$  is quantized,  $p^+ = N/R$ . The tension of the membrane is  $T_2 = \frac{1}{(2\pi)^2 l_p^3}$  (in the conventions of [33]). We adopt units which lead to a canonical Yang-Mills action:  $2\pi l_p^3 = R$ .

light-front gauge the fields  $X^+$ ,  $X^-$  are constrained:

$$\begin{aligned} X^+ &= t \\ \dot{X}^- &= \frac{1}{2}\dot{X}^i\dot{X}^i + \frac{2\gamma}{N^2} \\ &= \frac{1}{2}\dot{X}^i\dot{X}^i + \frac{1}{N^2}\{X^i, X^j\}\{X^i, X^j\} \\ \partial_a X^- &= \dot{X}^i \partial_a X^i \end{aligned} \tag{2.2}$$

where  $\gamma \equiv \det \gamma_{ab} = \frac{1}{2}\{X^i, X^j\}\{X^i, X^j\}$ . The transverse coordinates  $X^i$  have dynamical equations of motion

$$\begin{aligned} \ddot{X}^i &= \frac{4}{N^2}\partial_a(\gamma\gamma^{ab}\partial_b X^i) \\ &= \frac{4}{N^2}\{\{X^i, X^j\}, X^j\} \end{aligned}$$

which follow from the Hamiltonian

$$\begin{aligned} H &= \frac{N}{4\pi R} \int d^2\sigma \left( \frac{1}{2}\dot{X}^i\dot{X}^i + \frac{2\gamma}{N^2} \right) \\ &= \frac{N}{4\pi R} \int d^2\sigma \left( \frac{1}{2}\dot{X}^i\dot{X}^i + \frac{1}{N^2}\{X^i, X^j\}\{X^i, X^j\} \right). \end{aligned} \tag{2.3}$$

The transverse coordinates must obey the condition

$$\{\dot{X}^i, X^i\} = 0 \tag{2.4}$$

so that the constraint on  $\partial_a X^-$  can be satisfied. Note that the velocity in  $X^-$  is essentially the energy density on the membrane. The density of momentum conjugate to  $X^-$  is constant on the membrane, and we have chosen a normalization so that the total momentum takes on the expected value.

$$p^+ = T_2 \int d^2\sigma \sqrt{-\gamma/g_{00}} \dot{X}^+ = \frac{N}{R}$$

Here  $g_{00} \equiv \eta_{\mu\nu}\dot{X}^\mu\dot{X}^\nu$ . Then using the matrix correspondence

$$x_A \leftrightarrow \frac{2}{N}J_A \quad \{\cdot, \cdot\} \leftrightarrow \frac{-iN}{2}[\cdot, \cdot] \quad \frac{N}{4\pi} \int d^2\sigma \leftrightarrow \text{Tr}$$

we see that the membrane Hamiltonian becomes the (bosonic part of) the Hamiltonian of Matrix theory

$$H = \frac{1}{R} \text{Tr} \left( \frac{1}{2}\dot{\mathbf{X}}^i\dot{\mathbf{X}}^i - \frac{1}{4}[\mathbf{X}^i, \mathbf{X}^j][\mathbf{X}^i, \mathbf{X}^j] \right). \tag{2.5}$$

This Hamiltonian gives rise to the Matrix equations of motion

$$\ddot{\mathbf{X}}^i + [[\mathbf{X}^i, \mathbf{X}^j], \mathbf{X}^j] = 0$$

which must be supplemented with the Gauss constraint

$$[\dot{\mathbf{X}}^i, \mathbf{X}^i] = 0. \quad (2.6)$$

As a particularly simple example of this formalism, which will be useful to us later, we consider the dynamics of an “ellipsoidal” membrane, in which the worldvolume sphere is mapped linearly into the first three transverse coordinates. First we introduce an equal-area parameterization of the unit sphere, for example by coordinates  $\sigma^a = (x, \theta)$  with  $-1 \leq x \leq 1$ ,  $\theta \approx \theta + 2\pi$ . The Cartesian coordinates on the sphere

$$\begin{aligned} x_1 &= x \\ x_2 &= \sqrt{1-x^2} \sin \theta \\ x_3 &= \sqrt{1-x^2} \cos \theta \end{aligned}$$

obey the expected  $SU(2)$  algebra:  $\{x_A, x_B\} = \epsilon_{ABC} x_C$ . The ellipsoid is described by setting

$$X_i(t, \sigma^a) = r_i(t) x_i(\sigma^a), \quad i \in \{1, 2, 3\}. \quad (2.7)$$

By taking the velocities  $\dot{r}_i$  to vanish at some initial time, we are guaranteed to satisfy the constraint (2.4). The equations of motion for the radii are

$$\begin{aligned} \ddot{r}_1 &= -\alpha r_1 (r_2^2 + r_3^2) \\ \ddot{r}_2 &= -\alpha r_2 (r_1^2 + r_3^2) \\ \ddot{r}_3 &= -\alpha r_3 (r_1^2 + r_2^2) \end{aligned} \quad (2.8)$$

where  $\alpha = 4/N^2$ . As expected, an initially static membrane will start to contract. If the membrane is initially an isotropic sphere of radius  $r_0$ , energy conservation gives

$$\dot{r}^2(t) + \alpha r^4 = \alpha r_0^4.$$

Classically, after a time

$$t = \frac{1}{r_0} \frac{N\Gamma(1/4)^2}{\sqrt{128\pi}},$$

a spherical membrane contracts to a point and begins to expand again with the opposite orientation. This solution was originally described by Collins and Tucker [35].

It is also straightforward to solve the constraint equations (2.2) to find the behavior of the general ellipsoidal membrane in  $X^-$ . One finds that

$$X^-(t, \sigma^a) = \frac{p^-}{p^+} t + \frac{1}{2} \sum_A r_A \dot{r}_A \left( x_A^2(\sigma^a) - \frac{1}{3} \right).$$

Note that an initially spherical membrane will remain spherical as it contracts. But in general an ellipsoidal membrane will begin to fluctuate in  $X^-$ .

For linearly embedded membranes, the correspondence with matrix theory is both simple and exact, even at finite  $N$ . The matrices associated with the ellipsoidal membrane

$$\mathbf{X}_i(t) = \frac{2}{N} r_i(t) J_i, \quad i \in \{1, 2, 3\} \quad (2.9)$$

obey commutation relations which are proportional to the Poisson brackets obeyed by the classical membrane coordinates. So the equations of motion will agree exactly, even at finite  $N$ .

This exact agreement is a special property of ellipsoidal membranes, or more generally of membranes with linear transverse embeddings. Generally speaking, the Poisson bracket of two polynomials of degree  $m$  and  $n$  is a polynomial of degree  $m + n - 1$ . Thus, if  $X_i$  contains terms of degree larger than 1, over time terms of arbitrarily large degree will be excited, which clearly makes an exact correspondence between membranes and finite- $N$  Matrix theory impossible. This will be discussed further in section 2.3.

