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Minimal-mass blow-up solutions for inhomogeneous nonlinear Schrödinger equations with growing potentials

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Abstract. In this paper, we consider the following equation:

$$i\frac{\partial u}{\partial t}+\Delta u+g(x)|u|^{4/N}u-Wu=0.$$

We construct a critical-mass solution that blows up at a finite time and describe the behaviour of the solution in the neighbourhood of the blow-up time. Banica-Carles-Duyckaerts (2011) have shown the existence of a critical-mass blow-up solution under the assumptions that $N\leq 2$, that g and W are sufficiently smooth and that each derivative of these is bounded. In this paper, we show the existence of a critical-mass blow-up solution under weaker assumptions regarding smoothness and boundedness of g and W. In particular, it includes the cases where W is unbounded at spatial infinity or not Lipschitz continuous.

1. Introduction

We consider the following nonlinear Schrödinger equation with potentials:

$$\text{(NLS)}\qquad \begin{cases} i\frac{\partial u}{\partial t}+\Delta u+g(x)|u|^{4/N}u-Wu=0, \\ u(t_0)=u_0 \end{cases}$$

in $\mathbf{R}^N$, where $g\in L^\infty(\mathbf{R}^N)$ and W is the sum of potentials satisfying one of the following conditions:

$$\text{(W1)}\qquad W\in C^\infty(\mathbf{R}^N),\quad W\geq 0,\quad \left(\frac{\partial}{\partial x}\right)^\alpha W\in L^\infty(\mathbf{R}^N)\quad (|\alpha|\geq 2),$$

$$\text{(1)}\qquad W\in L^p(\mathbf{R}^N)+L^\infty(\mathbf{R}^N)\quad \left(p\geq 1 \text{ and } p>\frac{N}{2}\right).$$

We define Hilbert spaces Σ^k by

$$\Sigma^k := \left\{u \in H^k(\mathbf{R}^N) \mid |x|^k u \in L^2(\mathbf{R}^N)\right\}, \quad \|u\|_{\Sigma^k}^2 := \|u\|_{H^k}^2 + \||x|^k u\|_2^2,$$

where k is an integer.

It is well known that (NLS) is locally well-posed in Σ^1 (see, e.g., [5] and [6]). This means that for any $u_0 \in \Sigma^1$, there exists a unique maximal solution $u \in C((T_*, T^*), \Sigma^1) \cap C^1((T_*, T^*), \Sigma^{-1})$. Moreover, the mass (i.e., L^2-norm) and energy E of the solution are conserved by the flow, where

$$E(u) := \frac{1}{2}\|\nabla u\|_2^2 - \frac{1}{2+\frac{4}{N}}\int_{\mathbf{R}^N} g(x)|u(x)|^{2+4/N}\, dx + \frac{1}{2}\int_{\mathbf{R}^N} W(x)|u(x)|^2\, dx.$$

Furthermore, there is a blow-up alternative

$$T^* < \infty \text{ implies } \lim_{t \nearrow T^*}\|\nabla u(t)\|_2 = \infty.$$

Moreover, we consider the following condition instead of (1):

$$W \in L^p(\mathbf{R}^N) + L^\infty(\mathbf{R}^N) \quad \left(p \geq 2 \text{ and } p > \frac{N}{2}\right). \tag{2}$$

Under this condition, if $u_0 \in \Sigma^2$, then the corresponding solution u belongs to $u \in C((T_*, T^*), \Sigma^2) \cap C^1((T_*, T^*), L^2(\mathbf{R}^N))$. To show this, we first ensure the regularity of the solution using [5, Theorem 5.7.1]. Next, we show that $t \mapsto |x|^2 u(t)$ belongs to $C((T_*, T^*), L^2(\mathbf{R}^N))$ if $|x|^2 u_0 \in L^2(\mathbf{R}^N)$ using [5, Lemma 6.5.2]. Strictly speaking, [5, Lemma 6.5.2] claims that $t \mapsto |x| u(t)$ belongs to $C((T_*, T^*), L^2(\mathbf{R}^N))$ if $|x| u_0 \in L^2(\mathbf{R}^N)$, but this can be justified by modifying the proof.

In this paper, we investigate the conditions for the inhomogeneity and the potential related with the existence of minimal-mass blow-up solution.

1.1. Critical problem

Firstly, we describe the results regarding the mass-critical problem:

$$i\frac{\partial u}{\partial t} + \Delta u + |u|^{4/N} u = 0, \quad (t,x) \in \mathbf{R} \times \mathbf{R}^N. \tag{CNLS}$$

It is well known ([2], [7], and [17]) that there exists a unique classical solution Q for

$$-\Delta Q + Q - |Q|^{4/N} Q = 0, \quad Q \in H^1(\mathbf{R}^N), \quad Q > 0, \quad Q \text{ is radial},$$

which is called the ground state. If $\|u\|_2 = \|Q\|_2$ ($\|u\|_2 < \|Q\|_2$, $\|u\|_2 > \|Q\|_2$), we say that u has the *critical mass* (*subcritical mass*, *supercritical mass*, respectively).

We note that $E_{\mathrm{crit}}(Q)=0$, where E_{crit} is the energy associated to (CNLS). Moreover, the ground state Q attains the best constant in the Gagliardo-Nirenberg inequality

$$\|v\|_{2+4/N}^{2+4/N} \leq \left(1+\frac{2}{N}\right)\left(\frac{\|v\|_2}{\|Q\|_2}\right)^{4/N}\|\nabla v\|_2^2 \quad \text{for } v\in H^1(\mathbf{R}^N).$$

Therefore, for all $v\in H^1(\mathbf{R}^N)$,

$$E_{\mathrm{crit}}(v)\geq \frac{1}{2}\|\nabla v\|_2^2\left(1-\left(\frac{\|v\|_2}{\|Q\|_2}\right)^{4/N}\right)$$

holds. This inequality and the mass and energy conservations imply that any subcritical-mass solution for (CNLS) is global and bounded in $H^1(\mathbf{R}^N)$.

Regarding the critical mass case, we apply the pseudo-conformal transformation

$$u(t,x) \longmapsto \frac{1}{|t|^{N/2}}u\left(-\frac{1}{t},\pm\frac{x}{t}\right)e^{i|x|^2/4t}$$

to the solitary wave solution $u(t,x):=Q(x)e^{it}$. Then we obtain

$$S(t,x):=\frac{1}{|t|^{N/2}}Q\left(\frac{x}{t}\right)e^{i(|x|^2-4)/4t},$$

which is also a solution for (CNLS) and satisfies

$$\|S(t)\|_2=\|Q\|_2, \quad \|\nabla S(t)\|_2\sim\frac{1}{|t|} \quad (t\nearrow 0).$$

Namely, S is a minimal-mass blow-up solution for (CNLS). Moreover, S is the only finite time blow-up solution for (CNLS) with critical mass, up to the symmetries of the flow (see [10]).

Regarding the supercritical mass case, there exists a solution u for (CNLS) such that

$$\|\nabla u(t)\|_2\sim\sqrt{\frac{\log|\log|T^*-t||}{T^*-t}} \quad (t\nearrow T^*)$$

(see [13], [14], and [15]).

1.2. Previous results

We describe previous results regarding the following nonlinear Schrödinger equation with a real-valued potential:

$$\text{(PNLS)} \qquad i\frac{\partial u}{\partial t}+\Delta u+|u|^{4/N}u-W(x)u=0, \quad (t,x)\in\mathbf{R}\times\mathbf{R}^N.$$

At first, [3] and [4] give results for unbounded potentials. Carles and Nakamura [4] deal with the case where W is a Stark potential, i.e., $W(x)=\xi\cdot x$ for some $\xi\in\mathbf{R}^N$. Carles [3] deals with the case where $W(x)=\pm\omega^2|x|^2$ for $\omega\in\mathbf{R}^N$. By using the Avron-Herbst formula for the former and the generalised lens transform for the latter, solutions for (CNLS) can be transformed into solutions for (PNLS). Therefore, in these cases, the minimal-mass blow-up solution for (PNLS) can be constructed from the critical-mass blow-up solution S for (CNLS). More generally, if (PNLS) can be reduced to (CNLS) (e.g., when W is easy to handle algebraically), then (PNLS) may have a critical-mass blow-up solution with a blow-up rate of t^{-1}.

Merle [11] and Raphaël and Szeftel [16] consider

$$\text{(ICNLS)} \qquad i\frac{\partial u}{\partial t}+\Delta u+g(x)|u|^{4/N}u=0, \quad (t,x)\in\mathbf{R}\times\mathbf{R}^N.$$

Firstly, [11] showed non-existent results:

Theorem 1.1. ([11]) *Assume the following for g:*

(i)

$$g_1\le g\le 1 \quad \textit{for some } g_1>0,$$

(ii)

$$g\in C^1(\mathbf{R}^N)\cap W^{1,\infty}(\mathbf{R}^N), \quad x\cdot\nabla g\in L^\infty(\mathbf{R}^N),$$

(iii)

$$g(x_0)=1 \quad \textit{for some } x_0\in\mathbf{R}^N,$$

(iv) *There exist $\delta_0, R_0>0$ such that for all $|x|>R_0$, $g(x)\le 1-\delta_0$,*

(v) *$g^{-1}(\{1\})$ is finite,*

(vi) *There exist $\rho_0>0$ and $\alpha_0\in(0,1)$ such that for all $|x-x_0|\le\rho_0$, $(x-x_0)\cdot\nabla g(x)\le-|x-x_0|^{1+\alpha_0}$.*

Then there is no blow-up solutions with critical mass.

