Adv. Theor. Math. Phys. **2** (1998) 155–180

On Inner Product in Modular Tensor Categories II: Inner Product on Conformal Blocks and Affine Inner Product Identities[1,2]

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Abstract

This is the second part of the paper (the first part is published in *Journal of AMS* **9** 1135). In the first part, we defined for every modular tensor category (MTC) inner products on the spaces of morphisms and proved that the inner product on the space $\mathrm{Hom}(\bigoplus X_i \otimes X_i^*, U)$ is modular invariant. Also, we have shown that in the case of the MTC arising from the representations of the quantum group $U_q\mathfrak{sl}_n$ at roots of unity and U being a symmetric power of the fundamental representation, this inner product coincides with so-called Macdonald's inner product on symmetric polynomials. In this paper, we apply the same construction to the MTC coming from the integrable representations of affine Lie algebras. In this case our construction immediately gives a hermitian form on the spaces of conformal blocks, and this form is modular invariant (Warning: we cannot prove that it is positive definite). We show that this form can be rewritten in terms of asymptotics

[1]1991 *Mathematics Subject Classification.* Primary 81R50, 05E35, 18D10; Secondary 57M99.

[2]*Key words and phrases.* Modular tensor categories, conformal field theory, conformal blocks, inner product identities.

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[4]The author was supported in part by the NSF grant DMS-9610201.

of KZ equations, and calculate it for sl_2, in which case the formula is a natural affine analogue of Macdonald's inner product identities. We also formulate as a conjecture similar formula for sl_n.

Introduction

This is the second part of the paper [1], and we freely use the notations from there. In [1], we defined for every modular tensor category (MTC) inner products on the spaces of morphisms and proved that the inner product on the space $\mathrm{Hom}(\bigoplus X_i \otimes X_i^*, U)$ is modular invariant. Also, we have shown that in the case of the MTC arising from the representations of the quantum group $U_q\mathfrak{sl}_n$ at roots of unity and U being a symmetric power of the fundamental representation, the action of modular group on the space of intertwiners $\mathrm{Hom}(\bigoplus X_i \otimes X_i^*, U)$ can be written explicitly in terms of Macdonald's polynomials at roots of unity.

In this paper, we apply the same construction to the MTC $\mathcal{O}_\varkappa^{int}$ coming from the integrable representations of affine Lie algebras or, equivalently, from Wess-Zumino-Witten model of conformal field theory. In Sections 8, 9 we briefly recall the construction of this category, first suggested by Moore and Seiberg (see [2,3]) and later refined by Kazhdan and Lusztig [4–7] and Finkelberg [8]. In particular, spaces of morphisms in this category are the spaces of conformal blocks of the WZW model. We prove in Section 10 that $\mathcal{O}_\varkappa^{int}$ is hermitian, i.e. can be endowed with a suitable complex conjugation. Thus, the general theory developed in Section 2 of [1] gives us an nondegenrate inner product on the spaces of conformal blocks, and so defined inner product is modular invariant. We show in Section 11 that this definition of the inner product is constructive: it can be rewritten so that it only involves Drinfeld associator, or, equivalently, asymptotics of solutions of the Knizhnik-Zamolodchikov equations.

We conjecture that this inner product is positive definite. In the $\mathfrak{sl}_2$ case, it can be proved using explicit formulas (see below); in general, this seems to be a very hard problem. Our motivation for this conjecture comes from physics, since the spaces of conformal blocks are state spaces for the Chern-Simons theory and thus must carry a positive definite inner product.

Since there are integral formulas for the solutions of the KZ equations ([9]), this shows that the inner product on the space of conformal blocks can be written in terms of asymptotics of certain integrals. In the case $\mathfrak{g} = \mathfrak{sl}_2$ these asymptotics can be calculated (see [10]), using Selberg integral, and the answer is given by certain products of Γ-functions. Thus, in this case we can write explicit formulas for the inner product on the space of conformal blocks; we do it in Section 12. These expressions are closely related with those suggested by Gawedzki et al. ([11, 12]), though their approach is completely different from ours. There is little doubt that the same correlation holds for arbitrary Lie algebras.

The norms of intertwiners for $\widehat{\mathfrak{sl}_2}$ are given by a formula which is very much similar to the Macdonald inner product formula, which gives the norms of intertwining operators for $U_q\mathfrak{sl}_2$ (see [1]). Motivated by this, we formulate a conjecture about the norms of the intertwining operators (in some special cases, which are related with Macdonald's theory) for $\widehat{\mathfrak{sl}_n}$.

It was proved recently in [4–8] (though this statement was widely believed long before) that as a braided category $\mathcal{O}^{int}_{\varkappa}$ is equivalent to the reduced category of representations of $U_q\mathfrak{g}$ with q being a root of unity, which was discussed in the first part of this paper. Since in both categories the action of modular group is written in terms of braiding, they are also equivalent as modular tensor categories. In particular, this implies that if we let $\mathfrak{g} = \mathfrak{sl}_n$ and consider the space of conformal blocks on the torus with one puncture, to which a symmetric power of fundamental representation is assigned, then in some basis the action of the modular group on this space is given by special values of Macdonald's polynomials. Thus, for $\mathfrak{g} = \mathfrak{sl}_2$ the action of modular group on any space of conformal blocks on a torus can be written in some basis in terms of q-ultraspherical polynomials, and this basis is related to the standard one by a matrix of gamma-functions.

8 Integrable Representations of Affine Lie Algebras and Conformal Blocks

Here we briefly recall the main definitions, referring the reader to [2,4–7,13] for details. This section is completely expository.

8.1 Integrable Modules

As before, let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ with the invariant bilinear form $(,)$ normalized so that $(\theta,\theta) = 2$ where θ is the highest root. Let $\tilde{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ be the corresponding affine Lie algebra; as usual, we denote $x[n] = x \otimes t^n$. We also denote by $\hat{\mathfrak{g}}$ its completion: $\hat{\mathfrak{g}} = \mathfrak{g} \otimes \mathbb{C}((t)) \oplus \mathbb{C}c$. In this whole paper, we fix a non-negative integer k (level). Let $\varkappa = k + h^\vee$, and let $\mathcal{O}^{int}_{\varkappa}$ be the category of finite-length integrable highest-weight $\hat{\mathfrak{g}}$-modules of level k. This category is semisimple and the simple objects in this category are the irreducible modules $L_{\lambda,k}, \lambda \in C$, where

$$C = \{\lambda \in P^+ | \langle \lambda, \theta^\vee \rangle \leq k\} = \{\lambda \in P^+ | \langle \lambda + \rho, \theta^\vee \rangle < \varkappa\}, \tag{8.1}$$

(see [14]).

There exists a natural notion of duality in this category: for $V \in \mathcal{O}_{\varkappa}^{int}$, let DV be the restricted dual space to V (that is, DV is direct sum of spaces dual to weight subspaces of V) and the action of $\tilde{\mathfrak{g}}$ is defined as the usual action in the dual space twisted by the automorphism $\sharp$ defined by

$$x[n]^{\sharp} = (-1)^n x[-n], \quad c^{\sharp} = -c.$$

It is easy to check that $DV \in \mathcal{O}_{\varkappa}^{int}$, and $D(DV)$ is canonically isomorphic to V: the usual isomorphism of vector spaces $V^{**} \simeq V$ is $\hat{\mathfrak{g}}$-isomorphism. Also, $DL_{\lambda,k} \simeq L_{\lambda^*,k}$, though the isomorphism is not canonical.

8.2 Conformal Blocks

Let $\mathcal{X}$ denote the following collection of data:

(1) X – a non-singular compact complex curve,

(2) $z_1, \ldots, z_n$ – distinct point on X divided into two sets In and Out,

(3) w_i – local parameter near the point z_i. i.e. a holomorphic function in a neighborhood of z_i such that $w_i(z_i) = 0, w_i'(z_i) \neq 0$.

We assume that on each connected component of X there is at least one of the points z_i.

With each point z_i we associate a Lie algebra $\mathfrak{g} \otimes \mathbb{C}((w_i))$. Let

$$\begin{aligned} \hat{\mathfrak{g}}_{in} &= \left(\bigoplus_{i \in In} \mathfrak{g} \otimes \mathbb{C}((w_i)) \right) \oplus \mathbb{C}c \\ \hat{\mathfrak{g}}_{out} &= \left(\bigoplus_{i \in Out} \mathfrak{g} \otimes \mathbb{C}((w_i)) \right) \oplus \mathbb{C}c. \end{aligned} \tag{8.2}$$

and the cocycle defining the central extension in $\hat{\mathfrak{g}}_{in}$ (respectively, $\hat{\mathfrak{g}}_{out}$) is the sum of standard cocycles on each of $\mathfrak{g} \otimes \mathbb{C}((w_i)), i \in In$ (respectively, $i \in Out$).

