Ark. Mat., **61** (2023), 343–374
DOI: 10.4310/ARKIV.2023.v61.n2.a5
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The spectral picture of Bergman-Toeplitz operators with harmonic polynomial symbols

Kunyu Guo, Xianfeng Zhao and Dechao Zheng

Abstract. This paper shows some new phenomenon in the spectral theory of Toeplitz operators on the Bergman space, which is considerably different from that of Toeplitz operators on the Hardy space. On the one hand, we prove that the spectrum of the Toeplitz operator with symbol $\overline{z}+p$ is always connected for every polynomial p with degree less than 3. On the other hand, we show that for each integer k greater than 2, there exists a polynomial p of degree k such that the spectrum of the Toeplitz operator with symbol $\overline{z}+p$ is a nonempty finite set. Then these results are applied to obtain a new class of non-hyponormal Toeplitz operators with bounded harmonic symbols on the Bergman space for which Weyl's theorem holds.

Contents

1. Introduction

Let dA denote the Lebesgue measure on the open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$, normalized so that the measure of the disk $\mathbb{D}$ is 1. The complex space $L^2(\mathbb{D}, dA)$ is a Hilbert space with the inner product

$$\langle f, g\rangle = \int_{\mathbb{D}} f(z)\overline{g(z)}\, dA(z).$$

Key words and phrases: Bergman space, Toeplitz operator, harmonic polynomial, spectrum.
2010 Mathematics Subject Classification: primary 47B35; secondary 47B38, 47A10.

The Bergman space L_a^2 is the set of those functions in $L^2(\mathbb{D}, dA)$ that are analytic on $\mathbb{D}$. Thus the Bergman space is a closed subspace of $L^2(\mathbb{D}, dA)$ and so there is an orthogonal projection P from $L^2(\mathbb{D}, dA)$ onto L_a^2. For $\varphi \in L^\infty(\mathbb{D}, dA)$, the Toeplitz operator T_φ with symbol φ on the Bergman space (or "Bergman-Toeplitz operator") is defined by

$$T_\varphi f = P(\varphi f)$$

for f in the Bergman space L_a^2.

In general, the behaviour of these operators may be quite different from that of the Toeplitz operators on the Hardy space. For example, there are many nontrivial compact Toeplitz operators on the Bergman space [3], [4], [23] and [29]. But there is not a nontrivial compact Toeplitz operator on the Hardy space [11]. On the other hand, there is not a nontrivial compact Toeplitz operator with a bounded harmonic symbol on the Bergman space [4] and the harmonic extension gives a natural corresponding relationship between the bounded functions on the unit circle and the bounded harmonic functions on the unit disk [12]. Thus Toeplitz operators on the Bergman space with harmonic symbols behave quite similarly to those on the Hardy space [15]. As a fundamental problem concerning Toeplitz operators is to determine the spectra in terms of the properties of their symbols, it is natural to study the spectra of Toeplitz operators with bounded harmonic symbols on the Bergman space.

An important result about Toeplitz operators on the Hardy space is the Widom theorem [26] and [27], which states that the spectrum of a bounded Toeplitz operator is always connected. Moreover, by means of some techniques in complex analysis and differential equation, Douglas showed that the essential spectrum of a bounded Toeplitz operator on the Hardy space is also connected [11, Theorem 7.45]. On the one hand, as there are many nontrivial compact Toeplitz operators on the Bergman space, one can easily construct a Toeplitz operator whose spectrum has isolated points and hence is disconnected. On the other hand, McDonald and Sundberg showed in [15] that the spectrum and essential spectrum of T_φ on the Bergman space are both connected for φ a bounded and real-valued harmonic function on $\mathbb{D}$. Moreover, they also showed that the essential spectrum of T_φ is connected if φ is harmonic on $\mathbb{D}$ and piecewise continuous on the boundary of the disk $\mathbb{D}$. These suggested the conjecture that a Toeplitz operator on the Bergman space with harmonic symbol has a connected spectrum [14, p. 320]. In 1979, McDonald and Sundberg asked the question in [15] whether the essential spectrum of a Toeplitz operator with bounded harmonic symbol is connected. About 30 years later, Sundberg and the third author gave a negative answer to the above question and disproved the above conjecture in [25]. Indeed, they first constructed a rational function q on $\mathbb{D}$ via two conformal mappings and showed that the spectrum of the

Toeplitz operator $T_{\overline{z}+q}$ has at least one isolated point and hence is disconnected. Based on a characterization of the essential spectrum for a certain Toeplitz algebra established in [24] and some techniques in function algebra theory, the authors used the function q quoted above to construct a bounded harmonic function h such that the essential spectrum of the Toeplitz operator T_h also has an isolated point [25]. Despite considerable effort devoted to studying more about the invertibility and spectra of Bergman-Toeplitz operators with harmonic symbols, little progress has been made on this topic in recent 10 years.

