

# D-Branes in Curved Space

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## Abstract

We obtain actions for  $N$  D-branes occupying points in a manifold with arbitrary Kähler metric. In one complex dimension, the action is uniquely determined (up to second order in commutators) by the requirement that it reproduce the masses of stretched strings and by imposing supersymmetry. These conditions are very restrictive in higher dimensions as well. The results provide a noncommutative extension of Kähler geometry.

## 1 Introduction

In this note we give some results for the world-volume theories of D-branes in curved space. D-branes were first defined in weakly coupled string theory terms [1], and as such their world-volume action is defined by computations in two-dimensional world-sheet field theory [2]. Their importance stretches beyond this; for example they are the lightest states in the M-theory limit [3, 4]. Computations in weakly coupled string theory are not obviously a good way to get at their physics in other limits, and in situations where the extrapolation from weak to strong coupling is not determined by supersymmetry numerous subtleties seem to be emerging, for example in [5–7].

Here, we approach the problem as a purely mathematical one: namely, given a particular manifold with metric  $\mathcal{M}$ , we describe a class of  $U(N)$  gauge theories with classical moduli space  $\mathcal{M}^N/S_N$ , such that the small

fluctuations have the expected masses of strings stretched between branes, in other words proportional to the length of the shortest geodesic between the branes. It was noticed in examples in [5] that the second condition does not follow from the first. As we will see, a solution to this problem defines an interesting noncommutative analog of Riemannian geometry. (We might call it “D-geometry.”)

In general this problem is underconstrained as stated; computations in string theory (or further consistency conditions) are required to get a unique answer. However, requiring supersymmetry for the D-brane world-volume action brings additional constraints. It turns out that four real supersymmetries ( $d = 4$ ,  $\mathcal{N} = 1$  supersymmetry), for which the metric must be Kähler, are enough to get interesting results.

We begin with the simplest non-trivial case – a curved target space having two real dimensions, or (by supersymmetry) one complex, and we show that the constraints have a unique solution, determining the terms in the action with up to two commutators.

We then discuss higher complex dimensions. Given a superpotential, we again find strong constraints on the Kähler potential. This case will be discussed in detail in [8].

Finally, as a first step in making contact with relevant mathematics, we make a coordinate-free definition of the algebra of gauge-invariant functions on the D-brane configuration space.

## 2 Kinematics

We consider a complex manifold  $\mathcal{M}$  with coordinates  $z^\mu$  and Kähler potential  $K_0(z, \bar{z})$ . A single D-brane sitting at a point in  $\mathcal{M}$  and extended in  $\mathbb{R}^k$  is described by the supersymmetrized Nambu-Born-Infeld action. In the low energy, low field strength ( $\alpha' F \ll 1$  and  $|\partial z| \ll 1$ ) limit this reduces to decoupled supersymmetric sigma model and  $U(1)$  gauge theory Lagrangians

$$\mathcal{L} = \int d^4\theta K_0(z, \bar{z}) + \sum_{\mu} |X^\alpha|^2 + \text{Re} \int d^2\theta W^2. \quad (2.1)$$

The flat coordinates  $X^\alpha$  play no role in our discussion and we henceforth drop them.

To describe  $N$  D-branes on  $\mathcal{M}$ , we promote the coordinates  $z^\mu$  to  $N \times N$  matrices  $Z_{ij}^\mu$ , whose components are coordinates on a complex manifold  $\mathcal{M}_N$ . We define  $\bar{Z}^\mu = Z^{\mu+}$ . As in flat space, the moduli space modulo gauge transformations will be parameterized by diagonal matrices whose eigenvalues give the positions of  $N$  branes. The off-diagonal component  $Z_{ij}^\mu$  is a superfield whose excitations are strings stretched between branes  $i$  and

$j$ . Each string has a spin (from fermion zero modes) labeled by the index  $\mu$  and component within the superfield.

For manifolds with non-trivial fundamental group  $\Gamma$ , we would want to allow stretched strings for every element of  $\Gamma$ , as in the toroidal case [1,4,9]. However, we will only consider a single coordinate patch here.

