D0 Branes on T^n and Matrix Theory

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Abstract

The Hamiltonian describing Matrix theory on T^n is identified with the Hamiltonian describing the dynamics of D0-branes on T^n in an appropriate weak coupling limit for all n up to 5. New subtleties arise in taking this weak coupling limit for $n = 6$, since the transverse size of the D0 brane system blows up in this limit. This can be attributed to the appearance of extra light states in the theory from wrapped D6 branes. This subtlety is related to the difficulty in finding a Matrix formulation of M-theory on T^6.

During the last year great progress has been made in finding a non-perturbative formulation of M-theory [1–4] and its toroidal compactification [1–19]. This formulation, known as Matrix theory, can be summarized into the following recipe:

- Begin with type IIA string theory on an n dimensional torus T^n. For simplicity we shall take the torus to be rectangular, without any background anti-symmetric tensor fields, but these can easily be introduced. Let m_S^2 be the string tension, R_i $(1 \leq i \leq n)$ be the radii of the n circles on T^n (measured in physical units), and g_S be the string coupling constant. By using the usual duality between M-theory on S^1 and type IIA string theory [25, 26], this theory can be regarded as M-theory on

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- We consider the dynamics of N D0 branes in this type IIA string theory, and consider the weak coupling limit $g_S \to 0$ keeping the following quantities fixed:

$$r_i \equiv m_p R_i = g_S^{-1/3} m_S R_i, \qquad M \equiv m_p^2 R = m_S g_S^{1/3}. \qquad (2)$$

Keeping r_i fixed corrresponds to keeping the size of T^n, measured in units of m_p^{-1}, fixed. As we shall see, keeping M fixed guarantees that we keep the 'correct' degrees of freedom − necessary for obtaining the Matrix theory Hamiltonian − on the D0 brane world volume, throwing away the rest. In this limit, the Hamiltonian describing the dynamics of N D0 branes can be parametrized by the parameters M and r_i. Let us denote this by

$$H_N^{(n)}(M, \{r_i\}). \qquad (3)$$

- Let us now consider M-theory on T^n, with radii L_i. According to the Matrix theory conjecture as stated in [2], the Discrete Light Cone Quantization (DLCQ) of this theory, in which we compactify another light-like direction on a circle of radius L, and study the sector carrying N units of momentum in the light like direction, is described *exactly* by the Hamiltonian:

$$H_N^{(n)}(M_p^2 L, \{M_p L_i\}). \qquad (4)$$

Here M_p is the Planck mass of this new M-theory whose matrix description we are seeking. This prescription allows us to have a complete non-perturbative formulation of M-theory in terms of weak coupling dynamics of $D0$ branes in type IIA string theory.

Since this recipe appears to differ from the conventional recipe for the Matrix formulation of M-theory on T^n, we shall first show that this is equivalent to the conventional recipe for all n up to 5. We shall then examine the difficulty in extending this recipe to give a Matrix formulation of M-theory on T^6 [20–24]. In showing the equivalence of this recipe with the conventional recipe, we shall make use of various known U-duality transformations in type II string theory to map the dynamics of D0 branes in type IIA on T^n to the dynamics of various other systems. Let us call the definition of $H_N^{(n)}$ that we have given as description A. In this description the parameters m_S and R_i are related to the finite parameters M and r_i as follows:

$$m_S = M g_S^{-1/3}, \qquad R_i = M^{-1} r_i g_S^{2/3}. \qquad (5)$$

We shall now give a second description of $H_N^{(n)}$, which we shall call description B, by making an $R \to 1/R$ duality transformation in all the n directions on T^n. This converts the type IIA theory to type IIA/IIB theory on a dual torus $\widetilde{T}^n$ depending on whether n is even or odd, and maps the system of N D0 branes to a system of N Dn-branes wrapped on $\widetilde{T}^n$. The string mass $\widetilde{m}_S$, the radii $\widetilde{R}_i$ of $\widetilde{T}^n$, and the string coupling $\widetilde{g}_S$ in this new theory are given by:

$$\widetilde{m}_S = m_S = Mg_S^{-1/3}, \qquad \widetilde{R}_i = m_S^{-2} R_i^{-1} = M^{-1} r_i^{-1},$$

$$\widetilde{g}_S = g_S / \prod_{i=1}^{n} (m_S R_i) = g_S^{1-\frac{n}{3}} \Big(\prod_{i=1}^{n} r_i^{-1} \Big). \tag{6}$$

Thus $H_N^{(n)}$ can also be regarded as the Hamiltonian describing the dynamics of N wrapped Dn branes in this theory in the $g_S \to 0$ limit.

