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On the sum of a prime power and a power in short intervals

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Abstract. Let $R_{k,\ell}(N)$ be the representation function for the sum of the k-th power of a prime and the ℓ-th power of a positive integer. Languasco and Zaccagnini (2017) proved an asymptotic formula for the average of $R_{1,2}(N)$ over short intervals $(X,X+H]$ of the length H slightly shorter than $X^{\frac{1}{2}}$, which is shorter than the length $H=X^{\frac{1}{2}+\varepsilon}$ in the exceptional set estimates of Mikawa (1993) and of Perelli and Pintz (1995). In this paper, we prove that the same asymptotic formula for $R_{1,2}(N)$ holds for H of the size $X^{0.337}$. Recently, Languasco and Zaccagnini (2018) extended their result to more general (k,ℓ). We also consider this general case and as a corollary, we prove a conditional result of Languasco and Zaccagnini (2018) for the case $\ell=2$ unconditionally up to some small factors.

1. Introduction

Let $R(N)$ be the representation function for a given additive problem with prime numbers. For example, in this paper, we consider the binary additive problem with prime numbers given by the equation

$$N=p^k+n^\ell, \tag{1}$$

where k,ℓ are given positive integers, p denotes a variable for prime numbers and n denotes a variable for positive integers. Then the representation function for the equation (1) with logarithmic weight is given by

$$R(N)=R_{k,\ell}(N)=\sum_{p^k+n^\ell=N}\log p, \tag{2}$$

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which counts the solutions (p,n) of (1). In this paper, we consider the short interval average of such representation function

$$\sum_{X<N\leq X+H} R(N), \tag{3}$$

where $4\leq H\leq X$. Recently, Languasco and Zaccagnini gave extensive research (e.g. see [5]–[9] and [10]) on the short interval average (3) for various additive problems with prime numbers and in the case $k=1$ of (1), they obtained short interval asymptotic formulas for the average (3) with H shorter than in the known exceptional set estimates in short intervals.

For example, let us consider the Hardy–Littlewood equation

$$N=p+n^2,$$

which is the case $(k,\ell)=(1,2)$ of our equation (1). In their famous paper Partitio Numerorum III, Hardy and Littlewood [1, Conjecture H] applied their circle method formally to obtain a hypothetical asymptotic formula

$$R_{1,2}(N)=\mathfrak{S}(N)\sqrt{N}+(\text{error}), \quad (N\colon \text{not square}) \tag{4}$$

as $N\to\infty$, where the singular series $\mathfrak{S}(N)$ is given by

$$\mathfrak{S}(N)=\prod_{p>2}\left(1-\frac{(N/p)}{p-1}\right), \quad (N/p)\colon \text{Legendre symbol.}$$

This asymptotic formula (4) itself seems still far beyond our current technology, but we can prove (4) on average. Let $A>0$ be an arbitrary constant and introduce

$$E(X)=\#\left\{N\leq X \;\middle|\; \left|R_{1,2}(N)-\mathfrak{S}(N)\sqrt{N}\right|\geq\sqrt{N}(\log N)^{-A},\ N\colon \text{not square}\right\},$$

where $X\geq 2$ is a real number. This function $E(X)$ counts the number of positive integers $\leq X$ for which the hypothetical asymptotic formula (4) fails. Miech [11] proved a non-trivial bound

$$E(X)\ll XL^{-A}, \quad L=\log X \tag{5}$$

for any $A>0$, where the implicit constant depends on A. Thus, Miech proved that the asymptotic formula (4) holds for almost all integer N. The short interval version of Miech's result (5) was obtained by Mikawa [12] and by Perelli and Pintz

[14] independently. Their result gives a non-trivial bound

$$E(X+H)-E(X)\ll HL^{-A} \tag{6}$$

for any $A>0$ provided

$$X^{\frac{1}{2}+\varepsilon}\le H\le X, \tag{7}$$

where X,H,ε are real numbers with $4\le H\le X, \varepsilon>0$ and the implicit constant may depend on A and ε. One of the aims in this problem is to obtain the same bound (6) for shorter H. Although the range (7) is still the best possible result today for the estimate (6), Languasco and Zaccagnini [5] showed that if we consider the direct average (3) instead, then we can deal with shorter H than (7). After some minor modification, Theorem 2 of [5] gives the following. In this paper, the letter B denotes the quantity given by

$$B=\exp\left(c\left(\frac{\log X}{\log\log X}\right)^{\frac{1}{3}}\right), \tag{8}$$

where c is some small positive constant which may depend on k,ℓ and ε.

Theorem LZ1. (Languasco and Zaccagnini [5, Theorem 2]) *For real numbers X,H and ε with $4\le H\le X$ and $\varepsilon>0$, we have*

$$\sum_{X<N\le X+H} R_{1,2}(N)=HX^{\frac{1}{2}}+O(HX^{\frac{1}{2}}B^{-1}) \tag{9}$$

provided $X^{\frac{1}{2}}B^{-1}\le H\le X^{1-\varepsilon}$, where the implicit constant depends on ε.

Thus, Languasco and Zaccagnini obtained the asymptotic formula (9) for H shorter than (7) up to the factor B^{-1}. However, we still have the same exponent $\frac{1}{2}$ of X. In this paper, we improve this exponent from $\frac{1}{2}$ to 0.336899....

Theorem 1. *For real numbers X,H,ε with $4\le H\le X$ and $\varepsilon>0$, we have the asymptotic formula (9) provided*

$$X^{\Theta(1,2)+\varepsilon}\le H\le X^{1-\varepsilon},\quad \Theta(1,2)=\frac{32-4\sqrt{15}}{49}=0.336899...,$$

where the implicit constant depends on ε.

Recently, Languasco and Zaccagnini [9] and [10] dealt with other cases of (1):

Theorem LZ2. (Languasco and Zaccagnini [10, Theorem 1.3]) *For positive integers $k,\ell\geq 2$ and real numbers X,H,ε with $4\leq H\leq X$ and $\varepsilon>0$, we have*

$$\sum_{X<N\leq X+H} R_{k,\ell}(N)=\frac{1}{k\ell}\frac{\Gamma(\frac{1}{k})\Gamma(\frac{1}{\ell})}{\Gamma(\frac{1}{k}+\frac{1}{\ell})}HX^{\frac{1}{k}+\frac{1}{\ell}-1}+O(HX^{\frac{1}{k}+\frac{1}{\ell}-1}B^{-1}) \tag{10}$$

provided

$$X^{\Theta_{LZ}(k,\ell)+\varepsilon}\leq H\leq X^{1-\varepsilon},$$

where

$$\Theta_{LZ}(k,\ell)=1-\theta_{\mathrm{LZ}}(k,\ell),\quad \theta_{\mathrm{LZ}}(k,\ell)=\min\left(\frac{5}{6k},\frac{1}{\ell}\right)$$

and the implicit constant depends on k,ℓ and ε.

Remark 2. In [10], Languasco and Zaccagnini considered

$$\widetilde{R}_{k,\ell}(N)=\sum_{\substack{p^k+n^\ell=N\\ N/B<p^k,n^\ell\leq N}}\log p$$

instead of (2). However, new restrictions

$$N/B<p^k,n^\ell\leq N$$

are introduced just for some technical simplicity of the proof. Indeed, it is easy to replace $\widetilde{R}_{k,\ell}(N)$ by $R_{k,\ell}(N)$ assuming $X^{1-\min(\frac{1}{k},\frac{1}{\ell})}\leq H\leq X$ as follows. If we remove the restriction $p^k>N/B$, then the resulting error is bounded by

$$\ll \sum_{\substack{X<p^k+m^\ell\leq X+H\\ p^k\leq 2X/B}}\log p\ll L\sum_{p^k\leq 2X/B}\ \sum_{X-p^k<m^\ell\leq X+H-p^k}1.$$

By Lemma 5 below and the assumption $H\geq X^{1-\frac{1}{\ell}}$, this is

$$\ll HL\sum_{p^k\leq 2X/B}(X-p^k)^{\frac{1}{\ell}-1}\ll HX^{\frac{1}{\ell}-1}(X/B)^{\frac{1}{k}}\ll HX^{\frac{1}{k}+\frac{1}{\ell}-1}B^{-\frac{1}{k}},$$

which is bounded by the error term of Theorem LZ2 up to replacing the constant c in (8). The restriction $n^k>N/B$ can be removed in the same way.

Actually, Theorem 1 above is a special case of the following general result:

Theorem 3. *For positive integers* k,ℓ *with* $\ell\ge 2$ *and real numbers* X,H,ε *with* $4\le H\le X$ *and* $\varepsilon>0$, *we have the asymptotic formula* (10) *provided*

$$X^{\Theta(k,\ell)+\varepsilon}\le H\le X^{1-\varepsilon},$$

where $\Theta(k,\ell)$ *is defined by*

$$\Theta(k,\ell)=1-\theta(k,\ell),\quad \theta(k,\ell)=\max(\theta_A(k,\ell),\theta_B(k,\ell)),$$
$$\theta_A(k,\ell)=\min\left(\frac{\lambda_1(\ell)}{k},\frac{\lambda_2(k,\ell)}{k},\frac{k}{\ell(k-1)}\right),$$
$$\theta_B(k,\ell)=\min\left(\frac{5}{12k},\frac{k}{\ell(k-1)}\right),$$

(11)
$$\lambda_1(\ell)=\begin{cases}\dfrac{\ell}{2(\ell-1)} & \text{if } 2\le\ell\le 3,\\[1ex] \dfrac{3\ell^2+2\sqrt{3}\ell^{\frac{3}{2}}+\ell}{(3\ell-1)^2} & \text{if } 3\le\ell\le\frac{25}{3},\\[1ex] \dfrac{5\ell}{4(3\ell-5)} & \text{if } \ell\ge\frac{25}{3},\end{cases}$$

$$\lambda_2(k,\ell)=\begin{cases}\dfrac{2}{3}\left(\dfrac{k}{\ell}+\dfrac{1}{2}\right) & \text{if } \frac{5}{8}\ell\le k,\\[1ex] \dfrac{10}{49}+\dfrac{2k}{7\ell}+\dfrac{4}{7}\sqrt{\dfrac{6}{7}\left(\dfrac{k}{\ell}-\dfrac{1}{7}\right)} & \text{if } \frac{31}{96}\ell\le k\le\frac{5}{8}\ell,\\[1ex] \dfrac{10}{11}\left(\dfrac{k}{\ell}+\dfrac{1}{4}\right) & \text{if } k\le\frac{31}{96}\ell\end{cases}$$

and the implicit constant depends on k,ℓ *and* ε.

We prove Theorem 3 at the end of Section 6.

The mainly concerned case of Theorem 3 is the case

(12)
$$\theta_A(k,\ell)>\theta_B(k,\ell),\theta_{LZ}(k,\ell).$$

We compare these three exponents in Section 7. By Lemma 28 in Section 6 and Lemma 31 in Section 7, it turns out that (12) occurs for

(13)
$$\begin{aligned}&\ell=2, \quad \text{or}\\ &3\le\ell\le 9 \quad \text{and} \quad \frac{5}{24}\ell<k<\lambda_1(\ell)\ell, \quad \text{or}\\ &\ell\ge 10 \quad \text{and} \quad \frac{5}{24}\ell+\frac{1}{24}\sqrt{\ell(25\ell-240)}<k<\lambda_1(\ell)\ell.\end{aligned}$$

Table 1. The best exponents in $\theta_A,\theta_B,\theta_{LZ}$.

$\ell\backslash k$	1	2	3	4	5	6	7	8	9	10
2	A	A	A	A	A	A	A	A	A	A
3	A	A	LZ	LZ	LZ	LZ	LZ	LZ	LZ	LZ
4	A	A	LZ	LZ	LZ	LZ	LZ	LZ	LZ	LZ
5	B	A	A	LZ	LZ	LZ	LZ	LZ	LZ	LZ
6	B	A	A	LZ	LZ	LZ	LZ	LZ	LZ	LZ
7	B	A	A	LZ	LZ	LZ	LZ	LZ	LZ	LZ
8	B	A	A	A	LZ	LZ	LZ	LZ	LZ	LZ
9	B	A	A	A	LZ	LZ	LZ	LZ	LZ	LZ
10	B	B	A	A	LZ	LZ	LZ	LZ	LZ	LZ
11	B	B	B	A	A	LZ	LZ	LZ	LZ	LZ
12	B	B	B	A	A	LZ	LZ	LZ	LZ	LZ
13	B	B	B	B	A	A	LZ	LZ	LZ	LZ
14	B	B	B	B	A	A	LZ	LZ	LZ	LZ
15	B	B	B	B	B	A	A	LZ	LZ	LZ
16	B	B	B	B	B	A	A	LZ	LZ	LZ
17	B	B	B	B	B	A	A	LZ	LZ	LZ
18	B	B	B	B	B	B	A	A	LZ	LZ
19	B	B	B	B	B	B	A	A	LZ	LZ
20	B	B	B	B	B	B	B	A	A	LZ
21	B	B	B	B	B	B	B	A	A	LZ
22	B	B	B	B	B	B	B	B	A	LZ

Furthermore, in Section 7, we also see that

$$\theta(k,\ell)=\begin{cases} \dfrac{\lambda_2(k,\ell)}{k} & \text{for } (k,\ell)=(1,2),(1,3),(1,4) \\ & \qquad (2,5),(2,6),(2,7),(2,8),(2,9), \\ \theta_B(k,\ell) & \text{for } k=1 \text{ and } \ell\geq 5, \\ \min\left(\dfrac{\lambda_1(\ell)}{k},\dfrac{k}{\ell(k-1)}\right) & \text{otherwise.} \end{cases} \tag{14}$$

