

Branes and Toric Geometry

Naichung Conan Leung^a, Cumrun Vafa^b

^aSchool of Mathematics, University of Minnesota
Minneapolis, MN 55455, USA

^bLyman Laboratory of Physics, Harvard University
Cambridge, MA 02138, USA

Abstract

We show that toric geometry can be used rather effectively to translate a brane configuration to geometry. Roughly speaking the skeletons of toric space are identified with the brane configurations. The cases where the local geometry involves hypersurfaces in toric varieties (such as \mathbf{P}^2 blown up at more than 3 points) presents a challenge for the brane picture. We also find a simple physical explanation of Batyrev's construction of mirror pairs of Calabi-Yau manifolds using T-duality.

1 Introduction

One of the main new physical insights we have recently gained from string theory is that geometric singularities of the internal compactification manifold encode a great deal of information about quantum field theories. Turning things around we can engineer quantum field theories by suitably choosing singularities under consideration and use them to gain insight into quantum field theories (see for example [1–6]). This program of studying QFT's is called *geometric engineering*.

On the other hand there has been another direction of construction of field theories involving branes (see for example [7, 8]). Some of these cases are already known to be equivalent, by T-dualities to the geometrical cases (see for example [9–12]). Here we try to extend this dictionary to a more general class of theories and in particular to 5 dimensional critical theories

constructed by Hanany and Aharony [13] and studied further in [14–16]. The approach we follow will also lead to a simple geometric realization of the Riemann surface for $N = 2$ theories appearing in [4,9] through fivebranes of type IIA. This is also related to the recent observations made in [14–16]¹. Even though here we limit ourselves to few examples, the approach we take is quite general and can be applied to many other cases.

The summary of our results is as follows: Toric geometry involves viewing manifolds as roughly speaking products of some space with a torus. The only non-triviality involves the fact that on some loci certain cycles of tori can shrink. Toric geometry is a way to encode this combinatoric data as to which cycles shrink where. This constitutes faces of the polytope describing the toric spaces. On the other hand vanishing cycles have been known to be associated with branes. This connection thus identifies these toric skeletons directly with branes of appropriate types!

Toric geometry, however, can be used in a more general way to get interesting geometries, namely by going to a higher dimensional space and imposing equations. This presents a major challenge for the brane picture and it is not clear how to modify the brane story to accommodate this simple geometric idea. This is precisely the flexibility that the geometric constructions enjoy over the brane picture; It would be interesting to try to find a way of adapting the brane picture to such cases as well.

The organization of this paper is as follows: In section two we give a very simple overview of toric geometry (intended for physicists unfamiliar with it). In section 3 we describe the relation between toric geometry and branes of various types. Finally in section 4 we use $R \rightarrow 1/R$ duality in the context of toric geometry to give a simple intuitive explanation of Batyrev's construction of Calabi-Yau mirror pairs [17] (see also [18]).

2 Review of Certain Aspects of Toric Geometry

In this section we review certain aspects of toric geometry, intended mainly for physicists unfamiliar with the ideas in toric geometry. We aim to give a very simple treatment of the ideas of toric geometry. For a detailed pedagogical review emphasizing other aspects of toric constructions see [19,20].

Toroidal compactifications are among the most special classes of compactifications in string theory. They preserve the maximal amount of symmetry a lower dimensional theory can possibly have starting from a higher

¹We note that the notion of 'grid diagrams' discussed by those authors in this context has been well known as a standard construction in toric geometry and was extensively discussed and used already in [4]. In this paper we will also relate the physics of the situation discussed in [4] to the brane realization of the same theories.

dimensional one. For example a d -dimensional torus admits a $U(1)^d$ translational symmetry. Even though it would be easiest to analyze the physical systems under such compactifications, they would typically have too much symmetry for many applications of interest in physics. The next best thing in physics is compactifications of something that comes close to being toroidal. Toric geometry basically studies geometries where there is a $U(1)^d$ action, as in the T^d case, but unlike the toroidal case, the $U(1)^d$ action is allowed to have fixed points. The basic idea in characterizing such geometries is to isolate the fixed point structure and use that to encode the geometry.

It is best to start with some examples:

Example i) Consider the complex plane \mathbf{C} . This manifold admits a $U(1)$ action

$$z \rightarrow z \exp(i\theta), \tag{2.1}$$

with a fixed point at $z = 0$. The geometry of the plane can be represented by a half-line, corresponding to $|z|$ above which there is a circle. Moreover the circle shrinks at the end of the half-line.

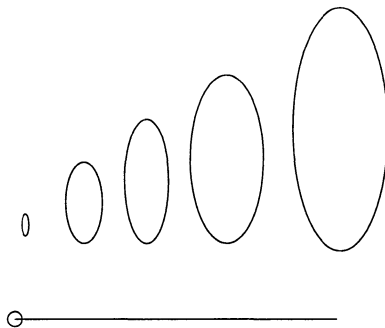


Figure 1: Complex plane can be viewed as a half-line with a circle on top, which shrinks at the end.

Example ii) Consider the 2-sphere \mathbf{P}^1 , which can be viewed as the compactified complex plane z . Again there is a $U(1)$ action, just as above, in terms of which the 2-sphere can be represented as an interval times a circle, where the circle shrinks at the two ends, corresponding to north and south poles of the sphere. The coordinates on the interval can be identified with a function of $r = |z|$. The length of this interval is determined by the size of the 2-sphere. More precisely, if the 2-sphere has a metric which is α times the standard metric on the 2-sphere, namely the Fubini-Study metric $\frac{|dz|^2}{(1+|z|^2)^2}$ then the coordinate on the interval is given by

$$x = \frac{\alpha |z|^2}{(1 + |z|^2)}, \tag{2.2}$$

which runs from 0 to α . Notice that the origin of the interval is not relevant and we can perform a coordinate transformation of the interval by translation $x \rightarrow x + x_0$ without changing our picture.

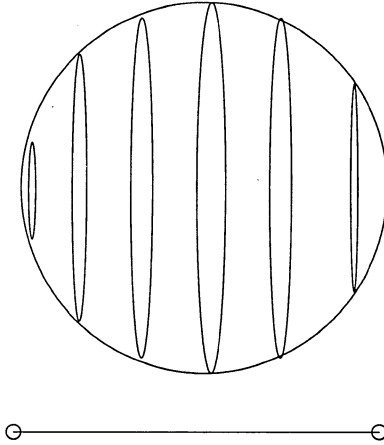


Figure 2: The 2-sphere can be viewed as an interval with a circle on top, where the circle shrinks to zero size at the two ends.

In various applications it is important to consider in addition a holomorphic line bundles on the 2-sphere. Then the first Chern class c_1 of the bundle is a $(1, 1)$ form which can be taken to be the volume element corresponding to the Kahler form for the Fubini-Study metric $c_1 = n$. In this case, however, we will have to use *integral* α giving integral volume of the sphere because

$$\int_{\mathbf{P}^1} c_1 = n, \quad (2.3)$$

for some integer n and we can identify $\alpha = n$. If we wish to consider the sphere together with the bundle on top it is then convenient to choose the interval to go from 0 to n . Such an interval will be called integral. Note that the number of holomorphic sections of the bundle is then related to the number of integral points on the interval. Thinking of the volume form on the 2-sphere as a symplectic form, with x giving the radial direction, the number of sections of the bundle is related to the dimension of the Hilbert-space, which is the same as the Bohr-Sommerfeld quantization rule for a compact phase space. Note that one can identify the holomorphic sections in this case with z^i where $i = 0, \dots, n$.

Note that near each of the two ends the geometry is the same as the example 1, which is just the statement that near the north pole or the south pole we can view the sphere as a patch which is just \mathbf{C} .