It is also shown that solutions for (ICNLS) with subcritical mass are globally in time if g satisfies (i) and (ii). Moreover, it is additionally shown that if k satisfies (iii) and (vi), then there is a blow-up solution with supercritical mass less than $\|Q\|_2+\varepsilon$ for some $\varepsilon>0$. Thus, Theorem 1.1 means that there is no minimal-mass blow-up solution at a finite time.

In contrast, [16] obtains results for existence:

Theorem 1.2. ([16]) *Assume $N=2$ and the following for g:*

$$g\in C^5(\mathbf{R}^2)\cap W^{1,\infty}(\mathbf{R}^2),$$
$$g_1\le g\le 1 \quad \text{for some } g_1>0 \quad \text{and} \quad g(x_0)=1 \quad \text{for some } x_0\in\mathbf{R}^N,$$
$$\nabla^2 g(x_0)<0.$$

Then for any E_0 such that

$$E_0>\frac{1}{8}\int_{\mathbf{R}^2}\nabla^2 g(x_0)(y,y)Q(y)^4\,dy>0,$$

there exist $t_0<0$ and a unique up to phase shift $u\in C([t_0,0),H^1(\mathbf{R}^2))$ that is solution for (ICNLS) with critical mass and energy E_0 and blows up at $t=0$.

The result differs from results in [3] and [4] in that it does not use the classical method of pseudo-conformal transformation to construct the blow-up solution. Le Coz, Martel, and Raphaël [8], based on the methodology of [16], obtain results for

$$\text{(DPNLS)} \qquad i\frac{\partial u}{\partial t}+\Delta u+|u|^{4/N}u\pm|u|^{p-1}u=0, \quad (t,x)\in\mathbf{R}\times\mathbf{R}^N.$$

Banica, Carles, and Duyckaerts [1] present the following result for

$$\text{(INLS)} \qquad i\frac{\partial u}{\partial t}+\Delta u+g(x)|u|^{4/N}u-W(x)u=0, \quad (t,x)\in\mathbf{R}\times\mathbf{R}^N.$$

Theorem 1.3. ([1]) *Let $N=1$ or 2, $W\in C^2(\mathbf{R}^N,\mathbf{R})$, and $g\in C^4(\mathbf{R}^N,\mathbf{R})$. Assume $\left(\frac{\partial}{\partial x}\right)^\beta W\in L^\infty(\mathbf{R}^N)$ $(|\beta|\le 2)$, $\left(\frac{\partial}{\partial x}\right)^\beta g\in L^\infty(\mathbf{R}^N)$ $(|\beta|\le 4)$, and*

$$g(0)=1, \quad \frac{\partial g}{\partial x_j}(0)=\frac{\partial^2 g}{\partial x_j\partial x_k}(0)=0 \quad (1\le j,k\le N).$$

Then there exist $T>0$ and a solution $u\in C((0,T),\Sigma^1)$ for (INLS) such that

$$\left\|u(t)-\frac{1}{\lambda(t)^{N/2}}Q\left(\frac{x-x(t)}{\lambda(t)}\right)e^{i|x|^2/4t-i\theta(1/t)-itV(0)}\right\|_{\Sigma^1}\longrightarrow 0 \quad (t\searrow 0),$$

where θ and λ are continuous real-valued functions and x is a continuous $\mathbf{R}^N$-valued function such that

$$\theta(\tau)=\tau+o(\tau) \quad \text{as } \tau\longrightarrow+\infty,$$
$$\lambda(t)\sim t \text{ and } |x(t)|=o(t) \quad \text{as } t\searrow 0.$$

[9] obtains the following result, which partially extends the result of [1] using the method of [8].

Theorem 1.4. ([9]) *Let the potential W satisfy*

$$W \in C^{1,1}_{\mathrm{loc}}(\mathbf{R}^N),$$
$$\nabla W, \nabla^2 W \in L^q(\mathbf{R}^N) + L^\infty(\mathbf{R}^N) \quad (q \geq 2 \text{ and } q > N).$$

Then there exist $t_0 < 0$ and a initial value $u_0 \in \Sigma^1$ with $\|u_0\|_2 = \|Q\|_2$ such that the corresponding solution u for (PNLS) *with $u(t_0) = u_0$ blows up at $t = 0$. Moreover,*

$$\left\| u(t) - \frac{1}{\lambda(t)^{N/2}} Q\left(\frac{x + w(t)}{\lambda(t)}\right) e^{-ib(t)|x+w(t)|^2/4\lambda(t)^2 + i\gamma(t)} \right\|_{\Sigma^1} \longrightarrow 0 \quad (t \nearrow 0)$$

holds for some C^1 functions $\lambda\colon (t_0, 0) \to (0, \infty)$, $b, \gamma\colon (t_0, 0) \to \mathbf{R}$, and $w\colon (t_0, 0) \to \mathbf{R}^N$ such that

$$\lambda(t) = |t|\,(1 + o(1)), \quad b(t) = |t|\,(1 + o(1)), \quad \gamma(t) \sim |t|^{-1}, \quad |w(t)| = O(|t|^2)$$

as $t \nearrow 0$.

1.3. Main result

In the main result, the following conditions are assumed.

Assumption 1.5. *The inhomogeneous function g satisfies the following conditions*:

(G1) $$g \in W^{1,\infty}(\mathbf{R}^N), \quad x \cdot \nabla g \in L^\infty(\mathbf{R}^N),$$

(G2) $$|g(x) - 1| \lesssim |x|^{2+r_1}, \quad |\nabla g(x)| \lesssim |x|^{1+r_1} \quad (|x| \leq 1)$$

for some $r_1 > 0$.

We use the following notation

$$X(f) := \{g : \text{measurable} \mid |g| \leq Cf \text{ for some } C > 0\}.$$

Assumption 1.6. *The potential W is the sum of potentials satisfying* (W1) *or the following conditions*:

(W2) $$\begin{cases} W \in L^{p_1}(\mathbf{R}^N) + L^\infty(\mathbf{R}^N) & \left(p_1 \geq 2 \text{ and } p_1 > \frac{N}{2}\right), \\ \nabla W \in L^{p_2}(\mathbf{R}^N) + X(1 + |x|) & (p_2 \geq 2 \text{ and } p_2 > N), \end{cases}$$

and furthermore satisfies one of the followings:

(W2-1) *W is locally Lipschitz continuous,*

(W2-2) $W \in X(|x|^{r_2} e^{C|x|})$ *for some $C, r_2 > 0$.*

Namely, we can write $W=W_1+W_2$ and $W_2=W_{21}+W_{22}$ using W_1, W_2, W_{21}, and W_{22} satisfying (W1), (W2), (W2-1), and (W2-2), respectively.

Theorem 1.7. (Existence of a critical-mass blow-up solution) *Assume Assumptions 1.5 and 1.6. For any energy level $E_0>0$, there exist $t_0<0$ and a initial value $u_0\in\Sigma^1$ with $\|u_0\|_2=\|Q\|_2$ and $E(u_0)=E_0$ such that the corresponding solution u for (NLS) with $u(t_0)=u_0$ blows up at $t=0$. Moreover,*

$$\left\|u(t,x)-\frac{1}{\lambda(t)^{N/2}}Q\left(\frac{x+w(t)}{\lambda(t)}\right)e^{-ib(t)|x+w(t)|^2/4\lambda(t)^2+i\gamma(t)}\right\|_{\Sigma^1}\longrightarrow 0 \quad (t\nearrow 0)$$

holds for some C^1 functions $\lambda\colon(t_0,0)\to(0,\infty)$, $b,\gamma\colon(t_0,0)\to\mathbf{R}$, and $w\colon(t_0,0)\to\mathbf{R}^N$ such that

$$\lambda(t)=\sqrt{\frac{8E_0}{\|yQ\|_2^2}}|t|\left(1+o(1)\right),\quad b(t)=\frac{8E_0}{\|yQ\|_2^2}|t|\left(1+o(1)\right),$$
$$\gamma(t)\sim|t|^{-1},\quad |w(t)|=o(|t|)$$

as $t\nearrow 0$.

Remark 1.8. In contrast, if $g\le 1$ and W satisfies (W1) or (1), then any subcritical-mass solution for (NLS) exists globally in time and is bounded in H^1. This can be proved easily by the Gagliardo-Nirenberg inequality and the Sobolev embedding theorem. Therefore, the solution in Theorem 1.7 is a minimal-mass blow-up solution if $g\le 1$.