In Table 1, we show which exponent gives the best result in the range $1\leq k\leq 10$ and $2\leq\ell\leq 22$, where if some exponents coincide, then we use the order of priority $\theta_{LZ},\theta_B,\theta_A$ from high to low. We also plot the value of $\theta_A,\theta_B,\theta_{LZ}$ with $k=1,2,3,4$ in Figure 1 and the value of $\theta_A,\theta_B,\theta_{LZ}$ with $\ell=2$ in Figure 2. We can see that the case (12) occurs for at least one k for each $\ell\geq 2$ as follows: For $2\leq\ell\leq 9$, we can check the existence of such k by Table 1. For $\ell\geq 10$, by (13), we have (12) if

$$k\in\left(\frac{5}{24}\ell+\frac{1}{24}\sqrt{\ell(25\ell-240)},\lambda_1(\ell)\ell\right).$$

If the interval on the right has length >1, we can take an integer from the interval, which is our desired k. We have

$$\frac{5}{24}\ell+\frac{1}{24}\sqrt{\ell(25\ell-240)}=\frac{5}{24}\ell+\frac{5}{24}\ell\sqrt{1-\frac{48}{5\ell}}\leq\frac{5}{12}\ell-1 \quad\text{and}\quad \lambda_1(\ell)\ell>\frac{5}{12}\ell$$

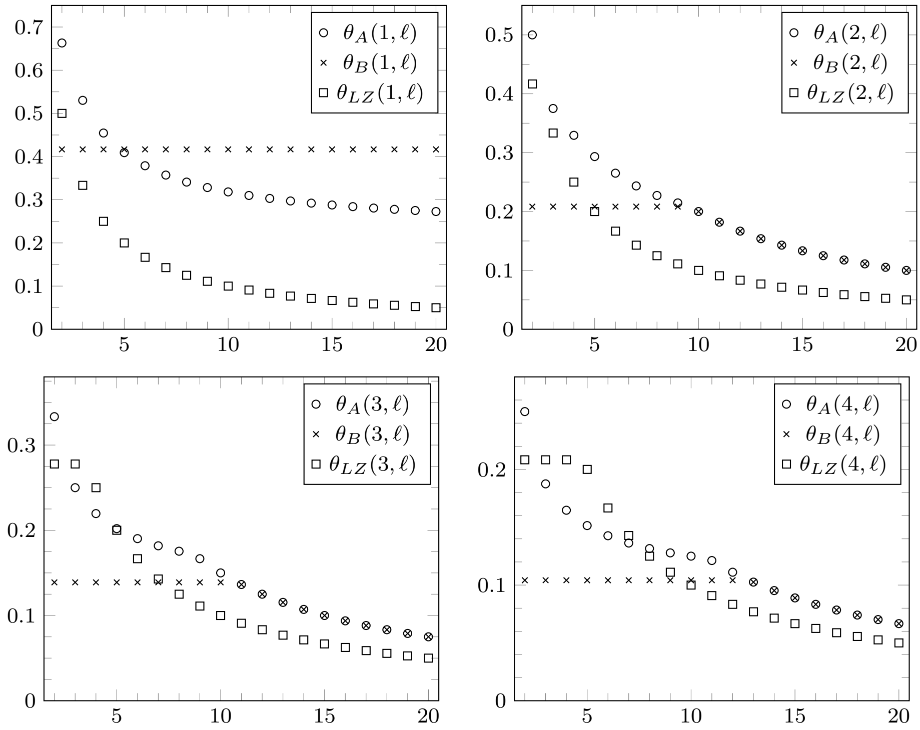

Figure 1. $\theta_A(k,\ell), \theta_B(k,\ell), \theta_{LZ}(k,\ell)$ with $k=1,2,3,4$. (The horizontal axes are for the variable $\ell\geq 2$.)

and so the length of the interval is estimated from below strictly by 1. However, (12) occurs only in a small neighborhood of the line $k=\frac{5}{12}\ell$ for $\ell\geq 3$. In contrast, for $\ell=2$, (12) is always the case. In particular, as a corollary of Theorem 3, we can obtain the exponent $1-\frac{1}{k}$ unconditionally for the case $k\geq 2$ and $\ell=2$, which was obtained under the Riemann hypothesis by Languasco and Zaccagnini [9, Theorem 1.4] up to a small factor X^{ε}:

Theorem 4. *For positive integer k with $k\geq 2$ and real numbers X, H, ε with $4\leq H\leq X$ and $\varepsilon>0$, we have the asymptotic formula (10) with $\ell=2$ provided*

$$X^{1-\frac{1}{k}+\varepsilon}\leq H\leq X^{1-\varepsilon},$$

where the implicit constant depends on k and ε.

We prove Theorem 4 at the end of Section 6.

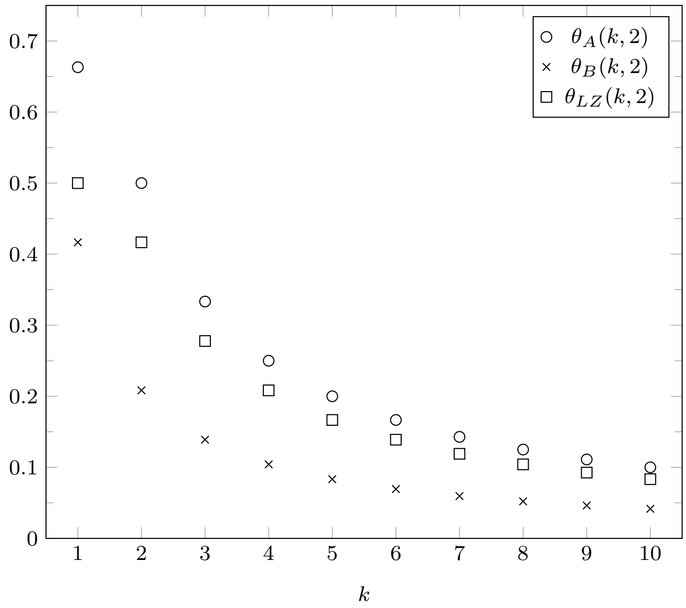

Figure 2. $\theta_A(k,\ell), \theta_B(k,\ell), \theta_{LZ}(k,\ell)$ with $\ell{=}2$.

Languasco and Zaccagnini applied the circle method to prove Theorem LZ1 and Theorem LZ2. In this paper, we deal with the average (3) rather more directly. Our argument is similar to the classical proof of the prime number theorem in short intervals. We first insert the von Mangoldt explicit formula. Then, we apply the Poisson summation formula in order to detect the cancellations over the sequence n^ℓ, which is not involved in the proof of Languasco and Zaccagnini. Finally, we estimate the sum over non-trivial zeros of the Riemann zeta function by using the Huxley–Ingham zero density estimate.

2. Notations and conventions

We use the following notations and conventions.

As usual, let $\Lambda(n)$ be the von Mangoldt function and

$$\psi(x) = \sum_{m \le x} \Lambda(m). \tag{15}$$

We denote the Riemann zeta function by $\zeta(s)$. By $\rho{=}\beta{+}i\gamma$, we denote non-trivial zeros of $\zeta(s)$ with the real part β and the imaginary part γ. For a real number α

and T with $T\geq 0$, let $N(\alpha,T)$ be the number of non-trivial zeros $\rho=\beta+i\gamma$ of $\zeta(s)$ in the rectangle $\alpha\leq\beta\leq 1$ and $|\gamma|\leq T$ counted with multiplicity.

For a complex valued function f defined over an interval $[a,b]$, let $V_{[a,b]}(f)$ be the total variation of f over $[a,b]$ and

$$\|f\| = \|f\|_{BV([a,b])} = \sup_{x\in[a,b]} |f(x)| + V_{[a,b]}(f). \tag{16}$$

For a real number x, let $e(x)=\exp(2\pi i x)$, $[x]$ be the largest integer not exceeding x and $\{x\}=x-[x]$.

The letters X, H, Q denote real numbers and they are always assumed to satisfy

$$4 \leq H \leq X, \quad X \leq Q \leq X+H.$$

The letters $c_0, c_1>0$ denote some small absolute constants and c denotes a constant with $0<c\leq 1$ which may depend on k, ℓ and ε. The letters B and L are used for the quantities

$$B = \exp\left(c\left(\frac{\log X}{\log\log X}\right)^{\frac{1}{3}}\right), \quad L = \log X.$$

For positive integers k,ℓ and a non-zero complex number α, we let

$$S_\alpha(Q) = S_{\alpha,k,\ell}(Q;X) = \frac{1}{\alpha}\sum_{n^\ell\leq X}(Q-n^\ell)^{\frac{\alpha}{k}}, \quad S(Q)=S_1(Q). \tag{17}$$

Let $\phi(\lambda)$ be a function defined over $[0,+\infty)$ by

$$\phi(\lambda) = \begin{cases} \dfrac{3}{5}\lambda+\dfrac{3}{4} & \text{if } 0\leq\lambda\leq\frac{25}{48}, \\ 3\lambda+2(1-\sqrt{3\lambda}) & \text{if } \frac{25}{48}\leq\lambda\leq\frac{3}{4}, \\ \lambda+\dfrac{1}{2} & \text{if } \frac{3}{4}\leq\lambda\leq 1. \end{cases} \tag{18}$$

This function will be used for estimating sums over non-trivial zeros of $\zeta(s)$. For real numbers k,ℓ with $k\geq 1$ and $\ell\geq 2$, we also introduce two real-valued functions $\lambda_1(\ell)$ and $\lambda_2(k,\ell)$ as in (11). These functions are used in the exponent of the admissible ranges for X and H.

We use a convention $\min(A,\infty)=A$.

If Theorem or Lemma is stated with the phrase “where the implicit constant depends on $a,b,c,\ldots$”, then every implicit constant in the corresponding proof may also depend on $a,b,c,\ldots$ even without special mentions.

3. Preliminary lemmas

In this section, we prepare some lemmas for the proof of Theorem 3. We start with some simple estimates for short interval sums without prime numbers.

Lemma 5. *For positive integer ℓ and real numbers X,H with $X,H\geq 2$,*

$$\sum_{X<n^\ell\leq X+H} 1 \ll HX^{\frac{1}{\ell}-1}+1,$$

where the implicit constant is absolute.

Proof. By using $x-1<[x]\leq x$, we see that

$$\sum_{X<n^\ell\leq X+H} 1 \leq (X+H)^{\frac{1}{\ell}}-X^{\frac{1}{\ell}}+1=\frac{1}{\ell}\int_X^{X+H} u^{\frac{1}{\ell}-1}\,du+1\leq HX^{\frac{1}{\ell}-1}+1.$$

This completes the proof. □

Lemma 6. *For positive integers k,ℓ and real numbers X,H with $4\leq H\leq X$,*

$$\begin{aligned}&\frac{1}{k\ell}\frac{\Gamma(\frac{1}{k})\Gamma(\frac{1}{\ell})}{\Gamma(\frac{1}{k}+\frac{1}{\ell}+1)}\left((X+H)^{\frac{1}{k}+\frac{1}{\ell}}-X^{\frac{1}{k}+\frac{1}{\ell}}\right)\\&=\frac{1}{k\ell}\frac{\Gamma(\frac{1}{k})\Gamma(\frac{1}{\ell})}{\Gamma(\frac{1}{k}+\frac{1}{\ell})}HX^{\frac{1}{k}+\frac{1}{\ell}-1}+O(H^2X^{\frac{1}{k}+\frac{1}{\ell}-2}),\end{aligned}$$

where the implicit constant is absolute.

Proof. By the fundamental theorem of calculus,

$$\begin{aligned}&\frac{1}{k\ell}\frac{\Gamma(\frac{1}{k})\Gamma(\frac{1}{\ell})}{\Gamma(\frac{1}{k}+\frac{1}{\ell}+1)}\left((X+H)^{\frac{1}{k}+\frac{1}{\ell}}-X^{\frac{1}{k}+\frac{1}{\ell}}\right)\\&=\frac{1}{k\ell}\frac{\Gamma(\frac{1}{k})\Gamma(\frac{1}{\ell})}{\Gamma(\frac{1}{k}+\frac{1}{\ell})}\int_X^{X+H}u^{\frac{1}{k}+\frac{1}{\ell}-1}\,du.\end{aligned}\tag{19}$$

For $X<u\leq X+H$, by using the mean value theorem, we have

$$u^{\frac{1}{k}+\frac{1}{\ell}-1}=X^{\frac{1}{k}+\frac{1}{\ell}-1}+O\left(HX^{\frac{1}{k}+\frac{1}{\ell}-2}\right).$$

Thus, the integral in (19) can be rewritten as

$$\int_X^{X+H}u^{\frac{1}{k}+\frac{1}{\ell}-1}\,du=HX^{\frac{1}{k}+\frac{1}{\ell}-1}+O\left(H^2X^{\frac{1}{k}+\frac{1}{\ell}-2}\right).$$

On inserting this formula into (19) and noting that

$$\frac{1}{k\ell}\frac{\Gamma(\frac{1}{k})\Gamma(\frac{1}{\ell})}{\Gamma(\frac{1}{k}+\frac{1}{\ell})} \ll \frac{1}{k\ell}\frac{(\frac{1}{k})^{-1}(\frac{1}{\ell})^{-1}}{(\frac{1}{k}+\frac{1}{\ell})^{-1}} \ll \frac{1}{k}+\frac{1}{\ell} \ll 1, \tag{20}$$

we arrive at the lemma. □

Lemma 7. *For positive integers k,ℓ and real numbers X,H with $4\le H\le X$,*

$$S(X+H)-S(X)=\frac{1}{k\ell}\frac{\Gamma(\frac{1}{k})\Gamma(\frac{1}{\ell})}{\Gamma(\frac{1}{k}+\frac{1}{\ell})}HX^{\frac{1}{k}+\frac{1}{\ell}-1}+O\left(H^{1+\frac{1}{k}}X^{\frac{1}{\ell}-1}+H^{\frac{1}{k}}\right),$$

where $S(Q)$ is defined as in (17) and the implicit constant is absolute.