For $n = 5$, we shall give yet another description of $H_N^{(n)}$, which we shall call description C, by making an S-duality transformation in the type IIB string theory [27]. This converts the wrapped D5-branes to wrapped NS five branes, and gives the following new set of parameters labelling string mass scale $\widehat{m}_S$, radii $\widehat{R}_i$ of the five torus $\widehat{T}^5$, and the string coupling $\widehat{g}_S$ respectively:

$$\widehat{m}_S = \widetilde{m}_S \widetilde{g}_S^{-1/2} = M \Big(\prod_{i=1}^{5} r_i^{1/2} \Big), \qquad \widehat{R}_i = \widetilde{R}_i = M^{-1} r_i^{-1},$$

$$\widehat{g}_S = \widetilde{g}_S^{-1} = g_S^{\frac{2}{3}} \Big(\prod_{i=1}^{5} r_i \Big). \tag{7}$$

At this stage we are ready to compare $H_N^{(n)}$ as defined above with the conventional description of Matrix theory on T^n for $n \leq 5$. First of all, note that for $n = 5$, as we take the $g_S \to 0$ limit in description C, the new string coupling $\widehat{g}_S$ vanishes, with $\widehat{m}_S$ and $\widehat{R}_i$ approaching finite value. Thus $H_N^{(5)}$ corresponds to the Hamiltonian describing the dynamics of wrapped NS five branes in type IIB on T^5 in the limit of zero string coupling. This is precisely the Matrix formulation of M-theory on T^5 as proposed in [19]. The relationship between the parameters of this new IIB theory, and the original variables also work out correctly if we identify (4) as the Matrix theory Hamiltonian. Incidentally, this analysis can also be repeated for D0 branes moving on $K3 \times S^1$ instead of T^5, and the weak coupling limit of this theory correctly reproduces the proposed Matrix description of M-theory on $K3 \times S^1$ [28, 29]. The only difference is that in going from the description A to description B, making $R \to (1/R)$ duality on the first four directions need to be replaced by making a mirror transformation on $K3$.

Once we have found agreement for $n = 5$, the agreement for all other $n \leq 4$ is guaranteed, since these can be obtained from the $n = 5$ Hamiltonian by taking one or more r_i's to infinity. However, one can also verify this explicitly. For example, for $n \leq 3$ we can use the description B and take the $g_S \to 0$ limit. In this limit, $\widetilde{m}_S$ approaches infinity and $\widetilde{g}_S$ either approaches 0 (for $n \leq 2$) or remains finite (for $n = 3$). $\widetilde{R}_i$ remain finite. Thus we can safely ignore the effect of the massive string modes, as well as higher derivative terms in the effective action, and the effective dynamics of the wrapped Dn-branes is described by $(n + 1)$ dimensional supersymmetric $U(N)$ gauge theory compactified on $\widetilde{T}^n$. The bosonic part of this action is given by:

$$
\frac{\widetilde{m}_S^{n-3}}{\widetilde{g}_S} \int dt \int d^n x \Big(Tr(F_{\mu\nu}F^{\mu\nu}) + \sum_{\alpha=1}^{9-n} Tr(D_\mu \Phi^\alpha D^\mu \Phi^\alpha)
$$
$$
+ \sum_{\alpha,\beta} Tr\big([\Phi^\alpha, \Phi^\beta]^2\big)\Big)
$$
$$
= M^{n-3}\Big(\prod_{i=1}^{n} r_i\Big) \int dt \int d^n x \Big(Tr(F_{\mu\nu}F^{\mu\nu}) + \sum_{\alpha=1}^{9-n} Tr(D_\mu \Phi^\alpha D^\mu \Phi^\alpha)
$$
$$
+ \sum_{\alpha,\beta} Tr\big([\Phi^\alpha, \Phi^\beta]^2\big)\Big) \,, \quad (8)
$$

where Φ^α are scalar fields in the adjoint of $U(N)$, normalized so as to have mass dimension unity. This is precisely the proposed Hamiltonian for Matrix theory on T^n for $n \leq 3$.