Example iii) Our next example is \mathbf{P}^2 . As is well known \mathbf{P}^2 is the space of three complex numbers (z_1, z_2, z_3) not all zero, modulo identifying them

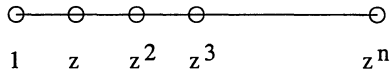


Figure 3: The integral lattice can be used to summarize the information about sections of bundle on the sphere. Each point on the lattice corresponds to a section of the bundle.

up to multiplication by a non-zero complex number. In this case we have a $U(1)^2$ action, consisting of the $U(1)^3$ action on the phases of the z_i , modulo the action of the diagonal $U(1)$ which acts trivially on \mathbf{P}^2 . We can consider a basis of the $U(1)^2$ action to consist of

$$(z_1, z_2, z_3) \rightarrow (z_1 \exp(i\theta), z_2 \exp(i\phi), z_3). \tag{2.4}$$

The fixed point of θ action consists of $(0, z_2, z_3)$ up to an overall rescaling, which gives a \mathbf{P}^1 parameterized by z_2/z_3 . Similarly the fixed point of the ϕ action is a \mathbf{P}^1 parameterized by z_1/z_3 . Also if we consider $\theta = \phi$ diagonal $U(1)$ we get another fixed point locus being the \mathbf{P}^1 parameterized by z_1/z_2 . Moreover, there are three fixed points where both $U(1)$ actions have fixed points (when any pair of $z_i, z_j = 0$) corresponding to the intersection of any pair of these \mathbf{P}^1 's. We are interested in viewing \mathbf{P}^2 as a space having generic T^2 fibers parametrized by the action of (θ, ϕ) introduced above. We have to choose coordinates for a two real dimensional base which is invariant under the T^2 action. Note that $r_1 = |z_1/z_3|$ and $r_2 = |z_2/z_3|$ are such coordinates. Again we can map it to finite regions by considering appropriate functions of r_1, r_2 . We thus can represent the \mathbf{P}^2 according to the figure:

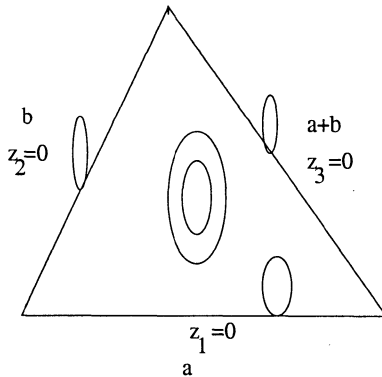


Figure 4: The toric realization of \mathbf{P}^2 involves a triangle over each point of which there is a 2-torus, which shrinks to a circle at each edge, and where it shrinks to a point at each vertex of the triangle. Each edge of the triangle, with the circle on top, corresponds to a $\mathbf{P}^1 \in \mathbf{P}^2$.

where this represents the base of the \mathbf{P}^2 . Above each point in the interior

of the triangle we have a T^2 fiber (θ, ϕ) . Let us denote the cycles of this torus corresponding to θ, ϕ by a, b respectively. The T^2 fibration degenerates near the edges of the triangle, where over one edge a shrinks (corresponding to $z_1 = 0$), over the other b shrinks (corresponding to $z_2 = 0$) and over the other $a + b$ shrinks (corresponding to $z_3 = 0$). On the vertices of the triangle both a and b shrink. For various applications it turns out to be convenient to introduce the following construction. One realizes the base of the \mathbf{P}^2 in \mathbf{R}^2 , where we orient each face so that the normal vector to that face corresponds to the cycle direction of the fiber T^2 which it shrinks. For example for the \mathbf{P}^2 example above we draw the \mathbf{P}^2 base as follows:

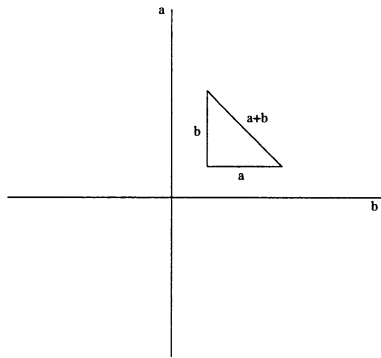


Figure 5: It is natural to draw the faces of the toric space angled in such a way that is normal to the direction of the cycle which vanishes over it.

Now if we wish to emphasize the bundle structure, all we have to do is to choose the vertices to lie on integral points \mathbf{Z}^2 . Given the geometry of \mathbf{P}^2 , the angles of all the three edges are fixed and all we can vary is an overall size. This is in accord with the fact that line bundles on \mathbf{P}^2 are characterized by the choice of an integer n . In this case the line bundle restricted to each of the three \mathbf{P}^1 's will correspond to the number of integral points on the interval and is identified with this n . Moreover, the totality of the points in the triangle (the integral points in the interior as well as points on the boundary) will correspond to the number of holomorphic sections of the bundle over \mathbf{P}^2 . In fact if we denote a point in the triangle by (a, b) we associate to this the section $z_1^a z_2^b$ (note that $a + b \leq n$, $a \geq 0$ and $b \geq 0$). The example of \mathbf{P}^2 with degree 3 bundle is shown below:

Example iv) We can also describe “blowing up” of \mathbf{P}^2 at some number of generic points $n \leq 3$ in a similar manner. What blowing up means in this context is to replace a point on \mathbf{P}^2 by a sphere \mathbf{P}^1 . With no loss of generality we can take the point to be at any of the three vertices of the toric triangle (by the $SL(3)$ symmetry of \mathbf{P}^2). Since blowing up means replacing a point by a \mathbf{P}^1 and that is realized in toric language by an interval, as discussed in

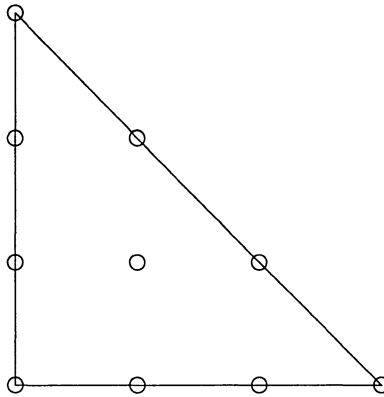


Figure 6: The integral toric realization of \mathbf{P}^2 with a line bundle of degree 3. All the integral points shown in the figure correspond to sections of that bundle.

example ii), this implies that \mathbf{P}^2 with one point blown up will be torically given by:

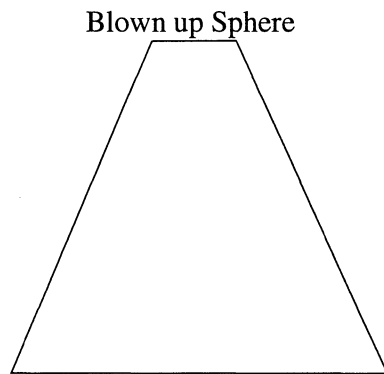


Figure 7: Blowing up can be realized very easily using toric geometry. Here we are drawing the blowing up of \mathbf{P}^2 at one point (what used to be the top vertex of \mathbf{P}^2) which has the effect of replacing it by a \mathbf{P}^1 (shown as the top interval in the above figure.) The size of the interval is a direct measure of the size of the blown up \mathbf{P}^1 .

where again one can work out what cycle of T^2 vanishes over the new \mathbf{P}^1 . Note that blowing up, up to three generic points can be realized torically, because using $SL(3)$ symmetry of \mathbf{P}^2 we can map any three points to the three vertices of the triangle above. Beyond three points we can still blow up, but that cannot be realized torically for generic points. Only if we choose special points which lie at the corner of the toric base can we continue blowing up torically. However it is known that the manifold one gets by blowing up

more than 3 points on \mathbf{P}^2 will depend on where the points are chosen. If they are done generically, then we get what is called a del-Pezzo surface (up to blowing up 8 points) and has a positive first Chern class $c_1 > 0$. But if the points are not generic, as will be the case in the toric realization where we choose more than 3 points to blow up, the manifold we get will not have $c_1 > 0$. This will prove important for certain comparisons with physical realization via 5-branes which we will discuss later in this paper. We emphasize that this is a limitation of toric realization of blown up \mathbf{P}^2 and not a reflection of any intrinsic problem with the geometry of blown up \mathbf{P}^2 .

Example v) We can now generalize easily to \mathbf{P}^n , where we have an n -dimensional toroidal fiber T^n over an n -dimensional base, which is identified with an n -dimensional simplex. Let us denote a basis of the cycles of T^n by a_1, \dots, a_n . The simplex has $n + 1$ boundary faces, over each of which a 1-cycle of T^n shrinks. These can be taken to correspond to $a_1, \dots, a_n, \sum a_i$.

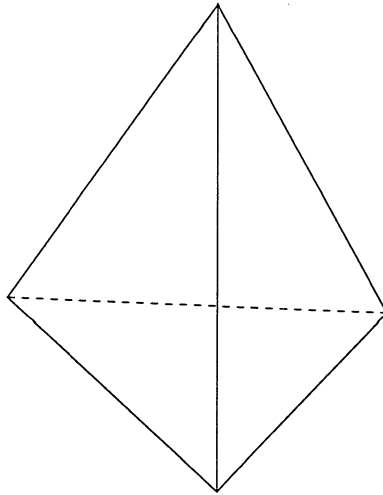


Figure 8: Here we are showing the toric realization of \mathbf{P}^3 .