Proof. The left-hand side of the assertion is

$$=\frac{1}{k}\sum_{n^\ell\le X}\int_X^{X+H}(u-n^\ell)^{\frac{1}{k}-1}\,du=\int_X^{X+H}\frac{1}{k}\sum_{n^\ell\le X}(u-n^\ell)^{\frac{1}{k}-1}\,du. \tag{21}$$

Since the function $(u-w^\ell)^{\frac{1}{k}-1}$ is non-decreasing over $0\le w\le X^{\frac{1}{\ell}}$,

$$\begin{aligned}\frac{1}{k}\sum_{n^\ell\le X}(u-n^\ell)^{\frac{1}{k}-1}&=\frac{1}{k}\int_0^{X^{\frac{1}{\ell}}}(u-w^\ell)^{\frac{1}{k}-1}\,dw+O\left(\frac{1}{k}(u-X)^{\frac{1}{k}-1}\right)\\&=\frac{1}{k\ell}\int_0^X(u-w)^{\frac{1}{k}-1}w^{\frac{1}{\ell}-1}\,dw+O\left(\frac{1}{k}(u-X)^{\frac{1}{k}-1}\right)\end{aligned} \tag{22}$$

for $X<u\le X+H$. Note that the second term on the right-hand side may tend to ∞ as $u\to X+0$, but this term is integrable over $(X,X+H]$. We next extend the integral on the right-hand side. For $X<u\le X+H$, by changing the variable,

$$\begin{aligned}\frac{1}{k\ell}\int_X^u(u-w)^{\frac{1}{k}-1}w^{\frac{1}{\ell}-1}\,dw&\ll\frac{1}{k\ell}X^{\frac{1}{\ell}-1}\int_0^{u-X}w^{\frac{1}{k}-1}\,dw\\&\ll\frac{1}{\ell}X^{\frac{1}{\ell}-1}(u-X)^{\frac{1}{k}}\ll\frac{1}{\ell}H^{\frac{1}{k}}X^{\frac{1}{\ell}-1}.\end{aligned}$$

Hence, we can extend the integral in (22) as

$$\begin{aligned}&\frac{1}{k}\sum_{n^\ell\le X}(u-n^\ell)^{\frac{1}{k}-1}\\&=\frac{1}{k\ell}\int_0^u(u-w)^{\frac{1}{k}-1}w^{\frac{1}{\ell}-1}\,dw+O\left(\frac{1}{k}(u-X)^{\frac{1}{k}-1}+H^{\frac{1}{k}}X^{\frac{1}{\ell}-1}\right).\end{aligned}$$

The last integral on the right-hand side is

$$\frac{1}{k\ell}\int_0^u (u-w)^{\frac{1}{k}-1}w^{\frac{1}{\ell}-1}\,dw=\frac{1}{k\ell}\frac{\Gamma(\frac{1}{k})\Gamma(\frac{1}{\ell})}{\Gamma(\frac{1}{k}+\frac{1}{\ell})}u^{\frac{1}{k}+\frac{1}{\ell}-1}.$$

Therefore,

$$\frac{1}{k}\sum_{n^\ell\le X}(u-n^\ell)^{\frac{1}{k}-1}=\frac{1}{k\ell}\frac{\Gamma(\frac{1}{k})\Gamma(\frac{1}{\ell})}{\Gamma(\frac{1}{k}+\frac{1}{\ell})}u^{\frac{1}{k}+\frac{1}{\ell}-1}+O\left(\frac{1}{k}(u-X)^{\frac{1}{k}-1}+H^{\frac{1}{k}}X^{\frac{1}{\ell}-1}\right).$$

On inserting this formula into (21), the left-hand side of the assertion is

$$=\frac{1}{k\ell}\frac{\Gamma(\frac{1}{k})\Gamma(\frac{1}{\ell})}{\Gamma(\frac{1}{k}+\frac{1}{\ell})}\int_X^{X+H}u^{\frac{1}{k}+\frac{1}{\ell}-1}\,du+O\left(H^{\frac{1}{k}}+H^{1+\frac{1}{k}}X^{\frac{1}{\ell}-1}\right).$$

By Lemma 6, this is

$$=\frac{1}{k\ell}\frac{\Gamma(\frac{1}{k})\Gamma(\frac{1}{\ell})}{\Gamma(\frac{1}{k}+\frac{1}{\ell})}HX^{\frac{1}{k}+\frac{1}{\ell}-1}+O\left(H^2X^{\frac{1}{k}+\frac{1}{\ell}-2}+H^{1+\frac{1}{k}}X^{\frac{1}{\ell}-1}+H^{\frac{1}{k}}\right).$$

Since $H\le X$, we can estimate the first error term as

$$H^2X^{\frac{1}{k}+\frac{1}{\ell}-2}=H^2X^{-(1-\frac{1}{k})}X^{\frac{1}{\ell}-1}\le H^{1+\frac{1}{k}}X^{\frac{1}{\ell}-1}.$$

This completes the proof. □

Lemma 8. *For positive integers k,ℓ and real numbers X,H with $4\le H\le X$,*

$$\sum_{X<m^k+n^\ell\le X+H}1\ll HX^{\frac{1}{k}+\frac{1}{\ell}-1}+H^{\frac{1}{k}}+X^{\frac{1}{\ell}},$$

where the implicit constant is absolute.

Proof. We rewrite the left-hand side as

$$\sum_{X<m^k+n^\ell\le X+H}1=\sum_{n^\ell\le X+H}\ \sum_{X-n^\ell<m^k\le X+H-n^\ell}1.$$

We next truncate the outer summation over n^ℓ. By using Lemma 5,

$$\sum_{X<n^\ell\le X+H}\ \sum_{X-n^\ell<m^k\le X+H-n^\ell}1\ll\sum_{X<n^\ell\le X+H}\ \sum_{m^k\le H}1\ll H^{1+\frac{1}{k}}X^{\frac{1}{\ell}-1}+H^{\frac{1}{k}}.$$

Thus, by using the assumption $H\le X$,

$$\sum_{X<m^k+n^\ell\le X+H}1=\sum_{n^\ell\le X}\ \sum_{X-n^\ell<m^k\le X+H-n^\ell}1+O\left(HX^{\frac{1}{k}+\frac{1}{\ell}-1}+H^{\frac{1}{k}}\right). \tag{23}$$

The sum on the right-hand side is

$$\sum_{n^\ell \le X} \sum_{X-n^\ell < m^k \le X+H-n^\ell} 1 = \sum_{n^\ell \le X} \big((X+H-n^\ell)^{\frac{1}{k}} - (X-n^\ell)^{\frac{1}{k}} + O(1)\big)$$
$$= S(X+H) - S(X) + O(X^{\frac{1}{\ell}}).$$

By using Lemma 7 with the bound (20) and the assumption $H \le X$,

$$\sum_{n^\ell \le X} \sum_{X-n^\ell < m^k \le X+H-n^\ell} 1 \ll HX^{\frac{1}{k}+\frac{1}{\ell}-1} + H^{\frac{1}{k}} + X^{\frac{1}{\ell}}.$$

On inserting this estimate into (23), we obtain the lemma. □

We next recall some standard lemmas in prime number theory.

Lemma 9. *For real numbers X, T, x with $2 \le T \le 2X$ and $0 \le x \le X$, we have*

$$\psi(x) = x - \sum_{\substack{\rho \\ |\gamma| \le T}} \frac{x^\rho}{\rho} + O(XT^{-1}L^2),$$

where the implicit constant is absolute.

Proof. In the case $2 \le x \le X$, this follows from Theorem 12.5 of [13]. In the case $0 \le x \le 2$, the lemma trivially follows since $XT^{-1}L^2 \gg L^2$ by $T \le 2X$ and

$$\sum_{\substack{\rho \\ |\gamma| \le T}} \frac{x^\rho}{\rho} \ll \sum_{\substack{\rho \\ |\gamma| \le T}} \frac{1}{|\rho|} \ll (\log T)^2 \ll L^2$$

for the case $0 \le x \le 2$. This completes the proof. □

Lemma 10. (The Korobov–Vinogradov zero-free region) *We have $\zeta(s) \ne 0$ for*

$$\sigma > 1 - c_0 (\log \tau)^{-\frac{2}{3}} (\log\log \tau)^{-\frac{1}{3}}, \quad s = \sigma + it, \quad \tau = |t| + 4,$$

where $c_0 > 0$ is some absolute constant.

Proof. See [4, Theorem 6.1, p. 143]. □

Lemma 11. (The Huxley–Ingham zero density estimate [2] and [3]) *For real numbers α and T with $\frac{1}{2} \le \alpha \le 1$ and $T \ge 2$,*

$$N(\alpha, T) \ll T^{c(\alpha)} (\log T)^A, \quad c(\alpha) = \begin{cases} \frac{3(1-\alpha)}{3\alpha-1} & \text{if } \frac{3}{4} \le \alpha \le 1, \\ \frac{3(1-\alpha)}{2-\alpha} & \text{if } \frac{1}{2} \le \alpha \le \frac{3}{4}, \end{cases}$$

where the constant A and the implicit constant are absolute.

Proof. See (11.26) and (11.27) of [4, p. 275]. □

Lemma 12. *For real numbers X, H, ε with $4 \le H \le X$ and $\varepsilon > 0$,*

$$\psi(X+H) - \psi(X) = H + O(HB^{-1})$$

provided $X^{\frac{7}{12}+\varepsilon} \le H \le X$, where the implicit constant depends on ε.

Proof. This follows by Lemma 10 and Lemma 11 through the standard argument. □

In the proof of Theorem 3, we need to estimate several sums over non-trivial zeros of the Riemann zeta function. Our next several lemmas deal with such sums and the exponents in the resulting estimates.

Lemma 13. *For real numbers K, X, Y with $1 \le K \le Y \le X^2$ and $X \ge 4$,*

$$\sum_{\substack{\rho \\ K < |\gamma| \le 2K}} Y^\beta \ll \left(Y^{\phi(\lambda)} + Y^{1-\eta+2\eta\lambda} \right) L^A,$$

where the function $\phi(\lambda)$ is defined by (18),

$$\eta = c_1 (\log X)^{-\frac{2}{3}} (\log\log X)^{-\frac{1}{3}}, \qquad \lambda = \frac{\log K}{\log Y},$$

and constants $A, c_1 > 0$ and the implicit constant are absolute.

Proof. By Lemma 10 and Lemma 11, the left-hand side is bounded by

$$\sum_{\substack{\rho \\ K < |\gamma| \le 2K \\ \beta \ge \frac{1}{2}}} Y^\beta = -\int_{\frac{1}{2}}^{1-\eta} Y^\alpha \, dN(\alpha, 2K) \ll KY^{\frac{1}{2}} L + L^A \int_{\frac{1}{2}}^{1-\eta} K^{c(\alpha)} Y^\alpha \, d\alpha \tag{24}$$

for sufficiently small $c_1 > 0$. We determine the maximum value of

$$K^{c(\alpha)} Y^\alpha = Y^{\lambda c(\alpha) + \alpha}$$

over $\alpha \in [\frac{1}{2}, 1-\eta]$. Let $h(\alpha) = \lambda c(\alpha) + \alpha$. For $\alpha \in [\frac{1}{2}, \frac{3}{4}]$, we have

$$h(\alpha) = \frac{3\lambda(1-\alpha)}{2-\alpha} + \alpha = 3\lambda - \frac{3\lambda}{2-\alpha} + \alpha.$$

By taking the derivative,

$$h'(\alpha) = -\frac{3\lambda}{(2-\alpha)^2} + 1.$$

Thus, in the range $\alpha\in(-\infty,2)$,

$$h'(\alpha)=0 \quad\Longleftrightarrow\quad \alpha=2-\sqrt{3\lambda},$$

so $h(\alpha)$ is increasing for $\alpha<2-\sqrt{3\lambda}$ and decreasing for $2-\sqrt{3\lambda}<\alpha<2$. Hence,

$$\max_{\alpha\in[\frac{1}{2},\frac{3}{4}]} h(\alpha)=\begin{cases} \dfrac{3}{5}\lambda+\dfrac{3}{4} & \text{if } 0\le\lambda\le\frac{25}{48},\\ 3\lambda+2(1-\sqrt{3\lambda}) & \text{if } \frac{25}{48}\le\lambda\le\frac{3}{4},\\ \lambda+\dfrac{1}{2} & \text{if } \frac{3}{4}\le\lambda\le 1.\end{cases}$$

For $\alpha\in[\frac{3}{4},1-\eta]$, we have

$$h(\alpha)=\frac{3\lambda(1-\alpha)}{3\alpha-1}+\alpha=-\lambda+\frac{2\lambda}{3\alpha-1}+\alpha.$$

By taking the derivative twice, in the range $\alpha\in[\frac{3}{4},1-\eta]$,

$$h''(\alpha)=\frac{18\lambda}{(3\alpha-1)^3}>0$$

so that $h(\alpha)$ is convex downwards in this range. Thus, for small c_1,

$$\max_{\alpha\in[\frac{3}{4},1-\eta]} h(\alpha)=\max\left(h\left(\frac{3}{4}\right),h\left(1-\eta\right)\right)\le\max\left(h\left(\frac{3}{4}\right),1-\eta+2\eta\lambda\right).$$

By using the above observations for $h(\alpha)$ in (24), we obtain the lemma. □

Lemma 14. *Let $\phi(\lambda)$ be the function given by (18). Then,*

$$\frac{3}{5}\le\phi'(\lambda)\le 1$$

for $\lambda\ge 0$. In particular, $\phi(\lambda)$ is increasing.