We will take the  $U(N)$  gauge action to be

$$Z^\mu \rightarrow U^+ Z^\mu U, \quad (2.2)$$

which implies  $\bar{Z}^\mu \rightarrow U^+ \bar{Z}^\mu U$ . At least locally, this should not be regarded as an assumption but rather as a choice of coordinate system for the off-diagonal elements of  $Z$ . What we know a priori about the  $U(N)$  action is that it generates orbits generically isomorphic to the orbits of 2.2, and in one complex dimension, we could instead take as coordinates the D-brane positions and coordinates on the orbit of the complexified gauge group, e.g.  $Z = g^+ z_i \delta_{ij} g$ , for which 2.2 is clear. In higher dimensions, we are assuming a result similar to the Frobenius theorem.

The precise definition of the off-diagonal components provided by 2.2 depends on the choice of coordinate system, and we will return to this point.

The action will be written in terms of gauge invariant functions, which are products of traces of products of the matrices  $Z^\mu$  and  $\bar{Z}^\mu$ . The classical action produced by string theory will satisfy a stronger condition – each term will be a single trace of a product of matrices. This is well-known and follows from the definition of Chan-Paton factors and the disk topology of the world-sheet. It is a non-trivial constraint and was used for example in [10] to restrict the possibilities for the non-abelian Born-Infeld action. The single trace condition is also required in the application of [4], to get a sensible large  $N$  limit describing free particles.

Another evident property of the action derived from the string theory is that, considered as a function of formal variables  $Z^\mu$  and  $\bar{Z}^\mu$ , the action takes the same form for any  $N$ . Indeed, we do not need to specify  $N$  to compute a specific amplitude. Now the answer for larger  $N$  clearly determines the answer for smaller  $N$  in numerous ways – we can take one D-brane far from the others; we can group the D-branes in pairs and derive an action for  $N/2$  subsystems, and so on. This “stability” guarantees consistency under these reductions in  $N$ . Thus we can regard the coordinates  $Z^\mu$  and  $\bar{Z}^\mu$  as free variables satisfying no relations, as in [11]. The action will be largely determined by a single function of free variables, the Kähler potential  $\text{tr } K(Z, \bar{Z})$ .

On the other hand, there is a clear sense in which commutators  $[Z^\mu, \bar{Z}^\nu]$  are subleading in the action. This is essentially to say that we can count the number of stretched strings in any given process. The considerations here only involve a single stretched string and will only determine  $K$  up to

terms involving two commutators. It will be very interesting to go further by computing (or postulating) multistring interactions and proposing an action which summarizes these.

### 3 D-Brane Action in One Complex Dimension

We begin with this case to illustrate the ideas. It is not clear to us whether the result will have a physical interpretation in superstring theory, but we defer discussion of this point to the end of the section.

The low energy Lagrangian for  $N$  D-branes at points in  $\mathcal{M}$  is

$$\mathcal{L} = \int d^4\theta K(Z, \bar{Z}) + \text{Re} \int d^2\theta \text{tr} W^2, \quad (3.1)$$

where  $K$  is a single trace  $\text{tr} K_N(Z, \bar{Z})$ .

We can exclude the possibility of a superpotential. By holomorphy, gauge invariance and the single trace condition, the only superpotentials we can write are  $\mathcal{W} = \text{tr} \mathcal{W}(Z)$ , and the corresponding conditions on supersymmetric vacua  $\mathcal{W}'(Z) = 0$  would only have isolated solutions.

A more general gauge kinetic term would have been

$$\sum_i \int d^2\theta \text{tr} a_i(Z) \times W b_i(Z) W.$$

Assuming that well separated branes are described by 2.1, the gauge kinetic term for each brane in this limit is the trivial  $\int d^2\theta W^2$ . Combining this with holomorphy would then determine the kinetic term of 3.1. We comment on possible generalizations at the end.

Reproducing the metric for each of the  $N$  branes requires that the Kähler potential on diagonal matrices  $Z_D$  with eigenvalues  $z_i$  be  $K(Z_D, \bar{Z}_D) = \sum_i K_0(z_i, \bar{z}_i)$ . Thus  $K$  must have the same expansion in powers of  $Z$  and  $\bar{Z}$  as  $K_0$ , but with some precise ordering for the  $Z$ 's and  $\bar{Z}$ 's in each factor. We will express this by choosing a standard ordering for  $\text{tr} K_0(Z, \bar{Z})$ , and represent other possibilities by adding terms with commutators.

As mentioned above, the terms with up to two commutators will be constrained by the condition on masses of stretched strings. The reason they can be determined without doing string theory computations is that the potential comes entirely from  $D$ -terms, and thus these masses are entirely determined by the gauge action and the choice of  $K$ .

