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Evolution of eigenvalue of the Wentzell-Laplace operator along the conformal mean curvature flow

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Abstract. In this paper, we investigate continuity, differentiability and monotonicity for the first nonzero eigenvalue of the Wentzell-Laplace operator along the conformal mean curvature flow on n-dimensional compact manifolds with boundary for $n \geq 3$ under a boundary condition. In especial, we show that the first nonzero eigenvalue of the Wentzell-Laplace operator is monotonic under the conformal mean curvature flow and we find some monotonic quantities dependent to the first nonzero eigenvalue along the conformal mean curvature flow.

1. Introduction

Let (M, g_0) be an n-dimensional compact Riemannian manifold with smooth boundary ∂M, where $n \geq 3$. Yamabe problem is a generalization of uniformization theorem. For a closed manifold M, the Yamabe conjecture [26] asserts that in each conformal class of g_0 there is a metric g of constant scalar curvature R_g. R. Hamilton ([2] and [14]) introduced the Yamabe flow

$$\frac{\partial g}{\partial t} = -(R_g - \overline{R}_g)g \quad g(0) = g_0,$$

for find such a metric, where R is a scalar curvature of M and $\overline{R}_g = \frac{\int_M R_g \, dV_g}{\int_M dV_g}$ is the mean value of the scalar curvature on M. There are two generalization of the Yamabe problem for manifolds with boundary. The first case finds a conformal metric which has constant scalar curvature in the interior of M and vanishing mean curvature on ∂M. The second case tries to get a conformal metric with vanishing

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scalar curvature in the interior of M and constant mean curvature on ∂M. These generalization of the Yamabe problem were studied by many authors, see [4], [10], [11], [15] and [16] and references therein. Brendle [4] has introduced some geometric flows to study the Yamabe problem with boundary. For the first case, Brendle [4], introduced the Yamabe flow with boundary as follows

$$\frac{\partial g}{\partial t}=-(R_g-\overline{R}_g)g \text{ in } M, \quad \text{and} \quad H_g=0 \text{ on } \partial M, \tag{1}$$

where H is the mean curvature of ∂M. Almaraz and Sun [10] studied the convergence of Yamabe flow with boundary (1). For second case, Brendle considered the normalized conformal mean curvature flow, and is defined as

$$\frac{\partial g}{\partial t}=-2(H_g-\overline{H}_g)g \text{ on } \partial M, \quad \text{and} \quad R_g=0 \text{ in } M, \tag{2}$$

where $\overline{H}_g=\frac{\int_{\partial M} H_g\,dA_g}{\int_{\partial M} dA_g}$ is the average of the mean curvature H_g. The uniqueness and existence of the flow (2) has been studied in [1].

On the other hand, the study of the eigenvalue problem with Wentzell boundary has recently attacked a lot of attention (see [9], [7], [13] and [25]). Let $\overline{\Delta}$ and Δ be the Laplace-Beltrami operators on M and ∂M, respectively. Assume that τ is a real number. The eigenvalue problem for Wentzell boundary conditions given by

$$\begin{cases} \overline{\Delta} u=0, & \text{in } M, \\ -\tau\Delta u+\frac{\partial u}{\partial \nu}=\lambda u & \text{on } \partial M, \end{cases} \tag{3}$$

where ν denotes the outward unit normal vector field of ∂M. When $\tau=0$, the eigenvalue problem (3) becomes the following Steklov eigenvalue problem

$$\begin{cases} \overline{\Delta} u=0, & \text{in } M, \\ \frac{\partial u}{\partial \nu}=\sigma u & \text{on } \partial M. \end{cases}$$

When $\tau\geq 0$, the spectrum of the Wentzell-Laplace problem (3) consist in an increasing countable sequence of eigenvalue

$$0=\lambda_0<\lambda_1\leq\lambda_2\leq \ldots\leq\lambda_k\leq\ldots$$

with corresponding real orthonormal eigenfunction $u_0, u_1, u_2, \ldots$. Throughout the paper, we assume that $\tau\geq 0$. We have the following variational characterization for the first nonzero eigenvalue λ_1 of Wentzell-Laplace operator

$$\lambda=\inf_{u\in C^\infty(\overline{M})\setminus\{0\}}\left\{\frac{\int_M |\overline{\nabla}_g u|^2\,dV_g+\tau\int_{\partial M}|\nabla_g u|^2\,dA_g}{\int_{\partial M} u^2\,dA_g}:\int_{\partial M} u\,dA_g=0, \quad \overline{\Delta}_g u=0\right\}.$$

There are many papers on the evolution of the eigenvalue of geometric operators under geometric flows, see [3], [8], [12], [20], [22], [23] and the references therein. In [19], Ho and Koo studied the evolution of Steklov eigenvalue along the geodesic curvature on two-dimensional compact Riemannian manifold with smooth boundary ∂M. Also, Ho [18] investigated the evolution of the Dirichlet eigenvalue on manifolds with boundary along the Yamabe flow with boundary (1) and he [17] studied the evolution of the Steklov eigenvalue under the conformal mean curvature flow. The main results of this paper are as follows:

Theorem 1.1. *Let $\lambda(t)$ be the nonzero first eigenvalue of Wentzell-Laplace operator* (3)*. Then, we have the following cases*:

(i) *if* $\min_{\partial M} H_g \geq \frac{n-2}{n-1} \max_{\partial M} H_g \geq 0$ *then $\lambda(t)$ is nondecreasing and differentiable almost everywhere along the unnormalized conformal mean curvature flow on* $[0,T)$.

(ii) *If* $\max_{\partial M} H_g \leq \frac{n-2}{n-1} \min_{\partial M} H_g \leq 0$ *then $\lambda(t)$ is nonincreasing and differentiable almost everywhere along the unnormalized conformal mean curvature flow on* $[0,T)$.

(iii) *If* $\max_{\partial M} H_g \geq 0$ *then the quantity*

$$\lambda(t) - \int_0^t \left((n-1) \min_{\partial M} H_g - (n-2) \max_{\partial M} H_g \right) \lambda(s)\, ds$$

is nondecreasing along the unnormalized conformal mean curvature flow on $[0,T)$.

(iv) *If* $\min_{\partial M} H_g \leq 0$ *then the quantity*

$$\lambda(t) - \int_0^t \left((n-1) \max_{\partial M} H_g - (n-2) \min_{\partial M} H_g \right) \lambda(s)\, ds$$

is nonincreasing along the unnormalized conformal mean curvature flow on $[0,T)$.