Also, let us consider the Lie algebra Γ:

$$\Gamma(\mathcal{X}) = \{\mathfrak{g}\text{-valued meromorphic functions on } X \text{ regular outside of } z_1, \ldots, z_n\} \tag{8.3}$$

and its central extension $\widehat{\Gamma} = \Gamma \oplus \mathbb{C}c$ with the defining cocycle given by

$$c(f,g) = \sum_{i \in In} \operatorname{Res}_{z_i}(g, df).$$

Expanding a function $f \in \Gamma$ near a point z_i in a Laurent series in w_i we get a Lie algebra homomorphism $\pi_i : \Gamma(\mathcal{X}) \to \mathfrak{g}\otimes\mathbb{C}((w_i))$ (if $\mathcal{X}$ is connected, this is an embedding).

Taking direct sum over all $i \in In$ (respectively, $i \in Out$), we get embeddings

$$\begin{aligned} \pi_{in} : \widehat{\Gamma}(\mathcal{X}) \subset \hat{\mathfrak{g}}_{in} : \qquad & f \mapsto \oplus_{i\in In}\pi_i(f), \qquad c \mapsto c, \\ \pi_{out} : \widehat{\Gamma}(\mathcal{X}) \subset \hat{\mathfrak{g}}_{out} : \qquad & f \mapsto \oplus_{i\in Out}\pi_i(f), \qquad c \mapsto -c. \end{aligned} \tag{8.4}$$

One can easily check that these embeddings are Lie algebra homomorphisms.

Definition 8.1. Let $\mathcal{X}$ be as above, and assume that we are given integrable modules $V_1, \dots, V_n \in \mathcal{O}^{int}_{\varkappa}$ assigned to the points $z_1, \dots, z_n$ respectively. Let us consider V_i as a module over $\mathfrak{g} \otimes \mathbb{C}((w_i)) \oplus \mathbb{C}c$. Then the corresponding space of conformal blocks is defined by

$$\begin{aligned} W(\mathcal{X}; V_1, \dots, V_n) = \{\Phi : \otimes_{i\in In} V_i \to \widehat{\otimes}_{i\in Out} V_i \big| (\pi_{out}(f))^{\sharp}\Phi = \Phi\pi_{in}(f) \\ \text{for all } f \in \Gamma(\mathcal{X})\} \end{aligned} \tag{8.5}$$

where $\widehat{\otimes}$ is the completion of the tensor product with respect to the homogeneous grading.

Remark 8.2. If $i \in Out$ then denote by $\mathcal{X}'$ the same data as $\mathcal{X}$ except that now we consider i as an element of In: $In' = In \cup \{i\}, Out' = Out \setminus \{i\}$. Then we have a canonical isomorphism:

$$W(\mathcal{X}; V_1, \dots, V_n) = W(\mathcal{X}'; V_1, \dots, DV_i, \dots V_n).$$

Thus, it suffices to consider conformal blocks when all points are incoming (or all are outgoing), as is done in [13]; however, it is more convenient to consider the general situation.

In a similar way, it can be shown that if $\mathcal{X}'$ is obtained from $\mathcal{X}$ by marking one more incoming point z_0 and assigning to it the representation $V_0 = L_{0,k}$ then the map

$$\begin{aligned} W(\mathcal{X}'; V_0, V_1, \dots, V_n) \to W(\mathcal{X}; V_1, \dots, V_n) \\ \Phi \mapsto \Phi(1, \dots), \end{aligned} \tag{8.6}$$

where 1 is the highest weight vector in $L_{0,k}$, is an isomorphism. Thus, the condition that there is at least one point on each connected component of X is inessential.

Remark 8.3. In fact, the definition of conformal blocks works in more general situation: we can allow X to be a semi-stable singular curve. This plays a crucial role in proving the gluing axiom ([13]). However, all of the results we need in this paper can be formulated without reference to singular curves.

Example 8.4. Let $n = 3$, $X = \mathbb{C}P^1$ with global coordinate w, $In = \{\infty\}$, $Out = \{0, z\}$ with local parameters $1/w$, $-w$, $z - w$ respectively. Let $\Phi : V_\infty \to V_0 \hat{\otimes} V_z$ be an element of the space of conformal blocks $W(\mathcal{X}; V_0, V_z, V_\infty)$. Then

$$\Phi x[n] = \left(x[n] \otimes 1 + \sum_{i \geq 0} \binom{-n}{i} z^{-n-i} 1 \otimes x[-i] \right) \Phi.$$

This slightly differs from the usual formulas in the physical literature where usually ∞ is considered as an outgoing and $0, z$ as incoming points.

The following result is well known; we refer the reader to [13] for the proof.

Proposition 8.5. *The spaces of conformal blocks are always finite-dimensional.*

8.3 Correlation Functions

For a simple module $L_{\lambda,k}$ let $L_{\lambda,k}[0]$ be the $\mathfrak{g}$-module generated by the highest weight vector; clearly, $L_{\lambda,k}[0] \simeq L_\lambda$. Since every object in $\mathcal{O}^{int}_{\varkappa}$ is isomorphic to a direct sum of $L_{\lambda,k}$, we can extend it to a map $V \mapsto V[0]$ from $\mathcal{O}^{int}_{\varkappa}$ to the category *Rep* $\mathfrak{g}$ of finite-dimensional representations of $\mathfrak{g}$. It is easy to show that this operation can be defined in invariant terms and thus, this map is a faithful functor. Note that we have canonical identifications $(DV)[0] \simeq (V[0])^*, L_{0,k}[0] \simeq \mathbb{C}$. Denote by $i : V[0] \to V$ and $p : V \to V[0]$ the canonical embedding and projection respectively.

For every $\Phi \in W(\mathcal{X}; V_1, \dots, V_n)$ define its correlation function (or, which is the same, its highest term) $\langle \Phi \rangle$ by

$$\langle \Phi \rangle = p \otimes \dots \otimes p \circ \Phi \circ (i \otimes \dots \otimes i) \in \mathrm{Hom}_{\mathfrak{g}}(\otimes_{In} V_i[0], \otimes_{Out} V_i[0]). \quad (8.7)$$

Proposition 8.6. *On a sphere, every $\Phi \in W(\mathcal{X}, V_1, \dots, V_n)$ is uniquely defined by its correlation function $\langle \Phi \rangle$. In other words, we have an embedding:*

$$\begin{aligned} W(\mathcal{X}; V_1, \dots, V_n) \subset \mathrm{Hom}_{\mathfrak{g}}(\bigotimes_{In} V_i[0], \bigotimes_{Out} V_i[0]) \\ \Phi \mapsto \langle \Phi \rangle. \end{aligned} \quad (8.8)$$

Remark. This statement is not true on higher genus surfaces.

Note that this embedding depends on the choice of points z_i and local parameters at these points (in fact, it depends only on 1-jet of the local parameter). For irreducible V_i the image of this embedding can be explicitly described; we refer the reader to [15, 16] (for the case of three-punctured sphere, this description also appears in [13]).

8.4 Flat Connection

The definition of conformal blocks which we gave depends on $\mathcal{X}$; thus, we can consider the space of conformal blocks as a finite-dimensional vector bundle over the corresponding moduli space. This vector bundle is called the bundle of conformal blocks.

It turns out that this vector bundle has a natural flat connection, which was first calculated by Knizhnik and Zamolodchikov on the sphere and by Bernard on a torus. This connection can be naturally defined using the Sugawara construction; we refer the reader to [5, 13] for details, giving here only the answer.

From now on we assume the following

$$\mathcal{X} = \begin{pmatrix} X = \mathbb{C}P^1, w - \text{global coordinate on } X \\ In = \{\infty\}, \quad w_\infty = 1/w, \\ Out = \{z_1, \dots, z_n\}, \quad w_i = z_i - w, \end{pmatrix} \tag{8.9}$$

By Proposition 8.6, for each $z_1, \dots, z_n$ the space $W(\mathcal{X}, V_1, \dots, V_n, V_\infty)$ can be identified with a certain subspace in $\mathrm{Hom}_{\mathfrak{g}}(V_\infty[0], V_1[0] \otimes \dots \otimes V_n[0])$. Thus, to define a flat connection on the space of conformal blocks it suffices to define a flat connection in the trivial vector bundle with the fiber $\mathrm{Hom}_{\mathfrak{g}}(V_\infty[0], V_1[0] \otimes \dots \otimes V_n[0]) \simeq (V_1[0] \otimes \dots \otimes V_n[0] \otimes V_\infty^*[0])^{\mathfrak{g}}$ which would preserve the subbundle of conformal blocks. Such a connection is obtained from the Knizhnik-Zamolodchikov connection on $V = V_1[0] \otimes \dots \otimes V_n[0]$

$$(k + h^\vee)\frac{\partial}{\partial z_i}\psi = \left(\sum_{\substack{j = 1 \dots n \\ j \neq i}} \frac{\Omega_{ij}}{z_i - z_j} \right) \psi, \tag{8.10}$$

where Ω is the standard $\mathfrak{g}$-invariant element in $\mathfrak{g} \otimes \mathfrak{g}$: if x_i, x^i are dual bases in $\mathfrak{g}$ with respect to the inner product $(\,,\,)$ then $\Omega = \sum x_i \otimes x^i$. As usual, we use the notation $\Omega_{ij} = \pi_i \otimes \pi_j(\Omega), i, j = 1, \dots, n$.