## 2.2 Membrane Geometry at Finite $N$

We now make a few observations which may help to clarify the geometrical interpretation of finite- $N$  spherical matrix membranes. For simplicity consider a spherical membrane embedded isotropically in the first three transverse space coordinates.

$$X_i(t, \sigma^a) = r(t) x_i(\sigma^a) \quad i \in \{1, 2, 3\}.$$

The corresponding matrices are

$$\mathbf{X}_i(t) = \frac{2}{N} r(t) J_i, \quad i \in \{1, 2, 3\}.$$

According to the usual Matrix theory description [1], the matrices  $\mathbf{X}_i$  describe a set of  $N$  0-branes (supergravitons with unit momentum in the longitudinal direction) which have been bound together with off-diagonal strings. If this matrix configuration is truly to describe a macroscopic spherical membrane, we would expect that the  $N$  0-branes should be uniformly distributed

on the surface of the sphere of radius  $r$ . A first check on this is to calculate the sum of the squares of the matrices.

$$\mathbf{X}_1^2 + \mathbf{X}_2^2 + \mathbf{X}_3^2 = r^2 \left( 1 - \frac{1}{N^2} \right) \mathbb{1}.$$

This indicates that for large  $N$  the 0-branes are in a noncommutative sense constrained to lie on the surface of a sphere of radius  $r$ . The radius of the matrix sphere, defined in this way, receives finite- $N$  corrections from its classical ( $N \rightarrow \infty$ ) value. This is a general feature: different definitions of the geometry will not agree at order  $1/N$ , reflecting the intrinsic fuzziness of a finite- $N$  matrix sphere.

A related geometrical check can be performed by diagonalizing any one of the matrices  $\mathbf{X}_i$ . If the matrix configuration is indeed to describe a sphere, each matrix should have a distribution of eigenvalues which matches the density of a uniform sphere projected onto a coordinate axis. This gives a density which is uniformly distributed over the interval  $[-r, r]$ . Indeed, the eigenvalues of each of the  $SU(2)$  generators are given by

$$\frac{N-1}{2}, \frac{N-3}{2}, \dots, \frac{3-N}{2}, \frac{1-N}{2}$$

and thus the spectrum of eigenvalues of each matrix  $\mathbf{X}_i$  is given by

$$r(1 - 1/N), r(1 - 3/N), \dots, r(-1 + 3/N), r(-1 + 1/N).$$

If we associate a 0-brane with each eigenvalue, and we assume that each 0-brane corresponds to an equal quantum of area on the surface of the sphere, we see that the area associated with these 0-branes precisely covers the interval  $[-r, r]$  with a uniform distribution. In contrast to the above, this is consistent with the classical expectation, with no  $1/N$  corrections.

As another example of how the spherical membrane geometry can be understood from the matrices, we ask whether the 0-branes corresponding to a small patch of the sphere can be studied in isolation. For large  $r$  and  $N$ , we expect that we should be able to isolate a region which looks locally like an infinite planar membrane, by focusing on the part of the matrices in which one of the coordinates is close to its maximum value (for example,  $X_3 \approx r$ ).

Indeed, this is the case. If we work in an eigenbasis of  $J_3$  with eigenvalues



decreasing along the diagonal, then the matrices  $\mathbf{X}_1, \mathbf{X}_2$  satisfy

$$[\mathbf{X}_1, \mathbf{X}_2] = \frac{2ir^2}{N} \begin{pmatrix} 1 - \frac{1}{N} & 0 & 0 & \cdots & \\ 0 & 1 - \frac{3}{N} & 0 & 0 & \cdots \\ 0 & 0 & 1 - \frac{5}{N} & 0 & \cdots \\ \cdots & 0 & 0 & 1 - \frac{7}{N} & \cdots \\ & & \cdots & \cdots & \cdots \end{pmatrix}$$

Restricting attention to the upper left-hand corner, up to terms of order  $1/N$  this reduces to  $[\mathbf{X}_1, \mathbf{X}_2] = \frac{2ir^2}{N} \mathbb{1}$ . This is precisely the type of relation introduced in BFSS [1] to describe an infinite planar membrane stretched in the  $X^1 X^2$  plane. Thus, in a sense this subset of the 0-branes describes a geometrical “piece” of the full spherical membrane.

### 2.3 Equations of Motion at Finite $N$

As we have seen, finite- $N$  Matrix theory has states which can be identified geometrically with membranes of spherical topology. To complete the identification we must further establish that these states in Matrix theory evolve according to the equations of motion of the corresponding membranes.

The membrane and Matrix Hamiltonians (2.3), (2.5) are related by the Poisson bracket – matrix commutator correspondence

$$\{\cdot, \cdot\} \leftrightarrow \frac{-iN}{2} [\cdot, \cdot].$$

Note that  $2/N$  plays the role of  $\hbar$  in the usual classical-quantum correspondence. Thus in the large- $N$  limit one expects the two equations of motion to agree.

It is important, however, to analyze the corrections which result from finite  $N$ . Not surprisingly finite- $N$  Matrix theory places a restriction on the number of modes which can be excited on the membrane. Suppose  $Z_1(x_A)$  and  $Z_2(x_A)$  are two polynomials of degree  $k_1$  and  $k_2$ , respectively. Then their Poisson bracket  $Z_3 = \{Z_1, Z_2\}$  is a polynomial of degree  $k_3 = k_1 + k_2 - 1$ . The associated matrices obey

$$-\frac{iN}{2} [\mathbf{Z}_1, \mathbf{Z}_2] = \mathbf{Z}_3 (1 + \mathcal{O}(k_1 k_2 / N)).$$

So, in order for the matrix and membrane equations of motion to agree, the modes which are excited on the membrane must satisfy  $k \ll k_{\max} \sim \sqrt{N}$ . Note that this is a stronger condition than the requirement  $k < N$  which was necessary for the membrane – matrix correspondence (2.1) to generate linearly independent matrices.

This restriction, even if satisfied at some initial time, will not in general persist. The membrane equations of motion are non-linear, so higher and higher modes on the membrane will become excited. This is just a consequence of the fact that for  $k_1, k_2 > 1$  taking the Poisson bracket generates polynomials of increasing degree. But an initially large, smooth membrane should be well described by matrix theory for a long time before the nonlinearities become important. Note that the ellipsoidal membranes discussed above are an exception to this general rule: the classical matrix and membrane equations agree exactly for arbitrary  $N$  and for all time.