Each face is an $n - 1$ dimensional simplex. Two such faces meet over an $n - 2$ dimensional simplex, over which two cycles of the T^n shrink. More generally k such faces meet over an $n - k$ dimensional simplex over which k cycles of T^n shrink. In particular when n of them meet (which happens at $n + 1$ points) the whole T^n fiber has shrunk.

Example ii') The examples of \mathbf{P}^n discussed above cannot be used for string compactification as they are not solutions to Einstein's equations (they are not Ricci-flat). However they can be part of a local geometry of a Calabi-Yau near a singularity. For example consider the case of \mathbf{P}^1 . It is known that the cotangent space $T^*\mathbf{P}^1$ can appear as part of Calabi-Yau

compactifications (for example near an A_1 singularity of $K3$). This space is also toric. If we denote the coordinates of $T^*\mathbf{P}^1$ by (z, p) corresponding to \mathbf{P}^1 and the cotangent direction respectively, we can consider two circle actions on this space. The first one is the one induced from the action on the \mathbf{P}^1 base to the normal direction (taking into account that pdz is invariant)

$$(z, p) \rightarrow (\exp(i\theta)z, \exp(-i\theta)p) \tag{2.5}$$

and the other circle action is new and acts entirely on the fiber

$$(z, p) \rightarrow (z, \exp(i\phi)p). \tag{2.6}$$

Let the (a, b) cycles of the T^2 to correspond to the (θ, ϕ) actions respectively. Then as before we can use $|z|$ and $|p|$ as defining a base for a T^2 fibration, with a T^2 fiber corresponding to the (θ, ϕ) action. There are three fixed loci of this toric action over which an S^1 in the fiber shrinks. These correspond to $z = 0, z = \infty$ and $p = 0$ (note that the p direction is non-compact and so it does not have a point at infinity). At $z = 0$ the invariant direction corresponds to setting $\theta = \phi$. In other words the $a - b$ -cycle shrinks. At $p = 0$ the b -cycle shrinks. To find out what the toric action is near $z = \infty$ it is convenient to change the patch to $\tilde{z} = 1/z$. Noting that pdz is invariant this undergoes a transformation $\tilde{p} = -pz^2$, in terms of which the T^2 action becomes

$$(\tilde{z}, \tilde{p}) \rightarrow (\exp(-i\theta)\tilde{z}, \exp(i\phi + i\theta)\tilde{p}). \tag{2.7}$$

Thus at $\tilde{z} = 0$, if we set $\phi = -\theta$, the action on $T^*\mathbf{P}^1$ is trivial. In other words at $\tilde{z} = 0$ the cycle $a + b$ shrinks. Thus the base of the toric fibration is given by the geometry below:

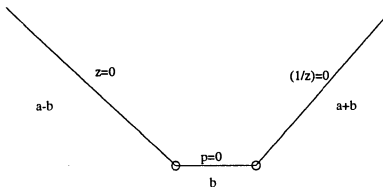


Figure 9: The toric realization of the blowing up of A_1 singularity in $K3$. The finite interval represents the blown up \mathbf{P}^1 . Note that a half line emanating from the interval going to the infinity corresponds to the cotangent bundle at that point, which is a copy of the complex plane \mathbf{C} .

Note that the boundary of this figure corresponds from left, to the $|p|$ over $z = 0$, the $p = 0$ which corresponds to the z -sphere and is represented by the interval, and finally the cotangent direction over \tilde{z} . The lines emanating from

the interval corresponding to the z -sphere going to infinity correspond to the non-compact cotangent direction over sphere. It is also easy to generalize this to when we have an A_n singularity. The geometry consists of n spheres intersecting according to the A_n Dynkin diagram. The toric geometry is summarized as follows:

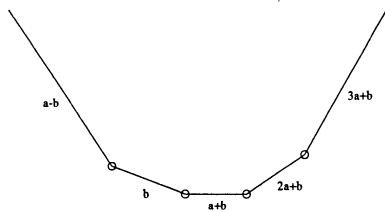


Figure 10: Shown in the figure is the blown up singularity A_3 . Note that the three finite size intervals in the middle denote the three blown up \mathbf{P}^1 's. Also note that the Dynkin diagram of A_3 is visibly seen here by the intersection of neighboring \mathbf{P}^1 's at a point.

Note that each \mathbf{P}^1 is given by an interval in the above figure, and the size of the \mathbf{P}^1 is represented by the size of the interval. In the limit that a \mathbf{P}^1 shrinks, the interval shrinks and we obtain a singular geometry. The geometry we have depicted above is a smooth geometry corresponding to “blowing up” the A_n singularity.

Example iii)' Similar to the above example, a \mathbf{P}^2 can appear in a Calabi-Yau, where there are some extra dimensions. In particular if this is embedded in a Calabi-Yau threefold there is a normal direction which corresponds to a line bundle on \mathbf{P}^2 . The condition that $c_1 = 0$ for the threefold implies that the normal bundle is the canonical line bundle (corresponding to $(2, 0)$ forms on \mathbf{P}^2), thus cancelling c_1 for the \mathbf{P}^2 . We will thus now have a 3-dimensional local toric geometry, where the extra circle action comes from the rotation on the phase of the normal line bundle. Going through the exercise just as we did for the case of \mathbf{P}^1 will give the toric data summarized below:

A copy of \mathbf{P}^2 is recognized at the bottom of the above figure and the lines over it correspond to the normal direction on \mathbf{P}^2 . If we call the extra circle direction c , then the zero section of the normal bundle, which gives a copy of \mathbf{P}^2 corresponds to c being shrunk. Similarly the cycles that vanish at the other faces can also be worked out and give the picture above. This example can be easily generalized to the case where we have a \mathbf{P}^n sitting in an $n + 1$ CY manifold, where again the normal direction to \mathbf{P}^n is identified with the space of $(n, 0)$ forms on \mathbf{P}^n .

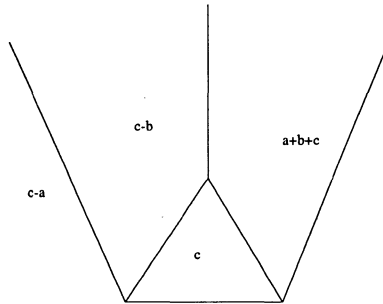


Figure 11: The toric realization of $N(\mathbf{P}^2)$. A copy of \mathbf{P}^2 is visible as the triangle at the bottom. Each half line emanating from any point on it, will correspond to the normal direction of \mathbf{P}^2 in the Calabi-Yau threefold.

2.1 Toric Varieties

From the above examples it should be clear how to generalize the notion of \mathbf{P}^n or normal bundles to them, to a more general class involving manifolds which admit toric action [21]. We are interested in manifolds admitting a T^n action, with an n -dimensional base. The n -dimensional base will have $n - 1$ dimensional boundary decomposed to various faces where a particular S^1 cycle of the fiber shrinks, corresponding to where the T^n action has fixed loci. Moreover when k of these faces meet a T^k has shrunk to zero size. The data defining the toric variety is precisely how these faces meet and which cycles vanish over which face.

In general toric varieties will have singularities. For example if you consider the case of A_{m-1} space, when we take the $m - 1$ finite size intervals to zero size geometrically we get a singular space. Torically the way to read this singularity is rather simple: The two edges that now meet correspond to shrinking $b - a$ and $b + (m - 1)a$ cycles. Note that the lattice of 1-cycles on T^2 are *not* generated by $b - a$ and $b + (m - 1)a$ for $m \geq 1$. Note that this is precisely the case where we have a $\mathbf{C}^2/\mathbf{Z}_m$ singularity. If we blow up the m -spheres then it is easy to see that whenever two edges meet the vanishing cycles form a basis for the lattice. This turns out to be the general consideration for a non-singular toric variety, namely whenever n faces meet we should get a basis for the n dimensional lattice dual to the T^n fiber coming from the vanishing cycles on each face. To be more precise, if we denote the lattice generated by all one cycles in the T^n fiber by M and we denote the sublattice generated by those shrinking one cycles at any face of the polytope by M_0 . Both M and M_0 will have the same rank when the face is of zero dimensional, namely a vertex. In this case the quotient $M/M_0 = G$ will be a finite group of reflexions. Locally the geometry looks like \mathbf{C}^n/G which is singular at the origin unless M and M_0 are the same. In the case

of A_{m-1} singularity we have $G = \mathbf{Z}_m$. It is not difficult to see that every points in a face which is adjacent to a smooth vertex point is also a smooth points. Therefore to check smoothness of the toric variety, it is sufficient to check only those vertex points.