This connection can be extended to a connection with values in $V \otimes V_\infty^*[0]$ with trivial action on the last factor. Furthermore, since this connection commutes with the action of $\mathfrak{g}$, it also defines a connection on the trivial vector bundle with the fiber $(V_1[0] \otimes \dots \otimes V_n[0] \otimes V_\infty^*[0])^{\mathfrak{g}}$, and it can be checked that it preserves the subbundle of conformal blocks (see [15, 16]).

Example 8.7. Let us consider the conformal blocks on a 3-punctured sphere: $In = \{\infty\}, Out = \{z_1, z_2\}$ and assume that the modules V_i are irreducible: $V_\infty = L_{\lambda,k}, V_1 = L_{\mu,k}, V_2 = L_{\nu,k}$. In this case it is easy to check that a section $\Phi(z_1, z_2)$ of the bundle of conformal blocks is flat iff $\langle\Phi(z_1, z_2)\rangle = (z_1 - z_2)^{\Delta_\lambda - \Delta_\mu - \Delta_\nu} g$, where $\Delta_\lambda = \frac{(\lambda, \lambda+2\rho)}{2(k+h^\vee)}$, and $g \in \mathrm{Hom}_{\mathfrak{g}}(L_\lambda, L_\nu \otimes L_\mu)$ does not depend on z_i.

In this case we will use the notation $\Phi^g(z), z = z_1 - z_2$ for such a flat section; the operators $\Phi(z)$ are called chiral vertex operators.

9 Category $\mathcal{O}^{int}_\varkappa$ as Modular Category

In this section we recall the construction of a tensor (and in fact, modular tensor) structure on the category $\mathcal{O}^{int}_\varkappa$. This section is again expository; we refer the reader to [2–7, 17, 18] for details and proofs.

As before, let us assume that $\mathcal{X}$ is $n+1$-punctured sphere (8.8) and $n \geq 1$. The moduli space of all such punctured spheres is the configuration space

$$X_n = \{(z_1, \dots, z_n) \in \mathbb{C}^n | z_i \neq z_j\}.$$

As was discussed above, the space of conformal blocks is a local system over this space with the connection given by the KZ equations (8.9). Note also that these equations imply that every flat section of this local system is invariant under translations $(z_1, \dots, z_n) \mapsto (z_1 + c, \dots, z_n + c)$ and thus consideration of this local system is can be reduced to consideration of a local system on $X_n^0 = \{\mathbf{z} \in X_n | z_1 = 0\}$. Define

$$\mathcal{D}_n = \{(z_1, \dots, z_n) \in \mathbb{R}^n z_1 < z_2 < \dots < z_n\} \subset X_n, \tag{9.1}$$

Note that $\mathcal{D}$ is contractible.

Now we can formulate the main theorem of this section, which is essentially due to Moore and Seiberg (see also [4–7]).

Theorem 9.1. *The category $\mathcal{O}^{int}_\varkappa$ can be endowed with the structure of a ribbon tensor category such that:*

(1) *If we denote the tensor product in this category by $\dot\otimes$ (to avoid confusion with the usual tensor product of vector spaces) then for any $n \geq 1$ and for any choice of representations $V_1, \dots, V_n, V_\infty \in \mathcal{O}^{int}_\varkappa$*

$$\mathrm{Hom}_{\mathcal{O}^{int}_\varkappa}(V_\infty, V_1 \dot\otimes \dots \dot\otimes V_n) = \Gamma(\mathcal{D}_n, W), \tag{9.2}$$

where $W = W(\mathcal{X}; V_1, \dots, V_\infty)$ is the local system of conformal blocks on $n+1$-punctured sphere (8.8), and Γ stands for the space of global flat sections of this local system over $\mathcal{D}_n$ defined by (9.1).

(2) *The unit object is $\mathbf{1} = L_{0,k}$ and the maps $V \simeq \mathbf{1} \dot\otimes V$ are constructed as in Remark 8.2.*

(3) *The dual object is given by $V^* = DV$, and the maps $i_V : \mathbf{1} \to V \dot{\otimes} V^*$ are defined so that $i_{L_{\lambda,k}} = \Phi^{i_\lambda}$, where Φ^g is defined in Example 8.7, i_λ is the canonical map of $\mathfrak{g}$-modules $\mathbb{C} \to L_\lambda \otimes L_\lambda^*$.*

(4) *The isomorphism $\delta_V : V \to V^{**} = D(DV)$ coincides with canonical identification of vector spaces $V \simeq V^{**}$ (recall that as a vector space $DV = V^*$ is the restricted dual to V), and the twist θ is given by $\theta = e^{2\pi i L_0}$, where L_0 is the Sugawara element in the completion of $U\hat{\mathfrak{g}}$; thus,*

$$\theta|_{L_{\lambda,k}} = e^{2\pi i \Delta_\lambda}, \qquad \Delta_\lambda = \frac{(\lambda, \lambda + 2\rho)}{2(k + h^\vee)}. \tag{9.3}$$

Remark. Since $\mathcal{D}_n$ is simply connected, we could just say that we fix some particular choice of points $z_1, \dots, z_n$, say, $\mathbf{z} = (0, 1, \dots, n-1)$ and let $\mathrm{Hom}_{\mathcal{O}_\varkappa^{int}}(V_\infty, V_1 \otimes \dots \otimes V_n) = W(\mathcal{X}; V_1, \dots, V_\infty)$.

We do not define here the associativity and commutativity isomorphisms, referring the reader to the original papers. However, it is necessary to mention that the construction of the associativity isomorphism

$$\mathrm{Hom}_{\mathcal{O}_\varkappa^{int}}(L_{\lambda,k}, (V_1 \dot{\otimes} V_2) \dot{\otimes} V_3) \simeq \mathrm{Hom}_{\mathcal{O}_\varkappa^{int}}(L_{\lambda,k}, V_1 \dot{\otimes} (V_2 \dot{\otimes} V_3)) \tag{9.4}$$

is based on the fact that we can identify each of these spaces with the space of conformal blocks on a 4-punctured sphere (see (9.2)). The identifications are obtained by considering the asymptotics of the flat sections in different asymptotic zones. Therefore, the associtivity morphism is written in terms of asymptotics of solutions of the KZ equations in 3 variables. This implies the following lemma.

Lemma 9.2. *If $V_1, V_2, V_3 \in \mathcal{O}_\varkappa^{int}$, $\lambda \in P_+$ are such that on the space of singular vectors of weight λ in $V_1[0] \otimes V_2[0] \otimes V_3[0]$ the operators Ω_{12}, Ω_{23} commute then the associativity isomorphism (9.4) is trivial, i.e. coincides with the restriction of the associativity isomorphism for vector spaces $V_1[0], V_2[0], V_3[0]$.*

As for any MTC, we can use the language of ribbon graphs for representing morphisms in the category $\mathcal{O}_\varkappa^{int}$; unfortunately, the associativity morphism (which is highly non-trivial) "does not show" in the pictures, so one must be careful when performing calculations (see [19] for an approach allowing to avoid this difficulty). In particular, we can define the "quantum dimension" of a module V in the same way as we did for an arbitrary MTC in Section 1; we will denote it by $\dim_\varkappa V$. Note that it has nothing to do with the usual dimension of V, which is infinite.

Example 9.3. The space of morphisms $\mathrm{Hom}_{\mathcal{O}_\varkappa^{int}}(V_\infty, V_1 \dot{\otimes} V_2)$ coincides with the space of chiral vertex operators (see Examples 8.4, 8.7) and can be identified with a subspace in the space of $\mathfrak{g}$-homomorphisms: it follows from Proposition 8.6 that we have an embedding

$$\begin{aligned} \mathrm{Hom}_{\mathcal{O}_\varkappa^{int}}(V_\infty, V_1\dot{\otimes}V_2) \subset \mathrm{Hom}_{\mathfrak{g}}(V_\infty[0], V_1[0]\otimes V_2[0]) \\ \Phi^g(z) \mapsto \langle \Phi^g(1)\rangle = g. \end{aligned} \tag{9.5}$$

Now we can describe the action of the modular group. Recall the object H defined in Section 1 for any MTC; in our case, it is given by $H = \bigoplus_\lambda DL_{\lambda,k}\dot{\otimes}L_{\lambda,k}$. The following result immediately follows from the gluing axiom for conformal blocks.

Lemma 9.4. *The space*

$$\mathrm{Hom}_{\mathcal{O}_\varkappa^{int}}(H, U) = \bigoplus_{\lambda\in C} \mathrm{Hom}_{\mathcal{O}_\varkappa^{int}}(L_{\lambda,k}, L_{\lambda,k}\dot{\otimes}U) \tag{9.6}$$

is isomorphic to the space of conformal blocks on a torus with one puncture to which the representation U is assigned. (Here C is the alcove (8.1).)

The following result clarifies the meaning of the action of $SL_2(\mathbb{Z})$ introduced in Section 1.