So we see that two conditions must be satisfied for a state in matrix theory to behave like the corresponding semiclassical membrane. First, the size of the membrane must be much larger than the Planck scale, so that the semiclassical approximation to the Matrix equations of motion is valid. Second, the modes which are excited on the membrane must obey the restriction  $k < \sqrt{N}$ . This latter restriction is analogous to the condition  $S < N$  which was found to be necessary to have an accurate finite- $N$  description of a matrix black hole [36–39]. The restriction can be understood heuristically in the present context by noting that finite- $N$  Matrix theory describes the dynamics of  $N$  partons, which matches the  $N$  degrees of freedom of a membrane that only has modes up to  $k \sim \sqrt{N}$  excited.

### 3 Matrix Theory Potential

As we have seen, Matrix theory correctly reproduces the classical dynamical equations which govern an M-theory membrane. Of course the spacetime interpretation requires that the membrane be surrounded by a gravitational field. We now wish to show that Matrix theory correctly reproduces this field, at least at large distances. To do this, we will compute the force exerted by a membrane on a D0-brane or on another membrane located far away. Closely related calculations have been performed by many authors; see for example [1, 7–17, 21].

It is natural at this point to make the discussion somewhat more general. Consider any state in Matrix theory which, from the spacetime point of view, is localized within a finite region. Then we expect a long-range gravitational field to be present, with leading long-distance falloff governed by a conserved quantity: the total mass of the system. This should also hold true in Matrix theory. That is, the leading long-distance potential seen by any probe should be governed by a conserved quantity, which can be identified with the mass of the system.

With this as motivation, we introduce a background

$$\langle \mathbf{X}_i(t) \rangle = \begin{bmatrix} \mathbf{Y}_i(t) & 0 \\ 0 & \tilde{\mathbf{Y}}_i(t) \end{bmatrix} \quad (3.1)$$

into the Matrix Yang-Mills theory. This background describes two systems.  $\mathbf{Y}_i(t)$   $i = 1, \dots, 9$  is an  $N \times N$  matrix representing the first system, while  $\tilde{\mathbf{Y}}_i(t)$  is an  $\tilde{N} \times \tilde{N}$  matrix representing the second system. In principle this background should be taken to satisfy the Matrix equations of motion, although for our purposes explicit solutions will not turn out to be necessary. We assume that the systems are localized, in that the spreads in the eigenvalues of the matrices  $\mathbf{Y}_i, \tilde{\mathbf{Y}}_i$  are all much smaller than the magnitude of the vector between their transverse center of mass positions, which we denote by  $b_i$ .

To compute the interaction potential we must integrate out the off-block-diagonal elements of the matrices. So we set

$$A_0 = \begin{bmatrix} 0 & a \\ a^\dagger & 0 \end{bmatrix} \quad \mathbf{X}_i = \langle \mathbf{X}_i \rangle + \begin{bmatrix} 0 & x_i \\ x_i^\dagger & 0 \end{bmatrix} \quad \Psi_a = \begin{bmatrix} 0 & \psi_a \\ \psi_a^\dagger & 0 \end{bmatrix}$$

where the fluctuations  $a, x_i, \psi_a$  are  $N \times \tilde{N}$  matrices. The Matrix action reads<sup>4</sup>

$$S = \frac{1}{g_{YM}^2} \int dt \text{Tr} \left\{ D_t \mathbf{X}_i D_t \mathbf{X}_i + \frac{1}{2} [\mathbf{X}_i, \mathbf{X}_j] [\mathbf{X}_i, \mathbf{X}_j] - \left( D_\mu^{\text{cl}} A^{\mu \text{qu}} \right)^2 + i \Psi_a D_t \Psi_a - \Psi_a \gamma_{ab}^i [\mathbf{X}_i, \Psi_b] \right\}$$

where we have added a covariant background-field gauge fixing term

$$D_\mu^{\text{cl}} A^{\mu \text{qu}} \equiv -\partial_t A_0 + i[\langle \mathbf{X}_i \rangle, \mathbf{X}_i].$$

The corresponding ghosts will be taken into account below. The Yang-Mills coupling is fixed by matching to the Hamiltonian (2.5):  $g_{YM}^2 = 2R$ .

We are interested in the interaction potential between the system and the probe at a given instant of time. This potential is simply the ground state energy of the off-block-diagonal fields  $a, x_i, \psi_a$ . The leading large-distance potential arises at one loop in Matrix theory. Thus, we expand the Matrix action to quadratic order in the off-block-diagonal fields. We end up with a system of harmonic oscillators, whose frequencies are determined by the background fields  $\mathbf{Y}_i(t)$  and  $\tilde{\mathbf{Y}}_i(t)$  (along with their time derivatives  $\partial_t \mathbf{Y}_i$ ,

<sup>4</sup>Conventions: All fields are Hermitian.  $i, j = 1, \dots, 9$  are  $SO(9)$  vector indices with metric  $\delta_{ij}$ , while  $a, b = 1, \dots, 16$  are  $SO(9)$  spinor indices. The  $SO(9)$  Dirac matrices  $\gamma^i$  are real, symmetric, and obey  $\{\gamma^i, \gamma^j\} = 2\delta^{ij}$ ,  $\gamma^1 \dots \gamma^9 = +\mathbb{1}$ . We define  $\gamma^{ij} = \frac{1}{2}[\gamma^i, \gamma^j]$ .

$\partial_t \tilde{\mathbf{Y}}_i$ ). The Yang-Mills plus gauge-fixing terms give rise to  $10N\tilde{N}$  complex bosonic oscillators with (frequency)<sup>2</sup> matrix

$$\begin{aligned} (\Omega_b)^2 &= M_{0b} + M_{1b} \\ M_{0b} &= \sum_i K_i^2 \otimes \mathbb{1}_{10 \times 10} \\ M_{1b} &= \begin{bmatrix} 0 & -2\partial_t K_j \\ 2\partial_t K_i & 2[K_i, K_j] \end{bmatrix} \end{aligned}$$

where

$$K_i \equiv \mathbf{Y}_i \otimes \mathbb{1}_{\tilde{N} \times \tilde{N}} - \mathbb{1}_{N \times N} \otimes \tilde{\mathbf{Y}}_i^T.$$

Similarly, the fermions give rise to  $16N\tilde{N}$  complex fermionic oscillators with (frequency)<sup>2</sup> matrix

$$\begin{aligned} (\Omega_f)^2 &= M_{0f} + M_{1f} \\ M_{0f} &= \sum_i K_i^2 \otimes \mathbb{1}_{16 \times 16} \\ M_{1f} &= i\partial_t K_i \otimes \gamma^i + \frac{1}{2}[K_i, K_j] \otimes \gamma^{ij} \end{aligned}$$

Finally we have (two identical sets of)  $N\tilde{N}$  complex scalar ghost oscillators with (frequency)<sup>2</sup> matrix

$$(\Omega_g)^2 = \sum_i K_i^2.$$

We are only interested in the leading large-distance potential. Then these oscillators have very large frequencies, on the order of the separation distance  $b$ , and we can treat the background as if it is quasi-static by comparison. That is, we can pretend that the background fields  $\mathbf{Y}_i$ ,  $\tilde{\mathbf{Y}}_i$  (as well as their time derivatives!) are constant matrices. Then the ground state energy, or equivalently the interaction potential, is given by the usual oscillator formula

$$V_{\text{matrix}} = \text{Tr} (\Omega_b) - \frac{1}{2}\text{Tr} (\Omega_f) - 2\text{Tr} (\Omega_g). \quad (3.2)$$

We are interested in the leading behavior of the potential as  $b \rightarrow \infty$ . Thus we can treat  $M_1$  as a perturbation, since its eigenvalues stay of order one in this limit, while the eigenvalues of  $M_0$  are of order  $b^2$ .