Proof. It suffices to consider the case $\frac{25}{48}\le\lambda\le\frac{3}{4}$. In this range,

$$\phi'(\lambda)=3-\sqrt{\frac{3}{\lambda}}.$$

Thus, the lemma easily follows. □

Lemma 15. *Let $\phi(\lambda)$ be the function given by (18). For real numbers k,ℓ with $k\ge 1$ and $\ell\ge 2$, consider the solutions λ_1 and λ_2 of the equations*

$$\phi(\lambda_1)-\frac{1}{\ell}\lambda_1=1,\quad \phi(\lambda_2)+\frac{1}{2}\lambda_2=1+\frac{k}{\ell}. \tag{25}$$

Then, these functions λ_1,λ_2 coincide with the functions given in (11).

Proof. By Lemma 14 and $\ell \geq 2$, both of the continuous functions

$$\phi(\lambda)-\frac{1}{\ell}\lambda, \quad \phi(\lambda)+\frac{1}{2}\lambda \tag{26}$$

are strictly increasing for $\lambda \geq 0$ and take the value from $3/4$ to $+\infty$. Thus, by the intermediate value theorem, λ_1 and λ_2 are well-defined.

We first consider λ_1. If $\phi(\frac{25}{48})-\frac{25}{48\ell} \geq 1$, i.e. $\ell \geq \frac{25}{3}$, then

$$1=\phi(\lambda_1)-\frac{1}{\ell}\lambda_1=\left(\frac{3}{5}-\frac{1}{\ell}\right)\lambda_1+\frac{3}{4}$$

so that

$$\lambda_1=\frac{5\ell}{4(3\ell-5)}.$$

If $\phi(\frac{25}{48})-\frac{25}{48\ell} \leq 1 \leq \phi(\frac{3}{4})-\frac{3}{4\ell}$, i.e. $3 \leq \ell \leq \frac{25}{3}$, then

$$1=\phi(\lambda_1)-\frac{1}{\ell}\lambda_1=\left(3-\frac{1}{\ell}\right)\lambda_1+2(1-\sqrt{3\lambda_1})$$

so that, by using $\frac{25}{48} \leq \lambda_1$ in the current case,

$$\lambda_1=\frac{3\ell^2+2\sqrt{3}\ell^{\frac{3}{2}}+\ell}{(3\ell-1)^2}.$$

Finally, if $\phi(\frac{3}{4})-\frac{3}{4\ell} \leq 1$, i.e. $2 \leq \ell \leq 3$, then

$$1=\phi(\lambda_1)-\frac{1}{\ell}\lambda_1=\left(1-\frac{1}{\ell}\right)\lambda_1+\frac{1}{2}$$

so that

$$\lambda_1=\frac{\ell}{2(\ell-1)}.$$

This completes the proof of the assertion for λ_1.

We next consider λ_2. If $1+\frac{k}{\ell} \leq \phi(\frac{25}{48})+\frac{1}{2}\cdot\frac{25}{48}$, i.e. $k \leq \frac{31}{96}\ell$, then

$$1+\frac{k}{\ell}=\phi(\lambda_2)+\frac{1}{2}\lambda_2=\frac{11}{10}\lambda_2+\frac{3}{4}$$

so that

$$\lambda_2=\frac{10}{11}\left(\frac{k}{\ell}+\frac{1}{4}\right).$$

If $\phi(\frac{25}{48})+\frac{1}{2}\cdot\frac{25}{48} \leq 1+\frac{k}{\ell} \leq \phi(\frac{3}{4})+\frac{1}{2}\cdot\frac{3}{4}$, i.e. $\frac{31}{96}\ell \leq k \leq \frac{5}{8}\ell$, then

$$1+\frac{k}{\ell}=\phi(\lambda_2)+\frac{1}{2}\lambda_2=\frac{7}{2}\lambda_2+2(1-\sqrt{3\lambda_2})$$

so that, by using $\frac{25}{48} \le \lambda_2$ in the current case,

$$\lambda_2 = \frac{10}{49} + \frac{2k}{7\ell} + \frac{4}{7}\sqrt{\frac{6}{7}\left(\frac{k}{\ell} - \frac{1}{7}\right)}.$$

Finally, if $\phi(\frac{3}{4}) + \frac{1}{2} \cdot \frac{3}{4} \le 1 + \frac{k}{\ell}$, i.e. $\frac{5}{8}\ell \le k$, then

$$1 + \frac{k}{\ell} = \phi(\lambda_2) + \frac{1}{2}\lambda_2 = \frac{3}{2}\lambda_2 + \frac{1}{2}$$

so that

$$\lambda_2 = \frac{2}{3}\left(\frac{k}{\ell} + \frac{1}{2}\right).$$

This completes the proof of the assertion for λ_2. $\square$

Lemma 16. *For positive integers k, ℓ with $\ell \ge 2$ and a real number ε with $\varepsilon > 0$,*

$$\phi(\lambda) - \frac{1}{\ell}\lambda \le 1 - \frac{\varepsilon}{10} \quad \textit{and} \quad \phi(\lambda) + \frac{1}{2}\lambda \le 1 + \frac{k}{\ell} - \frac{\varepsilon}{10}$$

provided

$$0 \le \lambda \le \min(\lambda_1, \lambda_2) - \varepsilon, \tag{27}$$

where λ_1, λ_2 are the solutions of (25) *or equivalently, defined by* (11).

Proof. By the assumption $\ell \ge 2$ and Lemma 14, both of the functions (26) have the derivative of the size $\ge \frac{1}{10}$. Thus, the mean value theorem and (27) give

$$\phi(\lambda) - \frac{1}{\ell}\lambda \le \phi(\lambda_1) - \frac{1}{\ell}\lambda_1 - \frac{1}{10}(\lambda_1 - \lambda) \le 1 - \frac{\varepsilon}{10}$$

and

$$\phi(\lambda) + \frac{1}{2}\lambda \le \phi(\lambda_2) + \frac{1}{2}\lambda_2 - \frac{1}{10}(\lambda_2 - \lambda) \le 1 + \frac{k}{\ell} - \frac{\varepsilon}{10}.$$

This completes the proof. $\square$

Lemma 17. *The functions $\lambda_1(\ell), \lambda_2(k, \ell)$ are strictly decreasing with respect to ℓ.*

Proof. By Lemma 15, we $\lambda_1(\ell)$ and $\lambda_2(k, \ell)$ can be regarded as the solutions of the equations (25). Then, the lemma follows since the functions (26) are increasing. $\square$

As we mentioned in Section 1, we shall apply the Poisson summation formula in order to detect some cancellation over the sequence n^ℓ. In order to estimate the resulting exponential integrals, we recall the next two standard estimates.

Lemma 18. (First derivative estimate) *Let λ be a positive real number and f,g be real-valued functions defined over an interval $[a,b]$ satisfying*

(A) *f is continuously differentiable on the interval $[a,b]$,*

(B) *f' is monotonic on the interval $[a,b]$ and*

(C) *f' satisfies $|f'(x)|\geq\lambda$ on the interval $[a,b]$.*

Then, by using notation (16), we have

$$\int_a^b g(x)e(f(x))\,dx \ll \|g\|\lambda^{-1},$$

where the implicit constant is absolute.

Proof. See [4, Lemma 2.1, p. 56]. □

Lemma 19. (Second derivative estimate) *Let λ be a positive real number and f,g be real-valued functions defined over an interval $[a,b]$ satisfying*

(A) *f is twice continuously differentiable on the interval $[a,b]$,*

(B) *f'' satisfies $|f''(x)|\geq\lambda$ on the interval $[a,b]$.*

Then, by using notation (16), we have

$$\int_a^b g(x)e(f(x))\,dx \ll \|g\|\lambda^{-\frac{1}{2}},$$

where the implicit constant is absolute.

Proof. See [4, Lemma 2.2, p. 56]. □

4. Preliminary calculations

In this section, we carry out preliminary calculations for the proof of Theorem 3. We first replace $\log p$ in (2) by the von Mangoldt function.

Lemma 20. *For positive integers k,ℓ and real numbers X,H,ε with $4\leq H\leq X$ and $\varepsilon>0$, we have*

$$\sum_{X<N\leq X+H} R(N) = \sum_{X<m^k+n^\ell\leq X+H} \Lambda(m) + O(HX^{\frac{1}{k}+\frac{1}{\ell}-1}B^{-1})$$

provided

$$X^{1-\min(\frac{1}{k},\frac{k}{\ell(k-1)})+\varepsilon} \leq H \leq X^{1-\varepsilon}, \tag{28}$$

where the implicit constant depends on k,ℓ and ε.

Proof. By definition (2) of $R(N)$,

$$\sum_{X<N\le X+H} R(N) = \sum_{X<p^k+n^\ell\le X+H} \log p$$
$$= \sum_{X<m^k+n^\ell\le X+H} \Lambda(m) - \sum_{\nu=2}^{O(L)} \sum_{X<p^{\nu k}+n^\ell\le X+H} \log p.$$

By Lemma 8, the second term on the right hand side is bounded by

$$\ll L \sum_{\nu=2}^{O(L)} (HX^{\frac{1}{\nu k}+\frac{1}{\ell}-1} + H^{\frac{1}{\nu k}} + X^{\frac{1}{\ell}}) \ll (HX^{\frac{1}{2k}+\frac{1}{\ell}-1} + H^{\frac{1}{k}} + X^{\frac{1}{\ell}})L^2,$$

which is $\ll HX^{\frac{1}{k}+\frac{1}{\ell}-1}B^{-1}$ provided (28). This completes the proof. ☐

We then modify the sum on the right-hand side of Lemma 20.

Lemma 21. *For positive integers k,ℓ and real numbers X,H,ε with $4\le H\le X$ and $\varepsilon>0$, we have*

$$\sum_{X<N\le X+H} R(N)$$
$$= \sum_{n^\ell\le X} \left(\psi\left((X+H-n^\ell)^{\frac{1}{k}}\right) - \psi\left((X-n^\ell)^{\frac{1}{k}}\right)\right) + O(HX^{\frac{1}{k}+\frac{1}{\ell}-1}B^{-1})$$

provided

$$X^{1-\min(\frac{1}{k},\frac{k}{\ell(k-1)})+\varepsilon} \le H \le X^{1-\varepsilon}, \tag{29}$$

where the implicit constant depends on k,ℓ and ε.

Proof. We truncate the summation over n in Lemma 20. By using Lemma 5 and the argument similar to the beginning of the proof of Lemma 8,

$$\sum_{X<n^\ell\le X+H} \sum_{X-n^\ell<m^k\le X+H-n^\ell} \Lambda(m) \ll H^{1+\frac{1}{k}}X^{\frac{1}{\ell}-1} + H^{\frac{1}{k}} \ll HX^{\frac{1}{k}+\frac{1}{\ell}-1}B^{-1}$$

provided (29). Thus we can employ the truncation as

$$\sum_{X<m^k+n^\ell\le X+H} \Lambda(m) = \sum_{n^\ell\le X} \sum_{X-n^\ell<m^k\le X+H-n^\ell} \Lambda(m) + O(HX^{\frac{1}{k}+\frac{1}{\ell}-1}B^{-1}).$$

By recalling the notation (15), this is

$$= \sum_{n^\ell\le X} \left(\psi\left((X+H-n^\ell)^{\frac{1}{k}}\right) - \psi\left((X-n^\ell)^{\frac{1}{k}}\right)\right) + O(HX^{\frac{1}{k}+\frac{1}{\ell}-1}B^{-1}).$$

By substituting this formula into Lemma 20, we obtain the lemma. ☐

5. Detection of the cancellation over the ℓ-th powers

In this section, we derive an expansion for

$$\sum_{n^\ell \le X} \left(\psi\left((X+H-n^\ell)^{\frac{1}{k}}\right)-\psi\left((X-n^\ell)^{\frac{1}{k}}\right)\right)$$

by which we try to detect some cancellation caused by the average over n^ℓ. This expansion will be given by Lemma 22 and Lemma 24. We first substitute Lemma 9.

Lemma 22. *Let k,ℓ be positive integers and X,H,Q,T be real numbers satisfying $4\le H\le X$, $X\le Q\le X+H$ and $2\le T\le X^{\frac{1}{k}}$. Then,*

$$\sum_{n^\ell\le X}\psi\left((Q-n^\ell)^{\frac{1}{k}}\right)=S(Q)-\sum_{\substack{\rho\\ |\gamma|\le T}}S_\rho(Q)+O(X^{\frac{1}{k}+\frac{1}{\ell}}T^{-1}L^2),$$

where $S(Q)$ and $S_\rho(Q)$ are defined as in (17) and the implicit constant is absolute.

Proof. This follows immediately by inserting Lemma 9. □

Our next task is to detect the cancellation in the sum $S_\rho(Q)$. We prepare the next lemma in order to estimate exponential integrals.