Theorem 9.5 ([3]). *The category $\mathcal{O}_\varkappa^{int}$ is a modular category in the sense of Definition 1.3, and the action of $SL_2(\mathbb{Z})$ on $\mathrm{Hom}_{\mathcal{O}_\varkappa^{int}}(H, U)$ defined in Theorem 1.10 coincides with the natural geometric action of $SL_2(\mathbb{Z})$ on the space of conformal blocks on a torus.*

This theorem can be proved in the general context of 2-dimensional modular functor, using Kirby calculus; see [20] for this approach. In fact, it is known that in any modular tensor category we can define an action of the mapping class group of any punctured Riemann surface on the appropriate space of conformal blocks: this automatically follows from the possibility to define the action of $SL_2(\mathbb{Z})$.

9.1 Equivalence of Categories $\mathcal{O}_\varkappa^{int}$ and $\mathcal{C}(\mathfrak{g}, \varkappa)$

The following important theorem, which was widely believed for several years, was proved by M. Finkelberg in his thesis [8].

Theorem 9.6. *There exist numbers $n(\mathfrak{g})$ such that for $k \geq n(\mathfrak{g})$ the functor $V \mapsto V[0]$ is an equivalence of modular tensor categories $\mathcal{O}_\varkappa^{int}$ and the reduced category $\mathcal{C}(\mathfrak{g}, \varkappa)$ of representations of the quantum group $U_q\mathfrak{g}$ at root of unity, described in Section 3, with $\varkappa = k + h^\vee$.*

The restriction $k \geq n(\mathfrak{g})$ mentioned in the theorem appears only for exceptional root systems (see table in [8]), and for all $\mathfrak{g}$, $n(\mathfrak{g}) \leq 6$; thus, this theorem is automatically satisfied if $k \geq 6$. From now on, we assume that k is chosen to satisfy these conditions.

In particular, this theorem implies that the quantum dimensions of objects in both categories coincide:

$$\dim_\varkappa V = \dim_q V[0], \quad q = e^{\pi i/m\varkappa}.$$

Another approach to the construction of an equivalence of these categories was initiated by Schechtman and Varchenko, who showed that one can identify Verma modules over the quantum group with the homologies of certain local systems on the configuration space (see [21]). However, for rational values of $\varkappa$ this approach requires use of intersection homology (or, equivalently, perverse sheaves), which is done in a recent series of papers by Schechtman and Finkelberg [22].

10 Hermitian Structure on $\mathcal{O}_\varkappa^{int}$ and Inner Product on the Space of Conformal Blocks

In this section we define a hermitian structure (which is a certain analogue of the complex conjugation) on the category $\mathcal{O}_\varkappa^{int}$ and use it to define an inner product on the spaces of morphisms in this category.

10.1 Hermitian Structure

Recall (see Section 1) that a hermitian structure on a tensor category is a system of maps $\overline{}$: $\mathrm{Hom}(V, W) \to \mathrm{Hom}(V^*, W^*)$ which satisfies certain compatibility conditions (1.22). When the category is defined over $\mathbb{C}$ we assume that these maps are $\mathbb{C}$-antilinear. Usually, to define such a structure we first define a functor $\overline{}$ which is antiequivalence of categories (i.e., we have canonical isomorphisms $\overline{V \otimes W} \simeq \overline{W} \otimes \overline{V}$) and then show that we have isomorphisms $\overline{V} \simeq V^*$. We can formalize this setup in the following lemma, proof of which is trivial.

Lemma 10.1. *Let $\mathcal{C}$ be a semisimple ribbon category over $\mathbb{C}$. Let us assume that we have the following data:*

(1) A functor $\omega : \mathcal{C} \to \mathcal{C}$ satisfying the following conditions:

- *It is antilinear: for every $\Phi \in \mathrm{Hom}_\mathcal{C}, \alpha \in \mathbb{C}$, we have $(\alpha\Phi)^\omega = \bar{\alpha}\Phi^\omega$;*

- *We have functorial isomorphisms* $V^{\omega\omega} \simeq V^{**}, (V \otimes W)^{\omega} \simeq W^{\omega} \otimes V^{\omega}, (V^{\omega})^{*} \simeq (V^{*})^{\omega}$ *and* $\mathbf{1}^{\omega} \simeq \mathbf{1}$ *compatible with each other in the natural way.*
- *If* α, R *are the associativity and commutativity isomorphisms in* $\mathcal{C}$ *then* $\alpha^{\omega} = \alpha^{-1}, R^{\omega} = R^{-1}$, *i.e. we have the following commutative diagram*

$$\begin{array}{ccc} (V \otimes W)^{\omega} & \xrightarrow{(R_{V,W})^{\omega}} & (W \otimes V)^{\omega} \\ \| & & \| \\ W^{\omega} \otimes V^{\omega} & \xrightarrow{R^{-1}_{V^{\omega},W^{\omega}}} & V^{\omega} \otimes W^{\omega}, \end{array}$$

and similarly for α.
- *If* $\theta : V \to V$ *is the universal twist in* $\mathcal{C}$ *then* $\theta^{\omega} = \theta^{-1}$; *similarly, if* $i_V : \mathbf{1} \to V \otimes V^{*}$ *and* $e_V : V^{*} \otimes V \to \mathbf{1}$ *are the duality maps then* $i^{\omega}_V = (1 \otimes \delta^{-1}) i_{V^{\omega *}}, e^{\omega}_V = e_{V^{\omega *}}(\delta^{-1} \otimes 1)$.

(2) Isomorphisms $X^{\omega}_i \simeq X^{*}_i$ *for all simple objects* X_i *(which, again, are compatible with* $V^{\omega\omega} \simeq V^{**}$*).*

Then the category $\mathcal{C}$ *can be uniquely endowed with a structure of hermitian ribbon category: there is a unique functorial isomorphism* $V^{\omega} \simeq V^{*}$ *so that it is compatible with all the structures of a ribbon category.*

Note that these conditions imply $\dim V \in \mathbb{R}$ for every object V; vice versa, if we know that $\dim V \in \mathbb{R}$ then we can omit the condition $e^{\omega} = e$ and replace part (2) by the following condition:

(2′) For every i, we are given non-zero homomorphisms
$\phi_i : \mathbf{1} \to X_i \otimes X^{\omega}_i$
such that $\phi^{\omega} = \phi$ (up to identification $V^{\omega\omega} \simeq V^{**} \simeq V$).

We have implicitly used this construction when defining the hermitian structure on the category of representations of a quantum group in Section 4. Now we do the same for the category $\mathcal{O}^{int}_{\varkappa}$. Let $\omega : \mathfrak{g} \to \mathfrak{g}$ be the $q = 1$ specialization of the antilinear involution defined in Section 4; on the generators it is given by

$$\begin{aligned} \omega : & e_i \mapsto e_{i^{\vee}}, \quad i = 1, \dots, r \\ & f_i \mapsto f_{i^{\vee}}, \quad i = 1, \dots, r \\ & h \mapsto -w_0(h), \quad h \in \mathfrak{h}. \end{aligned} \tag{10.1}$$

We extend it to an involution on $\hat{\mathfrak{g}}$ by

$$\omega(x[n]) = (-1)^n(\omega(x))[n], \quad \omega(c) = c. \tag{10.2}$$

Obviously, ω is an automorphism of Lie algebras. Now, for every $V \in \mathcal{O}_{\varkappa}^{int}$ define V^ω to be the same set as V but with a different structure of a complex vector space and of $\hat{\mathfrak{g}}$-module: if we for $v \in V$ we denote by v^ω the same vector V considred as an element of V^ω, then we define $\alpha v^\omega = (\bar{\alpha}v)^\omega, xv^\omega = (\omega(x)v)^\omega, \alpha \in \mathbb{C}, x \in \hat{\mathfrak{g}}$.

the action of $\hat{\mathfrak{g}}$ twisted by ω. Similarly, for every $\hat{\mathfrak{g}}$-homomorphism $\Phi : V \to W$ between modules in $\mathcal{O}_{\varkappa}^{int}$ let Φ^ω be the same map but considered as a homomorphism $V^\omega \to W^\omega$. Obviously, this operation preserves the composition of morphisms. One easily checks that we have canonical identifications $L_{0,k}^\omega \simeq L_{0,k}, (DV)^\omega \simeq D(V^\omega)$ (since ω commutes with the automorphism $\sharp$ used in the definition of the dual – see beginning of Section 8). Also, define the isomorphism $V^{\omega\omega} \simeq V$ by $v \mapsto Zv$, where Z is the central element in a completion of $U\mathfrak{g}$ which was constructed in Theorem 7.2 of [1].

The most difficult part is to prove that ω is a tensor functor, i.e. to construct isomorphisms $(V\dot{\otimes}W)^\omega = W^\omega\dot{\otimes}V^\omega$. To do it, let us return to definition of conformal blocks on Riemann surfaces.

Let $\mathcal{X}$ be a Riemann sphere with n marked points $z_1, \dots, z_n$ (see (8.8)) and representations $V_1, \dots, V_n, V_\infty \in \mathcal{O}_{\varkappa}^{int}$ assigned to these points, and let $\Phi : V_\infty \to V_1\widehat{\otimes}\dots\widehat{\otimes}V_n$ be a conformal block, i.e. a linear map satisfying commutation relations (8.5).