Supersymmetry requires the potential to take the form

$$V = \text{tr} D^2, \quad (3.2)$$

with [12]

$$D = [Z, \frac{\partial K}{\partial \bar{Z}}] = -[\bar{Z}, \frac{\partial K}{\partial Z}]. \quad (3.3)$$

The equality is guaranteed by gauge invariance of  $K$ .

The masses are then

$$m_{ij}^2 = \frac{\partial^2 V}{\partial Z_{ij} \partial \bar{Z}_{ji}} / \frac{\partial^2 K}{\partial Z_{ij} \partial \bar{Z}_{ji}}, \tag{3.4}$$

for each  $i$  and  $j$  with no summation implied. (One could take  $N = 2$ ,  $i = 1$  and  $j = 2$  to do this computation). Using  $D = 0$  at a supersymmetric minimum, this is

$$m_{ij}^2 = \text{tr} \frac{\partial D}{\partial Z_{ij}} \frac{\partial D}{\partial \bar{Z}_{ji}} / \frac{\partial^2 K}{\partial Z_{ij} \partial \bar{Z}_{ji}}. \tag{3.5}$$

Call the denominator of this expression  $\delta\bar{\delta}K$ .

We now show that, no matter what ordering we choose for  $K$ , we have

$$\frac{\partial D}{\partial Z_{ij}} = (\bar{Z}_{ii} - \bar{Z}_{jj}) \frac{\partial^2 K}{\partial Z_{ij} \partial \bar{Z}_{ji}}. \tag{3.6}$$

Consider a term in  $K$  with a specific ordering. The derivatives will produce a sum of terms, one for each appearance of a  $Z$  and each appearance of a  $\bar{Z}$ . Consider for example

$$K = \text{tr} Z \bar{Z} \delta Z \bar{Z} \bar{Z} Z \delta \bar{Z} Z, \tag{3.7}$$

where  $\delta$  and  $\bar{\delta}$  mark the appearance on which the partial derivatives will act. Since we are working around diagonal  $Z$ , this contribution is

$$\frac{\partial^2 K}{\partial Z_{ij} \partial \bar{Z}_{ji}} = Z_{ii} \bar{Z}_{ii} \bar{Z}_{jj} \bar{Z}_{jj} Z_{jj} Z_{ii}. \tag{3.8}$$

In general,  $A\delta Z B\bar{\delta}\bar{Z}$  contributes  $A(z_i, \bar{z}_i)B(z_j, \bar{z}_j)$  to  $\delta\bar{\delta}K$ .

Now  $\partial D/\partial Z_{ij}$  is also a sum over second derivatives. Using the representation  $D = -\text{tr} [\bar{Z}, \frac{\partial K}{\partial \bar{Z}}]$ , it is easy to see that each term in  $K$  produces a corresponding term in  $D$  with the form 3.6. The claim follows.

We thus have  $m_{ij}^2 = |z_i - z_j|^2 \delta\bar{\delta}K$  and to reproduce the masses, we require

$$\delta\bar{\delta}K = \frac{d^2(z_i, z_j)}{|z_i - z_j|^2}, \tag{3.9}$$

a simple ‘‘correction factor’’ to the flat space kinetic term. It is non-singular and approaches  $\partial\bar{\partial}K(z_j)$  as  $z_i - z_j \rightarrow 0$ .

We now show that there exists an ordering for  $K$  which will reproduce any desired  $\delta\bar{\delta}K$ . Consider an expansion with terms

$$K = \text{tr} K_0(Z, \bar{Z}) + \sum_a \text{tr} f_a(Z, \bar{Z})[Z, \bar{Z}]g_a(Z, \bar{Z})[Z, \bar{Z}] + \dots \tag{3.10}$$

where we take the convention that functions of two variables are ordered with all  $Z$ 's before all  $\bar{Z}$ 's, for example

$$K_0(Z, \bar{Z}) = \sum_{a,b} k_{a,b} Z^a \bar{Z}^b. \tag{3.11}$$

Terms with more commutators do not contribute to the second variation. Consider  $\partial^2 K / \partial Z_{12} \partial \bar{Z}_{21}$ , which is