Lemma 23. *For positive integers k,ℓ, an integer n not necessarily positive and real numbers α,γ,Q,U,V with $\alpha\le 1$, $|\gamma|\ge 1$ and $1\le U\le V\le Q$, we have*

$$\int_U^V u^{\alpha+\frac{i\gamma}{k}-1}e\left(n(Q-u)^{\frac{1}{\ell}}\right)du \ll \begin{cases} \dfrac{V^\alpha L}{|\gamma|^{\frac{1}{2}}} & \text{if } \alpha\ge 0,\\ \dfrac{U^\alpha L}{|\gamma|^{\frac{1}{2}}} & \text{if } \alpha\le 0,\\ \dfrac{Q^{1-\frac{1}{\ell}}}{|n|} & \text{if } |n|>\ell Q^{1-\frac{1}{\ell}}|\gamma|,\end{cases}$$

where the implicit constant depends on k,ℓ and ε.

Proof. We rewrite the left-hand side as

$$\int_U^V u^{\alpha+\frac{i\gamma}{k}-1}e\left(n(Q-u)^{\frac{1}{\ell}}\right)du=\int_U^V G(u)e(F(u))\,du, \tag{30}$$

where

$$F(u)=n(Q-u)^{\frac{1}{\ell}}+\frac{\gamma}{2\pi k}\log u,\quad G(u)=u^{\alpha-1}.$$

Then,

$$F'(u)=-\frac{1}{\ell}n(Q-u)^{\frac{1}{\ell}-1}+\frac{\gamma}{2\pi k u},\quad F''(u)=-\frac{\ell-1}{\ell^2}n(Q-u)^{\frac{1}{\ell}-2}-\frac{\gamma}{2\pi k u^2} \tag{31}$$

and since $G(u)$ is non-increasing, by using the notation (16),

$$\|G\|_{BV([R,R'])} \ll R^{\alpha-1}$$

for any subinterval $[R,R'] \subset [U,V]$.

For the former two estimates, we dissect the integral (30) dyadically as

$$\ll L \sup_{U<R\le V} \left| \int_R^{\min(2R,V)} G(u)e(F(u))\,du \right|. \tag{32}$$

If n and γ have the same signs, then we have

$$|F''(u)| \ge \frac{|\gamma|}{2\pi k(2R)^2}$$

for $u \in [R, \min(2R,V)]$. Therefore, by Lemma 19,

$$\int_R^{\min(2R,V)} u^{\alpha+\frac{i\gamma}{k}-1} e\left(n(Q-u)^{\frac{1}{\ell}}\right) du \ll R^{\alpha-1}\left(\frac{|\gamma|}{R^2}\right)^{-\frac{1}{2}} \ll \frac{R^\alpha}{|\gamma|^{\frac{1}{2}}}. \tag{33}$$

On the other hand, if n and γ have the opposite signs, then we have

$$|F'(u)| \ge \frac{|\gamma|}{2\pi k(2R)}$$

and $F''(u)$ has at most one zero in $[R, \min(2R,V)]$. Therefore, we may dissect $[R, \min(2R,V)]$ into at most two intervals, on each of which $F'(u)$ is monotonic. By applying Lemma 18,

$$\int_R^{\min(2R,V)} u^{\alpha+\frac{i\gamma}{k}-1} e\left(n(Q-u)^{\frac{1}{\ell}}\right) du \ll R^{\alpha-1}\left(\frac{|\gamma|}{R}\right)^{-1} = \frac{R^\alpha}{|\gamma|} \ll \frac{R^\alpha}{|\gamma|^{\frac{1}{2}}} \tag{34}$$

since $|\gamma| \ge 1$. Therefore, by (33) and (34), we have

$$\int_R^{\min(2R,V)} u^{\alpha+\frac{i\gamma}{k}-1} e\left(n(Q-u)^{\frac{1}{\ell}}\right) du \ll \frac{R^\alpha}{|\gamma|^{\frac{1}{2}}}$$

in any case. On inserting this estimate into (32), we obtain the first two estimates.

For the last estimate, we work without the dyadic dissection. We apply Lemma 18 to the integral (30). By assuming $|n| > \ell Q^{1-\frac{1}{\ell}}|\gamma|$,

$$|F'(u)| \ge \frac{|n|}{\ell Q^{1-\frac{1}{\ell}}} - \frac{|\gamma|}{2\pi k} \gg \frac{|n|}{Q^{1-\frac{1}{\ell}}}.$$

Also, by (31), we can dissect $[U,V]$ into at most two intervals, on each of which $F'(u)$ is monotonic. Thus, by Lemma 18,

$$\int_U^V u^{\alpha+\frac{i\gamma}{k}-1} e\left(n(Q-u)^{\frac{1}{\ell}}\right) du \ll U^{\alpha-1}\left(\frac{|n|}{Q^{1-\frac{1}{\ell}}}\right)^{-1} \ll \frac{Q^{1-\frac{1}{\ell}}}{|n|}$$

since $\alpha\le 1$. This completes the proof. □

We now apply the Poisson summation formula and detect the cancellation over the sequence n^ℓ.

Lemma 24. *For positive integers k,ℓ, real numbers X,H,ε with $4\le H\le X$ and $\varepsilon>0$ and a non-trivial zero $\rho=\beta+i\gamma$ of $\zeta(s)$ with $|\gamma|\le 2X$, we have*

$$\begin{aligned}&S_\rho(X+H)-S_\rho(X)\\&=\frac{1}{k\ell}\frac{\Gamma(\frac{\rho}{k})\Gamma(\frac{1}{\ell})}{\Gamma(\frac{\rho}{k}+\frac{1}{\ell}+1)}\left((X+H)^{\frac{\rho}{k}+\frac{1}{\ell}}-X^{\frac{\rho}{k}+\frac{1}{\ell}}\right)\\&\quad-\frac{(X+H)^{\frac{\rho}{k}}-X^{\frac{\rho}{k}}}{2\rho}+O\left(H^{\frac{\beta}{k}}|\gamma|^{\frac{\beta}{k}-\frac{1}{2}}L^2+\frac{HX^{\frac{1}{k}+\frac{1}{\ell}-1}B^{-2}}{|\gamma|}+L\right)\end{aligned}$$

provided

$$X^{1-\min(\frac{1}{k},\frac{k}{\ell(k-1)})+\varepsilon}\le H\le X^{1-\varepsilon}, \tag{35}$$

where the implicit constant depends on k,ℓ and ε.

Proof. By partial summation, for $X\le Q\le X+H$, we have

$$\begin{aligned}S_\rho(Q)&=\frac{1}{\rho}\int_0^X (Q-u)^{\frac{\rho}{k}}d[u^{\frac{1}{\ell}}]\\&=\frac{1}{\ell\rho}\int_0^X (Q-u)^{\frac{\rho}{k}}u^{\frac{1}{\ell}-1}\,du-\frac{1}{\rho}\int_0^X (Q-u)^{\frac{\rho}{k}}d\left(\{u^{\frac{1}{\ell}}\}-\frac{1}{2}\right).\end{aligned} \tag{36}$$

The first integral on the right-hand side of (36) is

$$\begin{aligned}\frac{1}{\ell\rho}\int_0^X (Q-u)^{\frac{\rho}{k}}u^{\frac{1}{\ell}-1}\,du&=\frac{1}{\ell\rho}\int_0^Q (Q-u)^{\frac{\rho}{k}}u^{\frac{1}{\ell}-1}\,du+O\left(\frac{H^{1+\frac{1}{k}}X^{\frac{1}{\ell}-1}}{|\gamma|}\right)\\&=\frac{1}{k\ell}\frac{\Gamma(\frac{\rho}{k})\Gamma(\frac{1}{\ell})}{\Gamma(\frac{\rho}{k}+\frac{1}{\ell}+1)}Q^{\frac{\rho}{k}+\frac{1}{\ell}}+O\left(\frac{HX^{\frac{1}{k}+\frac{1}{\ell}-1}B^{-2}}{|\gamma|}\right)\end{aligned}$$

provided (35). The second integral on the right-hand side of (36) is

$$-\frac{1}{\rho}\int_0^X (Q-u)^{\frac{\rho}{k}}d\left(\{u^{\frac{1}{\ell}}\}-\frac{1}{2}\right)$$

$$=-\frac{1}{k}\int_0^X (Q-u)^{\frac{\rho}{k}-1}\left(\{u^{\frac{1}{\ell}}\}-\frac{1}{2}\right)du-\frac{Q^{\frac{\rho}{k}}}{2\rho}+O\left(\frac{H^{\frac{1}{k}}}{|\gamma|}\right)$$
$$=-\frac{1}{k}\int_0^X (Q-u)^{\frac{\rho}{k}-1}\left(\{u^{\frac{1}{\ell}}\}-\frac{1}{2}\right)du-\frac{Q^{\frac{\rho}{k}}}{2\rho}+O\left(\frac{HX^{\frac{1}{k}+\frac{1}{\ell}-1}B^{-2}}{|\gamma|}\right)$$

provided (35). Recall the Fourier expansion

$$\{u\}-\frac{1}{2}=-\sum_{n\neq 0}\frac{e(nu)}{2\pi in},$$

which holds for $u\notin\mathbb{Z}$ and converges boundedly for $u\in\mathbb{R}$. Then since

$$-\frac{1}{k}\int_0^X (Q-u)^{\frac{\rho}{k}-1}\left(\{u^{\frac{1}{\ell}}\}-\frac{1}{2}\right)du$$
$$=-\frac{1}{k}\int_0^{X-1} (Q-u)^{\frac{\rho}{k}-1}\left(\{u^{\frac{1}{\ell}}\}-\frac{1}{2}\right)du+O\left(\frac{1}{k}\int_{X-1}^X (Q-u)^{\frac{\beta}{k}-1}\,du\right)$$
$$=-\frac{1}{k}\int_0^{X-1} (Q-u)^{\frac{\rho}{k}-1}\left(\{u^{\frac{1}{\ell}}\}-\frac{1}{2}\right)du+O\left(\frac{1}{\beta}\right),$$

by using Lemma 10 and the assumption $|\gamma|\leq X$, we have

$$S_\rho(Q)=\frac{1}{k\ell}\frac{\Gamma(\frac{\rho}{k})\Gamma(\frac{1}{\ell})}{\Gamma(\frac{\rho}{k}+\frac{1}{\ell}+1)}Q^{\frac{\rho}{k}+\frac{1}{\ell}}-\frac{Q^{\frac{\rho}{k}}}{2\rho}+R_\rho(Q)+O\left(\frac{HX^{\frac{1}{k}+\frac{1}{\ell}-1}B^{-2}}{|\gamma|}+L\right)$$

for $X\leq Q\leq X+H$, where we used the bound

$$\frac{1}{\beta}\ll L$$

obtained by Lemma 10 with the functional equation of $\zeta(s)$ and $R_\rho(Q)$ is defined by

$$R_\rho(Q)=R_{\rho,k,\ell}(Q)=\sum_{n\neq 0}\frac{I_\rho(Q,n)}{2\pi ikn},$$

$$I_\rho(Q,n)=I_{\rho,k,\ell}(Q,n)=\int_0^{X-1}(Q-u)^{\frac{\rho}{k}-1}e(nu^{\frac{1}{\ell}})\,du.$$

In order to prove the lemma, it suffices to estimate

$$R_\rho(X+H)-R_\rho(X).$$

We first estimate the difference of oscillating integrals

$$I_\rho(X+H,n)-I_\rho(X,n). \tag{37}$$

By changing the variable in the definiton of $I_\rho(Q,n)$, we obtain expressions

$$I_\rho(X+H,n)=\int_1^X (u+H)^{\frac{\rho}{k}-1}e\left(n(X-u)^{\frac{1}{\ell}}\right)\,du,$$
$$I_\rho(X,n)=\int_1^X u^{\frac{\rho}{k}-1}e\left(n(X-u)^{\frac{1}{\ell}}\right)\,du.$$

Let $U=\min(4H|\gamma|,X)$. Then we decompose (37) as

$$I_\rho(X+H,n)-I_\rho(X,n)=I+I_1-I_2,$$

where

$$I=\int_U^X \left((u+H)^{\frac{\rho}{k}-1}-u^{\frac{\rho}{k}-1}\right)e\left(n(X-u)^{\frac{1}{\ell}}\right)\,du,$$

$$I_1=\int_1^U (u+H)^{\frac{\rho}{k}-1}e\left(n(X-u)^{\frac{1}{\ell}}\right)\,du,\quad I_2=\int_1^U u^{\frac{\rho}{k}-1}e\left(n(X-u)^{\frac{1}{\ell}}\right)\,du.$$

For the integral I, we use the Taylor expansion

$$(u+H)^{\frac{\rho}{k}-1}-u^{\frac{\rho}{k}-1}=u^{\frac{\rho}{k}-1}\sum_{\nu=1}^{\infty}\binom{\frac{\rho}{k}-1}{\nu}\left(\frac{H}{u}\right)^{\nu}.$$

By substituting this expansion into the definition of I,

$$I=\sum_{\nu=1}^{\infty}\binom{\frac{\rho}{k}-1}{\nu}H^{\nu}\int_U^X u^{\frac{\rho}{k}-\nu-1}e\left(n(X-u)^{\frac{1}{\ell}}\right)\,du.$$