Theorem 10.2. *Let $\mathcal{X}, \Phi$ be as above and denote by Φ^ω the same Φ considered as a map $V_\infty^\omega \to V_1^\omega\widehat{\otimes}\dots\widehat{\otimes}V_n^\omega$. Then Φ^ω is a conformal block on the Riemann surface $\mathcal{X}^\omega = \mathbb{C}P^1$ with marked points $z_1' = -\bar{z}_1, \dots, z_n' = -\bar{z}_n, \infty$, local parameters $w_i = z_i' - w, w_\infty = 1/w$ at $z_1', \dots, z_n', \infty$ and representations $V_1^\omega, \dots, V_n^\omega, V_\infty^\omega$ assigned to these points.*

Proof. Follows from the fact that the map $f(w) \mapsto \omega(f(-\overline{w}))$ is an isomorphism of the algebras of rational functions $\Gamma(\mathcal{X}) \simeq \Gamma(\mathcal{X}^\omega)$ which were defined by (8.3). □

For example, if $\mathcal{X}$ is the $n+1$-punctured sphere (8.8), with $z_1 < z_2 < \dots < z_n$, then $\mathcal{X}^\omega$ is again an $n+1$-punctured sphere (8.8) with punctures at $\infty, -z_1, \dots, -z_n$; thus, the order of punctures on the real line is reversed.

Recalling that we have identified the space of morphisms $V_\infty \to V_1\dot{\otimes}V_2$ in $\mathcal{O}_{\varkappa}^{int}$ with the space of conformal blocks on the sphere with marked points $0, 1, \infty$ (see Theorem 9.1 and remark after it), we immediately deduce from Theorem 10.2 the following result:

Theorem 10.3. *The map $\Phi \mapsto \Phi^\omega$ defined in Theorem 10.2 gives rise to a functorial antilinear isomorphism*

$$\mathrm{Hom}_{\mathcal{O}^{int}_\varkappa}(V_\infty, V_1\dot{\otimes}V_2) \simeq \mathrm{Hom}_{\mathcal{O}^{int}_\varkappa}(V_\infty^\omega, V_2^\omega\dot{\otimes}V_1^\omega)$$

and thus gives rise to an isomorphism $(V_1\dot{\otimes}V_2)^\omega \simeq V_2^\omega\dot{\otimes}V_1^\omega$.

Example 10.4. Let $\Phi = \Phi^g \in \mathrm{Hom}_{\mathcal{O}^{int}_\varkappa}(V_\infty, V_1\dot{\otimes}V_2)$ (see Example 8.7). Then $(\Phi^g)^\omega = \Phi^{(g^\omega)}$, where

$$g^\omega : (V_\infty[0])^\omega \to V_2[0]^\omega \otimes V_1[0]^\omega$$

is defined using the involution ω on $\mathfrak{g}$.

Finally, let us fix for every $\lambda \in C$ a non-zero homomorphism of $\mathfrak{g}$-modules $\varphi_\lambda : \mathbb{C} \to L_\lambda \otimes L_\lambda^\omega$ such that it is symmetric: $\varphi_\lambda(1)^\omega = \varphi_\lambda(1)$. This defines φ_λ uniquely up to a real constant. Define $\mathcal{O}^{int}_\varkappa$-morphisms $\phi_\lambda : \mathbf{1} \to L_{\lambda,k}\dot{\otimes}L^\omega_{\lambda,k}$ by $\phi_\lambda = \Phi^{\varphi_\lambda}$.

Proposition 10.5. *The functor ω and the system of maps $\phi_\lambda : \mathbf{1} \to L_{\lambda,k}\dot{\otimes}L^\omega_{\lambda,k}$ defined above satisfy the assumptions of Lemma 10.1 and thus endow $\mathcal{O}^{int}_\varkappa$ with the structure of a hermitian category.*

Lemma 10.6. *Equivalence of categories $\mathcal{C}(\mathfrak{g}, \varkappa) \simeq \mathcal{O}^{int}_\varkappa$ constructed in [8] (see Theorem 9.6) preserves the hermitian structure.*

Proof. The proof is based on the analysis of formulas defining the equivalence of categories in [6]. □

Remark 10.7. So far, this definition depends on the normalizations of the maps ϕ_λ which are defined up to a real constant; however, as in the quantum group case, the involution defined above can be related with the usual compact involution on $\hat{\mathfrak{g}}$, which is much more usual in the physical literature, and this allows us to define the inner product up to a positive constant. We discuss this in the Appendix.

10.2 Existence Theorem

Now that we have defined the structure of a hermitian modular tensor category on $\mathcal{O}^{int}_\varkappa$, the general theory developed in Section 2 immediately yields the existence of a modular invariant hermitian form on the spaces of morphisms. We formulate this result as a theorem. As before, we denote

$$H = \bigoplus_{\lambda \in C} DL_{\lambda,k}\dot{\otimes}L_{\lambda,k}. \tag{10.3}$$

Let us introduce the following notation:

$$W_\lambda^{\mu\nu} = \mathrm{Hom}_{\mathcal{O}_\varkappa^{int}}(L_{\lambda,k}, L_{\mu,k}\dot{\otimes}L_{\nu,k}).$$

Theorem 10.8. *There exists a nondegenerate hermitian form on each of the spaces* $W_\lambda^{\mu\nu}$ *such that the resulting form on the space*

$$\mathrm{Hom}_{\mathcal{O}_\varkappa^{int}}(H, L_{\mu,k}) = \bigoplus_{\lambda\in C} W_{\lambda^*\lambda}^{\mu} = \bigoplus_{\lambda\in C} W_\lambda^{\lambda\mu} \tag{10.4}$$

is modular invariant.

Proof. Follows from the fact that $\mathcal{O}_\varkappa^{int}$ has a structure of modular tensor category and constructions of Section 2. □

Conjecture 10.9. *If we choose the normalizations of the maps* ϕ_λ *in the definition of the hermitian structure on* $\mathcal{O}_\varkappa^{int}$ *(see Proposition 10.5) as in the Appendix, then the above defined hermitian form is positive definite.*

This conjecture is quiet parallel (and in fact, equaivalent to) the similar conjecture in the setting of the theory of quantum groups (see [1, Conjecture 7.6]). So far, we have no proof of it except for $\mathfrak{sl}_2$ case where it can be checked by direct calculation (see, e.g.,

We will call this form "the inner product", even though, as was noted before, we have no proof that this form is positive definite. Note that the definition of this inner product depends on the choice of the maps $\phi_\lambda : \mathbf{1} \to L_{\lambda,k}\dot{\otimes}L^\omega_{\lambda,k}$ which were used in the definition of hermitian structure. However, the inner product on the space $\mathrm{Hom}_{\mathcal{O}_\varkappa^{int}}(H, L_{\mu,k})$ in fact depends only on the choice of identification ϕ_μ and does not depend on the choice of ϕ_λ for each λ. Thus, the inner product on this space is defined uniquely up to a constant factor.

Recall that (Theorem 9.1) the spaces $W_\lambda^{\mu\nu} = \mathrm{Hom}_{\mathcal{O}_\varkappa^{int}}(L_{\lambda,k}, L_{\mu,k}\dot{\otimes}L_{\nu,k})$ are identified with spaces of conformal blocks for a 3-punctured sphere; thus, the construction above defines an inner product on the spaces of conformal blocks on a 3-punctured sphere.

To define an inner product on other Riemann surfaces with marked points, we use the gluing, which allows us to represent the space of conformal blocks assigned to a Riemann surface as a sum of tensor products of the 3-point conformal blocks. Once we choose such a representation, we define the norm by the rule $\|\Phi_1 \otimes \Phi_2\| = \|\Phi_1\| \cdot \|\Phi_2\|$.

However, it is not clear why this inner product is well-defined. Even for the conformal blocks on a sphere with 3 punctures, we have so far defined the inner product not for all 3-punctured spheres but only for some contractible subspace $\mathcal{D}$ in the moduli space. For other Riemann surfaces, it is even more complicated, since there are different ways to cut a surface in trinions. For example, there are different ways to represent a torus with one puncture as a result of gluing together two holes of a three-punctured sphere. Each of these ways gives an identification of the space of conformal blocks with the space $\bigoplus_{\lambda\in C} W_\lambda^{\lambda\mu}$ (see Lemma 9.4), and different identifications give rise to the action of the modular group on this space – cf. Theorem 9.5.

Now we come to the main result of our paper:

Theorem 10.10. *There exists an inner product on the spaces of conformal blocks on arbitrary Riemann surface whihc is preserved by the natural flat connection on the space of conformal blocks.*

It is easy to see that Conjecture 10.9 implies positive definitedness of this inner product on arbitrary Riemann surface.

Proof. It follows from the general result of Moore and Seiberg (see [2, Section 5]) that it suffices to define such an inner product on the sphere with n punctures and on the torus with one puncture. Moreover, to define the latter inner products it suffices to check that the commutativity and associativity morphisms, as well as the action of the modular group, are unitary with respect to the inner product defined on $W_\lambda^{\mu\nu}$. But we have proved that in any modular tensor category, the inner product defined in Section 2 satisfies these conditions (see Theorems 2.4, 2.5). □

This theorem is very important for Conformal Field Theory, since existence of such an inner product is one of the axioms of CFT; thus, Theorem 10.10 claims that this axiom is satisfied in the WZW model. To the best of my knowledge, this has not been proven in general case so far. We refer the reader to the paper [23] for review of known results and explicit constructions related to the inner products in higher genera.