2.2 Hypersurfaces in Toric Varieties

So far we have talked about the manifolds being the toric varieties themselves. However many interesting geometries are not of this type. For example, no compact Ricci-flat manifold is toric—the above examples gave some non-compact examples of Ricci-flat manifolds. In order to remedy this, but still be close to the nice toric situation one can start with a higher dimensional toric variety and impose some equations. The simplest set of such cases involve degree $n + 2$ hypersurfaces in \mathbf{P}^{n+1} manifolds which give rise to Calabi-Yau n -folds. We have a polynomial

$$W(z_i) = 0, \tag{2.8}$$

where i runs from 1 to $n + 2$ and this is a homogeneous equation of degree $n + 2$. Note that we can view W as a section of a line bundle on \mathbf{P}^{n+1} (of degree $n + 2$); as discussed before it is natural to associate in such cases an integral polytope which has in addition the information of the monomials W in it. For every point on the integral polytope and its interior we can write a monomial deformation for W . What this means is as follows: Consider for each point $r = (r_1, \dots, r_{n+1})$ in the integral polytope, a monomial $z^r = z_1^{r_1} \dots z_{n+1}^{r_{n+1}}$. Then the manifold hypersurface is described by an equation

$$\sum_{r \in \text{polytope}} a_r z^r = 0, \tag{2.9}$$

for some coefficients a_r , where this is a local description of the manifold in a patch. In particular shifting the points by an integral shift does not change the local geometry (as long as we keep away from $z = 0, \infty$). It is often convenient to choose the integral polytope to contain the origin, in which case there would exist a monomial deformation corresponding to addition of 1 to the equation.

Note that this hypersurface given by $W = 0$ will not have any toric symmetry, because for a generic choice of W the torus actions is not compatible with the equation—i.e. W is not invariant under them. There is, however, a degenerate limit of W in which the space does become toric. Consider in the homogeneous variables

$$W(z_i) = W_0(z_i) + \psi z_1 z_2 \cdots z_{n+2} = 0. \tag{2.10}$$

The deformation corresponding to $\prod z_i$ in the above homogeneous variables gets mapped to the deformation given by 1 in the above patch description of the manifold (2.9). Now we consider the limit $\psi \rightarrow \infty$. In this limit the equation for the hypersurface becomes approximately

$$\prod_i z_i = 0. \tag{2.11}$$

This consists of the union of the boundary faces of the polytope each of which corresponds to $z_i = 0$. Thus roughly speaking the Calabi-Yau n -fold consists of an n dimensional real base over which we have T^n fibers (one circle has already shrunk on each face $T^{n+1} \rightarrow T^n$). Note that in this limit where k faces of the polytope meet the fiber is T^{n+1-k} .

More generally we can obtain a Calabi-Yau space as a hypersurface in toric variety by considering what is known as “reflexive polytope” as we will explain now. Instead of the standard $n + 1$ simplex which corresponds to \mathbf{P}^{n+1} , we can use arbitrary polytope Δ in \mathbf{R}^{n+1} to construct a corresponding toric variety \mathbf{P}_Δ which might be singular. Each of the boundary faces of Δ will give a hypersurface in \mathbf{P}_Δ which itself is a toric variety of one lower dimension. Equivalently we can view it as the zero locus of a section s of a line bundle L corresponding to this face. We assume that there are m boundary faces and $s_i = 0$ define them, where $i = 1, \dots, m$. Then $s_1 s_2 \cdots s_m$ becomes a section of $\otimes_{i=1}^m L_i$.

Just as in the \mathbf{P}^{n+1} case, we want to perturb $s_1 s_2 \cdots s_m$ by a general section $W_0(z)$ of $\otimes_{i=1}^m L_i$ to obtain a smooth hypersurface (or with mild singularities):

$$W(z) = W_0(z) + \psi s_1 s_2 \cdots s_m = 0 \tag{2.12}$$

and we will recover the union of the boundary hypersurfaces by taking the $\psi \rightarrow \infty$ limit. Note that since we are considering a hypersurface inside the toric variety as zeros of a section, we can again give the toric polytope an integral structure compatible with the choice of the line bundle. In order for this hypersurface to be Calabi-Yau we need $\otimes_{i=1}^m L_i$ to be the same as the inverse of the canonical line bundle $K_{\mathbf{P}_\Delta}^{-1}$ of \mathbf{P}_Δ in order to cancel the c_1 . It turns out that this condition can be rephrased in terms of the integral polytope Δ .

To do that we need to first introduce the idea of dual polytope. For any integral polytope Δ containing the origin as an interior point, its dual polytope ∇ is roughly the convex polytope bounded by hyperplanes $a_1 y_1 + a_2 y_2 + \dots + a_n y_n = 1$ for (a_1, a_2, \dots, a_n) any vertex of Δ . More precisely the dual polytope is defined by

$$\nabla = \{(y_1, \dots, y_n) : x_1 y_1 + x_2 y_2 + \dots + x_n y_n \leq 1 \text{ for } (x_1, \dots, x_n) \in \Delta\}. \tag{2.13}$$

For example let Δ be the polytope with vertexes $(-1, -1)$, $(-1, 2)$ and $(2, -1)$ in the plane corresponding to \mathbf{P}^2 with a degree 3 bundle on it, then its dual polytope ∇ is bounded by $(1, 1)$, $(-1, 0)$ and $(0, -1)$ as shown below:

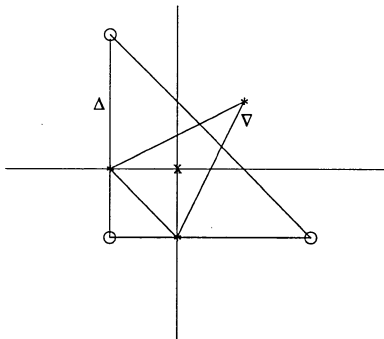


Figure 12: Here we are showing the integral reflexive polytope corresponding to \mathbf{P}^2 (denoted by Δ) and its dual denoted by ∇ . Note that for every face of Δ we get a vertex of ∇ and vice-versa. The origin connected to each vertex of ∇ is orthogonal to the corresponding dual face in Δ . Also note that the origin is the only interior integral point.

In general the dual polytope of an integral polytope may not be integral again, when this is the case, Δ will be called a reflexive polytope. In this case, its dual polytope is also a reflexive polytope. For a reflexive polytope, each vertex of Δ will correspond to a boundary face of ∇ and vice versa [17] (this is illustrated in the figure for the \mathbf{P}^2 example above). More generally for every k dimensional face of the polytope Δ there is a dual $n - k - 1$ dimensional face of ∇ .

In fact Δ being reflexive is equivalent to $K_{\mathbf{P}^n_\Delta}^{-1} = \bigotimes_{i=1}^m L_i$ which guarantees the corresponding hypersurface to be Calabi-Yau. The deformation which we denoted by $s_1 \dots s_m$ will correspond to the origin for reflexive polytopes. This can be easily seen to be the case for the \mathbf{P}^2 example discussed before and turns out to be a general fact.

These constructions can be generalized to the case of varieties defined by more equations. Our discussions can be easily generalized to these cases as well. We will leave this to the reader.

2.3 The Dual Toric Constructions

If Δ is a reflexive polytope then ∇ is also reflexive. Therefore it defines another toric variety \mathbf{P}_∇ and the zero of a general section W'_0 of $K_{\mathbf{P}_\nabla}^{-1}$ will be Calabi-Yau. This construction is proposed by Batyrev to obtain the mirror of Calabi-Yau hypersurface in \mathbf{P}_Δ . Again we can take the limit as

$\psi' \rightarrow \infty$ of

$$W'(z) = W'_0(z) + \psi' t_1 t_2 \cdots t_{m'} = 0, \tag{2.14}$$

to obtain the union of the boundary faces of ∇ and each such face corresponds to some $t_i = 0$.