Theorem 1.2. *Let $\lambda(t)$ be the nonzero first eigenvalue of Wentzell-Laplace operator* (3)*. Then we get the following cases*:

(i) *if $\tau = 0$ and* $\min_{\partial M}(H_g - \overline{H}_g) \geq \frac{n-2}{n-1} \max_{\partial M}(H_g - \overline{H}_g) \geq 0$ *then $\lambda(t)$ is nondecreasing and differentiable almost everywhere along the normalized conformal mean curvature flow on* $[0,T)$.

(ii) *If $\tau = 0$ and* $\max_{\partial M}(H_g - \overline{H}_g) \leq \frac{n-2}{n-1} \min_{\partial M}(H_g - \overline{H}_g) \leq 0$ *then $\lambda(t)$ is nonincreasing and differentiable almost everywhere along the normalized conformal mean curvature flow on* $[0,T)$.

(iii) *If* $\max_{\partial M}(H_g - \overline{H}_g) \geq 0$ *then the quantity*

$$\lambda(t) - \int_0^t \left((n-1) \min_{\partial M}(H_g - \overline{H}_g) - (n-2) \max_{\partial M}(H_g - \overline{H}_g) \right) \lambda(s)\, ds$$

is nondecreasing along the normalized conformal mean curvature flow on $[0,T)$.

(iv) *If* $\min_{\partial M}(H_g-\overline{H}_g)\leq 0$ *then the quantity*

$$\lambda(t)-\int_0^t\left((n-1)\max_{\partial M}(H_g-\overline{H}_g)-(n-2)\min_{\partial M}(H_g-\overline{H}_g)\right)\lambda(s)\,ds$$

is nonincreasing along the normalized conformal mean curvature flow on $[0,T)$.

2. Evolution of eigenvalue under the unnormalized flow

Let (M,g_0) be an n-dimensional compact Riemannian manifold with smooth boundary ∂M with $n\geq 3$. From [4, p. 630] we can find a metric conformal to g_0 such that the scalar curvature in M is zero. Thus, without loss of generality we can assume that

$$R_{g_0}=0,\quad \text{in } M. \tag{4}$$

We consider the unnormalized conformal mean curvature flow

$$\frac{\partial g}{\partial t}=-2H_g g \text{ in } \partial M,\quad \text{and}\quad R_g=0 \text{ on } M,\quad g(0)=g_0. \tag{5}$$

Let $g=u^{\frac{4}{n-2}}g_0$, then Riemannian volume on M and on ∂M induced by the metric g, dV_g and dA_g respectively, satisfy that $dV_g=u^{\frac{2n}{n-2}}\,dV_{g_0}$ and $dA_g=u^{\frac{2(n-1)}{n-2}}\,dA_{g_0}$. We also, have

$$\begin{cases} -\frac{4(n-1)}{n-2}\overline{\Delta}_{g_0}u+R_{g_0}u=R_gu^{\frac{n+2}{n-2}}, & \text{in } M,\\ \frac{2(n-1)}{n-2}\frac{\partial u}{\partial \nu}+H_{g_0}u=H_gu^{\frac{n}{n-2}} & \text{on } \partial M, \end{cases} \tag{6}$$

From (4), (5), and (6) we infer

$$\begin{cases} \overline{\Delta}_{g_0}u=0, & \text{in } M,\\ \frac{\partial u}{\partial t}=-\frac{n-2}{2}H_gu & \text{on } \partial M. \end{cases} \tag{7}$$

Hence, the evolution of the volume form of ∂M with respect to the metric g satisfies

$$\frac{\partial}{\partial t}(dA_g)=\frac{2(n-1)}{n-2}u^{\frac{2(n-1)}{n-2}-1}\frac{\partial u}{\partial t}\,dA_{g_0}=-(n-1)H_g\,dA_g. \tag{8}$$

From [4, Lemma 3.8], using (6) and (7) we conclude

$$\frac{\partial}{\partial t}(H_g)=-(n-1)\frac{\partial \widehat{H}_g}{\partial \nu_g}+H_g^2,\quad \text{on } \partial M$$

where $\widehat{H}_g$ is the harmonic extension of H_g to M with respect to g, i.e.

$$\overline{\Delta}_{g_0}\widehat{H}_g = 0 \text{ in } M, \quad \widehat{H}_g = H_g \text{ on } \partial M.$$

Along the flow (5), from [17, Lemma 2.1] we get

$$\frac{\partial u}{\partial t} = -\frac{n-2}{2}\widehat{H}_g u \quad \text{in } M. \tag{9}$$

Although, we do not know the differentiability for the first eigenvalue $\lambda(t)$ of Wentzell-Laplace operator, in following lemma we show that $\lambda(t)$ is a continuous function respect to t-variable along the flow (5) on $[0,T)$ where T is taken to be the maximum time of existence for the flow.

Lemma 2.1. *If g_1 and g_2 are two metrics on Riemannian manifold M which satisfy*

$$\frac{1}{1+\varepsilon}g_1 \le g_2 \le (1+\varepsilon)g_1, \tag{10}$$

then

$$\lambda(g_2) - \lambda(g_1) \le \left((1+\varepsilon)^{n+\frac{1}{2}} - 1\right)\lambda(g_1). \tag{11}$$

In particular, $\lambda(t)$ is a continuous function in the t-variable.

Proof. Inequality (10) yields

$$(1+\varepsilon)^{-\frac{n}{2}}\, dV_{g_1} \le dV_{g_2} \le (1+\varepsilon)^{\frac{n}{2}}\, dV_{g_1}, \tag{12}$$

and

$$(1+\varepsilon)^{-\frac{n-1}{2}}\, dA_{g_1} \le dA_{g_2} \le (1+\varepsilon)^{\frac{n-1}{2}}\, dA_{g_1}. \tag{13}$$

Let

$$\begin{aligned} G(g(t), f(t)) &:= \int_M |\overline{\nabla}_g f(t)|^2_{g(t)}\, dV_{g(t)} + \tau \int_{\partial M} |\nabla_g f(t)|^2_{g(t)}\, dA_{g(t)} \\ &= \int_{\partial M} f(t)\left(\frac{\partial f(t)}{\partial \nu_{g(t)}} - \tau\Delta_{g(t)} f(t)\right) dA_{g(t)}. \end{aligned} \tag{14}$$