Another important corollary of the equivalence of categories discussed in Theorem 9.6 and results of Section 5 is the following theorem.

Theorem 10.11. *Let $\mathfrak{g} = \mathfrak{sl}_n$. Consider the category $\mathcal{O}_\varkappa^{int}$ with $\varkappa = K+kh^\vee$ for some $K, k \in \mathbb{Z}_+$. Consider the space of conformal blocks on a torus with one puncture to which the representation $L_{\mu,K+(k-1)h^\vee}, \mu = n(k-1)\omega_1$ is assigned (here ω_1 is the first fundamental weight). Then the action of $SL_2(\mathbb{Z})$ on this space in some basis is given by formulas (5.8); in particular, it is written in terms of special values of Macdonald's polynomials at roots of unity.*

This theorem also gives the action of $SL_2(\mathbb{Z})$ on the affine Jack polynomials, introduced in [24]. In this paper, for every $K, k \in \mathbb{Z}_+, \lambda \in C_K = \{\lambda \in P^+ | \langle\lambda, \theta^\vee\rangle \le K\}$ we defined a function $J_{\lambda,K}(h, u, \tau), h \in \mathfrak{h}, u \in \mathbb{C}, \tau \in \mathbb{C}, Im\ \tau > 0$ (see definition in [24, Section 8]) which we called the (normalized) affine Jack polynomial; this name can be misleading, since they are not polynomials but rather theta-functions, but it was chosen since they are natural generalizations of Jack polynomials. We proved that for $\mathfrak{g} = \mathfrak{sl}_n$, $J_{\lambda,K}$ can be calculated as suitable renormalized traces of intertwining operators for the corresponding affine Lie algebra, and that the space spanned by $J_{\lambda,K}$ (with fixed K, k) is invariant under the action of $SL_2(\mathbb{Z})$. Now, using the previous results, we can calculate this action for $\mathfrak{g} = \mathfrak{sl}_n$.

Theorem 10.12. *In the assumptions of the previous theorem, let $J_{\lambda,K}$ be the normalized affine Jack polynomials as in [24, Section 8]. Then*

$$\begin{gathered} J_{\lambda,K}(h,u,\tau+1) = q^{(\lambda+k\rho,\lambda+k\rho)-k\varkappa(\rho,\rho)/n}, \\ \tau^{-j} J_{\lambda,K}\Big(\frac{h}{\tau}, u - \frac{(h,h)}{2\tau}, -\frac{1}{\tau}\Big) = \sum_{\mu\in C_K} S_{\mu\lambda} J_{\mu,K}(h,u,\tau), \end{gathered} \tag{10.5}$$

where $q = e^{\pi i/\varkappa}, \varkappa = K + nk$, $S_{\lambda\mu}$ is given by formula (5.8) in [1] and $j = -K(k-1)(n-1)/2\varkappa$.

Proof. Since $J_{\lambda,K}$ are (up to a renormalization) the traces of the intertwining operators for $\widehat{\mathfrak{sl}_n}$, the action of the modular group on these polynomials is up to simple renormalization the same as the action on the space of intertwiners, or, equivalently, on the space of conformal blocks. Combining it with the previous theorem, we get formulas (10.4). The factor τ^{-j} appears in the formula because the the tori $E_\tau = \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}$ is, of course, isomorphic to $E_{-1/\tau}$, but the local parameters around zero, inherited from $\mathbb{C}$, are different in these two realizations. The factor τ^{-j} accounts for the dependence of the space of conformal blocks on the choice of local parameter. □

11 Explicit Formulas

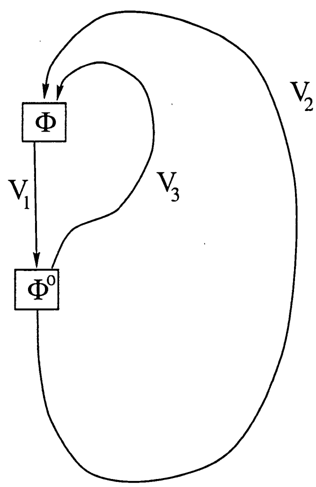
Our next goal is to give as explicit formulas as possible for this inner product. First, let us introduce some notations.

For every $\mathfrak{g}$-homomorphism $g : V_1 \to V_2 \otimes V_3$ denote by $g^o : V_2 \to V_1 \otimes V_3^*$ the image of g^ω (see (10.1)) under the canonical isomorphism $\mathrm{Hom}_\mathfrak{g}(V_1^*, V_3^* \otimes V_2^*) \simeq \mathrm{Hom}_\mathfrak{g}(V_2, V_1 \otimes V_3^*)$ (as before, we assume that we have chosen identifications $V^\omega \simeq V^*$). In a similar way, for every $\mathcal{O}_\varkappa^{int}$-homomorphism $\Phi : V_1 \to V_2 \dot{\otimes} V_3$ we define Φ^o by $(\Phi^g)^o = \Phi^{(g^o)}$ (see Example 8.7 for notations).

Now we can rewrite the definition of $\|\Phi\|^2$ given in Section 2 in the following form:

Lemma 11.1. *Let $\Phi \in \mathrm{Hom}_{\mathcal{O}_\varkappa^{int}}(V_1, V_2 \dot{\otimes} V_3)$. Then*

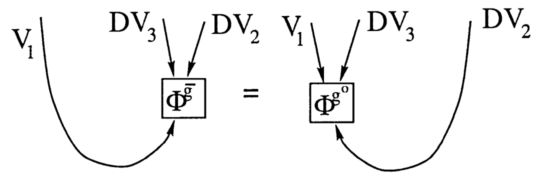
$$\|\Phi\|^2 = \frac{1}{(\dim_{\varkappa} V_1 \dim_{\varkappa} V_2 \dim_{\varkappa} V_3)^{1/2}}$$



. (11.1)

Proof. Recalling the definition (2.2) of the inner product and Example 10.4, we see that it suffices to prove the following lemma:

Lemma 11.2.



This Lemma follows from the fact that on the space $(V_1[0]\otimes V_2[0]\otimes V_3[0])^{\mathfrak{g}}$ the operators Ω_{12} and Ω_{23} commute and thus, according to Corollary 9.2, the associativity isomorphism is trivial. □

Let us additionally assume that V_1, V_2, V_3 are irreducible: $V_1 = L_{\lambda,k}, V_2 = L_{\mu,k}, V_3 = L_{\nu,k}$. Denote by Ψ composition of homomorphisms Φ, Φ^o : $\Psi = (\Phi\dot{\otimes}1)\Phi^o \in \mathrm{Hom}_{\mathcal{O}^{int}_{\varkappa}}(V_2, (V_2\dot{\otimes}V_3)\dot{\otimes}DV_3)$. It follows from the definition of the associativity isomorphism in $\mathcal{O}^{int}_{\varkappa}$ that Ψ can be considered as a flat section of the bundle of conformal blocks $W(\mathcal{X}, V_i)$, where $\mathcal{X}$ is $\mathbb{C}P^1$ with the marked points $0, z_1, z_2, \infty$ and representations V_2, V_3, DV_3, V_2 assigned to these points respectively. This flat section is uniquely defined by the following condition: if we denote by $\langle\Psi(z_1, z_2)\rangle$ the corresponding correlation function then it has the following asymptotic as $z_1/z_2 \to 0$: if $\Phi = \Phi^g$ then

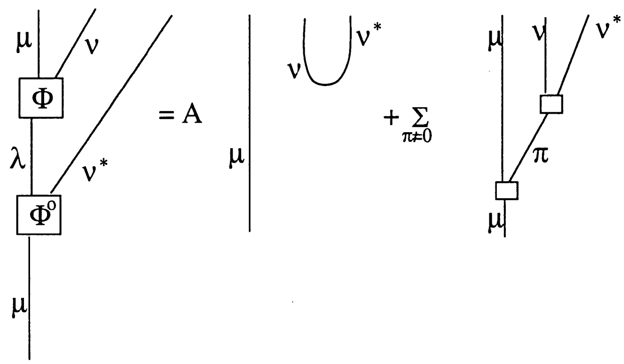
$$\langle\Psi(z_1, z_2)\rangle \sim (g\otimes 1)g^o z_1^{\Delta_\lambda-\Delta_\mu-\Delta_\nu} z_2^{\Delta_\mu-\Delta_\lambda-\Delta_\nu}$$

Theorem 11.3. *In the notations above, we have the following identity of* $\mathfrak{g}$*-homomorphisms* $L_\mu \to L_\mu \otimes L_\nu \otimes L_\nu^*$*:*

$$\|\Phi\|^2 \operatorname{Id}_{L_\mu} \otimes i_\nu = \left(\frac{\dim_q L_\mu \dim_q L_\nu}{\dim_q L_\lambda}\right)^{1/2} \lim_{z_1, z_2 \to 1} (z_1 - z_2)^{2\Delta_\nu} \langle \Psi(z_1, z_2) \rangle, \tag{11.2}$$

where $i_\nu : \mathbb{C} \to L_\nu \otimes L_\nu^*$ *is the canonical map of* $\mathfrak{g}$*-modules.*