In the limit of ψ and $\psi' \rightarrow \infty$, the geometries of the Calabi-Yau hypersurfaces are described by the boundary of Δ and ∇ . Each boundary face of Δ will correspond to a vertex in ∇ and vice versa. This is a manifestation of the $R \rightarrow 1/R$ duality of tori as we will see later in this paper.

2.4 Open Toric Varieties and their Dual

As in our earlier discussions, we are also interested in non-compact cases such as $N(\mathbf{P}^2)$, as they can form a local piece of a Calabi-Yau threefold. Instead of just one space like $N(\mathbf{P}_\Delta)$, we can also have a union of several toric varieties intersecting each other along toric subvarieties. Examples of this kind include the blown up of A_n singularity where $n \mathbf{P}^1$ intersecting each other in a linear manner as we already discussed. A large number of such examples has been considered in [4].

One can extend the definition of duality for open toric varieties as well. We first look at the dual polytope similar to the global case. Put the origin inside the polytope Δ and consider all the rays emanating from the origin and orthogonal to each face. If the boundary face is defined by $a_1 x_1 + \dots + a_n x_n = c$, then the corresponding ray will pass through the point (a_1, \dots, a_n) . If the polytope has m faces then the collection of rays is characterized by m integral points (a_1, \dots, a_n) . For example for the A_n case discussed before we get the collection of $(n + 2)$ points in \mathbf{Z}^2 given by $[(-1, 1), (0, 1), (1, 1), \dots, (n, 1)]$. Similarly for $N(\mathbf{P}^2)$ we get the four points $[(1, 1, 1), (0, -1, 1), (-1, 0, 1), (0, 0, 1)]$ Note that the last entry in all these cases is 1. This reflects the fact that if we fix the normal direction circle, to shrink at a given point, as we go from one face to another, which circle shrinks gets modified only by addition with a circle describing the compact pieces. Thus the data of the dual object given by rays will contain the same information as a collection of points in a 1 dimensional lower integral lattice consisting of m integral points. For the case of A_n this will give us $[(-1), (0), (1), \dots, (n)]$ which consists of $n + 2$ integral points along the line. Similarly for the $N(\mathbf{P}^2)$ case we get the points $[(1, 1), (0, -1), (-1, 0), (0, 0)]$. Note that these are exactly the same points defining the dual polytope for \mathbf{P}^2 together with the interior point $(0, 0)$.

3 Branes and Toric Geometry

There are some examples known where geometry can be replaced with a configuration of branes. These include M-theory with A_n singularity which correspond to n D6 branes of type IIA [22] type IIA/B over an A_n singularity which corresponds to type IIB/A with n NS 5-branes of type IIA/B [10], type IIA/B over a conifold, which is equivalent to 2 intersecting NS 5 branes [23] (see also [24, 25]). It is natural to try to extend this dictionary to other singularities of geometry (see for example for one such extension), and as it turns out toric geometry is the right language for this purpose. In fact those geometries which are locally a toric space, can be realized via branes. However as we have mentioned before, and will see below again, not all geometries have toric realization. In particular, in order to see some interesting geometries we have to consider hypersurfaces (or complete intersections) in a higher dimensional toric variety. In such cases in general there is no known way to associate a configuration of branes. Thus it appears that in geometry we have a more general approach in engineering physical systems.

We have seen that toric geometries are essentially trivial except for the fact that on some loci some circles shrink. The shrinking circles are a source of charge of branes and so these loci are naturally identified with branes of appropriate type. This is the basic link between toric geometry and branes.

3.1 M-theory on S^1 and D6 Branes

Let us consider the simple example of the A_1 singularity $T^*\mathbf{P}^1$. As noted above M-theory on this space is equivalent to type IIA with 2 units of D6 branes. To see this note that the KK monopoles of M-theory on a circle being equivalent to D6 branes means that if we consider a supersymmetric geometry in which the circle of M-theory shrinks to zero size at some loci, we obtain the D6 branes. Consider the toric geometry above and identify the S^1 circle of M-theory with the θ action on $T^*\mathbf{P}^1$ discussed in example ii'). If we consider modding out this space by the circle action

$$T^*\mathbf{P}^1/S_\theta, \tag{3.1}$$

where the S_θ denotes the circle action associated with the θ direction, we obtain a 3-dimensional geometry which we can identify with the type IIA space. Moreover the points on the geometry where the S_θ has zero size, correspond to D6 branes. These are the points where the a cycle vanishes and from the toric diagram it is clear that this happens only at two points on the ends of the interval representing the \mathbf{P}^1 . Similarly for M-theory on the A_n geometry we find it is equivalent to type IIA in the presence of $n+1$ D6 branes, where the $n+1$ points correspond to the $n+1$ end points of the n compact \mathbf{P}^1 's represented by the n -intervals in the toric diagram.

3.2 M-theory on T^2 and (p, q) 5-branes

Above we have considered M-theory on S^1 . Let us now consider M-theory on T^2 . This theory is equivalent to type IIB on S^1 . There are two cycles on the T^2 , and the Kaluza-Klein monopoles associated to the (p, q) cycle of the T^2 corresponds to (p, q) 5-branes of type IIB (transverse to the compactified S^1 of type IIB).

This relation allows us to realize the local geometry of Calabi-Yau threefolds which have toric realization in terms of type IIB (p, q) 5-branes. Such geometries will have a compact T^2 action corresponding to the 4-dimensional compact local model, which we can mod out, and just as in the above example realize them in terms of (p, q) 5-branes. For example let us consider M-theory on the Calabi-Yau threefold with a small \mathbf{P}^2 . Then the local model is as in example iii') above. Considering modding out by the T^2 action corresponding to the two finite circle actions on the \mathbf{P}^2 . Let $N(\mathbf{P}^2)$ denote the \mathbf{P}^2 with the normal bundle on top of it, and consider the 4-dimensional space

$$N(\mathbf{P}^2)/T^2, \tag{3.2}$$

where the T^2 denotes the lift of the action from the one on \mathbf{P}^2 . This 4-dimensional space is trivial except for the loci where a circle action of T^2 has fixed points—This happens when the a and b cycle vanish and from the diagram in Fig. 11 we see that this occurs on the subspace shown below (with the corresponding shrinking cycle (a, b) indicated). Note that when two faces in Fig. 11 meet, any combination of vanishing cycles on either side vanishes on the intersection.

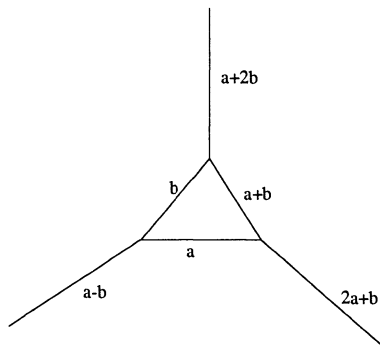


Figure 13: The brane realization of $N(\mathbf{P}^2)$ in Calabi-Yau threefold. All that has happened is that we have replaced the toric skeletons with the corresponding (p, q) 5-branes.

We thus interpret this as a type IIB geometry with the above diagram giving the configuration of (p, q) 5-branes (where one extra S^1 has an ar-

bitrary size on the type IIB side). This is exactly the geometry proposed in [13, 16] for a critical theory dual to the \mathbf{P}^2 shrinking in the Calabi-Yau. Here we have explained this duality. Moreover for every geometry proposed in [13] we can write down the geometric analog.

Note that the condition of (p, q) charge conservation in [13] gets mapped to the condition of $c_1 = 0$ for the local 3-fold. At each vertex, using the $SL(2, \mathbf{Z})$ symmetry, we can bring the local configuration to the standard basis $(1, 0)$ and $(0, 1)$ in \mathbf{Z}^2 together with an external leg with label (p, q) . For $N(\mathbf{P}^2)$ to be Calabi-Yau, as a line bundle over \mathbf{P}^2 , it must be $K_{\mathbf{P}^2}^{-1}$, i.e., the tensor product of the three line bundles corresponding to the three edges of the standard 2 simplex. Now the origin is given by the intersection of the two edges $z_1 = 0$ and $z_2 = 0$. Therefore the T^2 action on the fiber over the origin is given by $p \rightarrow \exp(-i\theta - i\phi)p$ as in iii). Hence the projection of this fiber to the (x, y) plane is the half line corresponding to the label $(-1, -1)$ which is the same as the (p, q) charge conservation.