By using Lemma 23 and the definition of U, if $4H|\gamma|\le X$,

$$I\ll\frac{U^{\frac{\beta}{k}}L}{|\gamma|^{\frac{1}{2}}}\sum_{\nu=1}^{\infty}\left|\binom{\frac{\rho}{k}-1}{\nu}\right|\left(\frac{H}{U}\right)^{\nu}\ll\frac{U^{\frac{\beta}{k}}L}{|\gamma|^{\frac{1}{2}}}\sum_{\nu=1}^{\infty}\prod_{\mu=1}^{\nu}\left(\frac{|\gamma|+2\mu}{4\mu|\gamma|}\right)\ll\frac{U^{\frac{\beta}{k}}L}{|\gamma|^{\frac{1}{2}}}$$

since $|\gamma|\ge 2$. If $4H|\gamma|>X$, then I is an empty integral, so the same estimate holds trivially. For the integrals I_1 and I_2, we may use Lemma 23 directly to obtain

$$I_1,I_2\ll\frac{U^{\frac{\beta}{k}}L}{|\gamma|^{\frac{1}{2}}}$$

since we can choose $Q=X+H$ for the integral

$$\begin{aligned}I_1&=\int_1^U (u+H)^{\frac{\beta}{k}+\frac{i\gamma}{k}-1}e\left(n(X-u)^{\frac{1}{\ell}}\right)\,du\\&=\int_{1+H}^{U+H} u^{\frac{\beta}{k}+\frac{i\gamma}{k}-1}e\left(n(X+H-u)^{\frac{1}{\ell}}\right)\,du.\end{aligned}$$

Therefore, we have

$$I_\rho(X+H,n)-I_\rho(X,n)\ll\frac{U^{\frac{\beta}{k}}L}{|\gamma|^{\frac12}}\ll H^{\frac{\beta}{k}}|\gamma|^{\frac{\beta}{k}-\frac12}L.$$

On the other hand, if $|n|>\ell(X+H)^{1-\frac1\ell}|\gamma|$, Lemma 23 gives

$$I_\rho(X+H,n)-I_\rho(X,n)\ll\frac{X^{1-\frac1\ell}}{|n|}.$$

Thus we have

$$\begin{aligned}&R_\rho(X+H)-R_\rho(X)\\&\ll H^{\frac{\beta}{k}}|\gamma|^{\frac{\beta}{k}-\frac12}L\sum_{n\le\ell(X+H)^{1-\frac1\ell}|\gamma|}\frac1n+X^{1-\frac1\ell}\sum_{n>\ell(X+H)^{1-\frac1\ell}|\gamma|}\frac{1}{n^2}\\&\ll H^{\frac{\beta}{k}}|\gamma|^{\frac{\beta}{k}-\frac12}L^2+1.\end{aligned}$$

This completes the proof. □

6. Completion of the proof

In this section, we complete the proof of main theorems. However, before the main part of the proof of Theorem 3, we check the direct consequence of Lemma 12.

Lemma 25. *For positive integers k,ℓ with $\ell\ge2$ and real numbers X,H,ε with $4\le H\le X$ and $\varepsilon>0$, we have the asymptotic formula (10) provided*

$$X^{1-\theta_B(k,\ell)+\varepsilon}\le H\le X^{1-\varepsilon}, \tag{38}$$

where $\theta_B(k,\ell)$ is defined by

$$\theta_B(k,\ell)=\min\left(\frac{5}{12k},\frac{k}{\ell(k-1)}\right)$$

as in Theorem 3 and the implicit constant depends on k,ℓ and ε.

Proof. We may assume that X is larger than some constant depends only on k,ℓ and ε since otherwise the assertion trivially holds. We use Lemma 12 in Lemma 21. If $n^\ell\le X$ and $(X+H-n^\ell)\le2(X-n^\ell)$,

$$(X+H-n^\ell)^{\frac1k}-(X-n^\ell)^{\frac1k}=\frac1k\int_{X-n^\ell}^{X+H-n^\ell}u^{\frac1k-1}\,du\ge\frac{1}{2k}H(X-n^\ell)^{\frac1k-1}$$

$$\geq \frac{1}{2k} X^{1-\frac{5}{12k}+\varepsilon}(X-n^\ell)^{\frac{1}{k}-1} \geq \left((X-n^\ell)^{\frac{1}{k}}\right)^{\frac{7}{12}+\frac{\varepsilon}{2}}$$

provided (38) and X is large. Thus, in this case, Lemma 12 gives

$$\begin{aligned}(39)\qquad &\psi\left((X+H-n^\ell)^{\frac{1}{k}}\right)-\psi\left((X-n^\ell)^{\frac{1}{k}}\right)\\ &=(X+H-n^\ell)^{\frac{1}{k}}-(X-n^\ell)^{\frac{1}{k}}+O(((X+H-n^\ell)^{\frac{1}{k}}-(X-n^\ell)^{\frac{1}{k}})B^{-1})\end{aligned}$$

by making the constant c smaller since

$$(X-n^\ell)^{\frac{1}{k}} \gg (X+H-n^\ell)^{\frac{1}{k}} \gg H^{\frac{1}{k}}$$

in the current case. If $n^\ell \leq X$ and $(X+H-n^\ell)>2(X-n^\ell)$, then we may apply the usual prime number theorem to obtain the same estimate (39) since in this case

$$(X+H-n^\ell)^{\frac{1}{k}}-(X-n^\ell)^{\frac{1}{k}} \asymp (X+H-n^\ell)^{\frac{1}{k}}.$$

By using (39) in Lemma 21 and using Lemma 7, we arrive at the lemma. □

We now prove the main part of Theorem 3.

Lemma 26. *For positive integers k,ℓ with $\ell\geq 2$ and real numbers X,H,ε with $4\leq H\leq X$ and $\varepsilon>0$, we have the asymptotic formula* (10) *provided*

$$X^{1-\theta_C(k,\ell)+\varepsilon} \leq H \leq X^{1-\varepsilon},$$

where $\theta_C(k,\ell)$ is defined by

$$\theta_C(k,\ell)=\begin{cases}\min\left(\dfrac{\lambda_1(\ell)}{k},\dfrac{\lambda_2(k,\ell)}{k},\dfrac{2}{\ell}\right) & \text{if } k=1,\\[2ex] \min\left(\dfrac{\lambda_1(\ell)}{k},\dfrac{\lambda_2(k,\ell)}{k},\dfrac{k}{\ell(k-1)}\right) & \text{if } k\geq 2,\end{cases}$$

and the implicit constant depends on k,ℓ and ε.

Proof. We may assume that X is larger than some constant depends only on k,ℓ and ε since otherwise the assertion trivially holds. By Lemma 21, Lemma 22 and Lemma 24,

$$\begin{aligned}(40)\qquad &\sum_{X<N\leq X+H} R(N)\\ &=M+R_1+R_2+O((R_3+X^{\frac{1}{k}+\frac{1}{\ell}}T^{-1}+T)L^2+HX^{\frac{1}{k}+\frac{1}{\ell}-1}B^{-1})\end{aligned}$$

provided

$$(41)\qquad X^{1-\min(\frac{1}{k},\frac{k}{\ell(k-1)})+\varepsilon} \leq H \leq X^{1-\varepsilon}, \quad 2\leq T\leq X^{\frac{1}{k}},$$

where

$$M = S(X+H) - S(X),$$

$$R_1 = -\sum_{\substack{\rho \\ |\gamma|\le T}} \frac{1}{k\ell} \frac{\Gamma(\frac{\rho}{k})\Gamma(\frac{1}{\ell})}{\Gamma(\frac{\rho}{k}+\frac{1}{\ell}+1)} \left((X+H)^{\frac{\rho}{k}+\frac{1}{\ell}} - X^{\frac{\rho}{k}+\frac{1}{\ell}}\right),$$

$$R_2 = \sum_{\substack{\rho \\ |\gamma|\le T}} \frac{(X+H)^{\frac{\rho}{k}} - X^{\frac{\rho}{k}}}{2\rho}, \quad R_3 = \sum_{|\gamma|\le T} H^{\frac{\beta}{k}} |\gamma|^{\frac{\beta}{k}-\frac{1}{2}}.$$

In order to control the size of the error $X^{\frac{1}{k}+\frac{1}{\ell}}T^{-1}L^2$, we choose T by

$$T = X^{1+\frac{\varepsilon_1}{k}} H^{-1}, \quad 0 < \varepsilon_1 \le \frac{\varepsilon}{2}, \tag{42}$$

where we choose ε_1 later (our choice will be $\varepsilon_1 = \frac{\varepsilon}{80}$). This choice is admissible since the former inequality of (41) implies

$$X^{\varepsilon} \le T \le X^{\frac{1}{k}-\frac{\varepsilon}{2}}. \tag{43}$$

If we assume further

$$X^{1-\frac{1}{2}(\frac{1}{k}+\frac{1}{\ell})+\varepsilon} \le H, \tag{44}$$

then

$$TL^2 = X^{1+\frac{\varepsilon_1}{k}} H^{-1} L^2 = HX^{1+\frac{\varepsilon_1}{k}} H^{-2} L^2 \le HX^{\frac{1}{k}+\frac{1}{\ell}-1-\varepsilon} L^2 \ll HX^{\frac{1}{k}+\frac{1}{\ell}-1} B^{-1}.$$

Thus,

$$(X^{\frac{1}{k}+\frac{1}{\ell}} T^{-1} + T) L^2 \ll HX^{\frac{1}{k}+\frac{1}{\ell}-1} B^{-1} \tag{45}$$

provided (44). By Lemma 7, the main term M can be evaluated as

$$M = \frac{1}{k\ell} \frac{\Gamma(\frac{1}{k})\Gamma(\frac{1}{\ell})}{\Gamma(\frac{1}{k}+\frac{1}{\ell})} HX^{\frac{1}{k}+\frac{1}{\ell}-1} + O\left(HX^{\frac{1}{k}+\frac{1}{\ell}-1} B^{-1}\right) \tag{46}$$

provided (41). The remaining task is to estimate R_1, R_2 and R_3.

We first estimate the sum R_1. By the fundamental theorem of calculus,

$$(X+H)^{\frac{\rho}{k}+\frac{1}{\ell}} - X^{\frac{\rho}{k}+\frac{1}{\ell}} = \left(\frac{\rho}{k} + \frac{1}{\ell}\right) \int_X^{X+H} u^{\frac{\rho}{k}+\frac{1}{\ell}-1}\, du \ll |\gamma| H X^{\frac{\beta}{k}+\frac{1}{\ell}-1}.$$

Then, by using Stirling's formula and dissecting dyadically,

$$R_1 \ll HX^{\frac{1}{\ell}-1} \sum_{|\gamma|\le T} \frac{X^{\frac{\beta}{k}}}{|\gamma|^{\frac{1}{\ell}}} \ll HX^{\frac{1}{\ell}-1} L \sup_{1\le K\le T} K^{-\frac{1}{\ell}} \sum_{K<|\gamma|\le 2K} X^{\frac{\beta}{k}}. \tag{47}$$

For $1 \le K \le T$, we write $K = X^{\frac{\delta}{k}}$. Further, we write

$$XH^{-1} = X^{\frac{\Delta}{k}}. \tag{48}$$

Then, by (42), δ moves in the range

$$0 \le \delta \le \Delta + \varepsilon_1. \tag{49}$$

By Lemma 13,

$$K^{-\frac{1}{\ell}} \sum_{K < |\gamma| \le 2K} X^{\frac{\beta}{k}} \ll \left(X^{\frac{1}{k}(\phi(\delta) - \frac{1}{\ell}\delta)} + X^{\frac{1}{k}(1-\eta+(2\eta-\frac{1}{\ell})\delta)} \right) L^{A}. \tag{50}$$

By Lemma 14 and the assumption $\ell \ge 2$, for sufficiently large X,

$$\frac{d}{d\delta}\left(\phi(\delta) - \frac{1}{\ell}\delta\right) > 0, \quad 2\eta - \frac{1}{\ell} < 0.$$

Therefore, by (47), (49) and (50),

$$\begin{aligned} R_1 &\ll HX^{\frac{1}{\ell}-1}\left(X^{\frac{1}{k}(\phi(\Delta+\varepsilon_1) - \frac{1}{\ell}(\Delta+\varepsilon_1))} + X^{\frac{1}{k}(1-\eta)} \right) L^{A+1} \\ &\ll HX^{\frac{1}{\ell}-1+\frac{1}{k}(\phi(\Delta+\varepsilon_1)-\frac{1}{\ell}(\Delta+\varepsilon_1))} L^{A+1} + HX^{\frac{1}{k}+\frac{1}{\ell}-1} B^{-1}. \end{aligned} \tag{51}$$

By Lemma 14 and the mean value theorem,

$$\phi(\Delta+\varepsilon_1) - \frac{1}{\ell}(\Delta+\varepsilon_1) \le \phi(\Delta) - \frac{1}{\ell}\Delta + \varepsilon_1.$$

Thus, by (51), we obtain

$$R_1 \ll HX^{\frac{1}{\ell}-1+\frac{1}{k}(\phi(\Delta)-\frac{1}{\ell}\Delta)+2\varepsilon_1} + HX^{\frac{1}{k}+\frac{1}{\ell}-1} B^{-1}. \tag{52}$$

This completes the estimate of R_1.

We next estimate the sum R_2. We use

$$(X+H)^{\frac{\rho}{k}} - X^{\frac{\rho}{k}} = \frac{\rho}{k} \int_X^{X+H} u^{\frac{\rho}{k}-1}\, du \ll |\gamma| HX^{\frac{\beta}{k}-1}.$$

Then, since $X/|\gamma| \ge 1$ for $|\gamma| \le T \le X$,

$$R_2 \ll HX^{-1} \sum_{|\gamma| \le T} X^{\frac{\beta}{k}} \ll HX^{\frac{1}{\ell}-1} \sum_{|\gamma| \le T} \frac{X^{\frac{\beta}{k}}}{|\gamma|^{\frac{1}{\ell}}}.$$

This right-hand side is the same quantity appeared in (47). Thus,

$$R_2 \ll HX^{\frac{1}{\ell}-1+\frac{1}{k}(\phi(\Delta)-\frac{1}{\ell}\Delta)+2\varepsilon_1} + HX^{\frac{1}{k}+\frac{1}{\ell}-1} B^{-1}. \tag{53}$$

This completes the estimate of R_2.