1.4. Comments regarding the main result

Theorem 1.7 is a generalisation of Theorems 1.3 and 1.4. For example, if $W(x):=\sin(|x|^2)$, then ∇W is not bounded. Therefore, Theorems 1.3 and 1.4 cannot be applied. On the other hand, W satisfies (W2) and (W2-2), therefore Theorem 1.7 can be applied.

We may consider g is locally Lipschitz continuous. Thus, $|g(x)-1|\lesssim|x|^{2+r_1}$ in (G2) may be replaced by $g(0)=1$.

From the assumption (vi) in Theorem 1.1, which is the nonexistence result, we obtain

$$|x|^{\alpha_0}\le|\nabla g(x)|\quad \text{for } |x|\le\rho_0$$

for some $\alpha_0\in(0,1)$, where we assume $g(0)=1$. In contrast, Theorem 1.7, which is the existence result, assumes

$$|\nabla g(x)|\lesssim|x|^{1+r_1}\quad \text{for } |x|\le 1$$

for some $r_1>0$. Therefore, the threshold for the existence and non-existence of blow-up solutions with critical mass can be said to be $\alpha_0=1$ (i.e., $r_1=0$). The result in the case of the threshold has been obtained in part by Theorem 1.2. In this result, g has a nondegenerate maximum at the origin.

From the point of view of differentiability, it seems that neither g nor W need to be smooth over the whole $\mathbf{R}^N$, since blow-up is crucial for behaviour in the neighbourhood of the blow-up point. On the other hand, first-order differentiations are necessary for the technicality of the proof. Thus, the assumption that g and W are first-order weakly differentiable would be quite close to the limit.

Compared to Theorem 1.4, Theorem 1.7 requires less order of differentiation for the potential W. In [9], the bootstrap of λ and b is done by differentiating and then integrating, thus the condition $\frac{\partial b}{\partial s}+b^2=o(s^{-3})$ is required. Thus, [9] has required $C^{1,1}_{\mathrm{loc}}$ for W. However, in this paper, the condition is removed by using the property of energy. Consequently, we reduce the order of differentiation.

From the point of view of integrability, it would be possible to replace (G1) and (W2) with weaker conditions. In fact, a scrutiny of proofs of Proposition 2.1, Lemma 5.3, etc. shows that some of them can be substituted by other integrable conditions in their proofs. However, it would be complex to attempt to describe them exhaustively.

2. Notation and preliminaries

We define

$$(u,v)_2:=\mathrm{Re}\int_{\mathbf{R}^N}u(x)\overline{v}(x)\,dx,\qquad \|u\|_p:=\left(\int_{\mathbf{R}^N}|u(x)|^p\,dx\right)^{1/p},$$
$$f(z):=|z|^{2+4/N}z,\qquad F(z):=\frac{1}{2+\frac{4}{N}}|z|^{2+4/N}\quad\text{for } z\in\mathbf{C}.$$

By identifying $\mathbf{C}$ with $\mathbf{R}^2$, we denote the differentials of f and F by df and dF, respectively. For instance,

$$df(z)(w)=\frac{\partial f}{\partial x}(z)\,\mathrm{Re}\,w+\frac{\partial f}{\partial y}(z)\,\mathrm{Im}\,w$$

where $x=\mathrm{Re}\,z$, $y=\mathrm{Im}\,z$, and $w\in\mathbf{C}$. We define

$$\Lambda:=\frac{N}{2}+x\cdot\nabla,\quad L_+:=-\Delta+1-\left(1+\frac{4}{N}\right)Q^{2+4/N},\quad L_-:=-\Delta+1-Q^{2+4/N}.$$

Namely, Λ is the generator of L^2-scaling, and L_+ and L_- come from the linearised Schrödinger operator to close Q. Then

$$L_-Q=0,\quad L_+\Lambda Q=-2Q,\quad L_-|x|^2Q=-4\Lambda Q,\quad L_+\rho=|x|^2Q,$$
$$L_-xQ=-\nabla Q,\quad L_+\nabla Q=0$$

hold, where $\rho\in\mathcal{S}(\mathbf{R}^N)$ is the unique radial solution for $L_+\rho=|x|^2Q$. Note that there exist $C_\alpha, K_\alpha>0$ such that

$$\left|\left(\frac{\partial}{\partial x}\right)^\alpha Q(x)\right|\le C_\alpha Q(x),\qquad \left|\left(\frac{\partial}{\partial x}\right)^\alpha \rho(x)\right|\le C_\alpha(1+|x|)^{K_\alpha}Q(x).$$

for any multi-index α. Furthermore, there exists $\mu>0$ such that for any $u\in H^1(\mathbf{R}^N)$,

$$\begin{aligned}&\langle L_+\operatorname{Re}u,\operatorname{Re}u\rangle+\langle L_-\operatorname{Im}u,\operatorname{Im}u\rangle\\ &\ge \mu\,\|u\|_{H^1}^2-\frac{1}{\mu}\left((\operatorname{Re}u,Q)_2{}^2+|(\operatorname{Re}u,xQ)_2|^2+(\operatorname{Re}u,|x|^2Q)_2{}^2+(\operatorname{Im}u,\rho)_2{}^2\right)\end{aligned}\tag{3}$$

holds (see, e.g., [12], [13], [16], and [18]). Finally, we use the notation $\lesssim$ and $\gtrsim$ when the inequalities hold up to a positive constant. We also use the notation $\approx$ when $\lesssim$ and $\gtrsim$ hold.

We estimate the error terms Ψ that is defined by

$$\Psi(y;\lambda,w):=\lambda^2W(\lambda y-w)Q(y).$$

Moreover, we define K by

$$K:=\min\left\{1,2-\frac{N}{p_1},1-\frac{N}{p_2},r_1,r_2\right\}\in(0,1],$$

where p_j and r_j are from Assumptions 1.5 and 1.6.

Without loss of generality, we may in addition assume that $W_1(0)=0$ and $W_{21}(0)=0$. In particular,

$$W_1\in X(|x|+|x|^2),\quad \nabla W_1\in X(1+|x|)$$

holds.

Proposition 2.1. (Estimate of Ψ) *There exists a sufficiently small constant $\varepsilon'>0$ such that*

$$\left\|e^{\varepsilon'|y|}\Psi\right\|_{H^1}\lesssim\lambda^{1+K}(\lambda+|w|)$$

for $0<\lambda\ll1$ and $w\in\mathbf{R}^N$ such that $|w|\le1$.

Proof. Firstly, since $W_1 \in X(|x|+|x|^2)$ and $\nabla W_1 \in X(1+|x|)$, we obtain

$$\|e^{\varepsilon'|y|}\lambda^2 W_1(\lambda y-w)Q\|_2 \lesssim \|e^{\varepsilon'|y|}\lambda^2(\lambda|y|+\lambda^2|y|^2+|w|)Q\|_2 \lesssim \lambda^2(\lambda+|w|),$$
$$\|e^{\varepsilon'|y|}\lambda^3 \nabla W_1(\lambda y-w)Q\|_2 \lesssim \|e^{\varepsilon'|y|}\lambda^3(1+\lambda|y|+|w|)Q\|_2 \lesssim \lambda^3.$$

Secondly, since

$$W_{21}(\lambda y-w) = \int_0^1 (\lambda y-w)\cdot\nabla W_{21}(\tau(\lambda y-w))\, d\tau,$$

we obtain

$$\|e^{\varepsilon'|y|}\lambda^2 W_{21}(\lambda y-w)Q\|_2 \lesssim \lambda^{2-N/p_2}(\lambda+|w|)+\lambda^2(\lambda+|w|),$$
$$\|e^{\varepsilon'|y|}\lambda^3 \nabla W_{21}(\lambda y-w)Q\|_2 \lesssim \lambda^{3-N/p_2}+\lambda^3.$$

Finally,

$$\begin{aligned}\|e^{\varepsilon'|y|}\lambda^2 W_{22}(\lambda y-w)Q\|_2 &\lesssim \|e^{\varepsilon'|y|}\lambda^2(\lambda^{r_2}|y|^{r_2}+|w|^{r_2})e^{C(\lambda|y|+|w|)}Q\|_2\\ &\lesssim \lambda^2(\lambda^{r_2}+|w|^{r_2})\\ &\lesssim \lambda^{1+r_2}(\lambda+|w|),\end{aligned}$$
$$\|e^{\varepsilon'|y|}\lambda^3 \nabla W_{22}(\lambda y-w)Q\|_2 \lesssim \lambda^{3-N/p_2}. \quad \square$$

Remark 2.2. The estimate stated in Proposition 2.1 holds true even if Q is replaced by $|y|^2Q$, ρ, etc.

Furthermore, direct calculations yield the following properties:

Proposition 2.3. *Let*

$$Q_{\lambda,b,w,\gamma}(x) := \frac{1}{\lambda^{N/2}} Q\left(\frac{x+w}{\lambda}\right) e^{-ib|x+w|^2/4\lambda^2+i\gamma}.$$

Then

$$\left|8E(Q_{\lambda,b,w,\gamma}) - \frac{b^2}{\lambda^2}\|yQ\|_2^2\right| \lesssim \frac{\lambda^{2+K}+|w|^{2+K}}{\lambda^2}.$$

holds for $0<\lambda\ll 1$ *and* $w\in\mathbf{R}^N$ *such that* $|w|\leq 1$.