Then

$$\begin{aligned} &G(g(t_2), f(t_2))\int_{\partial M} f^2\, dA_{g(t_1)} - G(g(t_1), f(t_1))\int_{\partial M} f^2\, dA_{g(t_1)} \\ &= \left(\int_M |\overline{\nabla}_g f(t_2)|^2_{g(t_2)}\, dV_{g(t_2)} - \int_M |\overline{\nabla}_g f(t_1)|^2_{g(t_1)}\, dV_{g(t_1)}\right)\int_{\partial M} f^2\, dA_{g(t_1)} \end{aligned}$$

$$+\left(\int_{\partial M} f^2\, dA_{g(t_1)} - \int_{\partial M} f^2\, dA_{g(t_2)}\right) \int_M |\overline{\nabla}_g f(t_1)|^2_{g(t_1)}\, dV_{g(t_1)}$$
$$+\tau\left(\int_M |\nabla_g f(t_2)|^2_{g(t_2)}\, dA_{g(t_2)} - \int_M |\nabla_g f(t_1)|^2_{g(t_1)}\, dA_{g(t_1)}\right) \int_{\partial M} f^2\, dA_{g(t_1)}$$
$$+\tau\left(\int_{\partial M} f^2\, dA_{g(t_1)} - \int_{\partial M} f^2\, dA_{g(t_2)}\right) \int_M |\nabla_g f(t_1)|^2_{g(t_1)}\, dA_{g(t_1)}.$$

Using (12) and (13) we get

$$\begin{aligned}
&G(g(t_2), f(t_2)) \int_{\partial M} f^2\, dA_{g(t_1)} - G(g(t_1), f(t_1)) \int_{\partial M} f^2\, dA_{g(t_1)}\\
&\quad \le \left((1+\varepsilon)^{\frac{n}{2}+1} - 1\right) \int_M |\overline{\nabla}_g f(t_1)|^2_{g(t_1)}\, dV_{g(t_1)} \int_{\partial M} f^2\, dA_{g(t_1)}\\
&\quad\quad + \left(1 - (1+\varepsilon)^{-\frac{n-1}{2}}\right) \int_{\partial M} f^2\, dA_{g(t_1)} \int_M |\overline{\nabla}_g f(t_1)|^2_{g(t_1)}\, dV_{g(t_1)}\\
&\quad\quad + \tau\left((1+\varepsilon)^{\frac{n-1}{2}+1} - 1\right) \int_M |\nabla_g f(t_1)|^2_{g(t_1)}\, dA_{g(t_1)} \int_{\partial M} f^2\, dA_{g(t_1)}\\
&\quad\quad + \tau\left(1 - (1+\varepsilon)^{-\frac{n-1}{2}}\right) \int_{\partial M} f^2\, dA_{g(t_1)} \int_M |\nabla_g f(t_1)|^2_{g(t_1)}\, dA_{g(t_1)}\\
&\quad = \left((1+\varepsilon)^{\frac{n}{2}+1} - (1+\varepsilon)^{-\frac{n-1}{2}}\right) \int_M |\overline{\nabla}_g f(t_1)|^2_{g(t_1)}\, dV_{g(t_1)} \int_{\partial M} f^2\, dA_{g(t_1)}\\
&\quad\quad + \tau\left((1+\varepsilon)^{\frac{n-1}{2}+1} - (1+\varepsilon)^{-\frac{n-1}{2}}\right) \int_M |\nabla_g f(t_1)|^2_{g(t_1)}\, dA_{g(t_1)} \int_{\partial M} f^2\, dA_{g(t_1)}\\
&\quad \le \left((1+\varepsilon)^{\frac{n}{2}+1} - (1+\varepsilon)^{-\frac{n-1}{2}}\right) \int_M |\overline{\nabla}_g f(t_1)|^2_{g(t_1)}\, dV_{g(t_1)} \int_{\partial M} f^2\, dA_{g(t_1)}\\
&\quad\quad + \tau\left((1+\varepsilon)^{\frac{n}{2}+1} - (1+\varepsilon)^{-\frac{n-1}{2}}\right) \int_M |\nabla_g f(t_1)|^2_{g(t_1)}\, dA_{g(t_1)} \int_{\partial M} f^2\, dA_{g(t_1)}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
&\frac{G(g(t_2), f(t_2))}{\int_{\partial M} f^2\, dA_{g(t_2)}} - \frac{G(g(t_1), f(t_1))}{\int_{\partial M} f^2\, dA_{g(t_1)}}\\
&\quad \le \left((1+\varepsilon)^{\frac{n}{2}+1} - (1+\varepsilon)^{-\frac{n-1}{2}}\right) \frac{\int_M |\overline{\nabla}_g f(t_1)|^2_{g(t_1)}\, dV_{g(t_1)}}{\int_{\partial M} f^2\, dA_{g(t_2)}}\\
&\quad\quad + \tau\left((1+\varepsilon)^{\frac{n}{2}+1} - (1+\varepsilon)^{-\frac{n-1}{2}}\right) \frac{\int_M |\nabla_g f(t_1)|^2_{g(t_1)}\, dA_{g(t_1)}}{\int_{\partial M} f^2\, dA_{g(t_2)}}\\
&\quad \le \left((1+\varepsilon)^{\frac{n}{2}+1} - (1+\varepsilon)^{-\frac{n-1}{2}}\right) (1+\varepsilon)^{\frac{n-1}{2}} \frac{G(g(t_1), f(t_1))}{\int_{\partial M} f^2\, dA_{g(t_1)}}.
\end{aligned}$$

This shows that (11) is true and completes the proof of lemma. □

Proposition 2.2. *Let $g=g(t)$, $t\in[0,T)$, be the solution of the unnormalized conformal mean curvature flow (5), and $\lambda(t)$ be the corresponding first eigenvalue of Wentzell-Laplace operator (3). Then for any $t_1\leq t_2$ we have*

$$\lambda(t_2)\geq\lambda(t_1)+\int_{t_1}^{t_2}\mathcal{G}(g(t),f(t))\,dt, \tag{15}$$

where

(16)
$$\mathcal{G}(g(t),f(t))=2\int_{\partial M}\frac{\partial f(t)}{\partial t}\left(\frac{\partial f(t)}{\partial\nu_{g(t)}}-\tau\Delta_{g(t)}f(t)\right)dA_g-(n-2)\int_M\widehat{H}_g|\overline{\nabla}_g f(t)|_g^2\,dV_g$$
$$-(n-3)\tau\int_{\partial M}H_g|\nabla_g f(t)|_g^2\,dA_g,$$

$f(t)$ is a smooth function in $M\times[0,T)$ satisfying $\int_{\partial M}f^2\,dA_g=1$, $\int_{\partial M}f\,dA_g=0$, and $\overline{\Delta}_g f(t)=0$ in M, such that at time t_2, $f(t_2)$ is the corresponding eigenfunction for the eigenvalue $\lambda(t_2)$.