In order to check explicitly that the recipe also correctly reproduces the Hamiltonian for Matrix theory on T^4, we start with the description B, and regard this as M-theory compactified on $\widetilde{S}^1 \times \widetilde{T}^4$. The wrapped D4 branes can then be regarded as M-theory five branes wrapped on $\widetilde{S}^1 \times \widetilde{T}^4$. We shall call this description D. The radius $\widetilde{R}$ of $\widetilde{S}^1$ and the new Planck mass $\widetilde{m}_p$ are given by,

$$
\widetilde{R} = \widetilde{g}_S/\widetilde{m}_S = M^{-1}\Big(\prod_{i=1}^{4} r_i^{-1}\Big), \qquad \widetilde{m}_p = \widetilde{m}_S/\widetilde{g}_S^{1/3} = M g_S^{-2/9}\Big(\prod_{i=1}^{4} r_i^{1/3}\Big).
$$
$$
(9)
$$

Thus in the $g_S \to 0$ limit $\widetilde{R}$ and $\widetilde{R}_i$ remain finite, and $\widetilde{m}_p$ approaches ∞. $H_N^{(4)}$ is given by the Hamiltonian of wrapped five branes in this limit. This is precisely the Matrix theory Hamiltonian for M-theory on T^4 as proposed in [17–19].

The recipe for constructing $H_N^{(n)}$ given here can be applied to any compactification — toroidal or otherwise — although whether it gives a sensible

and correct matrix description of M-theory is an altogether different question. We shall now try to apply this to construct $H_N^{(n)}$ for $n = 6$. This can be done by compactifying one of the non-compact directions on a circle of radius R_6, and taking the $g_S \to 0$ limit keeping fixed

$$r_6 = m_p R_6 = M g_S^{-2/3} R_6. \tag{10}$$

We can use any of the four descriptions to do this; but it will be most convenient to start with the description C. Thus we have type IIB on a torus $\widehat{T}^6$ with parameters as given in (7), and the radius of the sixth circle given by

$$\widehat{R}_6 = R_6 = M^{-1} g_S^{2/3} r_6. \tag{11}$$

$H_N^{(6)}$ corresponds to the Hamiltonian of NS five branes wrapped on 1-5 directions in the $g_S \to 0$ limit. In order to take this limit, it will be convenient to go to a new description of $H_N^{(6)}$ by making an $R \to (1/R)$ duality in the 6th direction. This converts type IIB to type IIA and the NS five branes to Kaluza-Klein monopoles associated with the sixth direction. The parameters in this theory are:

$$\bar{m}_S = \widehat{m}_S = M \Big(\prod_{i=1}^{5} r_i^{1/2} \Big), \qquad \bar{R}_i = \widehat{R}_i = M^{-1} r_i^{-1} \quad \text{for} \quad 1 \le i \le 5,$$

$$\bar{R}_6 = \widehat{m}_S^{-2} \widehat{R}_6^{-1} = M^{-1} \Big(\prod_{i=1}^{5} r_i^{-1} \Big) r_6^{-1} g_S^{-2/3},$$

$$\bar{g}_S = \widehat{g}_S \widehat{m}_S^{-1} \widehat{R}_6^{-1} = \Big(\prod_{i=1}^{5} r_i^{1/2} \Big) r_6^{-1}. \tag{12}$$

We shall call this description E. These Kaluza-Klein monopoles can also be described as type IIA on $\bar{T}^5 \times MTN \times R$, where $\bar{T}^5$ denotes the torus labelled by 1-5 directions, MTN is the multi-Taub-NUT space and R denotes the usual time direction.

We now take the $g_S \to 0$ limit. Note that in this limit the radii of $\bar{T}^5$, $\bar{g}_S$ and $\bar{m}_S$ remain finite. Furthermore, the dimensionless parameters $\bar{m}_S \bar{R}_i$ for $(1 \le i \le 5)$ and $\bar{g}_S$ are all independent. Thus this theory is expected to have the full U-duality symmetry of type IIB on $\bar{T}^5$, and it is easy to see that indeed, when we put in the appropriate 3-form field background and the general flat metric on T^6 in the original description A, the moduli space of this theory has the structure of $E_{6(6)}(Z) \backslash E_{6(6)}(R) / Sp(4)$, which is the expected structure of the moduli space for M-theory on T^6. However, note that in the $g_S \to 0$ limit $\bar{R}_6 \to \infty$. Since $\bar{R}_6$ sets the overall size of the

Taub-NUT space, we see that in this limit the Taub-NUT space expands to infinite size! In other words, the Taub-NUT space becomes the ALE space with A_{N-1} singularity [30]. (This problem is clearly related to the result of [23]). For $N = 1$, we just get back four dimensional Euclidean space R^4. Since the transverse space is now non-compact, we effectively get a (4+1) dimensional theory instead of a (0+1) dimensional theory. We shall not address the question as to whether this theory can in any way be useful in finding a Matrix formulation of M-theory on T^6. Instead, we shall try to analyze the reason behind getting a (4+1) dimensional theory in the first place.