Moreover, if $s\mapsto(\lambda(s), b(s), w(s))$ *is a* C^1*-function,*

$$\left|\frac{d}{ds}E(Q_{\lambda,b,w,\gamma})\right|$$
$$\lesssim \frac{1}{\lambda^2}\left(\left(\lambda^{1+K}+|b|+|w|^{1+K}\right)\left(\left|\frac{1}{\lambda}\frac{\partial\lambda}{\partial s}+b\right|+\left|\frac{\partial b}{\partial s}+b^2\right|+\left|\frac{\partial w}{\partial s}\right|\right)+|b|(\lambda^{2+K}+|w|^{2+K})\right)$$

holds.

At the end of this section, we state the following standard result. For the proof, see [13].

Lemma 2.4. (Decomposition) *There exists* $\overline{C}>0$ *such that the following statement holds. Let* I *be an interval and* $\delta>0$ *be sufficiently small. We assume that* $u\in C(I,H^1(\mathbf{R}^N))\cap C^1(I,\Sigma^{-1})$ *satisfies*

$$\left\|\lambda(t)^{\frac{N}{2}}u\left(t,\lambda(t)y-w(t)\right)e^{i\gamma(t)}-Q\right\|_{H^1}<\delta \quad \text{for any } t\in I$$

for some functions $\lambda\colon I\to(0,\infty)$, $\gamma\colon I\to\mathbf{R}$, *and* $w\colon I\to\mathbf{R}^N$. *Then there exist unique functions* $\tilde{\lambda}\colon I\to(0,\infty)$, $\tilde{b}\colon I\to\mathbf{R}$, $\widetilde{\gamma}\colon I\to\mathbf{R}/2\pi\mathbf{Z}$, *and* $\widetilde{w}\colon I\to\mathbf{R}^N$ *such that*

$$\begin{aligned}&u(t,x)=\frac{1}{\tilde{\lambda}(t)^{\frac{N}{2}}}\left(Q+\tilde{\varepsilon}\right)\left(t,\frac{x+\widetilde{w}(t)}{\tilde{\lambda}(t)}\right)e^{-i\tilde{b}(t)|x+\widetilde{w}(t)|^2/4\tilde{\lambda}(t)^2+i\widetilde{\gamma}(t)},\\ &\left|\frac{\tilde{\lambda}(t)}{\lambda(t)}-1\right|+\left|\tilde{b}(t)\right|+|\widetilde{\gamma}(t)-\gamma(t)|_{\mathbf{R}/2\pi\mathbf{Z}}+\left|\frac{\widetilde{w}(t)-w(t)}{\tilde{\lambda}(t)}\right|<\overline{C}\end{aligned}\tag{4}$$

hold, where $|\cdot|_{\mathbf{R}/2\pi\mathbf{Z}}$ *is defined by*

$$|c|_{\mathbf{R}/2\pi\mathbf{Z}}:=\inf_{m\in\mathbf{Z}}|c+2\pi m|,$$

and $\tilde{\varepsilon}$ *satisfies the orthogonal conditions*

$$(\tilde{\varepsilon},i\Lambda Q)_2=\left(\tilde{\varepsilon},|y|^2Q\right)_2=(\tilde{\varepsilon},i\rho)_2=0,\quad (\tilde{\varepsilon},yQ)_2=0\tag{5}$$

on I. *In particular,* $\tilde{\lambda}$, $\tilde{b}$, $\widetilde{\gamma}$, *and* $\widetilde{w}$ *are* C^1 *functions and independent of* λ, γ, *and* w.

3. Proof of Theorem 1.7

For $t_1<0$ sufficiently close 0, let $s_1,\lambda_1,b_1>0$ be defined by

$$s_1:=-\frac{\|yQ\|_2^2}{8E_0}{t_1}^{-1},\quad \lambda_1:=\sqrt{\frac{\|yQ\|_2^2}{8E_0}}{s_1}^{-1},\quad E(Q_{\lambda_1,b_1,0,0})=E_0.$$

Then, from Proposition 2.3, $\lambda_1\approx b_1$.

Let $u(t)$ be the solution for (NLS) with an initial value

$$u(t_1,x):=\frac{1}{{\lambda_1}^{N/2}}Q\left(\frac{x}{\lambda_1}\right)e^{-ib_1|x|^2/4{\lambda_1}^2}.\tag{6}$$

Since b_1 is sufficiently small, u satisfies the assumption in Lemma 2.4 with $\lambda=\lambda_1$, $\gamma=0$, and $w=0$ in a neighbourhood I of t_1. Therefore, there exist decomposition parameters $\tilde{\lambda}_{t_1}$, $\tilde{b}_{t_1}$, $\widetilde{\gamma}_{t_1}$, $\widetilde{w}_{t_1}$, and $\tilde{\varepsilon}_{t_1}$ such that (4) and (5) hold on I. Moreover, there exists $t_0<0$ which is independent of t_1 such that the following lemma holds:

Lemma 3.1. (Conversion of estimates) *For* $t\in[t_0,t_1]$,

$$\tilde{\lambda}_{t_1}(t)=\sqrt{\frac{8E_0}{\|yQ\|_2^2}}|t|\left(1+\varepsilon_{\tilde{\lambda},t_1}(t)\right),\quad \tilde{b}_{t_1}(t)=\frac{8E_0}{\|yQ\|_2^2}|t|\left(1+\varepsilon_{\tilde{b},t_1}(t)\right),$$
$$|\widetilde{w}_{t_1}(t)|\lesssim|t|^{1+K},\quad \|\tilde{\varepsilon}_{t_1}(t)\|_{H^1}\lesssim|t|^{1+3K/4},\quad \|y\tilde{\varepsilon}_{t_1}(t)\|_2\lesssim|t|^{3K/4}$$

holds. Furthermore,

$$\sup_{t_1\in[t,0)}\left|\varepsilon_{\tilde{\lambda},t_1}(t)\right|\lesssim|t|^K,\quad \sup_{t_1\in[t,0)}\left|\varepsilon_{\tilde{b},t_1}(t)\right|\lesssim|t|^K.$$

Note that constants omitted in inequalities in Lemma 3.1 are independent of t_1. In this section, we prove Theorem 1.7 by assuming Lemma 3.1.

Proof of Theorem 1.7. Let $\{t_n\}_{n\in\mathbf{N}}\subset(t_0,0)$ be a monotonically increasing sequence such that $\lim_{n\nearrow\infty}t_n=0$. For each $n\in\mathbf{N}$, let u_n be the solution for (NLS) with an initial value

$$u_n(t_n,x):=\frac{1}{{\lambda_{1,n}}^{N/2}}Q\left(\frac{x}{\lambda_{1,n}}\right)e^{-ib_{1,n}|x|^2/4{\lambda_{1,n}}^2}$$

at t_n, where

$$s_n:=-\frac{\|yQ\|_2^2}{8E_0}{t_n}^{-1},\quad \lambda_n:=\sqrt{\frac{\|yQ\|_2^2}{8E_0}}{s_n}^{-1},\quad E(Q_{\lambda_n,b_n,0,0})=E_0.$$

According to Lemma 2.4 with an initial value $\widetilde{\gamma}_n(t_n)=0$, there exists a decomposition

$$u_n(t,x)=\frac{1}{\tilde{\lambda}_n(t)^{\frac{N}{2}}}\left(Q+\tilde{\varepsilon}_n\right)\left(t,\frac{x+\widetilde{w}_n(t)}{\tilde{\lambda}_n(t)}\right)e^{-i\tilde{b}_n(t)|x+\widetilde{w}_n(t)|^2/4\tilde{\lambda}_n(t)^2+i\widetilde{\gamma}_n(t)}$$

on $[t_0,t_n]$. Up to a subsequence, there exists $u_\infty(t_0)\in\Sigma^1$ such that

$$u_n(t_0)\rightharpoonup u_\infty(t_0)\quad\text{weakly in }\Sigma^1,\quad u_n(t_0)\longrightarrow u_\infty(t_0)\quad\text{in }L^2(\mathbf{R}^N)\quad(n\longrightarrow\infty).$$

Moreover, since $u_n\colon[t_0,0)\to\Sigma^1$ is locally uniformly bounded,

$$u_n\longrightarrow u_\infty\quad\text{in }C([t_0,T'],L^2(\mathbf{R}^N)),\quad u_n(t)\rightharpoonup u_\infty(t)\text{ in }\Sigma^1\quad(n\longrightarrow\infty)$$

holds (see [9]). Particularly, we have $\|u_\infty(t)\|_2=\|Q\|_2$.