Proof. Motivated by the works of X.-D. Cao [5] and [6] and J. Y. Wu et al. [24], we first let at time t_2, $f_2=f(t_2)$ be the eigenfunction for the first eigenvalue $\lambda(t_2)$ of $g(t_2)$. We consider the following smooth function in ∂M defined by

$$h(t)=\frac{u(t_2)^{\frac{n-1}{n-2}}}{u(t)^{\frac{n-1}{n-2}}}f_2,$$

along the unnormalized conformal mean curvature flow (5) where $u=u(t)$ is the solution of (7) for $t\in[0,T)$. We assume that

$$f(t)=\frac{h(t)}{\left(\int_{\partial M}h(t)^2\,dA_g\right)^{\frac{1}{2}}},$$

which $f(t)$ is smooth function under the flow (5), satisfies $\int_{\partial M}f^2\,dA_g=1$, $\overline{\Delta}_g f(t)=0$ in M, and at time t_2, $f(t_2)$ is the corresponding eigenfunction for the eigenvalue $\lambda(t_2)$ of Wentzell-Laplace operator. Also, by definition of $h(t)$ and $f(t)$ we get

$$\begin{aligned}\int_{\partial M}f(t)\,dA_{g(t)}&=\frac{1}{\left(\int_{\partial M}h(t)^2\,dA_g\right)^{\frac{1}{2}}}\int_{\partial M}h(t)\,dA_{g(t)}\\&=\frac{1}{\left(\int_{\partial M}h(t)^2\,dA_g\right)^{\frac{1}{2}}}\int_{\partial M}\left(\frac{u(t_2)}{u(t)}\right)^{\frac{n-1}{n-2}}f(t_2)u^{\frac{(n-1)}{n-2}}\,dA_{g_0}\\&=\frac{1}{\left(\int_{\partial M}h(t)^2\,dA_g\right)^{\frac{1}{2}}}\int_{\partial M}f(t_2)\,dA_{g(t_2)}\end{aligned}$$

$$= 0.$$

We extended function $f(t)$ to a harmonic function in M with respect to g, which we still denote it by $f(t)$. Notice, $G(g(t),f(t))$ is a smooth function respect to t. Since $g{=}u^{\frac{4}{n-2}}g_0$, we have

$$\langle\overline{\nabla}_g\phi,\overline{\nabla}_g\psi\rangle_g = u^{-\frac{4}{n-2}}\langle\overline{\nabla}_{g_0}\phi,\overline{\nabla}_{g_0}\psi\rangle_{g_0},$$

for any smooth functions ϕ, ψ in M. This yields

$$G(g(t),f(t)) = \int_M u^2|\overline{\nabla}_g f(t)|^2_{g_0}\,dV_{g_0} + \tau\int_{\partial M}|\nabla_g f(t)|^2_g\,dA_g.$$

By derivation with respect to t, along the flow (5) we obtain

(17)

$$\begin{aligned}\mathcal{G}(g(t),f(t)) := {}& \frac{d}{dt}G(g(t),f(t))\\ ={}& 2\int_M\left\langle\overline{\nabla}_g f(t),\overline{\nabla}_g\frac{\partial f(t)}{\partial t}\right\rangle_g dV_g + 2\int_M u\frac{\partial u}{\partial t}|\overline{\nabla}_g f(t)|^2_{g_0}\,dV_{g_0}\\ &+2\tau\int_{\partial M}\left\langle\nabla_g f(t),\nabla_g\frac{\partial f(t)}{\partial t}\right\rangle_g dA_g + 2\tau\int_{\partial M}H_g|\nabla_g f(t)|^2_{g_0}\,dA_{g_0}\\ &-(n-1)\tau\int_{\partial M}H_g|\nabla_g f(t)|^2_g\,dA_g.\end{aligned}$$

In last line we use (8). Using (7) and (9), we get

$$\begin{aligned}\mathcal{G}(g(t),f(t)) = {}& 2\int_M\left\langle\overline{\nabla}_g f(t),\overline{\nabla}_g\frac{\partial f(t)}{\partial t}\right\rangle_g dV_g - (n-2)\int_M\widehat{H}_g|\overline{\nabla}_g f(t)|^2_{g(t)}\,dV_{g(t)}\\ &+2\tau\int_{\partial M}\left\langle\nabla_g f(t),\nabla_g\frac{\partial f(t)}{\partial t}\right\rangle_g dA_g - (n-3)\tau\int_{\partial M}H_g|\nabla_g f(t)|^2_g\,dA_g.\end{aligned}$$

From integration by parts we conclude

(18)

$$\begin{aligned}\mathcal{G}(g(t),f(t)) = {}& 2\int_{\partial M}\frac{\partial f(t)}{\partial t}\frac{\partial f(t)}{\partial\nu_{g(t)}}\,dA_g - (n-2)\int_M\widehat{H}_g|\overline{\nabla}_g f(t)|^2_{g(t)}\,dV_{g(t)}\\ &-2\tau\int_{\partial M}\frac{\partial f(t)}{\partial t}\Delta_g f(t)\,dA_g - (n-3)\tau\int_{\partial M}H_g|\nabla_g f(t)|^2_g\,dA_g.\\ ={}& 2\int_{\partial M}\frac{\partial f(t)}{\partial t}\left(\frac{\partial f(t)}{\partial\nu_{g(t)}} - \tau\Delta_{g(t)}f(t)\right)dA_g - (n-2)\int_M\widehat{H}_g|\overline{\nabla}_g f(t)|^2_g\,dV_g\\ &-(n-3)\tau\int_{\partial M}H_g|\nabla_g f(t)|^2_g\,dA_g.\end{aligned}$$

Taking integration on the both sides of (17) on interval $[t_1,t_2]\subset[0,T)$, we arrive at

$$G(g(t_2),f(t_2))-G(g(t_1),f(t_1))=\int_{t_1}^{t_2}\mathcal{G}(g(t),f(t))\,dt. \tag{19}$$