In fact for the models in [13] the angles at which the 5-branes intersected was related to the (p, q) charge. Basically (at $\tau = i$) the (p, q) 5-brane was placed along the (p, q) direction. As we have said before this is very natural from toric geometry view point as well. This is a beautiful interplay between branes and toric geometry.

This dictionary we have found between geometry and branes will explain some of the issues which were puzzling in [13, 16]: It was observed there that when one tries to construct the brane version of the critical theory in 5 dimensions corresponding to \mathbf{P}^2 blown up at more than 3 points one gets certain puzzles. One finds that the external lines become parallel or intersecting (for more than 5 points blown up) in such cases. This in particular prevents their interpretation as a critical 5 dimensional theory. This is puzzling because \mathbf{P}^2 blown up at up to 8 points should lead to critical theory [26–29]. Even though the case with parallel external lines appear less harmful than the one with intersecting external lines, some puzzles were raised even for this case in [16]. Note that this would limit the number of blow up points to 3, if we were to realize it torically.

In order to resolve this puzzle let us translate the condition of parallel external lines to geometry: If two parallel external lines bound an interval in the toric polytope, then the c_1 evaluated on the \mathbf{P}^1 which is represented by the interval is zero. To see this, without loss of generality we can denote the 5-brane charges on one end of the interval to consist of $(-1, 1)$ and $(0, 1)$ internal line 5-branes meeting the $(-1, 0)$ external 5-brane, and the other end the $(0, 1)$, $(1, 1)$ 5-brane meeting the $(-1, 0)$ external 5-brane which is parallel to the other external line. However the 2-dimensional complex piece of this geometry is *exactly* the same as the one we already studied, and the

geometry of \mathbf{P}^1 in \mathbf{P}_Δ is locally the same as $T^*\mathbf{P}^1$. This implies that

$$\int_{\mathbf{P}^1} c_1(\mathbf{P}_\Delta) = \mathbf{0}, \tag{3.3}$$

as is well known for the $T^*\mathbf{P}^1$ geometry. Similarly when external parallel lines intersect, one can show this implies that $c_1(\mathbf{P}_\Delta)$ integrated over the middle \mathbf{P}^1 is negative.

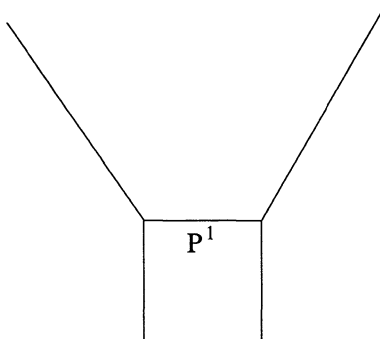


Figure 14: When we have parallel external lines the geometry in the neighborhood of the middle \mathbf{P}^1 in the 2-complex dimensional base, is essentially that of $T^*\mathbf{P}^1$.

However a submanifold can be shrunk within a background geometry if and only if its normal bundle is negative by Grauert's criterion [30]. When the background is a Calabi-Yau, that is $c_1 = 0$, then this is equivalent to the submanifold having c_1 strictly positive². This means that even though these geometries make sense locally they cannot be shrunk to zero at finite distance in moduli. In other words they do not lead to conformal theories in 5 dimensions.

As mentioned before and reviewed above, the fact that with more than 3 points blown up \mathbf{P}^2 cannot be realized torically is mathematically well known. However as noted before even between 3 and 8 points blown up \mathbf{P}^2 can shrink in a Calabi-Yau, and can be realized in a higher dimensional toric variety when we impose equations. We are just learning that the brane realization of quantum field theories appear to be more limited than geometric engineering approach. Or turning it around, we should try to understand what is the brane analog of going to higher dimensions in geometry and imposing equations to decrease the dimension back down. There are some hints how this may be possible: In particular there is a simple brane realization of \mathbf{P}^2 with 9 points blown up in terms of F-theory background, which involves (p, q) 7-branes compactified on a $\mathbf{P}^1 \times S^1$ [26,27,31] (see [32] for an extensive

²We would like to thank D. Morrison for a discussion on this point.

study of BPS states in this case). However, one should keep in mind that the main question is not whether a given geometry has a brane realization or not, but more importantly whether it has a useful brane realization. In the above brane realization one reverts to the geometric picture of M-theory on \mathbf{P}^2 with up to 8 points blown up to extract physical results [31, 32] (such as BPS spectrum). Another (and perhaps more useful) brane realization in this case may be to consider a knotted configuration of (p, q) 5-branes and (p, q) 7-branes piercing through them (perhaps corresponding to blowing up points *inside* the \mathbf{P}^2 triangle). It should be interesting to study such cases.

3.3 M-theory on T^3 and (p, q, r) 4-branes

There are many extensions of the above toric construction. We will limit ourselves just to one more example, though we believe our approach can be used in many different contexts.

Consider M-theory on T^3 . Then we have the $SL(3)$ symmetry as part of the U -duality group. The KK monopoles will now be labeled by a vector (p, q, r) and will correspond to a 4-brane in the 7-dimensional geometry. From the type IIB perspective this corresponds to compactification on T^2 where we consider a (p, q) 5-brane wrapped around one of the circles and bound to a KK monopole of charge r around that circle. There is also a type IIA description of the same object: It corresponds to compactification on T^2 with r units of $D6$ brane wrapped around T^2 and bound to KK monopoles of charge (p, q) on T^2 .

Now just as a simple application, consider M-theory on a local singularity of a Calabi-Yau 4-fold, yielding an $N = 2$ system in 3 dimensions.³ For example consider \mathbf{P}^3 shrinking inside a Calabi-Yau 4-fold. (similar shrinking spaces can be considered and some are equivalent to the models considered in [13]) Then the local model is the canonical bundle over \mathbf{P}^3 which we denote by $N(\mathbf{P}^3)$. Then just as in the case of \mathbf{P}^2 considered above we can realize the geometry of the toric polytope in terms of the (p, q, r) 4-brane intersecting in a particular geometry in 3-dimensions. Moreover the intersection angles are exactly dictated by the toric data, just as in the \mathbf{P}^2 case. The figure of intersections look as follows.

Reduction of such cases to 2 dimensions and local mirror symmetry in such cases has been recently considered in [6].

³These theories typically have superpotentials generated. Here we will not worry about whether there are such terms generated or not, and simply consider the “classical theory”. In fact for a shrinking \mathbf{P}^3 considered here there is a superpotential generated [33].

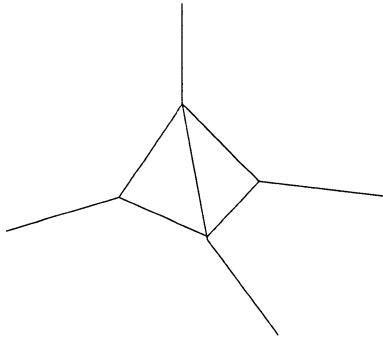


Figure 15: The brane realization of \mathbf{P}^3 . In addition to the 4 faces of the tetrahedron corresponding to certain (p, q, r) 4-branes, there are 6 external 4-branes ending on each of the 6-edges of the tetrahedron.

4 Intuitive Explanation of Mirror Symmetry

One of the most important applications of toric geometry has been to mirror symmetry. In fact starting with the work of Batyrev [17] the mirror symmetry conjecture has been made quite systematic using toric geometry. Many attempts have gone into proving mirror symmetry using this structure [34–36]. The basic idea being that the toric geometry means we have tori as fibers and $R \rightarrow 1/R$ duality symmetry of each circle should give rise to a simple description of mirror symmetry. Unfortunately none of these approaches have been made complete. An exception to these cases involve orbifolds for which mirror symmetry have already been rigorously proven to follow from $R \rightarrow 1/R$ symmetry [37]. Here we follow the same spirit of argument and find some intuitive explanation of mirror symmetry again reducing it to $R \rightarrow 1/R$. Our approach will not be rigorous, but we believe it makes mirror symmetry very plausible and intuitive. In particular we find a simple explanation of Batyrev’s construction for mirror pairs. We believe our approach can be generalized to $(0, 2)$ sigma models without much difficulty.