Based on weak convergence in $H^1(\mathbf{R}^N)$ and Lemma 2.4, we decompose u_∞ to

$$u_\infty(t,x)=\frac{1}{\tilde{\lambda}_\infty(t)^{\frac{N}{2}}}\left(Q+\tilde{\varepsilon}_\infty\right)\left(t,\frac{x+\widetilde{w}_\infty(t)}{\tilde{\lambda}_\infty(t)}\right)e^{-i\tilde{b}_\infty(t)|x+\widetilde{w}_\infty(t)|^2/4\tilde{\lambda}_\infty(t)^2+i\widetilde{\gamma}_\infty(t)}$$

on $[t_0,0)$. Furthermore, as $n\to\infty$,

$$\tilde{\lambda}_n(t)\longrightarrow\tilde{\lambda}_\infty(t),\quad \tilde{b}_n(t)\longrightarrow\tilde{b}_\infty(t),\quad \widetilde{w}_n(t)\longrightarrow\widetilde{w}_\infty(t),\quad e^{i\widetilde{\gamma}_n(t)}\longrightarrow e^{i\widetilde{\gamma}_\infty(t)},$$
$$\tilde{\varepsilon}_n(t)\rightharpoonup\tilde{\varepsilon}_\infty(t)\quad \text{in } \Sigma^1$$

holds for any $t\in[t_0,0)$. Therefore, we have

$$\tilde{\lambda}_\infty(t)=\sqrt{\frac{8E_0}{\|yQ\|_2^2}}\,|t|\,(1+\varepsilon_{\tilde{\lambda},0}(t)),\quad \tilde{b}_\infty(t)=\frac{8E_0}{\|yQ\|_2^2}\,|t|\,(1+\varepsilon_{\tilde{b},0}(t)),$$
$$|\widetilde{w}_\infty(t)|\lesssim|t|^{2L-1},\quad \|\tilde{\varepsilon}_\infty(t)\|_{H^1}\lesssim|t|^{L+K/4},\quad \|y\tilde{\varepsilon}_\infty(t)\|_2\lesssim|t|^{L+K/4-1},$$
$$\left|\varepsilon_{\tilde{\lambda},0}(t)\right|\lesssim|t|^K,\quad \left|\varepsilon_{\tilde{b},0}(t)\right|\lesssim|t|^K$$

from a uniform estimate of Lemma 3.1. Consequently, we obtain Theorem 1.7.

Finally, check the energy. Since $E'(w)=-\Delta w-g(x)|w|^{2+4/N}+Ww$, we obtain

$$E\left(u_n\right)-E\left(Q_{\tilde{\lambda}_n,\tilde{b}_n,\widetilde{w}_n,\widetilde{\gamma}_n}\right)=o_{t\nearrow 0}(1),\quad E\left(u_\infty\right)-E\left(P_{\tilde{\lambda}_\infty,\tilde{b}_\infty,\widetilde{w}_\infty,\widetilde{\gamma}_\infty}\right)=o_{t\nearrow 0}(1),$$

where $o_{t\nearrow 0}(1)$ is uniform with respect to n. From continuity of energy,

$$\lim_{n\to\infty}E\left(Q_{\tilde{\lambda}_n,\tilde{b}_n,\widetilde{w}_n,\widetilde{\gamma}_n}\right)=E\left(P_{\tilde{\lambda}_\infty,\tilde{b}_\infty,\widetilde{w}_\infty,\widetilde{\gamma}_\infty}\right)$$

holds and from conservation of energy,

$$E\left(u_n\right)=E\left(u_n(t_n)\right)=E\left(P_{\tilde{\lambda}_{1,n},\tilde{b}_{1,n},0,0}\right)=E_0$$

holds. Therefore, we obtain

$$E\left(u_\infty\right)=E_0+o_{t\nearrow 0}(1),$$

so that $E\left(u_\infty\right)=E_0$. □

4. Uniform estimates for modulation terms

From this section to Section 6, we prove Lemma 3.1.

Let $u(t)$ be the solution for (NLS) with an initial value (6). Note that $u\in C((T_*,T^*),\Sigma^2(\mathbf{R}^N))$ and $|x|\nabla u\in C((T_*,T^*),L^2(\mathbf{R}^N))$. Moreover,

$$\operatorname{Im}\int_{\mathbf{R}^N}u(t_1,x)\nabla\overline{u}(t_1,x)\;dx=0$$

holds.

Then there exist decomposition parameters $\tilde{\lambda}_{t_1}$, $\tilde{b}_{t_1}$, $\widetilde{\gamma}_{t_1}$, $\widetilde{w}_{t_1}$, and $\tilde{\varepsilon}_{t_1}$ such that (4) and (5) hold on a neighbourhood of t_1. We define the rescaled time s_{t_1} by

$$s_{t_1}(t) := s_1 - \int_t^{t_1} \frac{1}{\tilde{\lambda}_{t_1}(\tau)^2}\, d\tau. \tag{7}$$

Moreover, we define

$$t_{t_1} := {s_{t_1}}^{-1}, \quad \lambda_{t_1}(s) := \tilde{\lambda}_{t_1}(t_{t_1}(s)), \quad b_{t_1}(s) := \tilde{b}_{t_1}(t_{t_1}(s)),$$
$$\gamma_{t_1}(s) := \widetilde{\gamma}_{t_1}(t_{t_1}(s)), \quad w_{t_1}(s) := \widetilde{w}_{t_1}(t_{t_1}(s)), \quad \varepsilon_{t_1}(s,y) := \tilde{\varepsilon}_{t_1}(t_{t_1}(s), y).$$

In addition, although it is an abuse of the symbol, we define

$$\Psi(s,y) := \Psi(y; \lambda(s), w(s)).$$

For the sake of clarity in notation, we often omit the subscript t_1. Furthermore, let I_{t_1} be the maximal interval of the existence of the decomposition such that (4) and (5) hold and we define

$$J_{s_1} := s_{t_1}(I_{t_1}).$$

Then, from (4), (NLS), and (7), we obtain the equation of ε:

$$\begin{aligned} 0 = {} & i\frac{\partial \varepsilon}{\partial s} + \Delta\varepsilon - \varepsilon + g(\lambda y - w) f(Q+\varepsilon) - f(Q) - \lambda^2 W(\lambda y - w)\varepsilon \\ & - i\left(\frac{1}{\lambda}\frac{\partial \lambda}{\partial s} + b\right)\Lambda(Q+\varepsilon) + \left(1 - \frac{\partial \gamma}{\partial s}\right)(Q+\varepsilon) + \left(\frac{\partial b}{\partial s} + b^2\right)\frac{|y|^2}{4}(Q+\varepsilon) \\ & - \left(\frac{1}{\lambda}\frac{\partial \lambda}{\partial s} + b\right) b \frac{|y|^2}{2}(Q+\varepsilon) + i\frac{1}{\lambda}\frac{\partial w}{\partial s}\cdot\nabla(Q+\varepsilon) + \frac{1}{2}\frac{b}{\lambda}\frac{\partial w}{\partial s}\cdot y(Q+\varepsilon) - \Psi \end{aligned} \tag{8}$$

on J_{s_1}. Moreover, we define

$$\mathrm{Mod}(s) := \left(\frac{1}{\lambda}\frac{\partial \lambda}{\partial s} + b, \frac{\partial b}{\partial s} + b^2, 1 - \frac{\partial \gamma}{\partial s}, \frac{\partial w}{\partial s}\right).$$

We will show in this section that the second and third lines in (8) are small. Therefore, we show that Mod is small.

When Mod is small, λ, b, and w are expected to satisfy the following approximate equation:

$$\frac{1}{\lambda}\frac{\partial \lambda}{\partial s} + b = \frac{\partial b}{\partial s} + b^2 = \frac{\partial w}{\partial s} = 0.$$

Therefore, λ, b, and w can be considered to be approximated by the following solutions of the approximate equation:

$$\lambda_{\mathrm{app}}(s) = \mathcal{C}_\lambda s^{-1}, \quad b_{\mathrm{app}}(s) = s^{-1}, \quad w_{\mathrm{app}}(s) = 0$$

for some constant $\mathcal{C}_\lambda$. To adjust the energy of the blow-up solution to be constructed, we define $\mathcal{C}_\lambda := \sqrt{\frac{\|yQ\|_2^2}{8E_0}}$.

Let L be defined by

$$L := 1 + \frac{K}{2}$$

Moreover, let s_0 be sufficiently large, $s_1 > s_0$, and

$$s' := \max\{s_0, \inf J_{s_1}\}.$$

Furthermore, we define s_* by

$$s_* := \inf\{\sigma \in (s', s_1] \mid (9) \text{ holds on } [\sigma, s_1]\},$$

where

$$\|\varepsilon(s)\|_{H^1}^2 + b(s)^2\|y\varepsilon(s)\|_2^2 < s^{-2L},$$
$$\left|\frac{\lambda(s)}{\lambda_{\mathrm{app}}(s)} - 1\right| + \left|\frac{b(s)}{b_{\mathrm{app}}(s)} - 1\right| < s^{-K/2}, \quad |w(s)| < s^{-L}. \tag{9}$$

Then, from the definitions of λ_{app} and b_{app}, the following estimate holds on $(s_*, s_1]$:

$$\lambda(s) \approx \lambda_{\mathrm{app}}(s) \approx s^{-1}, \quad b(s) \approx b_{\mathrm{app}}(s) \approx s^{-1}.$$

The goal of this section is to estimate Mod.