We finally estimate the sum R_3. We dissect the sum dyadically to obtain

$$R_3 \ll L \sup_{1\le K\le T} K^{-\frac{1}{2}} \sum_{K<|\gamma|\le 2K} (HK)^{\frac{\beta}{k}}. \tag{54}$$

We again write $K = X^{\frac{\delta}{k}}$ and use the parameter Δ defined in (48). By (43),

$$X^{\frac{\Delta+\varepsilon_1}{k}} = X^{1+\frac{\varepsilon_1}{k}} H^{-1} = T \le X^{\frac{1}{k}}$$

so that

$$0 \le \Delta \le 1-\varepsilon_1. \tag{55}$$

Let

$$\lambda = \lambda(\delta) = \frac{\log K}{\log (HK)^{\frac{1}{k}}} = \frac{k\log K}{\log H + \log K} = \frac{\delta}{1-\frac{\Delta}{k}+\frac{\delta}{k}}.$$

By (55), this function $\lambda(\delta)$ is increasing with respect to δ. Note that

$$K = K^{1-\frac{1}{k}} K^{\frac{1}{k}} \le T^{1-\frac{1}{k}} K^{\frac{1}{k}} \le (X^{1-\frac{1}{k}} K)^{\frac{1}{k}} \le (HK)^{\frac{1}{k}}$$

by (43) provided (41). Thus, by using Lemma 13 with $Y = (HK)^{\frac{1}{k}}$,

$$K^{-\frac{1}{2}} \sum_{K<|\gamma|\le 2K} (HK)^{\frac{\beta}{k}} \ll \left((HK)^{\frac{1}{k}(\phi(\lambda)-\frac{1}{2}\lambda)} + (HK)^{\frac{1}{k}(1-\eta+(2\eta-\frac{1}{2})\lambda)} \right) L^A.$$

Since $HK = X(XH^{-1})^{-1}K = X^{1-\frac{\Delta}{k}+\frac{\delta}{k}}$, the last estimate is rewritten as

$$\begin{aligned} &K^{-\frac{1}{2}} \sum_{K<|\gamma|\le 2K} (HK)^{\frac{\beta}{k}} \\ &\ll \left(X^{\frac{1}{k}(1-\frac{\Delta}{k}+\frac{\delta}{k})(\phi(\lambda)-\frac{1}{2}\lambda)} + X^{\frac{1}{k}(1-\frac{\Delta}{k}+\frac{\delta}{k})(1-\eta+(2\eta-\frac{1}{2})\lambda)} \right) L^A. \end{aligned} \tag{56}$$

Since both of $\left(1-\frac{\Delta}{k}+\frac{\delta}{k}\right)$, $\left(\phi(\lambda)-\frac{1}{2}\lambda\right)$ are increasing function of δ, by (49),

$$\begin{aligned} X^{\frac{1}{k}(1-\frac{\Delta}{k}+\frac{\delta}{k})(\phi(\lambda)-\frac{1}{2}\lambda)} &\le X^{\frac{1}{k}(1+\varepsilon_1)(\phi(\lambda(\Delta+\varepsilon_1))-\frac{1}{2}\lambda(\Delta+\varepsilon_1))} \\ &\le X^{\frac{1}{k}(\phi(\lambda(\Delta+\varepsilon_1))-\frac{1}{2}\lambda(\Delta+\varepsilon_1))+\varepsilon_1}. \end{aligned}$$

Since

$$\lambda'(\delta) = \frac{1-\frac{\Delta}{k}}{(1-\frac{\Delta}{k}+\frac{\delta}{k})^2} \le 1-\frac{\Delta}{k} \le 1 \quad \text{for} \quad \Delta \le \delta \le \Delta+\varepsilon_1,$$

by Lemma 14 and the mean value theorem,

$$\phi(\lambda(\Delta+\varepsilon_1)) - \frac{1}{2}\lambda(\Delta+\varepsilon_1) \le \phi(\lambda(\Delta)) - \frac{1}{2}\lambda(\Delta) + \varepsilon_1 = \phi(\Delta) - \frac{1}{2}\Delta + \varepsilon_1.$$

Thus,

$$X^{\frac{1}{k}(1-\frac{\Delta}{k}+\frac{\delta}{k})(\phi(\lambda)-\frac{1}{2}\lambda)}\le X^{\frac{1}{k}(\phi(\Delta)-\frac{1}{2}\Delta)+2\varepsilon_1}. \tag{57}$$

Since

$$\begin{aligned}\left(1-\frac{\Delta}{k}+\frac{\delta}{k}\right)\left(1-\eta+\left(2\eta-\frac{1}{2}\right)\lambda\right)&=\left(1-\frac{\Delta}{k}+\frac{\delta}{k}\right)(1-\eta)+\left(2\eta-\frac{1}{2}\right)\delta\\&=\left(1-\frac{\Delta}{k}\right)(1-\eta)+\left(\frac{1-\eta}{k}+2\eta-\frac{1}{2}\right)\delta,\end{aligned}$$

we have

$$\begin{aligned}X^{\frac{1}{k}(1-\frac{\Delta}{k}+\frac{\delta}{k})(1-\eta+(\eta-\frac{1}{2})\lambda)}&\ll\begin{cases}H^{1-\eta}X^{(\frac{1}{2}+\eta)(\Delta+\varepsilon_1)} & \text{if } k=1,\\ H^{\frac{1-\eta}{k}}X^{\eta(\Delta+\varepsilon_1)} & \text{if } k\ge 2,\end{cases}\\&\ll\begin{cases}H^{\frac{1}{2}}X^{\frac{1}{2}+2\varepsilon_1} & \text{if } k=1,\\ H^{\frac{1}{k}}X^{2\varepsilon_1} & \text{if } k\ge 2,\end{cases}\end{aligned} \tag{58}$$

for sufficiently large X. By (54), (56), (57) and (58),

$$R_3\ll\begin{cases}X^{\frac{1}{k}(\phi(\Delta)-\frac{1}{2}\Delta)+3\varepsilon_1}+H^{\frac{1}{2}}X^{\frac{1}{2}+3\varepsilon_1} & (\text{if } k=1),\\ X^{\frac{1}{k}(\phi(\Delta)-\frac{1}{2}\Delta)+3\varepsilon_1}+H^{\frac{1}{k}}X^{3\varepsilon_1} & (\text{if } k\ge 2).\end{cases} \tag{59}$$

By combining (40), (45), (46), (52), (53) and (59), we have

$$\sum_{X<N\le X+H}R(N)=\frac{1}{k\ell}\frac{\Gamma(\frac{1}{k})\Gamma(\frac{1}{\ell})}{\Gamma(\frac{1}{k}+\frac{1}{\ell})}HX^{\frac{1}{k}+\frac{1}{\ell}-1}+O\left(HX^{\frac{1}{k}+\frac{1}{\ell}-1}B^{-1}+E\right)$$

provided

$$\begin{aligned}&X^{1-\min(\frac{1}{k},\frac{1}{2}(\frac{1}{k}+\frac{1}{\ell}),\frac{2}{\ell})+\varepsilon}\le H\le X^{1-\varepsilon} && \text{if } k=1,\\ &X^{1-\min(\frac{1}{k},\frac{1}{2}(\frac{1}{k}+\frac{1}{\ell}),\frac{k}{\ell(k-1)})+\varepsilon}\le H\le X^{1-\varepsilon} && \text{if } k\ge 2,\end{aligned}$$

and $\varepsilon_1\le\frac{\varepsilon}{16}$, where

$$E=HX^{\frac{1}{\ell}-1+\frac{1}{k}(\phi(\Delta)-\frac{1}{\ell}\Delta)+4\varepsilon_1}+X^{\frac{1}{k}(\phi(\Delta)-\frac{1}{2}\Delta)+4\varepsilon_1},\quad XH^{-1}=X^{\frac{\Delta}{k}}. \tag{60}$$

Let λ_1,λ_2 be the functions given by (11), or equivalently, given in Lemma 15. Then, by assuming further

$$X^{1-\frac{\min(\lambda_1,\lambda_2)}{k}+\varepsilon}\le H,$$

we have $0\le\Delta\le\min(\lambda_1,\lambda_2)-k\varepsilon$. Thus, Lemma 16 and (60) implies

$$E\ll HX^{\frac{1}{k}+\frac{1}{\ell}-1-\frac{\varepsilon}{10}+4\varepsilon_1}.$$

Thus, by taking $\varepsilon_1 = \frac{\varepsilon}{80}$, we obtain the asymptotic formula (10) provided

$$X^{1-\min(\frac{\lambda_1}{k},\frac{\lambda_2}{k},\frac{1}{k},\frac{1}{2}(\frac{1}{k}+\frac{1}{\ell}),\frac{2}{\ell})+\varepsilon} \le H \le X^{1-\varepsilon} \quad \text{if } k=1, \tag{61}$$
$$X^{1-\min(\frac{\lambda_1}{k},\frac{\lambda_2}{k},\frac{1}{k},\frac{1}{2}(\frac{1}{k}+\frac{1}{\ell}),\frac{k}{\ell(k-1)})+\varepsilon} \le H \le X^{1-\varepsilon} \quad \text{if } k\ge 2.$$

Our remaining task is to remove the exponents $\frac{1}{k}$ and $\frac{1}{2}(\frac{1}{k}+\frac{1}{\ell})$ in (61). Since $\lambda_1(\ell)\le 1$ for any $\ell\ge 2$, we have $\frac{1}{k}\ge\frac{\lambda_1(\ell)}{k}$. Thus, we can remove the exponent $\frac{1}{k}$ in (61). Note that

$$\frac{1}{2}\left(\frac{1}{k}+\frac{1}{\ell}\right)\ge\frac{\lambda_1(\ell)}{k} \quad\Longleftrightarrow\quad k\ge(2\lambda_1(\ell)-1)\ell =: \tilde{\lambda}_1(\ell).$$

For the function $\tilde{\lambda}_1(\ell)$, we can check $\tilde{\lambda}_1(2),\dots,\tilde{\lambda}_1(5)\le 2, \tilde{\lambda}_1(6),\dots,\tilde{\lambda}_1(9)\le 1$ numerically and we have

$$\ell\ge 10 \Longrightarrow \tilde{\lambda}_1(\ell)\le(2\lambda_1(10)-1)\ell = 0$$

since $\lambda_1(\ell)$ is decreasing. Thus,

$$\frac{1}{2}\left(\frac{1}{k}+\frac{1}{\ell}\right)\ge\frac{\lambda_1(\ell)}{k}$$

except the cases $(k,\ell)=(1,2),(1,3),(1,4),(1,5)$ for which we can check numerically

$$\frac{1}{2}\left(\frac{1}{k}+\frac{1}{\ell}\right)\ge\frac{\lambda_2(k,\ell)}{k}.$$

Thus, we can remove the exponent $\frac{1}{2}(\frac{1}{k}+\frac{1}{\ell})$ in (61). This completes the proof. □

We next replace the exponent $\theta_C(k,\ell)$ by $\theta_A(k,\ell)$ as in Theorem 3.

Lemma 27. *For positive integers k,ℓ with $\ell\ge 2$, we have*

$$\frac{5}{12k}\ge\frac{k}{\ell(k-1)}$$

if and only if

$$\ell\ge 10 \quad \textit{and} \quad \frac{5}{24}\ell-\frac{1}{24}\sqrt{\ell(25\ell-240)}\le k\le\frac{5}{24}\ell+\frac{1}{24}\sqrt{\ell(25\ell-240)}. \tag{62}$$

Proof. This lemma follows just by solving the quadratic inequality

$$\frac{5}{12k}\ge\frac{k}{\ell(k-1)} \quad\Longleftrightarrow\quad \left(k-\frac{5}{24}\ell\right)^2\le\left(\frac{1}{24}\right)^2\ell(25\ell-240)$$

for $k\ge 2$. Note that (62) never holds for $k=1$ since

$$\frac{5}{24}\ell-\frac{1}{24}\sqrt{\ell(25\ell-240)}=\frac{5}{24}\ell-\frac{5}{24}\ell\sqrt{1-\frac{48}{5\ell}}>\frac{5}{24}\ell-\frac{5}{24}\ell\left(1-\frac{24}{5\ell}\right)=1$$

for $\ell\ge 10$. This completes the proof. □

Lemma 28. *Let $\theta_A(k,\ell), \theta_B(k,\ell)$ be functions given in Theorem 3. Then, for positive integers k,ℓ with $\ell \geq 2$, we have*

$$\theta_B(k,\ell) < \theta_A(k,\ell) \iff \begin{cases} \ell \leq 9 \ \text{ and } \ \frac{5}{24}\ell < k, \\ \text{or} \quad \ell \geq 10 \ \text{ and } \ \frac{5}{24}\ell + \frac{1}{24}\sqrt{\ell(25\ell-240)} < k. \end{cases}$$

Proof. We first consider the case $k \leq \frac{5}{24}\ell$. In this case,

$$\theta_A(k,\ell) \leq \min\left(\frac{\lambda_2(k,\ell)}{k}, \frac{k}{\ell(k-1)}\right) \leq \min\left(\frac{\lambda_2(k,\frac{24}{5}k)}{k}, \frac{k}{\ell(k-1)}\right)$$
$$= \min\left(\frac{5}{12k}, \frac{k}{\ell(k-1)}\right) = \theta_B(k,\ell)$$

by Lemma 17. Thus, in the case $k \leq \frac{5}{24}\ell$, both hand sides of the assertion are false so that the assertion holds.