Proof. It follows from the existence of the associativity isomorphism that Ψ can be rewritten as the sum



(11.3)

for some $A \in \mathbb{C}$ and some intertwiners in the boxes that are left blank. Substituting this in the expression for $\|\Phi\|^2$ given in Lemma 11.1, we see that

$$\|\Phi\|^2 = \left(\frac{\dim_q L_\mu \ \dim_q L_\nu}{\dim_q L_\lambda}\right)^{1/2} A. \tag{11.4}$$

As was mentioned before, coefficients in the right hand side of (11.3) are related with asymptotic expansion of $\langle\Psi\rangle$ as $z_1 \to z_2$ (this is how the associativity morphism is defined). More precisely, the first term in (11.3) gives the asymptotics $A(z_1 - z_2)^{-2\Delta_\nu} \operatorname{Id}_{L_\mu} \otimes i_\nu$, where $i_\nu : \mathbb{C} \to L_\nu \otimes L_\nu^*$ is the canonical embedding and all other terms have highest term of the asymptotics $(z_1 - z_2)^{-2\Delta_\nu + \Delta_\pi}$. Since $\Delta_\pi > 0$ for $\pi \neq 0$, these terms give zero contribution to the limit, and therefore

$$\lim_{z_1, z_2 \to 1} (z_1 - z_2)^{2\Delta_\nu} \langle \Psi(z_1, z_2) \rangle = A \operatorname{Id}_{L_\mu} \otimes i_\nu.$$

□

Thus, the calculation of the norm reduces to computation of the limit (11.2). Since the asymptotics of $\langle\Psi\rangle$ in the limit $|z_2| \gg |z_1|$ is known, this is equivalent to calculation of Drinfeld's associator. Note that this shows that the structure of $L_{\lambda,k}$ as a module over $\hat{\mathfrak{g}}$ is not important here; in particular, this inner product can be as well calculated for $\varkappa \notin \mathbb{Q}$, and the answer is a meromorphic function of $\varkappa$, which may have poles or zeroes only at rational points. Since there are integral formulas for the solutions of the KZ equations, the answer can be always written in terms of asymptotics of some integrals. In the case of $\mathfrak{g} = \mathfrak{sl}_2$ these asymptotics reduce to the Selberg integrals and are given by certain products of Γ-functions (see [10]). For arbitrary Lie algebras, the inner product can also be written in terms of integrals of Selberg type, but in general can not be reduced to gamma-functions.

Remark. In the case where all the spaces of conformal blocks are zero or one dimensional (as happens for $\mathfrak{sl}_2$), formula (11.2) (in different form) for the inner product was suggested by Moore and Seiberg (see [3, Exercise 6.6]).

12 Example: $\mathfrak{g} = \mathfrak{sl}_2$

In this section we give explicit formulas for the norms of vertex operators for $\mathfrak{sl}_2$. As was noted in the end of the last section, it suffices to calculate these norms for $\varkappa \notin \mathbb{Q}$. This can be done using integral formulas for the solutions, given by Schechtman and Varchenko [9]. Moreover, in this case the asymptotics of the integrals can be calculated explicitly, using the Selberg integral, and the answer is written in terms of Γ-functions (see [10]). This can be used to find the matrices which relate asymptotics of solutions in different asymptotic zones. Fortunately, this work has already been done by Varchenko in [10], in which there is a construction of equivalence of categories $\mathcal{O}_\varkappa^{int}$ and *Rep* $\mathfrak{g}$ for $\mathfrak{sl}_2$.

In this section we only consider $\mathfrak{g} = \mathfrak{sl}_2$. We identify the Cartan subalgebra with $\mathbb{C}$ so that $\alpha \mapsto 1$, where α is the positive root; thus, $P \simeq \frac{1}{2}\mathbb{Z}$.

Let us assume that we are given $K \in \mathbb{C}\setminus\mathbb{Q}$, and we let $\varkappa = K+2$. Let $\mathcal{O}_\varkappa$ be the category of highest-weight modules of level K over $\widehat{\mathfrak{sl}_2}$ as in [4]. It is known that this category is semisimple, and simple objects in this category are precisely irreducible modules $L_{\lambda,K}$, which for $\varkappa \notin \mathbb{Q}$ coincide with Weyl (induced) modules: $L_{\lambda,k} = V_{\lambda,K}, \lambda \in P^+ \simeq \frac{1}{2}\mathbb{Z}_+$. The same constructions as before define on this category a structure of a ribbon category, which is hermitian for $K \in \mathbb{R}$. Note that since we assumed $\varkappa \notin \mathbb{Q}$, we have an isomorphism $\mathrm{Hom}_{\mathcal{O}_\varkappa}(V_1, V_2 \dot{\otimes} V_3) \simeq \mathrm{Hom}_{\mathfrak{sl}_2}(V_1[0], V_2[0] \otimes V_3[0])$ (compare with (9.5)). Thus, this category could be described without any reference to affine Lie algebra at all, just in terms of finite-dimensional representations of $\mathfrak{sl}_2$ and Knizhnik-Zamolodchikov equation, as was done in [25, 26].

For every triple $j, j_1, j_2 \in \frac{1}{2}\mathbb{Z}_+$ such that $|j_1 - j_2| \le j \le j_1 + j_2, j + j_1 + j_2 \in \mathbb{Z}$ there exists a unique up to a constant $\mathfrak{sl}_2$-morphism $g_j^{j_1 j_2} : V_j \to V_{j_1} \otimes V_{j_2}$, and corresponding $\mathcal{O}_\varkappa$-morphism $\Phi_j^{j_1 j_2} = \Phi^{g_j^{j_1 j_2}}$.

Let us for simplicity consider the case $j = j_1, j_2 = k$ for some j, k. Then we can fix normalization of g_j^{jk} by fixing some vector of weight zero $u \in V_k$ and requiring that $g_j^{jk} v_j = v_j \otimes u + \dots$, where v_j is a highest weight vector in V_j.

The main result of this section, which can be obtained by the use of explicit formulas in [10], is the following theorem:

Theorem 12.1. *Let $\varkappa \in \mathbb{R} \setminus \mathbb{Q}$. Then:*

$$\|\Phi_j^{jk}\|^2 = c(k, \varkappa) \prod_{i=1}^{k} \frac{[2j+1+i]}{[2j+1-i]} \frac{[\![2j+1+i]\!]^2}{[\![2j+1-i]\!]^2} \tag{12.1}$$

where the constant $c(k, \varkappa)$ does not depend on j and

$$[\![x]\!] = \Gamma\Big(-\frac{x}{\varkappa} + 1\Big). \tag{12.2}$$

Since $[\![x]\!]$ is well-defined and non-zero for $x \neq n\varkappa, n = 1, 2, ..$ we see that for $\varkappa \notin \mathbb{Q}$ the norm $\|\Phi_j^{jk}\|$ is well-defined and non-zero. One can also easily check that if $K \in \mathbb{Z}_+, j, k \leq K/2$ (this last condition ensures that $L_{j,K}, L_{k,K}$ are integrable) then $\|\Phi_j^{jk}\|$ is well-defined and non-zero iff $k \leq 2j \leq K - k$, which is exactly the condition for the space of intertwiners $\mathrm{Hom}_{\mathcal{O}_\varkappa^{int}}(L_{j,K}, L_{j,K} \dot{\otimes} L_{k,K})$ to be non-zero.

Note also that as $\varkappa \to \infty$, formula (12.7) coincides (up to a constant) with the Macdonald's inner product identity (5.7) for $\mathfrak{sl}_2$ (with $q = 1$).

This theorem justifies the following conjecture, which is a natural generalization of the Macdonald's inner product identities to affine root systems. Let $\mathfrak{g} = \mathfrak{sl}_n, k \in \mathbb{Z}_+, \varkappa \in \mathbb{R} \setminus \mathbb{Q}$. For $\lambda \in P_+$, choose $\hat{\Phi}_\lambda \in \mathrm{Hom}_{\mathcal{O}_\varkappa}(V, V \dot{\otimes} U)$, where $V = V_{\lambda+(k-1)\rho, \varkappa-h^\vee}, U = V_{n(k-1)\omega_1, \varkappa-h^\vee}$ so that $\hat{\Phi}(v) = v \otimes u_0 + \dots$, where v is a highest-weight vector in V and $u_0 \in U[0]$ is some fixed zero-weight vector. This is the natural affine analogue of the intertwiners Φ_λ used in the quantum group case to obtain Macdonald's polynomials (see [1, Section 5 and references therein]); as before, one can show that $\mathrm{Hom}_{\mathcal{O}_\varkappa}(V, V \dot{\otimes} U)$ is one-dimensional and thus the condition $\hat{\Phi}_\lambda(v) = v \otimes u_0 + \dots$ uniquely defines $\hat{\Phi}_\lambda$. The intertwiners $\hat{\Phi}_\lambda$ were used in [24] to construct an affine analogue of Jack's polynomials.