Lemma 4.1. *For* $s \in (s_*, s_1]$,

$$|(\mathrm{Im}\,\varepsilon(s), \nabla Q)_2| \lesssim s^{-(2L-1)}. \tag{10}$$

Proof. According to a direct calculation, we have

$$\frac{d}{dt}\mathrm{Im}\int_{\mathbf{R}^N} u(t,x)\nabla\overline{u}(t,x)\,dx$$
$$= \int_{\mathbf{R}^N}\left(-\frac{1}{1+\frac{2}{N}}\nabla g(x)|u(t,x)|^{2+4/N} + \frac{1}{2}\nabla W(x)|u(t,x)|^2\right)dx.$$

Since

$$\left|\lambda^2\int_{\mathbf{R}^N}\nabla g(x)|u(t(s),x)|^{2+4/N}\,dx\right| = \left|\int_{\mathbf{R}^N}\nabla g(\lambda y - w)|Q(y)+\varepsilon(s,y)|^{2+4/N}\,dy\right|$$
$$\lesssim \lambda^{1+K} + |w|^{1+K} + \|\varepsilon\|_{2+4/N}^{2+4/N},$$
$$\lambda^2\int_{\mathbf{R}^N}\nabla W(x)|u(t(s),x)|^2\,dx = \lambda^2\int_{\mathbf{R}^N}\nabla W(\lambda y - w)|Q(y)+\varepsilon(s,y)|^2\,dy,$$

$$\left|\lambda^2 \int_{\mathbf{R}^N} \nabla W(\lambda y - w) Q(y)^2 \, dy\right| \lesssim \frac{1}{\lambda}\|\Psi\|_{H^1},$$
$$\left|\lambda^2 \int_{\mathbf{R}^N} \nabla W_1(\lambda y - w)|\varepsilon(s,y)|^2 \, dy\right| \lesssim \lambda^2\|\varepsilon\|_2(\|\varepsilon\|_2 + b\|y\varepsilon\|_2),$$
$$\left|\lambda^2 \int_{\mathbf{R}^N} \nabla W_2(\lambda y - w)|\varepsilon(s,y)|^2 \, dy\right| \lesssim \lambda^{2-N/p_2}\|\varepsilon\|_{H^1}^2 + \lambda^2\|\varepsilon\|_2(\|\varepsilon\|_2 + b\|y\varepsilon\|_2),$$

we obtain

$$\left|\frac{d}{ds} \operatorname{Im} \int_{\mathbf{R}^N} u(t(s),x)\nabla\overline{u}(t(s),x) \, dx\right| \lesssim \lambda^2 \left|\frac{d}{dt} \operatorname{Im} \int_{\mathbf{R}^N} u(t,x)\nabla\overline{u}(t,x) \, dx\right| \lesssim s^{-(1+K)}.$$

Therefore, we obtain

$$\left|\operatorname{Im} \int_{\mathbf{R}^N} u(t(s),x)\nabla\overline{u}(t(s),x) \, dx\right| \lesssim s^{-K} = s^{-2(L-1)}.$$

The rest is shown in the same way as in [9, Lemma 3.2]. □

Lemma 4.2. (Estimation of modulation terms)*For* $s \in (s_*, s_1]$,

$$(\varepsilon(s), Q)_2 = -\frac{1}{2}\|\varepsilon(s)\|_2^2, \tag{11}$$
$$|\mathrm{Mod}(s)| \lesssim s^{-2L} \tag{12}$$

holds.

Proof. According to the mass conservation, we have

$$(\varepsilon, Q)_2 = \frac{1}{2}\left(\|u\|_2^2 - \|Q\|_2^2 - \|\varepsilon\|_2^2\right) = -\frac{1}{2}\|\varepsilon\|_2^2$$

meaning (11) holds.

For $v = \Lambda Q$, $i|y|^2 Q$, ρ, or $y_j Q$, the following estimates hold:

$$|(g(\lambda y - w) - 1) f(Q + \varepsilon)| \; |v| \lesssim (\lambda^{2+K} + |w|^{2+K})(Q + |\varepsilon|)|v|^{\frac{1}{2}}$$
$$|f(Q+\varepsilon) - f(Q) - df(Q)(\varepsilon)| \; |v| \lesssim |\varepsilon|^2,$$
$$|(\lambda^2 W(\lambda y - w)\varepsilon, v)_2| \lesssim \lambda^{1+K}(\lambda + |w|)\|\varepsilon\|_2.$$

Therefore, according to orthogonal conditions (5), Equation (8), Proposition 2.1, and (10), we see that

$$|\mathrm{Mod}(s)| \lesssim s^{-2L} + \varepsilon |\mathrm{Mod}(s)|.$$

Note that the constant omitted in the above inequality is independent of ε. For detail of the proof of the inequality, see [8, Lemma 4.1]. Consequently, we obtain (12). □

5. Modified energy function

In this section, we proceed with a modified version of the technique presented in Le Coz, Martel, and Raphaël [8] and Raphaël and Szeftel [16]. Let m and ε_j be defined by

$$
\begin{aligned}
&m := 2+\frac{K}{2}, \quad \varepsilon_1 := \frac{Km\mu}{32}, \quad \varepsilon_2 := \min\left\{\frac{\mu}{24}, \frac{K^2\mu}{24\times 64}\right\},\\
&\varepsilon_3 := \min\left\{\frac{m\mu}{24}, \frac{K^2 m\mu}{24\times 64}\right\}, \quad \varepsilon_4 := \frac{K}{8}
\end{aligned} \tag{13}
$$

where μ is from the coercivity (3) of L_+ and L_-. Moreover, we define

$$
\begin{aligned}
H(s,\varepsilon) :=& \frac{1}{2}\|\varepsilon\|_{H^1}^2 + \frac{\varepsilon_1 b^2}{2}\|y\varepsilon\|_2^2\\
&- \int_{\mathbf{R}^N} g(\lambda y - w)\left(F(Q(y)+\varepsilon(y)) - F(Q(y)) - dF(Q(y))(\varepsilon(y))\right)\, dy\\
&+ \frac{1}{2}\lambda^2 \int_{\mathbf{R}^N} W(\lambda y - w)|\varepsilon(y)|^2\, dy,\\
S(s,\varepsilon) :=& \frac{1}{\lambda^m} H(s,\varepsilon).
\end{aligned}
$$

Lemma 5.1. (Coercivity of H) *For $s\in(s_*, s_1]$,*

$$
H(s,\varepsilon) \geq \frac{\mu}{2}\|\varepsilon\|_{H^1}^2 + \frac{\varepsilon_1}{2} b^2\|y\varepsilon\|_2^2 - \varepsilon_2\left(\|\varepsilon\|_{H^1}^2 + b^2\|y\varepsilon\|_2^2\right)
$$

holds.

Proof. Firstly, we have

$$
\begin{aligned}
\left|\lambda^2\int_{\mathbf{R}^N} W_1(\lambda y - w)|\varepsilon|^2\, dy\right| &\lesssim \lambda^2\left(\|\varepsilon\|_{H^1}^2 + b^2\|y\varepsilon\|_2^2\right),\\
\left|\lambda^2\int_{\mathbf{R}^N} W_2(\lambda y - w)|\varepsilon|^2\, dy\right| &\lesssim \lambda^{2-N/p_1}\|\varepsilon\|_{H^1}^2 + \lambda^2\|\varepsilon\|_2^2.
\end{aligned}
$$

Secondly,

$$
\begin{aligned}
&\left|\int_{\mathbf{R}^N} g(\lambda y - w)\left(F(Q+\varepsilon) - F(Q) - dF(Q)(\varepsilon) - \frac{1}{2}d^2F(Q)(\varepsilon,\varepsilon)\right)\, dy\right|\\
&\lesssim \|\varepsilon\|_{H^1}^3 + \|\varepsilon\|_{H^1}^{2+4/N}.
\end{aligned}
$$

Thirdly,

$$
\left|\int_{\mathbf{R}^N} (g(\lambda y - w) - 1)\, d^2F(Q)(\varepsilon,\varepsilon)\, dy\right| \lesssim s^{-(2+K)}\|\varepsilon\|_{H^1}^2.
$$

Finally, from (3), (5), and (11) since

$$\|\varepsilon\|_{H^1}^2 - \int_{\mathbf{R}^N} d^2F(Q)(\varepsilon,\varepsilon)\ dy = (L_+ \operatorname{Re}\varepsilon, \operatorname{Re}\varepsilon)_2 + (L_- \operatorname{Im}\varepsilon, \operatorname{Im}\varepsilon)_2\,,$$

we have

$$H(s,\varepsilon) \geq \frac{\mu}{2}\|\varepsilon\|_{H^1}^2 + \frac{\varepsilon_1 b^2}{2}\|y\varepsilon\|_2^2 - \varepsilon_2\left(\|\varepsilon\|_2^2 + b^2\|y\varepsilon\|_2^2\right).$$

Consequently, we obtain Lemma 5.1. □

Corollary 5.2. (Estimation of S) *For* $s\in(s_*, s_1]$,

$$\frac{1}{\lambda^m}\left(\|\varepsilon\|_{H^1}^2 + b^2\|y\varepsilon\|_2^2\right) \lesssim S(s,\varepsilon) \lesssim \frac{1}{\lambda^m}\left(\|\varepsilon\|_{H^1}^2 + b^2\|y\varepsilon\|_2^2\right)$$

holds.