Since $f(t_2)$ is the corresponding eigenfunction of the eigenvalue $\lambda(t_2)$, we deduce

$$G(g(t_2),f(t_2))=\lambda(t_2)\int_{\partial M}f(t_2)^2\,dA_{g(t_2)}=\lambda(t_2).$$

Noticing, by definition of first eigenvalue for Wentzell-Laplace operator we have

$$G(g(t_1),f(t_1))\geq\lambda(t_1)\int_{\partial M}f(t_1)^2\,dA_{g(t_1)}=\lambda(t_1).$$

This completes the proof of Proposition. $\square$

Proposition 2.3. *Let $g=g(t)$, $t\in[0,T)$, be the solution of the unnormalized conformal mean curvature flow (5), and $\lambda(t)$ be the corresponding first eigenvalue of Wentzell-Laplace operator (3). Then for any $t_1\leq t_2$ we have*

$$\lambda(t_2)\leq\lambda(t_1)+\int_{t_1}^{t_2}\mathcal{G}(g(t),f(t))\,dt, \tag{20}$$

where $\mathcal{G}(g(t),f(t))$ defines in (16), $f(t)$ is a smooth function in $M\times[0,T)$ satisfying $\int_{\partial M}f^2\,dA_g=1$, $\int_{\partial M}f\,dA_g=0$, and $\overline{\Delta}_g f(t)=0$ in M, such that at time t_1, $f(t_1)$ is the corresponding eigenfunction for the eigenvalue $\lambda(t_1)$, and T is taken to be the maximum time of existence for the flow.

Proof. The proof is almost the same as the proof of Proposition 2.2. At time t_1, we consider $f_1=f(t_1)$ is the eigenfunction for the first eigenvalue $\lambda(t_1)$ of $g(t_1)$. We assume the smooth function $h(t)=\left(\frac{u(t_1)}{u(t)}\right)^{\frac{n-1}{n-2}}f_1$ in ∂M along the flow (5) where $u=u(t)$ is the solution of (7) for $t\in[0,T)$. We let $f(t)=\frac{h(t)}{(\int_{\partial M}h(t)^2\,dA_g)^{\frac{1}{2}}}$, which $f(t)$ is smooth function under the flow (5), satisfies $\int_{\partial M}f^2\,dA_g=1$, $\overline{\Delta}_g f(t)=0$ in M, and at time t_1, $f(t_1)$ is the corresponding eigenfunction for the eigenvalue $\lambda(t_1)$ of Wentzell-Laplace operator. We extended function $f(t)$ to a harmonic function in M with respect to g, which we still denote it by $f(t)$. We define $G(g(t),f(t))$ as (14). Then it is clear that (18) and (19) are still hold. Since $f(t_1)$ is the corresponding eigenfunction of the eigenvalue $\lambda(t_1)$, we get

$$G(g(t_1),f(t_1))=\lambda(t_1)\int_{\partial M}f(t_1)^2\,dA_{g(t_1)}=\lambda(t_1).$$

Also, by definition of first eigenvalue for Wentzell-Laplace operator we have

$$G(g(t_2), f(t_2)) \geq \lambda(t_2) \int_{\partial M} f(t_2)^2 \, dA_{g(t_2)} = \lambda(t_2).$$

By combining all these the proof of Proposition is complete $\square$

Proof of Theorem 1.1. To prove the theorem we will use the Propositions 2.2 and 2.3.

Case (i)
In this case, we show that in inequality (15) we $\mathcal{G}(g(t), f(t)) \geq 0$ when t is sufficiently closed to t_2. Notice that at time t_2, $f(t_2)$ is the corresponding eigenfunction for the eigenvalue $\lambda(t_2)$. Therefore, at time t_2 we have

$$\mathcal{G}(g(t_2), f(t_2)) = 2\lambda(t_2) \int_{\partial M} \frac{\partial f(t_2)}{\partial t} f(t_2) \, dA_g - (n-2) \int_M \widehat{H}_g |\overline{\nabla}_g f(t_2)|_g^2 \, dV_g - (n-3)\tau \int_{\partial M} H_g |\nabla_g f(t_2)|_g^2 \, dA_g. \tag{21}$$

Under the unnormalized conformal mean curvature flow, from the constraint condition

$$\frac{d}{dt} \int_{\partial M} f(t)^2 \, dA_g = 0,$$

we know that

$$2 \int_{\partial M} \frac{\partial f(t_2)}{\partial t} f(t_2) \, dA_g = (n-1) \int_{\partial M} f(t_2)^2 H_{g(t_2)} \, dA_g \geq (n-1) \min_{\partial M} H_g. \tag{22}$$

Combining (21) and (22), we obtain

$$\mathcal{G}(g(t_2), f(t_2)) \geq (n-1) \min_{\partial M} H_g \lambda(t_2) - (n-2) \int_M \widehat{H}_g |\overline{\nabla}_g f(t_2)|_g^2 \, dV_g - (n-3)\tau \int_{\partial M} H_g |\nabla_g f(t_2)|_g^2 \, dA_g.$$

Since $\widehat{H}_g$ is harmonic, the maximal principal implies that

$$\max_M \widehat{H}_g = \max_{\partial M} H_g \quad \text{and} \quad \min_M \widehat{H}_g = \min_{\partial M} H_g.$$

Hence,

$$\mathcal{G}(g(t_2), f(t_2)) \geq (n-1) \min_{\partial M} H_g \lambda(t_2) - (n-2) \max_{\partial M} H_g \int_M |\overline{\nabla}_g f(t_2)|_g^2 \, dV_g - (n-3)\tau \max_{\partial M} H_g \int_{\partial M} |\nabla_g f(t_2)|_g^2 \, dA_g.$$

Since $\max_{\partial M} H_g \geq 0$ we obtain

$$\mathcal{G}(g(t_2), f(t_2)) \geq \left((n-1) \min_{\partial M} H_g - (n-2) \max_{\partial M} H_g \right) \lambda(t_2).$$

The condition $\min_{\partial M} H_g \geq \frac{n-2}{n-1} \max_{\partial M} H_g \geq 0$ implies $\mathcal{G}(g(t_2), f(t_2)) \geq 0$. Notice $f(t)$ is a smooth function with respect to t-variable. Therefore, we have $\mathcal{G}(g(t), f(t)) \geq 0$ in any sufficient small neighborhood of t_2. Thus,

$$\int_{t_1}^{t_2} \mathcal{G}(g(t), f(t))\, dt \geq 0,$$

for any $t_1 < t_2$ sufficiently close to t_2. In the end, by (15), we conclude

$$\lambda(t_2) \geq \lambda(t_1)$$

for any $t_1 < t_2$ sufficiently close to t_2. Since $t_2 \in [0, T)$ is arbitrary, then $\lambda(t)$ is nondecreasing along the unnormalized conformal mean curvature flow.