Clearly the reason behind getting a (4+1) dimensional theory is the increase in the transverse size of the system, so we shall focus on this problem. For this we go back to the description A, and try to identify the various BPS states in the bulk theory. If any of them becomes light, then the interaction of these states with the D0-brane system might be responsible for the increase in the size of the system. The possible BPS states are as follows:

1. States carrying Kaluza-Klein momentum in the transverse direction. These states have masses of order:

$$R_i^{-1} = M g_S^{-2/3} r_i^{-1} \to \infty \quad \text{as} \quad g_S \to 0. \tag{13}$$

Thus these states disappear from the spectrum in the $g_S \to 0$ limit.

2. String winding modes. They have mass of order

$$m_S^2 R_i = M r_i. \tag{14}$$

These masses remain finite in the $g_S \to 0$ limit.

3. For $n \geq 5$, we can have NS five-branes wrapped on T^5. A five-brane wrapped in the first five directions has a mass of order:

$$g_S^{-2} m_S^6 \Big(\prod_{i=1}^{5} R_i \Big) = g_S^{-2/3} M \Big(\prod_{i=1}^{5} r_i \Big) \to \infty \quad \text{as} \quad g_S \to 0. \tag{15}$$

Thus these states also disappear from the spectrum as $g_S \to 0$.

4. Dirichlet p branes wrapped on a p-cycle. Of course since we are dealing with type IIA theory, p must be even. Also a given p-brane can appear only for $n \geq p$. Such a p-brane, wrapped on the first p cycles of T^n has mass of order:

$$g_S^{-1} m_S^{p+1} \Big(\prod_{i=1}^{p} R_i \Big) = g_S^{(p-4)/3} M \Big(\prod_{i=1}^{p} r_i \Big). \tag{16}$$

For $p < 4$, these states have infinite mass in the $g_S \to 0$ limit, and hence decouples from the theory. For $p = 4$ they have finite mass, representing new degrees of freedom from the bulk that might interact with the D0-brane system. Indeed, they form marginally stable bound state with the D0-brane system thereby opening up a whole new dimension [17, 18]. Finally, for $p = 6$, these states become massless, opening up the possibility that interaction of these states with the D0-brane system can effectively increase the transverse size of the system. Indeed, for finite but small g_S, these states have mass of order $g_S^{-2/3}$, precisely the inverse of the transverse size of the Multi-Taub-NUT space that we had seen in eq.(12).

One can make this more concrete by studying what a wrapped D6 brane corresponds to in description E. The series of duality transformations that leads us from the description A to description E transforms a wrapped D6 brane in description A to a state carrying Kaluza-Klein momentum in the 6th direction in description E. Let us now recall the reason as to why the transverse size of the Taub-NUT solution is related to the size $\bar{R}_6$ of the 6th direction. One can write down a solution of Einstein's equation, in which the transverse size is independent of $\bar{R}_6$; however, this solution will suffer from conical singularities unless the transverse size matches $\bar{R}_6$. These conical singularities can only be sensed by states carrying momentum along the 6th direction, since a state carrying no momentum in this direction will not sense the periodicity in this direction, and hence will not see the sigularity. Thus we see that it is indeed the interaction of the Taub-NUT space with the states carrying momentum in the 6th direction that is responsible for the large transverse size of the TN space.

It is now time to summarize our results. By adopting a uniform approach to the Hamiltonian for Matrix theory on T^n in terms of weak coupling dynamics of D0 branes on T^n, we can give a naive description of this Hamiltonian for any n. We have shown that this naive description produces a (4+1) dimensional Hamiltonian instead of a (0+1) dimensional Hamiltonian, − the Hamiltonian describing type IIB string theory on a dual torus $\bar{T}^5$ times the ALE space, at finite coupling. This clearly has the required U-duality symmetry $E_{6(6)}(Z)$ that is expected of a Matrix theory on T^6; however how a (4+1) dimensional theory could be useful in constructing a Matrix theory remains unclear. The reason for getting a (4+1) dimensional theory can be traced to the increase of the transverse size of the D0-brane system in the weak coupling limit. This, in turn, is due to the interaction of the D0-branes with wrapped D6-branes, which become massless in the weak coupling limit we are considering.

Note added

After the paper was sent to the archieve, another paper [31] containing very similar results appeared. This paper also provides an explanation of why this limit gives the correct quantum mechanical model for Matrix theory. It has been pointed out by M. Douglas that when the compact space is curved, there might be new subtleties due to the issues raised in [32, 33]. Finally, I wish to thank L. Motl for his comments on the manuscript.

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