As before, $\mathcal{O}_\varkappa$ has a natural structure of a hermitian category and thus we have an inner product on the spaces of morphisms.

Conjecture 12.2 (Affine inner product identities). *Let $\mathfrak{g} = \mathfrak{sl}_n, \varkappa \in \mathbb{R} \setminus \mathbb{Q}$, and let $\Phi_\lambda : V \to V \dot{\otimes} U$ be as above. Then:*

$$\|\hat{\Phi}_\lambda\|^2 = c(k, \varkappa) \prod_{\alpha \in R^+} \frac{[(\alpha, \lambda + k\rho) + i]}{[(\alpha, \lambda + k\rho) - i]} \left(\frac{[\![(\alpha, \lambda + k\rho) + i]\!]}{[\![(\alpha, \lambda + k\rho) - i]\!]} \right)^2, \tag{12.3}$$

where the constant $c(k, \varkappa)$ does not depend on λ.

In the case $\mathfrak{g} = \mathfrak{sl}_2$ this conjecture coincides with the statement of Theorem 12.1.

This formula is indeed a natural analogue of Macdonald's inner product identity (5.7) for the following reason: if we write weights for $\hat{\mathfrak{g}}$ in the form $\hat{\lambda} = \lambda + k\Lambda_0$ where Λ_0 is dual to the central element c, and denote $\hat{\rho} = \rho + h^\vee\Lambda_0$ then the right hand side of (12.9) is exactly the regularization of the meaningless expression

$$\prod_{\alpha \in \hat{R}^+_{re}} \prod_{i=1}^{k-1} \frac{(\alpha, \hat{\lambda} + k\hat{\rho}) + i}{(\alpha, \hat{\lambda} + k\hat{\rho}) - i}$$

obtained by replacing products of the form $\prod_{i=0}^{\infty} \frac{1}{x+i}$ with $\Gamma(x)$.

This conjecture is closely related to the expressions given by Gawędzki et al in [11, 12, 23]. In these papers they suggest a construction of the inner product on the spaces of conformal blocks based on the fact that these spaces are the state spaces of the quantum Chern-Simons theory. The inner product is obtained by regularization of certain infinite-dimensional integrals, and the final answer is given by a finite-dimensional integral of the following type

$$\int_{\mathbb{C}^k} |\omega|^2,$$

where ω is the differential form which appears in the integral formulas of Schechtman and Varchenko (up to $\varkappa \mapsto -\varkappa$).

There are reasons to believe that in the case $\mathfrak{g} = \mathfrak{sl}_2$ integrals of these type can be calculated explicitly (see [27] for a simplest example of computation of this type), and the answer can be expressed in terms of Γ-functions similarly to the Selberg integral. We expect that this procedure would yield the same answer as the one given by formula (12.7) above.

Appendix

Here we give another description of the hermitian structure on $\mathcal{O}^{int}_\varkappa$, based on compact involution of $\mathfrak{g}$. This construction is parallel to the one for $U_q\mathfrak{g}$ (see Section 7).

Let ω_c be the compact involution on $\mathfrak{g}$, i.e. an antilinear Lie algebra automorphism given on the generators by $\omega_c(e_i) = -f_i, \omega_c(f_i) = -e_i, \omega_c(h) = -h$. Let us extend it to an involution on $\hat{\mathfrak{g}}$ by letting $\omega_c(x[n]) = \omega_c(x)[n]$, $\omega_c(c) = c$ (**Warning: this is different from the standard compact involution on $\hat{\mathfrak{g}}$!**). We define $V^{\omega_c}, \Phi^{\omega_c}$ similar to above. Note that for any λ, $L^{\omega_c}_{\lambda,k} \simeq L_{\lambda^*,k}$, and thus, from complete reducibility, we have $V^{\omega_c} \simeq DV$ for any $V \in \mathcal{O}^{int}_\varkappa$, though these isomorphisms are not canonical.

For any $\mathcal{X}$ as in the beginning of Section 8, let $\overline{\mathcal{X}}$ be the following collection of data:

(1) The curve $\overline{X}$ which is the same curve as X but with the opposite complex structure (thus, if f is a meromorphic function on X then $\bar{f}$ is a meromorphic function on $\overline{X}$).

(2) The marked points z_i are the same as for $\mathcal{X}$.

(3) The new local parameters are $\overline{w_i}$ (here we consider w_i as holomorphic function on X vanishing at z_i).

Example. Let $\mathcal{X}$ be the Riemann sphere (8.8). Then $\overline{\mathcal{X}}$ is isomorphic to $\mathbb{C}P^1$ with global parameter w, marked points $\overline{z_i}, \infty$ and local parameters $\bar{z}_i - w, 1/w$ respectively.

Assume that we are given $\mathcal{X}$ as before and we have representations V_i assigned to the marked points. Let $\Phi \in W(\mathcal{X}; V_1, V_2, \dots, V_n)$ be a conformal block, i.e. a map

$$\otimes_{In} V_i \to \hat{\otimes}_{Out} V_i$$

which commutes with the action of the algebra Γ of meromorphic $\mathfrak{g}$-valued functions on $\mathcal{X}$ (see Definition 8.1).

Let us denote by Φ^{ω_c} the same map considered as a map

$$\otimes_{In} V_i^{\omega_c} \to \otimes_{Out} V_i^{\omega_c}.$$

Lemma A.1. *The mapping $\Phi \mapsto \Phi^{\omega_c}$ is an involutive isomorphism $W(\mathcal{X}; V_1, \dots, V_n) \to W(\overline{\mathcal{X}}; V_1^{\omega_c}, \dots, V_n^{\omega_c})$.*

Proof. This is obvious from the fact that $f(z) \mapsto \omega_c(f(z))$ identifies the algebras Γ of meromorphic functions on $\mathcal{X}$ and $\overline{\mathcal{X}}$. □

In particular, if $\mathcal{X}$ is the Riemann sphere (8.8) then Φ^{ω_c} will be a conformal block on the sphere with marked points $\bar{z}_1, \dots, \bar{z}_n$. This shows that we have a natural isomorphism $(V\dot{\otimes}W)^{\omega_c} \simeq V^{\omega_c}\dot{\otimes}W^{\omega_c}$.

Now we can define the hermitian structure on $\mathcal{O}_\varkappa^{int}$ similarly to the definitions before (see Lemma 10.1); the only non-trivial part is that we define isomorphisms $(V\dot{\otimes}W)^{\omega_c} \simeq W^{\omega_c}\dot{\otimes}V^{\omega_c}$ to be given by the composition of the natural isomorphism $(V\dot{\otimes}W)^{\omega_c} \simeq V^{\omega_c}\dot{\otimes}W^{\omega_c}$ and the commutativity morphism $Pe^{\pi i\Omega/\varkappa}$, where P is the transposition. Also, we define the maps $\phi_\lambda : \mathbf{1} \to L_{\lambda,k}\dot{\otimes}L_{\lambda,k}^{\omega_c}$ by $\phi_\lambda(1) = e^{\pi i\Delta_\lambda} v_{\lambda,k} \otimes v_{\lambda,k}^{\omega_c} + \dots$, where Δ_λ is defined by (9.3).

Lemma A.2. *So defined the functor ω_c satisfies all the conditions of Lemma 10.1 and thus gives rise to a hermitian structure on $\mathcal{O}_\varkappa^{int}$.*

In fact, it turns out that this hermitian strucutre is equivalent to the one described in Section 10. As in the classical case, this relies on the existence of an analogue of the longest element of Weyl group.

For every module $V \in \mathcal{O}_\varkappa^{int}$ define the map

$$\Omega = w_0 e^{-\pi i L_0} : V \to V, \tag{A.1}$$

where w_0 is the longest element of Weyl group for $\mathfrak{g}$, which acts on every finite-dimensional representation of $\mathfrak{g}$ (see, e.g. [6, (19.4)]); thus, it also acts on every module in $\mathcal{O}_\varkappa^{int}$. Its q-analogue (which we denoted also by Ω) was discussed in Section 7.

One can easily check that

$$\begin{aligned} \omega(x) = \Omega^{-1}\omega_c(x)\Omega = \omega_c(\Omega^{-1}x\Omega), \quad x \in \hat{\mathfrak{g}} \\ \Omega^2 = Ze^{-2\pi i L_0} \end{aligned} \tag{A.2}$$

where $Z = w_0^2$ was discussed in Section 7. Recall that Z is a central element which acts by ± 1 in any irreducible $\mathfrak{g}$-module and satisfies $Z|_{V\otimes W} = Z_V \otimes Z_W$; thus, it also acts by constant in any simple $\hat{\mathfrak{g}}$-module from category $\mathcal{O}_\varkappa^{int}$ and satisfies $Z_{V\dot{\otimes}W} = Z_V \dot{\otimes} Z_W, Z^2 = 1$.

Thus, for every representation $V \in \mathcal{O}_\varkappa^{int}$ the map $\Omega : V \to V$, considered as a map $V^\omega \to V^{\omega_c}$, is an isomorphism of $\hat{\mathfrak{g}}$-modules; similarly, we can identify $\overline{\Phi} = \Omega^{-1}\Phi^{\omega_c}\Omega$.

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