To make the expansion, it is convenient to first introduce an integral representation for the square root of a matrix, then use the standard Dyson

perturbation series.

$$\begin{aligned}
 & \text{Tr} \sqrt{M_0 + M_1} \\
 &= -\frac{1}{2\sqrt{\pi}} \text{Tr} \int_0^\infty \frac{d\tau}{\tau^{3/2}} e^{-\tau(M_0 + M_1)} \\
 &= -\frac{1}{2\sqrt{\pi}} \text{Tr}_G \left\{ \int_0^\infty \frac{d\tau_1}{\tau_1^{3/2}} e^{-\tau_1 M_0} \text{Tr}_L(\mathbb{1}) \right. \\
 &\quad - \int_0^\infty \int_0^\infty \frac{d\tau_1 d\tau_2}{(\tau_1 + \tau_2)^{3/2}} e^{-(\tau_1 + \tau_2)M_0} \text{Tr}_L(M_1(\tau_2)) \\
 &\quad + \int_0^\infty \int_0^\infty \int_0^\infty \frac{d\tau_1 d\tau_2 d\tau_3}{(\tau_1 + \tau_2 + \tau_3)^{3/2}} e^{-(\tau_1 + \tau_2 + \tau_3)M_0} \\
 &\quad \left. \cdot \text{Tr}_L(M_1(\tau_2 + \tau_3)M_1(\tau_3)) \mp \dots \right\}. \tag{3.3}
 \end{aligned}$$

In the last line we have broken the trace up into  $\text{Tr} = \text{Tr}_G \text{Tr}_L$ , where  $\text{Tr}_G$  is an  $N\tilde{N}$ -dimensional trace on group indices, and  $\text{Tr}_L$  is an appropriate trace on Lorentz indices. Also we have defined the ‘interaction picture’ field

$$M_1(\tau) \equiv e^{\tau M_0} M_1 e^{-\tau M_0}.$$

It is now straightforward to expand the potential. At zeroth order in the perturbation we get the expected supersymmetric cancellation of zero-point energy.

$$\text{Tr}_L(\mathbb{1}_{10 \times 10}) - \frac{1}{2} \text{Tr}_L(\mathbb{1}_{16 \times 16}) - 2 = 0.$$

Note that this cancellation arises from the trace over Lorentz indices, and does not depend on the detailed form of the matrices  $K_i$ . We shall see that this feature persists at higher orders. Also, note that if we were dealing with a configuration in which the background fields  $\langle \mathbf{X}_i \rangle$  commuted and were constant, then this lowest-order result would not be corrected: the perturbation  $M_1$  vanishes. This is the expected BPS result that there is no potential between a system of stationary D0-branes.

At first order in the perturbation, the bose and fermi contributions separately vanish:

$$\text{Tr}_L(M_{1b}(\tau)) = 0 \quad \text{Tr}_L(M_{1f}(\tau)) = 0.$$

At second order there is a cancellation:

$$\begin{aligned}
 & \text{Tr}_L(M_{1b}(\tau_1)M_{1b}(\tau_2)) = \frac{1}{2} \text{Tr}_L(M_{1f}(\tau_1)M_{1f}(\tau_2)) \\
 &= -8 \partial_t K_i|_{\tau_1} \partial_t K_i|_{\tau_2} - 4 [K_i, K_j]|_{\tau_1} [K_i, K_j]|_{\tau_2}
 \end{aligned}$$

where for example we have defined  $K_i|_\tau = e^{\tau M_0} K_i e^{-\tau M_0}$ . At third order the cancellation is less trivial:

$$\begin{aligned} \text{Tr}_L \left( M_{1b}(\tau_1) M_{1b}(\tau_2) M_{1b}(\tau_3) \right) &= \frac{1}{2} \text{Tr}_L \left( M_{1f}(\tau_1) M_{1f}(\tau_2) M_{1f}(\tau_3) \right) \\ &= -8 [K_i, K_j]|_{\tau_1} \partial_t K_j|_{\tau_2} \partial_t K_i|_{\tau_3} - 8 \partial_t K_i|_{\tau_1} [K_i, K_j]|_{\tau_2} \partial_t K_j|_{\tau_3} \\ &\quad - 8 \partial_t K_i|_{\tau_1} \partial_t K_j|_{\tau_2} [K_j, K_i]|_{\tau_3} + 8 [K_i, K_j]|_{\tau_1} [K_j, K_k]|_{\tau_2} [K_k, K_i]|_{\tau_3}. \end{aligned}$$

At fourth order we find a non-zero result. At large  $b$  the integral is dominated by the small- $\tau$  behavior of the integrand, so we can expand and find the leading term

$$\begin{aligned} V_{\text{matrix}} &= -\frac{1}{2\sqrt{\pi}} \text{Tr}_G \int_0^\infty \frac{d\tau_1 \cdots d\tau_5}{(\tau_1 + \cdots + \tau_5)^{3/2}} e^{-(\tau_1 + \cdots + \tau_5)b^2} \\ &\quad \cdot \left( \text{Tr}_L \left( M_{1b}^4 \right) - \frac{1}{2} \text{Tr}_L \left( M_{1f}^4 \right) \right) \\ &= -\frac{5}{128 b^7} W \end{aligned} \tag{3.4}$$

where the ‘‘gravitational coupling’’  $W$  is given by

$$\begin{aligned} W &\equiv \text{Tr}_G \left( \text{Tr}_L \left( M_{1b}^4 \right) - \frac{1}{2} \text{Tr}_L \left( M_{1f}^4 \right) \right) \\ &= \text{Tr}_G \left\{ 8 F^\mu{}_\nu F^\nu{}_\lambda F^\lambda{}_\sigma F^\sigma{}_\mu + 16 F_{\mu\nu} F^{\mu\lambda} F^{\nu\sigma} F_{\lambda\sigma} - 4 F_{\mu\nu} F^{\mu\nu} F_{\lambda\sigma} F^{\lambda\sigma} \right. \\ &\quad \left. - 2 F_{\mu\nu} F_{\lambda\sigma} F^{\mu\nu} F^{\lambda\sigma} \right\}. \end{aligned} \tag{3.5}$$

We have introduced the notation  $F_{0i} = -F_{i0} = \partial_t K_i$  and  $F_{ij} = i[K_i, K_j]$ . This general result is discussed in [16, 40]. A similar formula was discussed in [13, 17], but with the assumption that commutators of field strengths could be dropped.