In order to show the the differentiability for λ, since $\lambda(t)$ is nondecreasing on $[0, T)$ under the unnormalized conformal mean curvature flow, by the classical Lebesgue's theorem ([21, Chap. 4]), $\lambda(t)$ is differentiable almost everywhere on $[0, T)$.

Case (ii)

In this case, we show that in inequality (20) we $\mathcal{G}(g(t), f(t)) \leq 0$ when t is sufficiently closed to t_1. Note that at time t_1, $f(t_1)$ is the corresponding eigenfunction for the eigenvalue $\lambda(t_2)$. Hence, at time t_1 we get

$$\mathcal{G}(g(t_1), f(t_1)) = 2\lambda(t_1) \int_{\partial M} \frac{\partial f(t_1)}{\partial t} f(t_1)\, dA_g - (n-2) \int_M \widehat{H}_g |\overline{\nabla}_g f(t_2)|_g^2\, dV_g. \tag{23}$$

Under the unnormalized conformal mean curvature flow we have

$$2 \int_{\partial M} \frac{\partial f(t_1)}{\partial t} f(t_1)\, dA_g = (n-1) \int_{\partial M} f(t_1)^2 H_{g(t_1)}\, dA_g \leq (n-1) \max_{\partial M} H_g. \tag{24}$$

Combining (23) and (24), we infer

$$\mathcal{G}(g(t_1), f(t_1)) \leq \left((n-1) \max_{\partial M} H_g - (n-2) \min_{\partial M} H_g \right) \lambda(t_1).$$

The condition $\max_{\partial M} H_g \leq \frac{n-2}{n-1} \min_{\partial M} H_g \leq 0$ yields $\mathcal{G}(g(t_1), f(t_1)) \leq 0$. Thus, we arrive at $\mathcal{G}(g(t), f(t)) \leq 0$ in any sufficient small neighborhood of t_1. Therefore,

$$\int_{t_1}^{t_2} \mathcal{G}(g(t), f(t))\, dt \leq 0,$$

for any $t_2>t_1$ sufficiently close to t_1. In the end, by (20), we deduce

$$\lambda(t_2)\leq\lambda(t_1)$$

for any $t_2>t_1$ sufficiently close to t_1. Since $t_1\in[0,T)$ is arbitrary, then $\lambda(t)$ is non-increasing along the unnormalized conformal mean curvature flow. This completes the proof of case (ii).

Case (iii)
Assume that at time t_2, $f(t_2)$ is the corresponding eigenfunction for the eigenvalue $\lambda(t_2)$. Then, at time t_2, we get

$$\begin{aligned}\mathcal{G}(g(t_2),f(t_2))=2\lambda(t_2)\int_{\partial M}\frac{\partial f(t_2)}{\partial t}f(t_2)\,dA_g-(n-2)\int_M\widehat{H}_g|\overline{\nabla}_gf(t_2)|_g^2\,dV_g\\ -(n-3)\tau\int_{\partial M}H_g|\nabla_gf(t_2)|_g^2\,dA_g.\end{aligned}\tag{25}$$

From the condition

$$\int_{\partial M}f(t)^2\,dA_g=1,$$

we have

$$2\int_{\partial M}\frac{\partial f(t_2)}{\partial t}f(t_2)\,dA_g=(n-1)\int_{\partial M}f(t_2)^2H_{g(t_2)}\,dA_g\geq(n-1)\min_{\partial M}H_g.\tag{26}$$

Plugging (26) into (25) we obtain

$$\begin{aligned}\mathcal{G}(g(t_2),f(t_2))\geq(n-1)\min_{\partial M}H_g\lambda(t_2)-(n-2)\int_M\widehat{H}_g|\overline{\nabla}_gf(t_2)|_g^2\,dV_g\\ -(n-3)\tau\int_{\partial M}H_g|\nabla_gf(t_2)|_g^2\,dA_g.\end{aligned}$$

Due to $\max_M\widehat{H}_g=\max_{\partial M}H_g$, $\min_M\widehat{H}_g=\min_{\partial M}H_g$, and $\max_{\partial M}H_g\geq0$ we conclude

$$\begin{aligned}\mathcal{G}(g(t_2),f(t_2))&\geq\left((n-1)\min_{\partial M}H_g-(n-2)\max_{\partial M}H_g\right)\lambda(t_2)\\ &\quad+\tau\max_{\partial M}H_g\int_{\partial M}|\nabla_gf(t_2)|_g^2\,dA_g\\ &\geq\left((n-1)\min_{\partial M}H_g-(n-2)\max_{\partial M}H_g\right)\lambda(t_2).\end{aligned}$$

Therefore, in any sufficient small neighborhood of t_2 we have

$$\mathcal{G}(g(t),f(t))\geq\left((n-1)\min_{\partial M}H_g-(n-2)\max_{\partial M}H_g\right)\lambda(t).$$

Thus,

$$\int_{t_1}^{t_2} \mathcal{G}(g(t), f(t))\, dt \geq \int_{t_1}^{t_2} \left((n-1)\min_{\partial M} H_g - (n-2)\max_{\partial M} H_g\right)\lambda(s)\, ds,$$

for any $t_1 < t_2$ sufficiently close to t_2. Hence, by (15), we conclude

$$\begin{aligned}&\lambda(t_2) - \int_0^{t_2} \left((n-1)\min_{\partial M} H_g - (n-2)\max_{\partial M} H_g\right)\lambda(s)\, ds\\ &\geq \lambda(t_1) - \int_0^{t_1} \left((n-1)\min_{\partial M} H_g - (n-2)\max_{\partial M} H_g\right)\lambda(s)\, ds,\end{aligned}$$

for any $t_1 < t_2$ sufficiently close to t_2. Since $t_2 \in [0, T)$ is arbitrary, then the quantity

$$\lambda(t) - \int_0^{t} \left((n-1)\min_{\partial M} H_g - (n-2)\max_{\partial M} H_g\right)\lambda(s)\, ds,$$

is nondecreasing along the unnormalized conformal mean curvature flow.