We consider the remaining case $k > \frac{5}{24}\ell$. In this case, by Lemma 27,

$$\theta_B(k,\ell) = \begin{cases} \dfrac{k}{\ell(k-1)} & \text{if } \ell \geq 10 \text{ and } \frac{5}{24}\ell < k \leq \frac{5}{24}\ell + \frac{1}{24}\sqrt{\ell(25\ell-240)}, \\ \dfrac{5}{12k} & \text{otherwise.} \end{cases}$$

Therefore, in the former case, i.e. in the case

$$\ell \geq 10 \quad \text{and} \quad \frac{5}{24}\ell < k \leq \frac{5}{24}\ell + \frac{1}{24}\sqrt{\ell(25\ell-240)}, \tag{63}$$

we have

$$\theta_A(k,\ell) \leq \frac{k}{\ell(k-1)} = \theta_B(k,\ell).$$

This again makes both sides of the assertion false, which proves the assertion for the case (63).

In the remaining case, in which (63) does not hold but $k > \frac{5}{24}\ell$ holds, we have

$$\frac{\lambda_1(\ell)}{k} > \frac{5}{12k}, \qquad \frac{\lambda_2(k,\ell)}{k} > \frac{\lambda_2(k,\frac{24}{5}k)}{k} = \frac{5}{12k}, \qquad \frac{k}{\ell(k-1)} > \frac{5}{12k}$$

by Lemma 17 and Lemma 27. Thus,

$$\theta_A(k,\ell) > \frac{5}{12k} = \theta_B(k,\ell)$$

in the remaining case. This completes the proof. □

Lemma 29. *Let $\theta_B(k,\ell), \theta_C(k,\ell)$ be functions given in Lemma 25 and Lemma 26, respectively. Then, for positive integers k,ℓ with $\ell \geq 2$, we have*

$$\theta_B(k,\ell) < \theta_C(k,\ell) \iff \begin{cases} \ell \leq 9 \text{ and } \frac{5}{24}\ell < k, \\ \text{or} \quad \ell \geq 10 \text{ and } \frac{5}{24}\ell + \frac{1}{24}\sqrt{\ell(25\ell-240)} < k \end{cases}$$

and $\max(\theta_B(k,\ell), \theta_C(k,\ell)) = \max(\theta_A(k,\ell), \theta_B(k,\ell))$.

Proof. If $k \geq 2$, then this trivially holds since $\theta_A(k,\ell) = \theta_C(k,\ell)$ for $k \geq 2$. Thus we consider the case $k=1$. Since $\theta_C(1,\ell) \leq \theta_A(1,\ell)$ for any case, it suffices to prove that $\theta_A(1,\ell) \leq \frac{2}{\ell}$ if $\theta_B(1,\ell) < \theta_A(1,\ell)$. By Lemma 28, $\theta_B(1,\ell) < \theta_A(1,\ell)$ holds if and only if $\ell = 2,3,4$. For these cases, we have

$$\theta_A(1,2) = \frac{17+4\sqrt{15}}{49} \leq 1, \quad \theta_A(1,3) = \frac{44+24\sqrt{2}}{147} \leq \frac{2}{3}, \quad \theta_A(1,4) = \frac{5}{11} \leq \frac{1}{2}.$$

This completes the proof. □

We now complete the proof of main theorems. Since Theorem 1 is just a special case of Theorem 3, we prove only Theorem 3 and Theorem 4.

Proof of Theorem 3. By Lemma 25 and Lemma 26, we have (10) provided

$$X^{1-\max(\theta_B(k,\ell),\theta_C(k,\ell))+\varepsilon} \leq H \leq X^{1-\varepsilon}.$$

Then the theorem follows by Lemma 29. □

Proof of Theorem 4. By Theorem 3 and Lemma 28, it suffices to prove

$$\theta_A(k,\ell) = \frac{1}{k} \quad \text{for } k \geq 2 \text{ and } \ell = 2. \tag{64}$$

Since $k \geq \ell$, we have

$$\lambda_2(k,\ell) \geq \lambda_2(k,k) = 1, \quad \frac{k}{\ell(k-1)} \geq \frac{1}{\ell} \geq \frac{1}{k}.$$

Also, $\lambda_1(2) = 1$. Thus, we obtain (64) and arrive at the theorem. □

7. Comparison of the exponents

In this section, we compare three exponents θ_A, θ_B and θ_{LZ}. As a preparation, we prove (14), which determines the value $\theta = \max(\theta_A, \theta_B)$ more precisely.

Lemma 30. *Let $\theta(k,\ell)$ be the function given in Theorem 3. Then, for positive integers k, ℓ with $\ell \geq 2$, we have*

$$\theta(k,\ell) = \begin{cases} \dfrac{\lambda_2(k,\ell)}{k} & \text{for } (k,\ell) = (1,2), (1,3), (1,4) \\ & \quad (2,5), (2,6), (2,7), (2,8), (2,9), \\ \theta_B(k,\ell) & \text{for } k=1 \text{ and } \ell \geq 5, \\ \min\left(\dfrac{\lambda_1(\ell)}{k}, \dfrac{k}{\ell(k-1)}\right) & \text{otherwise.} \end{cases}$$

Proof. We first consider the case $k=1$. In this case, Lemma 28 implies $\theta(1,\ell) = \theta_B(1,\ell)$ for $\ell \geq 5$. Some numerical computation tells us $\theta(k,\ell) = \frac{\lambda_2(k,\ell)}{k}$ for the cases $(k,\ell) = (1,2), (1,3), (1,4)$.

We next consider the case $k > \ell$. In this case, we have

$$\frac{\lambda_2(k,\ell)}{k} = \frac{2}{3k}\left(\frac{k}{\ell} + \frac{1}{2}\right) > \frac{1}{k} \geq \frac{\lambda_1(\ell)}{k}.$$

Therefore,

$$\begin{aligned} \theta_A(k,\ell) &= \min\left(\frac{\lambda_1(\ell)}{k}, \frac{\lambda_2(k,\ell)}{k}, \frac{k}{\ell(k-1)}\right) \\ &= \min\left(\frac{\lambda_1(\ell)}{k}, \frac{k}{\ell(k-1)}\right) \geq \min\left(\frac{5}{12k}, \frac{k}{\ell(k-1)}\right) = \theta_B(k,\ell) \end{aligned}$$

so

$$\theta(k,\ell) = \theta_A(k,\ell) = \min\left(\frac{\lambda_1(\ell)}{k}, \frac{k}{\ell(k-1)}\right)$$

as in the assertion.

We further consider the case $k \geq 2$ and $\ell \geq 22$. If

$$k \leq \frac{5}{24}\ell + \frac{1}{24}\sqrt{\ell(25\ell - 240)},$$

then, since

$$\begin{aligned} \frac{5}{24}\ell - \frac{1}{24}\sqrt{\ell(25\ell - 240)} &= \frac{5}{24}\ell - \frac{5}{24}\ell\sqrt{1 - \frac{48}{5\ell}} \\ &< \frac{5}{24}\ell - \frac{5}{24}\ell\left(1 - \frac{48}{5\ell}\right) = 2 \leq k, \end{aligned}$$

Lemma 27 implies

$$\theta_B(k,\ell)=\frac{k}{\ell(k-1)}\geq\theta_A(k,\ell)\quad\text{and}\quad\frac{k}{\ell(k-1)}\leq\frac{5}{12k}<\frac{\lambda_1(\ell)}{k}$$

so that

$$\theta(k,\ell)=\theta_B(k,\ell)=\frac{k}{\ell(k-1)}=\min\left(\frac{\lambda_1(\ell)}{k},\frac{k}{\ell(k-1)}\right).$$

Therefore, for the case $k\geq 2$ and $\ell\geq 22$, it suffices to prove

$$\theta_A(k,\ell)=\min\left(\frac{\lambda_1(\ell)}{k},\frac{k}{\ell(k-1)}\right) \tag{65}$$

provided

$$k>\frac{5}{24}\ell+\frac{1}{24}\sqrt{\ell(25\ell-240)} \tag{66}$$

since $\min\left(\frac{\lambda_1(\ell)}{k},\frac{k}{\ell(k-1)}\right)\geq\theta_B(k,\ell)$. If $k>\frac{5}{8}\ell$ further holds, then

$$\lambda_2(k,\ell)=\frac{2}{3}\left(\frac{k}{\ell}+\frac{1}{2}\right)>\frac{3}{4}=\lambda_1(3)>\lambda_1(\ell)$$

so (65) holds. Thus we may assume $k\leq\frac{5}{8}\ell$. By (66), we have

$$\begin{aligned}k>\frac{5}{24}\ell+\frac{1}{24}\sqrt{\ell(25\ell-240)}&=\frac{5}{24}\ell+\frac{5}{24}\ell\sqrt{1-\frac{48}{5\ell}}\\&>\frac{5}{24}\ell+\frac{5}{24}\ell\left(1-\frac{48}{5\ell}\right)=\frac{5}{12}\ell-2.\end{aligned}$$

By using $\ell\geq 22$, we further find that

$$\frac{5}{8}\geq\frac{k}{\ell}>\frac{5}{12}-\frac{1}{11}=\frac{43}{132}>\frac{31}{96}. \tag{67}$$

Thus, by definition,

$$\lambda_2(k,\ell)=\frac{10}{49}+\frac{2k}{7\ell}+\frac{4}{7}\sqrt{\frac{6}{7}\left(\frac{k}{\ell}-\frac{1}{7}\right)}.$$

By using (67) and Lemma 17, we have

$$\lambda_2(k,\ell)\geq\lambda_2\left(k,\frac{132}{43}k\right)=\frac{961+156\sqrt{22}}{3234}>\frac{1}{2}=\lambda_1(10)>\lambda_1(\ell)$$

so (65) holds. Thus the assertion holds provided $k\geq 2$ and $\ell\geq 22$.

The remaining cases satisfy $2\leq k\leq\ell\leq 21$ so that only finitely many cases are remaining. Therefore, we can use some numerical calculation to check that the assertion holds even for the remaining cases. This completes the proof. □

We now prove that the case (12) occur if and only if (13) holds. Since the exponents θ_A and θ_B have been already compared in Lemma 28, it suffices to prove the next lemma. For completeness, we include the case $k=1$ as well.

Lemma 31. *Let $\theta_{LZ}(k,\ell),\theta(k,\ell)$ be functions given in Theorem LZ2 and Theorem 3, respectively. Then, for positive integers k,ℓ with $\ell\geq 2$, we have*

$$\theta_{LZ}(k,\ell)<\theta(k,\ell) \quad \Longleftrightarrow \quad \ell=2 \quad \text{or} \quad k<\lambda_1(\ell)\ell.$$

Proof. In the case $\ell=2$, we have

$$\theta(1,2)=\frac{17+4\sqrt{15}}{49}>\frac{1}{2}=\theta_{LZ}(1,2)$$

and as we have seen in the proof of Theorem 4,

$$\theta(k,2)=\theta_A(k,2)=\frac{1}{k}>\frac{5}{6k}\geq\theta_{LZ}(k,2) \quad \text{if } k\geq 2.$$

Thus, both sides of the assertion is true, so that the assertion itself is true.

We next consider the case $\ell\geq 3$ and $k\geq\lambda_1(\ell)\ell$. Since $\lambda_1(\ell)$ is decreasing,

$$\theta_A(k,\ell)\leq\frac{\lambda_1(\ell)}{k}=\min\left(\frac{\lambda_1(\ell)}{k},\frac{1}{\ell}\right)\leq\min\left(\frac{\lambda_1(3)}{k},\frac{1}{\ell}\right)\leq\theta_{LZ}(k,\ell).$$

Again, since $\lambda_1(\ell)$ is decreasing, the assumption $k\geq\lambda_1(\ell)\ell$ implies

$$k\geq\lambda_1(\ell)\ell\geq\frac{5}{12}\ell$$

so that

$$\theta_B(k,\ell)\leq\frac{5}{12k}\leq\min\left(\frac{5}{6k},\frac{1}{\ell}\right)=\theta_{\mathrm{LZ}}(k,\ell).$$

Combining two estimates above,

$$\theta(k,\ell)=\max(\theta_A(k,\ell),\theta_B(k,\ell))\leq\theta_{LZ}(k,\ell).$$

Thus, both sides of the assertion is false in this case so the assertion itself holds.

We finally consider the case $\ell\geq 3$ and $k<\lambda_1(\ell)\ell$. It suffices to prove

$$\theta(k,\ell)>\frac{1}{\ell} \tag{68}$$

since $\theta_{LZ}(k,\ell)\leq 1/\ell$. In the current case, we immediately have

$$\frac{\lambda_1(\ell)}{k}>\frac{1}{\ell},\quad \frac{k}{\ell(k-1)}>\frac{1}{\ell}.$$

Therefore, by Lemma 30, in order to prove (68), it suffices to consider the case

$$(k,\ell)=(1,3),(1,4),(2,5),(2,6),(2,7),(2,8),(2,9) \tag{69}$$

and the case

$$k=1 \quad \text{and} \quad \ell\geq 5. \tag{70}$$

In the case (69), we can check (68) numerically. For the case (70), it suffices to see

$$\frac{5}{12k}>\frac{1}{\ell}$$

which trivially holds. This completes the proof. □

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