## 4 Comparison to Supergravity

We now compare the general Matrix expression (3.4) to the naive supergravity potential, that is, to the potential arising from tree-level graviton exchange in discrete light-front supergravity.

### 4.1 Matrix and Gravitational Potentials

In general, the effective action from one-graviton exchange in a flat background metric  $\eta_{\mu\nu} = (- + \cdots +)$  is

$$S_{\text{eff}} = -\frac{1}{8} \int d^d x d^d y T_{\mu\nu}(x) D^{\mu\nu\lambda\sigma}(x-y) T_{\lambda\sigma}(y)$$

where the (harmonic gauge) graviton propagator in  $d$  spacetime dimensions is

$$D^{\mu\nu\lambda\sigma}(x-y) = 16\pi G \left( \eta^{\mu\lambda}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\lambda} - \frac{2}{d-2}\eta^{\mu\nu}\eta^{\lambda\sigma} \right) \int \frac{d^d k}{(2\pi)^d} \frac{e^{ik\cdot(x-y)}}{-k^2}.$$

We should use a Green's function which is periodic in  $X^-$  but this turns out not to affect the result. The stress tensor for an object with momentum  $p^\mu$  located at light-front coordinates  $(z^-, z^i)$  is

$$T^{\mu\nu}(x^+, x^-, x^i) = \frac{p^\mu p^\nu}{p^+} \delta(x^- - z^-) \delta(x^i - z^i). \quad (4.1)$$

With two objects the total stress tensor is a sum  $T^{\mu\nu}(x) + \tilde{T}^{\mu\nu}(x)$ . In accord with our quasi-static approximation in Matrix theory, we treat the positions  $z, \tilde{z}$  as if they are constants. To get the potential from the effective action we change the overall sign and strip off  $\int dx^+$ . Because no longitudinal momentum is exchanged, we must average  $\frac{1}{2\pi R} \int dx^-$ . This gives

$$V_{\text{gravity}} = -\frac{15}{4} \frac{R^4}{N\tilde{N}b^7} \left[ (p \cdot \tilde{p})^2 - \frac{1}{9} p^2 \tilde{p}^2 \right] \quad (4.2)$$

where we have specialized to eleven dimensions. Our normalizations are such that  $16\pi G = (2\pi)^8 l_P^9$  [33], and we use units in which  $2\pi l_P^3 = R$ .

We can compare this to the predictions of Matrix theory in two cases. First, consider the case where the second object is a massless particle with

$$\tilde{p}^+ = \tilde{N}/R \quad \tilde{p}^- = 0 \quad \tilde{p}^i = 0.$$

Then the gravitational potential simplifies to

$$V_{\text{gravity}} = -\frac{15}{4} \frac{R^2 \tilde{N}}{b^7 N} E^2 \quad (4.3)$$

where  $E$  is the light-front energy of the first object, given by the Matrix theory Hamiltonian

$$E = \frac{1}{R} \text{Tr} \left( \frac{1}{2} \dot{\mathbf{Y}}^i \dot{\mathbf{Y}}^i - \frac{1}{4} [\mathbf{Y}^i, \mathbf{Y}^j][\mathbf{Y}^i, \mathbf{Y}^j] \right). \quad (4.4)$$

The Matrix theory and supergravity predictions for the potential agree when

$$W = 96 \frac{R^2}{N} E^2. \quad (4.5)$$

Here  $W$  is to be constructed just from the matrix variables  $\mathbf{Y}_i$  describing the first object (*i.e.*  $F_{0i} = \dot{\mathbf{Y}}_i$  and  $F_{ij} = i[\mathbf{Y}_i, \mathbf{Y}_j]$  in (3.5)).

The second case is where both objects are massive. We take both of the objects to be at transverse rest. Then the Matrix and supergravity potentials agree when

$$W = 96 \frac{R^2}{N\tilde{N}} \left[ (\tilde{N}E)^2 + (N\tilde{E})^2 + \frac{14}{9}(\tilde{N}E)(N\tilde{E}) \right] \quad (4.6)$$

where  $W$  is constructed from  $K_i = \mathbf{Y}_i \otimes \mathbb{1}_{\tilde{N} \times \tilde{N}} - \mathbb{1}_{N \times N} \otimes \tilde{\mathbf{Y}}_i^T$ ,  $\mathbf{Y}_i$  and  $\tilde{\mathbf{Y}}_i$  are matrices describing the two objects, and the Matrix energies  $E, \tilde{E}$  are given by (4.4) evaluated on  $\mathbf{Y}_i, \tilde{\mathbf{Y}}_i$  respectively.

We emphasize that we are comparing the Matrix theory potential (3.4) to the gravitational potential (4.2) which we *evaluate using the Matrix expressions for  $p$  and  $\tilde{p}$* . This means we never have to specify a precise correspondence between a state in Matrix theory and a state in supergravity. This is fortunate because, as we discussed in section 2.2, these identifications are afflicted with  $1/N$  ambiguities. Instead, we can test whether the expected relationship (4.2) holds internally within Matrix theory for any  $N$ .

An important subtlety arises in verifying the relations (4.5) and (4.6) which express the correspondence between the Matrix and supergravity potentials. For many membrane configurations with nontrivial time dependence, the Matrix theory potential is time-dependent. Generally, this time-dependence arises from the outgoing gravitational radiation produced by the fluctuating membrane source(s). In order to compare with the supergravity calculation, where we assumed a stress-energy tensor of the form (4.1), we must neglect gravitational radiation. This can be done in Matrix theory by simply time-averaging the expression for  $W$  in (3.5). In the remainder of this paper we will time-average all Matrix potentials as needed in order to remove radiation effects. A more complete discussion of gravitational radiation in Matrix theory will appear in [41].

To summarize the discussion, in order to check that a single object described by matrices  $\mathbf{Y}_i$  has the proper interaction with a graviton it suffices to verify the matrix relation

$$\langle W \rangle = 96 \frac{R^2}{N} E^2. \quad (4.7)$$

where  $\langle \cdot \rangle$  denotes a time average. Similarly, to verify that two massive objects described by matrices  $\mathbf{Y}_i, \tilde{\mathbf{Y}}_i$  have the proper long-range potential it suffices to verify the relation

$$\langle W \rangle = 96 \frac{R^2}{N\tilde{N}} \left[ (\tilde{N}E)^2 + (N\tilde{E})^2 + \frac{14}{9}(\tilde{N}E)(N\tilde{E}) \right]. \quad (4.8)$$

The rest of this paper is devoted to checking these relations. In section 4.2 we consider a simple example: a finite  $N$  sphere interacting with a graviton.