Case (iv)

Let at time t_1, $f(t_1)$ be the corresponding eigenfunction for the eigenvalue $\lambda(t_1)$. Then, at time t_1, we obtain

$$\begin{aligned}\mathcal{G}(g(t_1), f(t_1)) = 2\lambda(t_1)\int_{\partial M} \frac{\partial f(t_1)}{\partial t} f(t_1)\, dA_g - (n-2)\int_M \widehat{H}_g |\overline{\nabla}_g f(t_1)|_g^2\, dV_g\\ -(n-3)\tau \int_{\partial M} H_g |\nabla_g f(t_1)|_g^2\, dA_g.\end{aligned} \tag{27}$$

From the condition $\int_{\partial M} f(t)^2\, dA_g = 1$, we get

$$2\int_{\partial M} \frac{\partial f(t_1)}{\partial t} f(t_1)\, dA_g = (n-1)\int_{\partial M} f(t_1)^2 H_{g(t_1)}\, dA_g \leq (n-1)\max_{\partial M} H_g. \tag{28}$$

Applying (28) into (27) we arrive at

$$\begin{aligned}\mathcal{G}(g(t_1), f(t_1)) \leq (n-1)\max_{\partial M} H_g \lambda(t_1) - (n-2)\int_M \widehat{H}_g |\overline{\nabla}_g f(t_1)|_g^2\, dV_g\\ -(n-3)\tau \int_{\partial M} H_g |\nabla_g f(t_1)|_g^2\, dA_g.\end{aligned}$$

Since $\max_M \widehat{H}_g = \max_{\partial M} H_g$, $\min_M \widehat{H}_g = \min_{\partial M} H_g$, and $\min_{\partial M} H_g \leq 0$, we deduce

$$\begin{aligned}\mathcal{G}(g(t_1), f(t_1)) \leq \left((n-1)\max_{\partial M} H_g - (n-2)\min_{\partial M} H_g\right)\lambda(t_1)\\ +\tau \min_{\partial M} H_g \int_{\partial M} |\nabla_g f(t_2)|_g^2\, dA_g\end{aligned} \tag{29}$$

$$\leq \left((n-1) \max_{\partial M} H_g - (2n-3) \min_{\partial M} H_g \right) \lambda(t_1).$$

Therefore, in any sufficient small neighborhood of t_1 we have

$$\mathcal{G}(g(t), f(t)) \leq \left((n-1) \max_{\partial M} H_g - (n-2) \min_{\partial M} H_g \right) \lambda(t).$$

Hence,

$$\int_{t_1}^{t_2} \mathcal{G}(g(t), f(t))\, dt \leq \int_{t_1}^{t_2} \left((n-1) \max_{\partial M} H_g - (n-2) \min_{\partial M} H_g \right) \lambda(s)\, ds,$$

for any $t_2 > t_1$ sufficiently close to t_1. Thus, by (20), we infer

$$\lambda(t_2) - \int_0^{t_2} \left((n-1) \max_{\partial M} H_g - (n-2) \min_{\partial M} H_g \right) \lambda(s)\, ds$$
$$\leq \lambda(t_1) - \int_0^{t_1} \left((n-1) \max_{\partial M} H_g - (n-2) \min_{\partial M} H_g \right) \lambda(s)\, ds,$$

for any $t_2 > t_1$ sufficiently close to t_1. Since $t_1 \in [0, T)$ is arbitrary, then the quantity

$$\lambda(t) - \int_0^{t} \left((n-1) \max_{\partial M} H_g - (n-2) \min_{\partial M} H_g \right) \lambda(s)\, ds,$$

is nonincreasing along the unnormalized conformal mean curvature flow. □

3. Evolution of eigenvalue under the normalized flow

Assume that (M, g_0) be an n-dimensional compact Riemannian manifold with smooth boundary ∂M with $n \geq 3$. We assume that

$$R_{g_0} = 0 \quad \text{in } M.$$

We consider the normalized conformal mean curvature flow

$$\frac{\partial g}{\partial t} = -2(H_g - \overline{H}_g) g \text{ on } \partial M, \quad \text{and} \quad R_g = 0 \text{ in } M, \quad g(0) = g_0. \tag{30}$$

If $g = u^{\frac{4}{n-2}} g_0$, then we have

$$\begin{cases} \overline{\Delta}_{g_0} u = 0, & \text{in } M, \\ \frac{\partial u}{\partial t} = -\frac{n-2}{n}(H_g - \overline{H}_g) u & \text{on } \partial M. \end{cases} \tag{31}$$

Hence, the evolution of the volume form of ∂M with respect to the metric g satisfies

$$\frac{\partial}{\partial t}(dA_g) = -(n-1)(H_g - \overline{H}_g)\, dA_g. \tag{32}$$

From [4, Lemma 3.8], using (6) and (31) we conclude

$$\frac{\partial}{\partial t}(H_g) = -(n-1)\frac{\partial \widehat{H}_g}{\partial \nu_g} + H_g(H_g - \overline{H}_g), \quad \text{on } \partial M.$$

Along the flow (30), we get

$$\frac{\partial u}{\partial t} = -\frac{n-2}{2}(\widehat{H}_g - \overline{H}_g)u \quad \text{in } M. \tag{33}$$

Proposition 3.1. *Let $g=g(t)$, $t\in[0,T)$, be the solution of the normalized conformal mean curvature flow (30), and $\lambda(t)$ be the corresponding first eigenvalue of Wentzell-Laplace operator (3). Then for any $t_1\le t_2$ we have*

$$\lambda(t_2) \ge \lambda(t_1) + \int_{t_1}^{t_2} \mathcal{B}(g(t), f(t))\, dt,$$

where

$$\begin{aligned} \mathcal{B}(g(t), f(t)) = 2\int_{\partial M} \frac{\partial f(t)}{\partial t}\left(\frac{\partial f(t)}{\partial \nu_{g(t)}} - \tau \Delta_{g(t)} f(t)\right) dA_g \\ -(n-2)\int_M (\widehat{H}_g - \overline{H}_g)|\overline{\nabla}_g f(t)|_g^2\, dV_g \\ -(n-3)\tau \int_{\partial M} (H_g - \overline{H}_g)|\nabla_g f(t)|_g^2\, dA_g, \end{aligned} \tag{34}$$

$f(t)$ is a smooth function in $M\times[0,T)$ satisfying $\int_{\partial M} f^2\, dA_g = 1$, $\int_{\partial M} f\, dA_g = 0$, and $\overline{\Delta}_g f(t) = 0$ in M, such that at time t_2, $f(t_2)$ is the corresponding eigenfunction for the eigenvalue $\lambda(t_2)$.