$$\begin{aligned} \delta \bar{\delta} K &= \sum_{a,b} k_{a,b} \frac{z_1^a - z_2^a}{z_1 - z_2} \frac{\bar{z}_1^b - \bar{z}_2^b}{\bar{z}_1 - \bar{z}_2} \\ &\quad + \sum_a |z_1 - z_2|^2 (f_a(z_1, \bar{z}_1) g_a(z_2, \bar{z}_2) + f_a(z_2, \bar{z}_2) g_a(z_1, \bar{z}_1)) \\ &= \frac{1}{|z_1 - z_2|^2} \left( K_0(z_1, \bar{z}_1) - K_0(z_1, \bar{z}_2) - K_0(z_2, \bar{z}_1) + K_0(z_2, \bar{z}_2) \right) \\ &\quad + |z_1 - z_2|^2 h(z_1, \bar{z}_1, z_2, \bar{z}_2), \end{aligned} \tag{3.12}$$

where  $h$  is a general function invariant under  $z_1 \leftrightarrow z_2$ . For small  $\epsilon = z_2 - z_1$ , this has the expansion

$$\delta \bar{\delta} K = \partial \bar{\partial} K(z_1) + O(\epsilon^2) \tag{3.13}$$

and the  $O(\epsilon^2)$  and higher terms are freely adjustable.

We thus have shown that there exists a Kähler potential in 3.1 which satisfies the requirements. Furthermore, it is uniquely determined up to terms with more than two commutators. This is to say that any term in 3.10 whose second variation is zero could be written as a product with more than two commutators. This can be checked using 3.12.

It is easy to check that adding dependence on the longitudinal coordinates  $X^\alpha$  in the obvious way (dimensionally reducing a higher dimensional world-volume theory) works as it should, because  $\delta \bar{\delta} K$  also multiplies the potential in this case.

We now consider the case of non-trivial gauge kinetic term, as could be produced by a non-constant dilaton background. The  $D$ -term potential becomes

$$V = D(\delta \bar{\delta} f)^{-1} D, \tag{3.14}$$

where  $\delta \bar{\delta} f$  is the second variation of the gauge kinetic term with respect to  $W$ . This is a diagonal matrix and easy to invert, so this leads to

$$m_{ij}^2 = |z_i - z_j|^2 \frac{\delta \bar{\delta} K(z_i, z_j)}{\delta \bar{\delta} f(z_i, z_j)}. \tag{3.15}$$

One can check that the gauge boson masses are given by the same formula.

### 3.1 Example: The Two-Sphere

A Kähler potential producing the rotationally symmetric metric in the usual stereographic coordinates is

$$K = \log(1 + z\bar{z}). \quad (3.16)$$

The shortest geodesic distance between two points is

$$d(z_1, z_2) = 2 \arctan \frac{|z_1 - z_2|}{(1 + z_1\bar{z}_2)^{1/2}(1 + z_2\bar{z}_1)^{1/2}}. \quad (3.17)$$

Using 3.9 and 3.12 we have

$$\begin{aligned} h(z_1, z_2) &= \frac{1}{|z_1 - z_2|^4} \left( d^2(z_1, z_2) - K_0(z_1, \bar{z}_1) + K_0(z_1, \bar{z}_2) + K_0(z_2, \bar{z}_1) \right. \\ &\quad \left. - K_0(z_2, \bar{z}_2) \right) \\ &= \frac{1}{|z_1 - z_2|^4} \left( d^2(z_1, z_2) - \log \frac{(1 + |z_1|^2)(1 + |z_2|^2)}{(1 + z_1\bar{z}_2)(1 + z_2\bar{z}_1)} \right) \\ &= \frac{1}{|z_1 - z_2|^4} \left( \log^2 \frac{1 + i\sqrt{u}}{1 - i\sqrt{u}} - \log(1 + u) \right). \end{aligned} \quad (3.18)$$

The parenthesized expression has a Taylor expansion in

$$u = |z_1 - z_2|^2 / (1 + z_1\bar{z}_2)(1 + z_2\bar{z}_1),$$

so this can be reproduced by 3.10.

Explicitly reproducing just the terms up to two commutators in this way leads to a complicated and unenlightening expression. It seems likely that choices exist for the higher commutator terms which lead to a simpler expression, which would help in finding a more geometric description of the result. However, the main point we want to make at present is that the result is determined.