We divide our discussion into two parts. One involving the global case, where we consider compact Calabi-Yau manifolds realized as hypersurfaces in toric geometry and the second one involving what is called ‘local mirror symmetry’ [2, 4]. This latter one is the one of most interest in ‘solving’ the $N = 2$ gauge theories in 4 dimensions. The intuitive explanation that we will find for this case makes the toric construction as simply related to the geometry of the $N = 2$ curves as the one recently found in [8] but the geometry appears to be distinct from it. However, just as was the case for \mathbf{P}^2 with more than 3 points blown up where we cannot just use a local toric model to realize all such models. In general we need to apply mirror symmetry in situations as in the global case where we need a higher dimensional space

with some equations imposed. Again finding a brane analog for such cases is challenging. Many such cases were addressed geometrically in [2] and are a special limit of the global case considered below.

4.1 Global Case

Let us start with an example: Consider \mathbf{P}^{n+1} with degree $n+2$ polynomial. As discussed before this gives rise to a Calabi-Yau n -fold. At a particular singular complex structure limit this Calabi-Yau becomes toric and is identified with the boundary of the $n+1$ dimensional polytope. There are $n+2$ faces, over each of which we have a T^n fiber (one of the circles have already shrunk by restricting to the boundary). Now, if we perform T-duality for each circle of T^n replacing it by $1/R$, we should get a mirror Calabi-Yau. However, note that where k faces meet the T^n shrinks to T^{n-k} and which T^{n-k} we get depends on which faces meet. Applying T-duality on such loci leads to the following picture: not only it restores the shrunk circles to big circles, but makes them more dominant than the finite size circles. In order to get a better picture, let us assume the zero size circles have finite size ϵ and at the end we let $\epsilon \rightarrow 0$. Moreover let us change the Kahler class of the Calabi-Yau so that all the lengths are rescaled by the large factor $1/\epsilon$. Now the cycles which were finite size become of order of $1/\epsilon$ and the ones which were shrinking become finite size. Now we apply the T-duality. In this case all the finite size circles shrink and all the ones which were previously finite size become of size one. One gets a new space which in fact by definition is the boundary of the dual polytope! Note that *the construction of the dual polytope where there is a Poincare duality on the boundary of polytope is now interpreted very physically as the exchange between the regions where the circles were small with regions where the circles are big.*

Note that in the original manifold we have taken *both* the Kahler and complex structures large. Mirror transform will tell us that we should land on the mirror manifold which is again corresponding to the large complex class and Kahler class. This is consistent with the fact that in order to get the boundary of the dual polytope as the mirror, we are at the large complex structure of the mirror which is mirror to the large Kahler class of the original manifold. It is no accident that we had to take the large Kahler structure of the original manifold to get a simple description of the mirror as a degenerate toric manifold.

At the large Kahler and complex structure limit, we can also identify, up to first order, the Kahler moduli of the original manifold with the complex moduli of its mirror and vice versa in more detail. This would give a realization of the monomial-divisor mirror map as proposed in [38]. The arguments we give are parallel to those recently given in [14–16] in connection with the M-theory realization of $N = 2$ gauge systems in 4 dimensions.

Let us illustrate how this works: Suppose that X is the Calabi-Yau hypersurface defined by $W(z) = 0$ inside a toric variety \mathbf{P}_Δ with Δ a reflexive polytope as in section 2.2. We consider Kahler metric on X which comes from restriction of Kahler metric on \mathbf{P}_Δ . Each boundary face of Δ will determine a line bundle whose curvature is a closed two form on \mathbf{P}_Δ . Suitable positive combination of them will give us general toric Kahler metric on \mathbf{P}_Δ . Explicit description of these Kahler forms are given in [39]. In general cohomology classes of these Kahler forms could be dependent. For example in the case of $K3$ surface in \mathbf{P}^3 , there is only one such Kahler form up to scaling factor.

In the $R \rightarrow 1/R$ duality, before we scale the Kahler class by the global factor $1/\epsilon$, if we take the Kahler metric to be one which is dominated by one of the boundary faces of Δ , then in the mirror, the corresponding face will become a vertex of ∇ . As noted before any integral point in the mirror reflexive polytope ∇ will correspond to a monomial which is in fact a section s of $K_{\mathbf{P}_\nabla}^{-1}$ and which can be used to deform the complex structure of the mirror. Let y_i with $i = 1, \dots, n$ denote the coordinates of the mirror toric variety. Then as noted the deformation corresponding to the origin of ∇ corresponds to the monomial 1. On the other hand let (a_1, \dots, a_n) denote a vertex on ∇ . To that we can associate the monomial deformation on the mirror of the form $y_1^{a_1} \dots y_n^{a_n}$. Now let us consider the limit in complex deformation of the mirror in which these two terms have large coefficients. Then the equation defining the manifold in the mirror is effectively dominated by

$$\alpha + \beta y_1^{a_1} \dots y_n^{a_n} = 0, \tag{4.1}$$

for some α, β . Notice that some of a_i 's can be negative and setting α goes to infinity gives the singular Calabi-Yau corresponding to the union of boundary faces of ∇ . Now when we deform the complex structure away from this singular point using the monomial deformation by $y_1^{a_1} \dots y_n^{a_n}$. then the manifold becomes dominated in this limit by

$$y_1^{a_1} \dots y_n^{a_n} = \text{const.} \tag{4.2}$$

Given that in our derivation of mirror symmetry this vertex was associated to a face of the original polytope Δ we would expect that this large complex structure limit should be mirror to large Kahler class limit for this particular divisor. But in this limit the manifold is dominated by that face, whose real section corresponds to

$$\sum_i a_i x_i = \text{const.} \tag{4.3}$$

As far as mirror symmetry is concerned the base of the geometry for the manifold and the mirror are identified, thus x_i should be related to real part

of y_i . In fact comparing (4.2) with (4.3) we see that the natural identification of

$$y_i = e^{x_i} \tag{4.4}$$

will make (4.3) and (4.2) identical. We have thus found a simple physical explanation of the divisor/monomial mirror map. In other words the complex deformation mirror to the Kahler deformation controlling the size of a divisor is the coefficient of the monomial associated to the vertex mirror to the divisor.

4.2 Local Case

Now consider a local toric model of Calabi-Yau, such as $N(\mathbf{P}^n)$. We wish to find its mirror, i.e. a geometry whose complex structure is mirror to the Kahler class of $N(\mathbf{P}^n)$. We can try to repeat an argument similar to the above as in the global case, but we will try to take a short cut. This arises because some of the data in the original manifold is not necessary in defining it. In fact this was already reflected in our discussion of the dual object associated in these cases.

The data characterizing the Kahler geometry of the model in these non-compact cases is concentrated on the subspace of polytope of dimension $n - 1$. For example for A_k this will be $k + 2$ points which is also related to the fact that we found that the natural dual object in this case will consist of $k + 2$ successive points on an integral lattice in one dimension. In the case of the $N(\mathbf{P}^2)$ this is a one dimensional graph together with three external lines, which constitute the (p, q) 5-branes, together with the S^1 fibers on top. In other words the data of the local geometry can be reconstructed by a 1-dimensional singular object consisting of $3\mathbf{P}^1$'s meeting along a triangle and with 3 half \mathbf{P}^1 's coming out from the vertices:

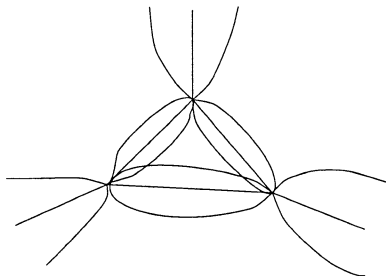


Figure 16: The toric realization of $N(\mathbf{P}^2)$ is captured by three spheres intersecting each other as well as three external half-spheres as shown. The elliptic curve is visible in the picture as the fattened triangle in the middle.

More generally this gives the $n - 1$ complex dimensional object which is a T^{n-1} toric fiber over an $n - 1$ dimensional collection of faces. We now apply mirror symmetry to this situation by acting on the T^{n-1} fibers with $R \rightarrow 1/R$. We can trust this action away from the singular points where the curves meet. Note that in the case of $N(\mathbf{P}^2)$ we get a Riemann surface and the endpoint of the external lines get mapped to a point on the surface (because the circle infinitely big is mapped to a tiny circle). So we obtain an elliptic curve with 3 punctures.