Proof. We first let at time t_2, $f_2 = f(t_2)$ is the eigenfunction for the first nonzero eigenvalue $\lambda(t_2)$ of $g(t_2)$. We assume the following smooth function in ∂M defined by

$$h(t) = \frac{u(t_2)^{\frac{n-1}{n-2}}}{u(t)^{\frac{n-1}{n-2}}} f_2,$$

along the unnormalized conformal mean curvature flow (5) where $u = u(t)$ is the solution of (7) for $t\in[0,T)$. We assume that

$$f(t) = \frac{h(t)}{\left(\int_{\partial M} h(t)^2\, dA_g\right)^{\frac{1}{2}}},$$

which $f(t)$ is smooth function under the flow (5), satisfies $\int_{\partial M} f^2\, dA_g = 1$, $\overline{\Delta}_g f(t) = 0$ in M, and at time t_2, $f(t_2)$ is the corresponding eigenfunction for the eigenvalue $\lambda(t_2)$ of Wentzell-Laplace operator. Also, by definition of $h(t)$ and $f(t)$ we get $\int_{\partial M} f(t)\, dA_{g(t)} = 0$.

We extended function $f(t)$ to a harmonic function in M with respect to g, which we still denote it by $f(t)$. The function $G(g(t), f(t))$ is a smooth function respect to t. By derivation of $G(g(t), f(t))$ with respect to t, along the flow (5) we get

$$\begin{aligned}\mathcal{B}(g(t), f(t)) &:= \frac{d}{dt} G(g(t), f(t))\\ &= 2\int_M \left\langle \overline{\nabla}_g f(t), \overline{\nabla}_g \frac{\partial f(t)}{\partial t}\right\rangle_g dV_g + 2\int_M u \frac{\partial u}{\partial t} |\overline{\nabla}_g f(t)|^2_{g_0}\, dV_{g_0}\\ &\quad + 2\tau \int_{\partial M} \left\langle \nabla_g f(t), \nabla_g \frac{\partial f(t)}{\partial t}\right\rangle_g dA_g + 2\tau \int_{\partial M} (H_g - \overline{H}_g) |\nabla_g f(t)|^2_{g_0}\, dA_{g_0}\\ &\quad - (n-1)\tau \int_{\partial M} (H_g - \overline{H}_g) |\nabla_g f(t)|^2_g\, dA_g.\end{aligned}$$

In last line we use (32). Using (31) and (33) we obtain

$$\begin{aligned}\mathcal{B}(g(t), f(t)) = 2\int_M \left\langle \overline{\nabla}_g f(t), \overline{\nabla}_g \frac{\partial f(t)}{\partial t}\right\rangle_g dV_g - (n-2)\int_M (\widehat{H}_g - \overline{H}_g) |\overline{\nabla}_g f(t)|^2_{g(t)}\, dV_{g(t)}\\ + 2\tau \int_{\partial M} \left\langle \nabla_g f(t), \nabla_g \frac{\partial f(t)}{\partial t}\right\rangle_g dA_g\\ - (n-3)\tau \int_{\partial M} (H_g - \overline{H}_g) |\nabla_g f(t)|^2_g\, dA_g.\end{aligned}$$

From integration by parts we arrive at

$$\begin{aligned}\mathcal{B}(g(t), f(t)) = 2\int_{\partial M} \frac{\partial f(t)}{\partial t} \left(\frac{\partial f(t)}{\partial \nu_{g(t)}} - \tau \Delta_{g(t)} f(t)\right) dA_g\\ - (n-2)\int_M (\widehat{H}_g - \overline{H}_g) |\overline{\nabla}_g f(t)|^2_g\, dV_g\\ - (n-3)\tau \int_{\partial M} (H_g - \overline{H}_g) |\nabla_g f(t)|^2_g\, dA_g.\end{aligned}$$

The definition of $\mathcal{B}(g(t), f(t))$ yields

$$G(g(t_2), f(t_2)) - G(g(t_1), f(t_1)) = \int_{t_1}^{t_2} \mathcal{B}(g(t), f(t))\, dt.$$

Since $f(t_2)$ is the corresponding eigenfunction of the eigenvalue $\lambda(t_2)$, we have

$$G(g(t_2),f(t_2))=\lambda(t_2)\int_{\partial M} f(t_2)^2\,dA_{g(t_2)}=\lambda(t_2).$$

Also, by definition of first eigenvalue for Wentzell-Laplace operator we deduce

$$G(g(t_1),f(t_1))\geq\lambda(t_1)\int_{\partial M} f(t_1)^2\,dA_{g(t_1)}=\lambda(t_1).$$

This completes the proof of Proposition. □

Proposition 3.2. *Let $g=g(t)$, $t\in[0,T)$, be the solution of the normalized conformal mean curvature flow (5), and $\lambda(t)$ be the corresponding first eigenvalue of Wentzell-Laplace operator (3). Then for any $t_1\leq t_2$ we have*

$$\lambda(t_2)\leq\lambda(t_1)+\int_{t_1}^{t_2}\mathcal{B}(g(t),f(t))\,dt,$$

where $\mathcal{B}(g(t),f(t))$ defines in (34), $f(t)$ is a smooth function in $M\times[0,T)$ satisfying $\int_{\partial M} f^2\,dA_g=1$, $\int_{\partial M} f\,dA_g=0$, and $\overline{\Delta}_g f(t)=0$ in M, such that at time t_1, $f(t_1)$ is the corresponding eigenfunction for the eigenvalue $\lambda(t_1)$, and T is taken to be the maximum time of existence for the flow.

Proof. The proof is almost the same as the proof of Proposition 2.3. It is enough to replace $(H_g-\overline{H}_g)$ and $(\widehat{H}_g-\overline{H}_g)$ instead of H_g and $\widehat{H}_g$, respectively. □

If in proof of Theorem 1.1, we substitute $(H_g-\overline{H}_g)$ and $(\widehat{H}_g-\overline{H}_g)$ instead of H_g and $\widehat{H}_g$, respectively, then we get the Theorem 1.2.

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