We will find that the leading large- $N$  potentials agree, but that there is a discrepancy at subleading orders in  $N$ . Then, in section 4.3, we prove a general result that (4.7) always holds in a formal large- $N$  limit. In section 4.4 we show that, even if  $N$  is finite, any two objects which separately satisfy (4.7) will jointly satisfy (4.8), as long as the internal dynamics of the two objects are not correlated over time.

## 4.2 Sphere-Graviton Interaction

The first thing we will examine is the long-range potential between a finite- $N$  spherical membrane and a graviton. As in section 2 we describe the sphere by matrices

$$\mathbf{Y}_i(t) = \frac{2r(t)}{N} J_i, \quad i \in \{1, 2, 3\}.$$

The corresponding background field strengths are

$$F_{0i} = \frac{2\dot{r}}{N} J_i, \quad F_{ij} = -\frac{4r^2}{N^2} \epsilon_{ijk} J_k.$$

The Matrix energy of the sphere is given by (4.4).

$$\begin{aligned} E &= \frac{1}{R} \left( \frac{2\dot{r}^2}{N^2} + \frac{8r^4}{N^4} \right) \text{Tr} (J_i J_i) \\ &= \frac{8r_0^4}{RN^3} c_2. \end{aligned}$$

Here  $c_2 = \frac{N^2-1}{4}$  is the quadratic Casimir, and  $r_0$  is the initial radius of the sphere:  $\dot{r}^2 + \frac{4}{N^2} r^4 = \frac{4}{N^2} r_0^4$ . Thus from (4.3) we expect the gravitational potential

$$V_{\text{gravity}} = -240 \frac{r_0^8}{N^7 b^7} c_2^2.$$

On the other hand we can compute the one-loop Matrix potential. Substituting the field strengths into (3.5) we find that the coupling  $W$  is given by

$$\begin{aligned} W &= \frac{2^{12} r_0^8}{N^8} \left[ \text{Tr} (J_i J_i J_j J_j) + \frac{1}{2} \text{Tr} (J_i J_j J_i J_j) \right] \\ &= \frac{2^{12} r_0^8}{N^8} \left( \frac{3}{2} N c_2^2 - \frac{1}{2} N c_2 \right) \end{aligned}$$

and thus from (3.4) that the Matrix potential is

$$V_{\text{matrix}} = -240 \frac{r_0^8}{N^7 b^7} \left( c_2^2 - \frac{1}{3} c_2 \right).$$

Since  $c_2$  is order  $N^2$ , we see that the large- $N$  behavior of the two potentials is identical. This supports the BFSS conjecture that the large- $N$  limit of Matrix quantum mechanics reproduces uncompactified M-theory and its low-energy limit, 11-dimensional supergravity.

Following the discrete light-front conjecture [28], however, we might also expect agreement at finite  $N$ . This is evidently not the case. The ordering of the matrices in the expression for the Matrix potential (3.4) gives rise to a term, involving the commutator of two field strengths, which is subleading in  $N$ . But this term is not present in the naive supergravity potential (4.2).

It seems unlikely that our lowest-order results will be corrected in a way that resolves this discrepancy. For example, there are higher-derivative corrections to 11-dimensional supergravity, which modify the graviton propagator. But such modifications produce effects which fall off more rapidly than  $1/b^7$ . Likewise, there are loop corrections in the Matrix gauge theory. But such corrections will bring in additional powers of the dimensionless quantities  $g_{YM}^2/b^3$  and  $g_{YM}^2/r^3$ .

Very convincing arguments have been advanced that finite- $N$  Matrix theory provides a description of DLCQ M-theory [24, 42]. So a plausible resolution seems to be that the DLCQ of M-theory is simply not described at low energies by the DLCQ of 11-dimensional supergravity [25, 29]. In particular, there is no reason to believe that DLCQ M-theory should have a low energy effective Lagrangian which is Lorentz invariant, or even local.

### 4.3 Large $N$ Interactions with a Graviton

We will now prove a general theorem stating that an arbitrary finite-size classical membrane configuration in Matrix theory has the correct long-distance interactions with a graviton at large  $N$ . We proceed as follows: Our goal is to establish that (4.7) holds at large  $N$ .<sup>5</sup> We do this formally, in two steps. First, we invert the correspondence between membranes and matrices (2.1).

$$\{\cdot, \cdot\} \leftrightarrow \frac{-iN}{2}[\cdot, \cdot] \quad \frac{N}{4\pi} \int d^2\sigma \leftrightarrow \text{Tr}$$

Then we make use of the classical membrane equations of motion discussed in section 2 to simplify the resulting expressions. Of course these formal manipulations are only valid on a limited class of matrices, namely those which converge to a smooth membrane in the large- $N$  limit. We leave it to future work to make the proof more general.

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<sup>5</sup>This procedure amounts to taking the limit of large separation first, then the limit of large  $N$ . It is a subtle question whether the opposite order of limits, which is what the IMF proposal of BFSS [1] calls for, will give the same result. We thank Tom Banks for emphasizing this point to us.

Translating back into membrane language, in the expression for  $W$  (3.5) we replace the matrices  $\mathbf{Y}_i(t)$  with functions  $Y_i(t, \sigma^a)$ , and identify

$$F_{0i} = \dot{Y}_i \quad F_{ij} = -\frac{2}{N}\{Y_i, Y_j\}.$$

The resulting expression simplifies, with the use of the identity

$$\{Y_i, Y_j\}\{Y_j, Y_k\} = -\gamma\gamma^{ab}\partial_a Y_i \partial_b Y_k,$$

to

$$\begin{aligned} W &= \frac{N}{4\pi} \int d^2\sigma \left( 24 \dot{Y}_i \dot{Y}_i \dot{Y}_j \dot{Y}_j + \frac{192}{N^2} \gamma \dot{Y}_i \dot{Y}_i + \frac{384}{N^4} \gamma^2 \right. \\ &\quad \left. - \frac{384}{N^2} \gamma \gamma^{ab} \dot{Y}_i (\partial_a Y_i) \dot{Y}_j (\partial_b Y_j) \right) \\ &= \frac{N}{4\pi} \int d^2\sigma \left( 96 (\dot{Y}^-)^2 - \frac{384}{N^2} \gamma \gamma^{ab} \partial_a Y^- \partial_b Y^- \right) \end{aligned} \quad (4.9)$$

where in the second line we made use of the constraint equations for  $Y^-$ . The constraints on  $Y^-$  also imply that

$$Y^-(t, \sigma^a) = \frac{R}{N} E t + \xi(t, \sigma^a)$$

where  $E$  is the classical light-front energy of the object, and the fluctuation satisfies  $\int d^2\sigma \xi = 0$ . In terms of  $\xi$  we have

$$W = \frac{N}{4\pi} \int d^2\sigma \left( 96 \frac{R^2}{N^2} E^2 + 96 \frac{\partial}{\partial t} (\xi \dot{\xi}) - 96 \xi \left[ \ddot{Y}^- - \frac{4}{N^2} \partial_a (\gamma \gamma^{ab} \partial_b Y^-) \right] \right).$$