The physics of the result is in the mixed components of the curvature on the moduli space. These are the second derivatives of the logarithm of the geodesic distance  $\partial^2 \log d^2(z_i, z_j) / \partial z_i \partial \bar{z}_j$  (we derive this below). They could be seen in the low energy scattering of transverse ripples on the brane.

On general grounds one expects this curvature to be non-singular except at special points in configuration space where massless degrees of freedom appear. In the present example, the special points in the moduli space are the points where a pair of D-branes sit at antipodal points of the sphere, and there is no longer a unique shortest geodesic connecting them. The resulting singularity in the curvature should be associated with the stretched string having acquiring a zero mode for rotations around the sphere.

### 3.2 Physical Interpretation

Although the mathematical problem is well-defined in one complex dimension, and serves to illustrate the general case, it is not clear whether the result can be directly interpreted as coming from a string theory.

First of all, these spaces (except for flat space) are not Ricci flat and these are not solutions of superstring theory or of M-theory. Conceivably, it might be possible to define a classical D-brane action in such a background, by requiring conformal invariance only for the boundary interactions.

A more serious problem is that there will be no covariantly constant spinor on  $\mathcal{M}$ , so string theory D-branes on  $\mathcal{M} \times \mathbb{R}^k$  will not have  $\mathcal{N} = 1$  supersymmetry. Rather than derive fermions by dimensional reduction, we have implicitly postulated fermions which are supersymmetry partners of the bosons.

The result for the bosonic part of the action does seem physically sensible, and to get this the supersymmetry is being used only as a device (indeed any two-dimensional Riemannian manifold admits complex coordinates in which the metric is Kähler), so it is quite possible that the result itself does not depend on supersymmetry, only this derivation.

Another one-dimensional problem (currently under investigation) which should have a superstring interpretation would be to allow a non-constant dilaton background, as in F-theory [13].

## 4 General Covariance

The Kähler potential 3.9 for the off-diagonal modes is not manifestly covariant — it depends on the choice of coordinate  $z$ . This can be traced back to the definition of the gauge action 2.2.

On the other hand, physical quantities are covariant. The masses of states are, by assumption. Let us check the Riemann curvature on the moduli space. The mixed components at  $\phi = 0$  (let  $\phi$  be an off-diagonal component) are

$$\begin{aligned}
 R_{\bar{z}z\phi}^{\phi} &= \partial_{\bar{z}}(g^{\phi\bar{\phi}}\partial_z g_{\phi\bar{\phi}}) \\
 &= \partial_{\bar{z}}\partial_z \log \delta\bar{\delta}K \\
 &= \partial_{\bar{z}}\partial_z (\log d^2(z_1, z_2) - \log |z_1 - z_2|^2) \\
 &= \partial_{\bar{z}}\partial_z \log d^2(z_1, z_2) ,
 \end{aligned}
 \tag{4.1}$$

except at  $z_1 = z_2$ , where the  $\log |z_1 - z_2|^2$  serves to cancel the short distance singularity. The components  $R_{\phi\phi}^{\phi\phi}$ , or the curvature at  $\phi \neq 0$ , require knowing



terms with more commutators to compute. Thus, to the extent we have computed the action here, it is covariant.

One way to restore manifest covariance, at generic points in moduli space, would be to absorb the coordinate dependence into the fields, by making the field redefinition  $\tilde{Z}_{ij} = Z_{ij}/(Z_{ii} - Z_{jj})$ . However this breaks down when  $Z_{ii} = Z_{jj}$  and is obviously not a good definition.

A better way to implement covariance is to postulate non-trivial transformation laws for the off-diagonal components of  $Z$ . Evidently they should transform like differences of coordinates.

The simplest treatment would be to postulate a transformation law for the entire matrix  $Z$ . In one dimension, there is only one possibility for a holomorphic coordinate transformation compatible with gauge invariance. We can only write

$$Z = f(Z'), \quad (4.2)$$

where  $z = f(z')$  is the coordinate transformation on  $z$ .