Lemma 5.3. *For all* $s\in(s_*, s_1]$,

$$|(g(\lambda y - w)\left(f(Q+\varepsilon) - f(Q)\right), \Lambda\varepsilon)_2| \lesssim \|\varepsilon\|_{H^1}^2, \tag{14}$$

$$|(g(\lambda y - w)\left(f(Q+\varepsilon) - f(Q)\right), \nabla\varepsilon)_2| \lesssim \|\varepsilon\|_{H^1}^2, \tag{15}$$

$$\left|\lambda^2\left(W(\lambda y - w)\varepsilon, \Lambda\varepsilon\right)_2\right| \lesssim s^{-1}\left(\|\varepsilon\|_{H^1}^2 + b^2\|y\varepsilon\|_2^2\right), \tag{16}$$

$$\left|\lambda^2\left(W(\lambda y - w)\varepsilon, \nabla\varepsilon\right)_2\right| \lesssim s^{-1}\left(\|\varepsilon\|_{H^1}^2 + b^2\|y\varepsilon\|_2^2\right). \tag{17}$$

Proof. For (16) and (17), see [9].

Firstly,

$$\begin{aligned}
&\nabla\left(g(\lambda y - w)\left(F(Q+\varepsilon) - F(Q) - dF(Q)(\varepsilon)\right)\right)\\
=&\lambda(\nabla g)(\lambda y - w)\left(F(Q+\varepsilon) - F(Q) - dF(Q)(\varepsilon)\right)\\
&\quad + g(\lambda y - w)\operatorname{Re}\left(f(Q+\varepsilon) - f(Q) - df(Q)(\varepsilon)\right)\nabla Q\\
&\quad\quad + g(\lambda y - w)\operatorname{Re}\left(\left(f(Q+\varepsilon) - f(Q)\right)\nabla\overline{\varepsilon}\right)
\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}
&g(\lambda y - w)\operatorname{Re}\left(\left(f(Q+\varepsilon) - f(Q)\right)\Lambda\overline{\varepsilon}\right)\\
=&\frac{N}{2}g(\lambda y - w)\operatorname{Re}\left(\left(f(Q+\varepsilon) - f(Q)\right)\overline{\varepsilon}\right)\\
&\quad + y\cdot\nabla\left(g(\lambda y - w)\left(F(Q+\varepsilon) - F(Q) - dF(Q)(\varepsilon)\right)\right)\\
&\quad\quad - w\cdot(\nabla g)(\lambda y - w)\left(F(Q+\varepsilon) - F(Q) - dF(Q)(\varepsilon)\right)\\
&\quad\quad\quad - (\lambda y - w)\cdot(\nabla g)(\lambda y - w)\left(F(Q+\varepsilon) - F(Q) - dF(Q)(\varepsilon)\right)\\
&\quad\quad\quad\quad + g(\lambda y - w)\operatorname{Re}\left(f(Q+\varepsilon) - f(Q) - df(Q)(\varepsilon)\right)y\cdot\nabla Q.
\end{aligned}$$

Therefore, we obtain

$$\left|\left(g(\lambda y-w)\left(f(Q+\varepsilon)-f(Q)\right),\Lambda\varepsilon\right)_2\right|\lesssim\|\varepsilon\|_{H^1}^2$$

so that (14) holds. (15) is also shown by similar calculations. □

Lemma 5.4. (Derivative of H in time) *For all* $s\in(s_*,s_1]$,

$$\frac{d}{ds}H(s,\varepsilon(s))\geq -b\left(\left(\frac{\varepsilon_1}{\varepsilon_4}+\varepsilon_3\right)\|\varepsilon\|_{H^1}^2+\left(1+\frac{\varepsilon_3}{\varepsilon_1}+\varepsilon_4\right)\varepsilon_1 b^2\|y\varepsilon\|_2^2+Cs^{-(2+K)}\right).$$

Proof. Outline the proofs. See [8] for details.
Firstly, we have

$$\frac{d}{ds}H(s,\varepsilon(s))=\frac{\partial H}{\partial s}(s,\varepsilon(s))+\left\langle i\frac{\partial H}{\partial\varepsilon},i\frac{\partial\varepsilon}{\partial s}\right\rangle.$$

Secondly, we have

$$\begin{aligned}\frac{\partial H}{\partial\varepsilon}&=-\Delta\varepsilon+\varepsilon+\varepsilon_1b^2|y|^2\varepsilon-g(\lambda y-w)(f(Q+\varepsilon)-f(Q))+\lambda^2W(\lambda y-w)\varepsilon\\&=L_+\operatorname{Re}\varepsilon+iL_-\operatorname{Im}\varepsilon+\varepsilon_1b^2|y|^2\varepsilon-(g(\lambda y-w)-1)\ df(Q)(\varepsilon)\\&\qquad-g(\lambda y-w)(f(Q+\varepsilon)-f(Q)-df(Q)(\varepsilon))+\lambda^2W(\lambda y-w)\varepsilon,\\i\frac{\partial\varepsilon}{\partial s}&=\frac{\partial H}{\partial\varepsilon}-\varepsilon_1b^2|y|^2\varepsilon-(g(\lambda y-w)-1)f(Q)+\operatorname{Mod}_{\mathrm{op}}(Q+\varepsilon)+\Psi,\end{aligned}$$

where

$$\begin{aligned}\operatorname{Mod}_{\mathrm{op}}v:=\ &i\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial s}+b\right)\Lambda v-\left(1-\frac{\partial\gamma}{\partial s}\right)v-\left(\frac{\partial b}{\partial s}+b^2\right)\frac{|y|^2}{4}v\\&+\left(\frac{1}{\lambda}\frac{\partial\lambda}{\partial s}+b\right)b\frac{|y|^2}{2}v-i\frac{1}{\lambda}\frac{\partial w}{\partial s}\cdot\nabla v-\frac{1}{2}\frac{b}{\lambda}\frac{\partial w}{\partial s}\cdot yv.\end{aligned}$$

For $\frac{\partial H}{\partial s}$, we have

$$\begin{aligned}\frac{\partial H}{\partial s}=\ &\varepsilon_1b\frac{\partial b}{\partial s}\|y\varepsilon\|_2^2-\int_{\mathbf{R}^N}\left(\frac{\partial\lambda}{\partial s}y-\frac{\partial w}{\partial s}\right)\\&\cdot(\nabla g)(\lambda y-w)\left(F(Q+\varepsilon)-F(Q)-dF(Q)(\varepsilon)\right)\ dy\\&+\lambda^2\frac{1}{\lambda}\frac{\partial\lambda}{\partial s}\int_{\mathbf{R}^N}W(\lambda y-w)|\varepsilon|^2\ dy\\&+\frac{1}{2}\lambda^2\int_{\mathbf{R}^N}\left(\frac{\partial\lambda}{\partial s}y-\frac{\partial w}{\partial s}\right)\cdot(\nabla W)(\lambda y-w)|\varepsilon|^2\ dy.\end{aligned}$$

Therefore, we obtain

$$\frac{\partial H}{\partial s} \geq -\varepsilon_1 b^3 \|y\varepsilon\|_2^2 + o\left(b\left(\|\varepsilon\|_{H^1} + b^2\|y\varepsilon\|_2^2\right)\right). \tag{18}$$

For $\langle i\frac{\partial H}{\partial \varepsilon}, i\frac{\partial \varepsilon}{\partial s}\rangle$, the following estimates hold:

$$\left|\left\langle i\frac{\partial H}{\partial \varepsilon}, \varepsilon_1 b^2 |y|^2 \varepsilon\right\rangle\right| \leq 2\varepsilon_1 b^2 \|\varepsilon\|_{H^1} \|y\varepsilon\|_2 + o(b\|\varepsilon\|_{H^1}^2), \tag{19}$$

$$\left|\left\langle i\frac{\partial H}{\partial \varepsilon}, (g(\lambda y - w) - 1) f(Q)\right\rangle\right| \lesssim s^{-(2+K+L)}, \tag{20}$$

$$\left|\left\langle i\frac{\partial H}{\partial \varepsilon}, \Psi\right\rangle\right| \lesssim s^{-(2+K+L)}, \tag{21}$$

$$\left|\left\langle i\frac{\partial H}{\partial \varepsilon}, \mathrm{Mod}_{\mathrm{op}}\, Q\right\rangle\right| \lesssim s^{-(4L-1)}, \tag{22}$$

$$\left|\left\langle i\frac{\partial H}{\partial \varepsilon}, \mathrm{Mod}_{\mathrm{op}}\, \varepsilon\right\rangle\right| \lesssim s^{-(4L-1)}. \tag{23}$$