The final term in  $W$  vanishes by the constraint equations for  $Y^-$ . As discussed in Section 4.1, time-dependent terms in  $W$  correspond to gravitational radiation. We wish to drop such terms in checking (4.7) to compare with supergravity. We remove radiation effects by integrating over time. The fluctuation  $\xi$  is of bounded variation, so the term  $\partial_t (\xi \dot{\xi})$  will not contribute when integrated over time, and can be dropped. Thus we have shown that after time-averaging

$$\langle W \rangle = 96 \frac{R^2}{N} E^2$$

which establishes (4.7).

#### 4.4 Interactions Between Massive Objects

We now prove the general result that any two objects which separately have the correct long-range potentials with a graviton in Matrix theory, also have the correct long-range potential between themselves. Mathematically, this amounts to proving that if two sets of matrices  $\mathbf{Y}_i, \tilde{\mathbf{Y}}_i$  each independently satisfy (4.7) then the two sets of matrices also satisfy (4.8). In order to prove this theorem, we will time-average the Matrix expression separately on each of the two sets of matrix variables. In the Matrix theory picture this corresponds to the assumption that the internal degrees of freedom describing the two objects are not correlated in time so that

$$\langle f(\mathbf{Y})g(\tilde{\mathbf{Y}}) \rangle = \langle f(\mathbf{Y}) \rangle \cdot \langle g(\tilde{\mathbf{Y}}) \rangle$$

where the brackets denote time-averaging. In general, this step is justified unless there are correlations between the dynamics of the two Matrix objects. Generically, there will be no such correlations. The behavior of the Matrix theory variables describing each object are classically chaotic [43]. It is possible to construct configurations with correlated dynamics, such as a pair of identical spheres. This corresponds physically to a pair of pulsating spherical membranes whose periods of pulsation are equal. Since the gravitational radiation from each membrane will have the same period, the interaction between each sphere and the radiation field of the other will not average to zero over time, so the gravitational calculation is more complicated.

We now proceed with the proof in the case where the objects are uncorrelated. The Matrix gravitational coupling  $W$  between the two objects can be broken into three terms, corresponding to the three terms on the RHS of (4.8). The terms which depend only on  $\mathbf{Y}_i$  or only on  $\tilde{\mathbf{Y}}_i$  are correctly reproduced, as follows directly from (4.7). The only term which needs to be checked in detail is the cross term containing matrices of both types. Expanding out the terms in this part of  $W$  we find

$$\begin{aligned} W_{\text{cross}} = \text{Tr} \left\{ 48 \dot{\mathbf{Y}}_i \dot{\mathbf{Y}}_i \dot{\tilde{\mathbf{Y}}}_j \dot{\tilde{\mathbf{Y}}}_j + 96 \dot{\mathbf{Y}}_i \dot{\mathbf{Y}}_j \dot{\tilde{\mathbf{Y}}}_i \dot{\tilde{\mathbf{Y}}}_j + 24 \dot{\mathbf{Y}}_i \dot{\mathbf{Y}}_i \tilde{F}_{jk} \tilde{F}_{jk} \right. \\ \left. + 24 F_{jk} F_{jk} \dot{\tilde{\mathbf{Y}}}_i \dot{\tilde{\mathbf{Y}}}_i + 96 \dot{\mathbf{Y}}_i \dot{\mathbf{Y}}_j \tilde{F}_{ik} \tilde{F}_{kj} + 96 F_{ik} F_{kj} \dot{\tilde{\mathbf{Y}}}_i \dot{\tilde{\mathbf{Y}}}_j \right. \\ \left. + 96 F_{ij} F_{jk} \tilde{F}_{kl} \tilde{F}_{li} - 12 F_{ij} F_{ij} \tilde{F}_{kl} \tilde{F}_{kl} \right\} \quad (4.10) \end{aligned}$$

where we have abbreviated  $F_{ij} = i[\mathbf{Y}_i, \mathbf{Y}_j]$ . In writing (4.10) we have already made several simplifications. A term proportional to  $\dot{\mathbf{Y}}_i F_{ij}$  has been dropped by the Gauss constraint (2.6):

$$\text{Tr} \left( \dot{\mathbf{Y}}_i [\mathbf{Y}_i, \mathbf{Y}_j] \right) = \text{Tr} \left( [[\dot{\mathbf{Y}}_i, \mathbf{Y}_i] \mathbf{Y}_j] \right) = 0.$$

A term proportional to  $\dot{\mathbf{Y}}_i F_{jk}$  has been dropped because it is a total time derivative:

$$\langle \text{Tr} \left( \dot{\mathbf{Y}}_i F_{jk} + \dot{\mathbf{Y}}_j F_{ki} + \dot{\mathbf{Y}}_k F_{ij} \right) \rangle = \langle i \frac{d}{dt} \text{Tr} \left( \mathbf{Y}_i [\mathbf{Y}_j, \mathbf{Y}_k] \right) \rangle \approx 0.$$

Finally a term proportional to  $F_{ij} F_{kl}$  has been dropped because it vanishes by the Jacobi identity:

$$\text{Tr} (F_{ij} F_{kl} + F_{ik} F_{lj} + F_{il} F_{jk}) = -\text{Tr} (\mathbf{Y}_i ([\mathbf{Y}_j, [\mathbf{Y}_k, \mathbf{Y}_l]] + \text{cyclic})) = 0.$$

We can use the following relations (and their counterparts for the  $\tilde{\mathbf{Y}}$  variables) to integrate by parts on the two separate membranes (this has no effect on the time-averaged quantities).

$$\frac{d}{dt} \text{Tr} (\mathbf{Y}_i \dot{\mathbf{Y}}_i) = \text{Tr} \left( \dot{\mathbf{Y}}_i^2 - \mathbf{Y}_i [[\mathbf{Y}_i, \mathbf{Y}_j], \mathbf{Y}_j] \right) = \text{Tr} \left( \dot{\mathbf{Y}}_i^2 - F_{ij}^2 \right) \approx 0 \quad (4.11)$$

$$\frac{d}{dt} \text{Tr} (\mathbf{Y}_i \dot{\mathbf{Y}}_j) = \text{Tr} \left( \dot{\mathbf{Y}}_i \dot{\mathbf{Y}}_j - \mathbf{Y}_i [[\mathbf{Y}_j, \mathbf{Y}_k], \mathbf{Y}_k] \right) = \text{Tr} \left( \dot{\mathbf{Y}}_i \dot{\mathbf{Y}}_j + F_{ik} F_{kj} \right) \approx 0 \quad (4.12)$$