Thus we ask whether the expression 3.10 is covariant under this definition of change of coordinate. A nice feature of the ordering prescription we used is that it is preserved under 4.2, so the functions transform as

$$K_0(Z, \bar{Z}) \rightarrow K_0(f(Z'), f^*(\bar{Z}')) \quad (4.3)$$

and so on. The second variation 3.12 with respect to the off-diagonal components of  $Z'$  becomes

$$\begin{aligned} \frac{\partial^2 K'}{\partial Z'_{12} \partial \bar{Z}'_{21}} &= \frac{1}{|z'_1 - z'_2|^2} \left( K_0(f(z'_1), f^*(\bar{z}'_1)) - K_0(f(z'_1), f^*(\bar{z}'_2)) \right. \\ &\quad \left. - K_0(f(z'_2), f^*(\bar{z}'_1)) + K_0(f(z'_2), f^*(\bar{z}'_2)) \right) \\ &\quad + \frac{|f(z'_1) - f(z'_2)|^4}{|z'_1 - z'_2|^2} h(f(z'_1), f^*(\bar{z}'_1), f(z'_2), f^*(\bar{z}'_2)) \\ &= \frac{|z_1 - z_2|^2}{|z'_1 - z'_2|^2} \frac{\partial^2 K}{\partial Z_{12} \partial \bar{Z}_{21}} \end{aligned} \quad (4.4)$$

and the masses  $m_{12}^2 = |z'_1 - z'_2|^2 \delta' \bar{\delta}' K'$  are invariant.

Since the action was determined by the fact that it reproduced the metric and masses, and these transform properly, we conclude that this definition of the action is indeed covariant under the simple extension of change of coordinates 4.2.

### 5 Higher Dimensions

We introduce complex matrix coordinates  $Z^\mu$  with  $1 \leq \mu \leq D$  and make no restriction on  $D$ . The strategy will again be to reproduce the masses of stretched strings by finding some correct ordering of the Kähler potential  $K(Z, \bar{Z})$ . We take the masses for every polarization of the stretched string to be  $m^2 = d^2$ . This seems quite plausible at least at leading order in  $\alpha'$  in string theory, since these differ only in the state of their fermion zero modes.

The  $D$ -terms are  $D = \sum_\mu [Z^\mu, \partial_\mu K]$  and  $D = 0$  combined with gauge quotient will leave a moduli space of complex dimension  $(d - 1)N^2 + N$ . Thus a superpotential is required to restrict the moduli space to commuting  $[Z^\mu, Z^\nu] = 0$ . This can be accomplished by a generic superpotential of the form

$$W = \frac{1}{2} \sum_{\mu, \nu} \text{tr } w_{\mu\nu}(Z) [Z^\mu, Z^\nu]. \tag{5.1}$$

Computing the mass matrix for the off-diagonal fields  $Z_{12}^\mu$  will again involve their variations,  $\delta_\mu X = \partial X / \partial Z_{12}^\mu$ . One finds

$$m_{1\mu, 2\nu}^2 = (\delta\bar{\delta}K)^{-1, \mu\bar{\rho}} (\bar{\delta}_{\bar{\rho}} D \delta_\nu D + (\delta_\lambda \bar{\delta}_{\bar{\lambda}} K)^{-1} \bar{\delta}_{\bar{\rho}} \bar{\delta}_{\bar{\lambda}} \bar{W} \delta_\nu \partial_\lambda W). \tag{5.2}$$

The result 3.6 generalizes in an obvious way:

$$\delta_\mu D = \sum_{\bar{\nu}} (\bar{Z}_{11}^{\bar{\nu}} - \bar{Z}_{22}^{\bar{\nu}}) \delta_\mu \bar{\delta}_{\bar{\nu}} K. \tag{5.3}$$

Explicit formulas for  $\delta_\mu \bar{\delta}_{\bar{\nu}} K$  and  $\delta_\nu \partial_\lambda W$  are complicated by the fact that now we have to choose an ordering among the holomorphic coordinates. One possibility is to totally symmetrize, writing

$$K = \oint \prod_\mu \frac{d\alpha^\mu}{\alpha^\mu} \oint \prod_\mu \frac{d\bar{\alpha}^\mu}{\bar{\alpha}^\mu} \text{tr} \frac{1}{1 - \sum_\mu Z^\mu / \alpha^\mu} \frac{1}{1 - \sum_\mu \bar{Z}^\mu / \bar{\alpha}^\mu} K_0(\alpha, \bar{\alpha}) \tag{5.4}$$

as the leading term, and then expressing corrections to this in terms of commutators. However this is not preserved by holomorphic coordinate transformations and it is not obvious that it is natural, or indeed that there is any natural universal ordering. Thus we refrain from writing explicit analogs of 3.10 and 3.12, and content ourselves with the observation that there is again enough freedom to produce