Combining inequalities (18), (19), (20), (21), (22), and (23), we obtain

$$\begin{aligned}
&\frac{d}{ds} H(s,\varepsilon(s)) = \frac{\partial H}{\partial s}(s,\varepsilon(s)) + \left\langle i\frac{\partial H}{\partial \varepsilon}, i\frac{\partial \varepsilon}{\partial s}\right\rangle \\
&\geq -\varepsilon_1 b^3 \|y\varepsilon\|_2^2 + o\left(b\left(\|\varepsilon\|_{H^1}^2 + b^2\|y\varepsilon\|_2^2\right)\right) - 2\varepsilon_1 b^2 \|\varepsilon\|_{H^1}\|y\varepsilon\|_2 + o(b\|\varepsilon\|_{H^1}^2) \\
&\qquad - C\left(s^{-(2+K+L)} + s^{-(4L-1)}\right) \\
&\geq -\varepsilon_1 b^3 \|y\varepsilon\|_2^2 - 2\varepsilon_1 b^2 \|\varepsilon\|_{H^1}\|y\varepsilon\|_2 - \varepsilon_3 b\left(\|\varepsilon\|_{H^1}^2 + b^2\|y\varepsilon\|_2^2\right) - bC\left(s^{-(2+\frac{3K}{2})} + s^{-(2+K)}\right) \\
&\geq -b\left(\left(\frac{\varepsilon_1}{\varepsilon_4} + \varepsilon_3\right)\|\varepsilon\|_{H^1}^2 + \left(1 + \frac{\varepsilon_3}{\varepsilon_1} + \varepsilon_4\right)\varepsilon_1 b^2 \|y\varepsilon\|_2^2 + Cs^{-(2+K)}\right). \quad \square
\end{aligned}$$

Lemma 5.5. (Derivative of S in time) *For all* $s \in (s_*, s_1]$,

$$\frac{d}{ds} S(s,\varepsilon(s)) \gtrsim \frac{b}{\lambda^m}\left(\|\varepsilon\|_{H^1}^2 + b^2\|y\varepsilon\|_2^2 - Cs^{-(2+K)}\right).$$

Proof. According to Lemma 5.1, Lemma 5.4, and (12), we have

$$\begin{aligned}
\frac{d}{ds} S(s,\varepsilon(s)) &= m\frac{b}{\lambda^m} H(s,\varepsilon(s)) - m\frac{1}{\lambda^m}\left(\frac{1}{\lambda}\frac{\partial \lambda}{\partial s} + b\right) H(s,\varepsilon(s)) + \frac{1}{\lambda^m}\frac{d}{ds} H(s,\varepsilon(s)) \\
&\geq \frac{b}{\lambda^m}\left(\left(\frac{m\mu}{2} - \frac{\varepsilon_1}{\varepsilon_4} - \varepsilon_2 m - 2\varepsilon_3\right)\|\varepsilon\|_{H^1}^2\right. \\
&\qquad \left. + \varepsilon_1\left(\frac{m}{2} - 1 - \frac{\varepsilon_2 m}{\varepsilon_1} - \frac{2\varepsilon_3}{\varepsilon_1} - \varepsilon_4\right) b^2 \|y\varepsilon\|_2^2 - Cs^{-(2+K)}\right)
\end{aligned}$$

From (13),

$$\frac{m\mu}{2}-\frac{\varepsilon_1}{\varepsilon_4}-\varepsilon_2 m-2\varepsilon_3 \geq \frac{m\mu}{2}-\frac{m\mu}{4}-\frac{m\mu}{24}-\frac{m\mu}{12}=\frac{m\mu}{8}$$
$$\frac{m}{2}-1-\frac{\varepsilon_2 m}{\varepsilon_1}-\frac{2\varepsilon_3}{\varepsilon_1}-\varepsilon_4 \geq \frac{K}{4}-\frac{K}{24\times 2}-\frac{K}{24}-\frac{K}{8}=\frac{K}{16}$$

hold. □

6. Bootstrap

In this section, we establish the estimates of the decomposition parameters by using a bootstrap argument and the estimates obtained in Section 5.

Lemma 6.1. *There exists a sufficiently small $\varepsilon_2>0$ such that for all $s\in(s_*, s_1]$,*

$$\|\varepsilon(s)\|_{H^1}^2+b(s)^2\,\|y\varepsilon(s)\|_2^2 \lesssim s^{-(2L+K/2)}, \tag{24}$$

$$\left|\frac{\lambda(s)}{\lambda_{\mathrm{app}}(s)}-1\right|+\left|\frac{b(s)}{b_{\mathrm{app}}(s)}-1\right| \lesssim s^{-2(L-1)}, \tag{25}$$

$$|w(s)| \lesssim s^{-(2L-1)}. \tag{26}$$

Proof. See [8] for the proof of (24).

From Proposition 2.3 and (12),

$$|E(Q_{\lambda,b,w,\gamma})-E_0| \leq \int_s^{s_1}\left|\left.\frac{d}{d\sigma}\right|_{\sigma=\tau} E(Q_{\lambda,b,w,\gamma}(\sigma))\right|\,d\tau \lesssim \int_s^{s_1}\tau^{-(1+K)}\,d\tau \lesssim s^{-K}$$

holds. Therefore, since

$$\begin{aligned}\left|b^2\|yQ\|_2^2-8\lambda^2E_0\right| &\leq \lambda^2\left(\left|\frac{b^2}{\lambda^2}\|yQ\|_2^2-8E(P_{\lambda,b,\gamma})\right|+8\left|E_0-E(P_{\lambda,b,\gamma})\right|\right)\\ &\lesssim s^{-(2+K)},\end{aligned}$$

we obtain

$$\begin{aligned}\left|\frac{\partial}{\partial s}\left(\sqrt{\frac{\|yQ\|_2^2}{8E_0}}\frac{1}{\lambda}-s\right)\right| &\leq \left|-\sqrt{\frac{\|yQ\|_2^2}{8E_0}}\frac{1}{\lambda^2}\frac{\partial\lambda}{\partial s}-1\right|\\ &\lesssim \frac{1}{\lambda}\left(\left|\frac{1}{\lambda}\frac{\partial\lambda}{\partial s}+b\right|+\left|b\|yQ\|-\sqrt{8E_0}\lambda\right|\right)\\ &\lesssim s^{-(2L-1)}+s^{-(1+K)}.\end{aligned}$$

Since $\sqrt{\frac{\|yQ\|_2^2}{8E_0}}\frac{1}{\lambda(s_1)}=s_1$, we obtain

$$\left|\sqrt{\frac{\|yQ\|_2^2}{8E_0}}\frac{1}{\lambda}-s\right|\lesssim s^{-2(L-1)},\quad \text{i.e.,}\quad \left|\frac{\lambda_{\mathrm{app}}(s)}{\lambda(s)}-1\right|\lesssim s^{-(2L-1)}.$$

Next, since

$$\left|b^2-b_{\mathrm{app}}{}^2\right|=\left|b^2-\frac{8E_0}{\|yQ\|_2^2}\lambda_{\mathrm{app}}{}^2\right|\lesssim\left|b^2-\frac{8E_0}{\|yQ\|_2^2}\lambda^2\right|+\left|\lambda^2-\lambda_{\mathrm{app}}{}^2\right|\lesssim s^{-(2+K)}+s^{-2L},$$

we obtain (25).

Finally, we prove (26). Since

$$|w(s)|\le\int_s^{s_1}|\operatorname{Mod}(\sigma)|\,d\sigma\lesssim s^{-(2L-1)},$$

(26) holds. □

From Lemma 6.1, we obtain the following corollary:

Corollary 6.2. *If s_0 is sufficiently large, then $s_*=s'=s_0$ for any $s_1>s_0$.*

Finally, we rewrite the estimates obtained for the time variable s in Lemma 6.1 into an estimates for the time variable t.

Lemma 6.3. (Interval) *If s_0 is sufficiently large, then there exists $t_0<0$ such that*

$$[t_0,t_1]\subset s_{t_1}{}^{-1}([s_0,s_1]),\quad \left|\mathcal{C}s_{t_1}(t)^{-1}-|t|\right|\lesssim|t|^{1+K}\ (t\in[t_0,t_1])$$

hold for $t_1\in(t_0,0)$, where $\mathcal{C}=\frac{\|yQ\|_2^2}{8E_0}$.

Proof. See [9] for the proof. □

Finally, we prove Lemma 3.1.

Proof of Lemma 3.1. Firstly, we define

$$\varepsilon_{\tilde{\lambda},t_1}(t):=\frac{\sqrt{\mathcal{C}}\tilde{\lambda}_{t_1}(t)}{|t|}-1.$$

From Lemma 6.3 and $\lambda_{\mathrm{app}}(s)=\sqrt{\mathcal{C}}s^{-1}$, we have

$$\left|\varepsilon_{\tilde{\lambda},t_1}(t)\right|=\left|\left(\frac{s_{t_1}(t)\tilde{\lambda}_{t_1}(t)}{\sqrt{\mathcal{C}}}-1\right)\frac{\mathcal{C}}{s_{t_1}(t)|t|}+\frac{\mathcal{C}}{s_{t_1}(t)|t|}-1\right|\lesssim|t|^K.$$

The same can be proved for $\tilde{b}_{t_1}$ and $\widetilde{w}_{t_1}$. □

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