From (4.12) we have

$$\langle \text{Tr} \left\{ \dot{\mathbf{Y}}_i \dot{\mathbf{Y}}_j \dot{\tilde{\mathbf{Y}}}_i \dot{\tilde{\mathbf{Y}}}_j + \dot{\mathbf{Y}}_i \dot{\mathbf{Y}}_j \tilde{F}_{ik} \tilde{F}_{kj} + F_{ik} F_{kj} \dot{\tilde{\mathbf{Y}}}_i \dot{\tilde{\mathbf{Y}}}_j + F_{ij} F_{jk} \tilde{F}_{kl} \tilde{F}_{li} \right\} \rangle = 0$$

so

$$\langle W_{\text{cross}} \rangle = \text{Tr} \left\{ 48 \dot{\mathbf{Y}}_i \dot{\mathbf{Y}}_i \dot{\tilde{\mathbf{Y}}}_j \dot{\tilde{\mathbf{Y}}}_j + 24 \dot{\mathbf{Y}}_i \dot{\mathbf{Y}}_i \tilde{F}_{jk} \tilde{F}_{jk} + 24 F_{jk} F_{jk} \dot{\tilde{\mathbf{Y}}}_i \dot{\tilde{\mathbf{Y}}}_i - 12 F_{ij} F_{ij} \tilde{F}_{kl} \tilde{F}_{kl} \right\}.$$

Applying (4.11) a number of times, we find

$$\begin{aligned} \langle W_{\text{cross}} \rangle &= \frac{7}{9} \text{Tr} \left\{ 48 \dot{\mathbf{Y}}_i \dot{\mathbf{Y}}_i \dot{\tilde{\mathbf{Y}}}_j \dot{\tilde{\mathbf{Y}}}_j + 24 \dot{\mathbf{Y}}_i \dot{\mathbf{Y}}_i \tilde{F}_{jk} \tilde{F}_{jk} + 24 F_{jk} F_{jk} \dot{\tilde{\mathbf{Y}}}_i \dot{\tilde{\mathbf{Y}}}_i \right. \\ &\quad \left. + 12 F_{ij} F_{ij} \tilde{F}_{kl} \tilde{F}_{kl} \right\} \\ &= \frac{96 \cdot 14}{9} \text{Tr} \left\{ \left( \frac{1}{2} \dot{\mathbf{Y}}_i^2 + \frac{1}{4} F_{ij}^2 \right) \cdot \left( \frac{1}{2} \dot{\tilde{\mathbf{Y}}}_k^2 + \frac{1}{4} \tilde{F}_{kl}^2 \right) \right\}. \end{aligned}$$

Comparing this to (4.8) we see that the theorem has been proven. Thus, we have shown that any two finite size matrix objects which have the correct leading long-range interaction with a graviton have the correct supergravity interaction with one another. Note that unlike the result in the previous subsection, this proof did not depend upon the matrix-membrane correspondence and therefore is valid for any  $N$ .

## 5 Conclusions

In this paper we have discussed a number of aspects of membrane solutions of Matrix theory. We showed that, in appropriate regimes, finite- $N$  Matrix theory has states which behave as semiclassical membranes. We calculated the potential between two objects, and established a general pair of theorems which show that in the large  $N$  limit the leading long-range force between any pair of classical Matrix states depends only upon the energies of the states, after radiation effects are averaged out. This is strong evidence for the conjecture that large- $N$  Matrix theory reproduces supergravity. In particular, it seems that the equivalence principle follows from algebraic manipulations in Matrix theory. At finite  $N$ , on the other hand, we found that the Matrix potential does not agree with the naive supergravity potential. This suggests that the low energy effective action of DLCQ M-theory is not given by DLCQ supergravity.

In this paper we only considered one-loop effects in Matrix theory. Recently, a number of papers have considered two-loop effects [16, 18–20]; it would be interesting to see how the results in this paper are modified at higher order.

There are a number of interesting directions in which the work in this paper might lead. Perhaps the most interesting questions about Matrix membranes involve the fate of a membrane as it shrinks. Work is in progress to give a more detailed description of this phenomenon; however, we give here a brief qualitative picture of the story. A very large spherical membrane is well described by a semiclassical large  $N$  Matrix theory configuration such as those we have considered in this paper. As the membrane begins to contract, it will emit quadrupole radiation since it is invariant only under an  $O(3)$  subgroup of the full  $O(10)$  rotation symmetry. It is possible to study this radiation quantitatively in the Matrix theory picture [41]. When the membrane contracts still further, it will eventually approach its Schwarzschild radius. At this point, from the supergravity point of view we would expect the membrane to become a black hole, and to see Hawking radiation start to emerge. Qualitatively, it is clear that this does correspond to what happens in the Matrix picture. The initial membrane state corresponds to a superposition of eigenstates of the Hamiltonian for  $N$  0-branes. If the initial membrane configuration is sufficiently large, a wavefunction localized around a classical configuration will have essentially zero overlap with any state in which a majority of the 0-branes are bound together. Thus, as the quantum state evolves over time in Matrix theory we expect that gravitons should emerge sporadically until the state has completely evaporated. It would be a significant coup for Matrix theory if it could be shown that the rate of graviton emission from such a state corresponds to the expected rate

of Hawking radiation. Recent works on Schwarzschild black holes in Matrix theory [36–38, 44–46] have focused on verifying the energy-entropy relation using thermodynamic or mean field arguments for Yang-Mills field theory and quantum mechanics. (Earlier work on non-Schwarzschild black holes appeared in [47–49]). The states we consider here provide a natural framework for a more detailed analysis of how these arguments work in noncompact space. In particular, in [36, 37] the thermodynamic analysis was performed by considering fluctuations around a state described by  $N$  0-branes uniformly distributed over a compactification torus. The spherical state we consider, which has 0-branes uniformly distributed over its surface, forms a natural background for an analogous calculation. In this background, the 0-branes are “tethered” to the membrane; as discussed in [46], this effectively distinguishes the degrees of freedom of the individual 0-branes, leading to the correct black hole entropy formula. A related picture was discussed in [38], where it was suggested that a black hole could be formed by assembling a large number of independent membrane configurations. The arguments of that paper seem to apply equally well to the situation considered here where a single collapsing membrane forms a black hole. Thus, at this point it seems that there are several related arguments indicating that the entropy-energy relations of a black hole are correctly reproduced for a gas of 0-branes attached to a classical configuration such as we consider here. The outstanding challenge is clearly to derive the correct rate of Hawking radiation from such a state.

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