$$\delta_\mu \bar{\delta}_{\bar{\nu}} K = \partial_\mu \bar{\delta}_{\bar{\nu}} K_0 + (z_1^\rho - z_2^\rho)(\bar{z}_1^\lambda - \bar{z}_2^\lambda) h_{\mu\rho\nu\lambda}(z_1, z_2), \tag{5.5}$$

with  $h_{\mu\nu\rho\lambda}(z_1, z_2)$  symmetric in  $z_1 \leftrightarrow z_2$  and in  $\mu\nu \leftrightarrow \rho\lambda$  but otherwise arbitrary, and

$$\delta_\nu \partial_\lambda W = (z_1^\rho - z_2^\rho) w_{\nu\lambda\rho}(z_1, z_2), \tag{5.6}$$

with  $w$  a holomorphic function satisfying  $w_{\nu\lambda\rho}(z_1, z_2) = -w_{\lambda\nu\rho}(z_2, z_1)$  and  $(z_1' - z_2')w_{\nu\lambda\rho}(z_1, z_2) = 0$  (this follows from gauge invariance).

Thus,  $\delta\bar{\delta}K$  already has enough freedom to reproduce the masses. The freedom in  $W$  is not enough to do it alone but does make the answer non-unique, in a way analogous to the non-uniqueness we found in one dimension if we introduced a general gauge kinetic term.

Of course we might have other physical constraints on the superpotential, for example that it be non-singular, which would be very constraining if  $\mathcal{M}$  is compact. Another interesting case is  $\mathcal{N} = 2$ ,  $d = 4$  supersymmetry, which requires  $\mathcal{M}$  to be hyperkähler, and for which the superpotential is also uniquely determined [12]. This suggests the possibility that the action is uniquely determined up to higher commutators in this case as well.

Additional knowledge about the Kähler potential would also fix this. For example, we might try the ansatz “ $\delta_\mu\bar{\delta}_\nu K \propto g_{\mu\bar{\nu}}$ ”. This is not sensible as it stands – we need a quantity depending both on  $z_i$  and  $z_j$ , which is likely to be an integral of a locally defined tensor along the geodesic. This is a point at which computation in superstring theory (or other underlying definition of D-brane) might be required to completely determine the action. One limit to which this action naturally applies is weak curvature  $\alpha'R \ll 1$  and long stretched strings,  $|z_i - z_j|^2 \gg \alpha'$ , and computations in this regime do not look difficult, but we leave this for future work.

## 6 Noncommutative Geometry

Clearly we are talking about some sort of “noncommutative geometry,” and it will be interesting to make contact with related work in mathematics, such as that of Connes [14].

As a start, let us define the algebra of gauge-invariant functions which we used here in a coordinate-free way. It would be very useful to have a coordinate-free version of the present discussion.

Let  $M_N(\mathbb{C})$  be the algebra of complex  $N \times N$  matrices, and  $\mathcal{M}_N$  the  $DN^2$ -dimensional configuration space of the D-brane theory admitting a  $U(N)$  action  $\pi$  with the general properties above. Then  $\mathcal{A}_N$  is the subalgebra of  $M_N(\mathbb{C}) \otimes C^\infty(\mathcal{M}_N)$  satisfying

$$g^{-1}ag = \pi^*(g)a. \quad (6.1)$$

In our earlier discussion, the action  $\pi^*$  was given by 2.2, and we used the statement that  $\mathcal{A}_N$  was generated by  $Z$  and  $\bar{Z}$ . The action is a  $U(N)$  invariant linear functional of an element of  $\mathcal{A}_N$ , i.e. a trace.

Another application of a precise definition would be to facilitate taking the large  $N$  limit — we would just replace  $M_N(\mathbb{C})$  with another algebra  $\mathcal{T}$ . We also need to replace  $\mathcal{M}_N$  with a space which locally is modelled on

$\mathcal{T} \otimes \mathbb{R}^D$ . This would be useful (for example) in describing membranes along the lines of [4, 15].

## Acknowledgements

I would like to thank Costas Bachas, Brian Greene, Akishi Kato, Hiroshi Ooguri, John Schwarz and Steve Shenker for discussions and the Caltech theoretical physics group for their hospitality. This research was supported by DOE grant DE-FG02-96ER40959.

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