To write the local complex geometry of the mirror we can use quite generally exactly the same idea as in the global case. Namely consider the set of integral points which will be m points in a lattice of dimension n . Recall that these are in one to one correspondence with rays orthogonal to faces. Consider the complex space

$$\sum_{r=(a_1, \dots, a_n) \in \text{space of rays}} c_r y_1^{a_1} \dots y_n^{a_n} = 0, \tag{4.5}$$

which gives a space of complex dimension $n - 1$ as expected above. To make this application of mirror symmetry more intuitive again we can extend the analog of $R \rightarrow 1/R$ duality face by face. Namely recall that the original polytope is characterized by the projections of its faces onto an n real dimensional space which consists of $n - 1$ dimensional skeletons. Consider one such face of the skeleton. This will be bounded by two n dimensional faces of the original polytope which are associated with two monomials $y_1^{a_1} \dots y_n^{a_n}$ and $y_1^{b_1} \dots y_n^{b_n}$. Now if we consider the limit of Kahler classes where a particular $n - 1$ dimensional skeleton is large. This will be mirror to the complex deformation where the two monomials have large complex coefficients. In this limit the equation for the mirror gets dominated by

$$\alpha y_1^{a_1} \dots y_n^{a_n} + \beta y_1^{b_1} \dots y_n^{b_n} = 0, \tag{4.6}$$

which gives

$$y_1^{a_1 - b_1} \dots y_n^{a_n - b_n} = \text{const}. \tag{4.7}$$

Again as in the global case since the $R \rightarrow 1/R$ does not act on the real part this should also give the real part of the space which is the skeleton. If we identify $y_i = \exp(x_i)$ we obtain the equation of the skeleton

$$\sum_i (a_i - b_i) x_i = \text{const}, \tag{4.8}$$

as expected. Just as an example if we consider the $N(\mathbf{P}^2)$ case we get (using the definition of the rays discussed in section 2)

$$a + by_1 + cy_2 + \frac{d}{y_1 y_2} = 0, \tag{4.9}$$

as the local model for the mirror manifold. Another example involves fibering A_n spaces over \mathbf{P}^1 (the generalization of which to many examples was already considered in detail in [4] and the complex moduli including the coupling can be viewed as the radius dependence of the 5 dimensional critical theories compactified on a circle [40]). In this case we get the figure

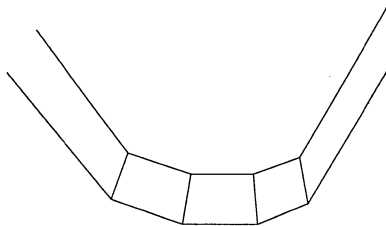


Figure 17: The toric realization of $SU(n)$ gauge theory produces a ‘visible’ genus $n - 1$ Riemann surface as its skeleton. The short direction of the ‘ladder’ in the above figure corresponds to a \mathbf{P}^1 and the long direction corresponds to the blown up of an A_{n-1} space.

Here we see the $N = 2$ Riemann surface very visibly as the skeleton of the toric graph. Similar observations have been recently made in connection with the M-theory approach [14–16].

We would like to thank S. Katz, P. Mayr and S.-T. Yau for valuable discussions.

The research of C.V. was supported in part by NSF grant PHY-92-18167.

References

- [1] S. Kachru and C. Vafa, *Nucl. Phys.* **B450** (1995) 69; S. Kachru et al., *Nucl. Phys.* **B459** (1996) 537; A. Klemm, W. Lerche, P. Mayr, C. Vafa and N. Warner, *Nucl. Phys.* **B477** (1996) 746 ; A. Brandhuber and S. Stieberger, *Nucl. Phys.* **B488** (1997) 199; J. Schulze and N. P. Warner, *Nucl. Phys.* **B498** (1997) 101; J. M. Rabin, *Phys. Lett.* **B411** (1997) 274.
- [2] S. Katz, A. Klemm and C. Vafa, *Nucl. Phys.* **B497** (1997) 173.
- [3] S. Katz and C. Vafa, *Nucl. Phys.* **B497** (1997) 196.
- [4] S. Katz, P. Mayr and C. Vafa, *Adv. Theor. Math. Phys.* **1** (1998) 53.
- [5] C. Vafa, *Adv. Theor. Math. Phys.* **1** (1998) 158.
- [6] W. Lerche, “Fayet-Iliopoulos Potentials from Four-Folds”, hep-th/9709146.

- [7] A. Hanany and E. Witten, *Nucl. Phys.* **B492** (1997) 152.
- [8] E. Witten, *Nucl. Phys.* **B500** (1997) 3.
- [9] A. Klemm, W. Lerche, P. Mayr, C. Vafa and N. Warner, *Nucl. Phys.* **B477** (1996) 746.
- [10] H. Ooguri and C. Vafa, *Nucl. Phys.* **B463** (1996) 55.
- [11] H. Ooguri and C. Vafa, *Nucl. Phys.* **B500** (1997) 62.
- [12] K. Hori, H. Ooguri and C. Vafa, *Nucl. Phys.* **B504** (1997) 147.
- [13] O. Aharony and A. Hanany, *Nucl. Phys.* **B504** (1997) 239.
- [14] B. Kol, “5d Field Theories and M-Theory”, hep-th/9705031.
- [15] A. Brandhuber, N. Itzhaki, J. Sonnenschein, S. Theisen and S. Yankielowicz, *Phys. Lett.* **B415** (1997) 127.
- [16] O. Aharony, A. Hanany and B. Kol, “Webs of (p,q) 5-branes, Five Dimensional Field Theories and Grid Diagrams”, hep-th/9710116.
- [17] V. Batyrev, *J. Alg. Geom.* **3** (1994) 493.
- [18] S.-S. Roan, *Internat. J. Math.* **2** (1991) 439.
- [19] W. Fulton, “Introduction to Toric Varieties”, *Annals of Math. Studies*, No. **131**, Princeton University Press 1993.
- [20] D.R. Morrison and M.R. Plesser, *Nucl. Phys.* **B440** (1995) 279.
- [21] V. Guillemin, “Moment Maps and Combinatorial Invariants of Hamiltonian T^n -spaces”, Birkhauser 1994.
- [22] P.K. Townsend, *Phys. Lett.* **B350** (1995) 184.
- [23] M. Bershadsky, V. Sadov and C. Vafa, *Nucl. Phys.* **B463** (1996) 398.
- [24] J.P. Gauntlett, G.W. Gibbons, G. Papadopoulos and P.K. Townsend, *Nucl. Phys.* **B500** (1997) 133; G. W. Gibbons, *Nucl. Phys. Proc. Suppl.* **62** (1998) 422.
- [25] A. Sen, “A Note on Enhanced Gauge Symmetries in M- and String Theory”, hep-th/9707123.
- [26] D. R. Morrison and C. Vafa, *Nucl. Phys.* **B476** (1996) 437.
- [27] E. Witten, *Nucl. Phys.* **B471** (1996) 195.

- [28] M. R. Douglas, S. Katz and C. Vafa, *Nucl. Phys.* **B497** (1997) 155.
- [29] D. R. Morrison and N. Seiberg, *Nucl. Phys.* **B483** (1997) 229.
- [30] H. Grauert, *Math. Annalen* **146** (1962) 331.
- [31] A. Klemm, P. Mayr and C. Vafa, “BPS States of Exceptional Non-Critical Strings”, hep-th/9607139.
- [32] J. A. Minahan, D. Nemeschansky and N. P. Warner, *Adv. Theor. Math. Phys.* **1** (1998) 167.
- [33] E. Witten, *Nucl. Phys.* **B474** (1996) 343.
- [34] D. R. Morrison and M. R. Plesser, *Nucl. Phys. Proc. Suppl.* **46** (1996) 177.
- [35] A. Strominger, S.-T. Yau and E. Zaslow, *Nucl. Phys.* **B479** (1996) 243.
- [36] D.R. Morrison, “The Geometry Underlying Mirror Symmetry” in “Proceedings of European Algebraic Geometry Conference”, Warwick, 1996.
- [37] C. Vafa and E. Witten, *J. Geom. Phys.* **15** (1995) 189.
- [38] P.S. Aspinwall, B.R. Greene and D.R. Morrison, *Internat. Math. Res. Notices* (1993) 319.
- [39] V. Guillemin, *J. Diff. Geom.* **40** (1994) 285.
- [40] A. Lawrence and N. Nekrasov, “Instanton Sums and Five-dimensional Gauge Theories”, hep-th/9706025.