An especially important but quite difficult problem in operator theory is to determine the spectrum of a bounded linear operator. Note that the spectral structure is closely related to the invariant subspaces of bounded linear operators. Indeed, for each bounded linear operator T with disconnected spectrum, we can apply the Riesz decomposition theorem to construct a hyperinvariant subspace of T, see [5, Lemma 1.19] or Proposition 4.11 of Chapter VII in [9]. However, there is little characterization for the topological structure of the spectrum of the Toeplitz operator with a bounded harmonic symbol, even if the symbol is the harmonic function $\overline{z}+p$ for an analytic polynomial p. In this paper, we will investigate the structure of the spectrum of the Toeplitz operator $T_{\overline{z}+p}$ via certain analytic properties of polynomials. The main idea is to show how eigenvalues of $T_{\overline{z}+p}$ depend on p by solving the first order complex differential equation for eigenvectors. But nontrivial solutions of the differential equation in the Bergman space will be subject to zeros in the unit disk of some analytic polynomials. In order to estimate the modulus of these complex zeros, our approach here is to use the theorem of the zeros of a polynomial depend continuously on its coefficients [17] and [19].

For a function φ bounded and analytic on the unit disk $\mathbb{D}$, it is well-known that the spectrum of the Toeplitz operator T_φ equals the closure of the image of the unit disk under φ. In [28], it was shown that if the symbol φ is an affine function of z and $\overline{z}$, then it is also true that the spectrum of T_φ equals the closure of the image of $\mathbb{D}$ under φ, and hence it is a connected set. In this paper, we obtain a characterization on the point spectra of Toeplitz operators with certain bounded harmonic symbols, and then establish a necessary and sufficient condition for this class of Toeplitz operators to be invertible on the Bergman space, see Theorems 2.4 and 2.5 in the next section. On the one hand, we prove in Theorem 3.1 that the spectrum of the Bergman Toeplitz operator $T_{\overline{z}+p}$ is connected for every polynomial p with degree less than or equal to 2. On the other hand, for each integer k greater than 2, we construct an analytic polynomial p with degree k such that the spectrum of the Toeplitz operator with symbol $\overline{z}+p$ has at least one isolated point but has at most finitely many isolated points, see Theorems 4.1 and 4.3 for the details. The significance of further discussing the spectral structure of these operators is that isolated spectral points are closely related to their nontrivial invariant subspaces and hypercyclicity [5].

In addition, for a bounded linear operator on a Hilbert space, the topological structure of the spectrum plays an important role in the study of its *Weyl spectrum* (which will be introduced in Section 5), see [6]–[8] and [16] for the classical results. It is known that the Weyl spectra of every hermitian operator and every normal operator consist precisely of all points in the spectra except the isolated eigenvalues of finite geometric multiplicity. "Weyl's theorem for an operator" was first introduced by Coburn [8] in 1966, which says that the complement in the spectrum of the Weyl spectrum coincides with the isolated points of the spectrum which are eigenvalues of finite geometric multiplicity. Moreover, Coburn showed that Weyl's theorem holds for all hyponormal operators and Hardy-Toeplitz operators [8]. Weyl type theorems with respect to isolated points of the spectrum of an operator were investigated for many cases and many classes of operators. Based on the characterizations for the spectra of Toeplitz operators in Theorems 2.4 and 4.1, we show in Theorem 5.2 that the Bergman-Toeplitz operator $T_{\overline{z}+q}$ satisfies Weyl's theorem, where q is an arbitrary function in the disk algebra $H^\infty \cap C(\overline{\mathbb{D}})$.

2. Preliminary

In this section, we first introduce some notations and include some lemmas. As usual, we use $\sigma(T_\varphi)$, $\sigma_p(T_\varphi)$ and $\sigma_e(T_\varphi)$ to denote the spectrum, point spectrum (or the set of eigenvalues) and essential spectrum of the Toeplitz operator T_φ, respectively. Let $\mathbb{N}$ denote the set of nonnegative integers. The following lemma is useful for us to look for eigenvalues of Toeplitz operators with some harmonic symbols [25, Lemma 2.1].

Lemma 2.1. *For each function $f \in L_a^2$, we have*

$$T_{\overline{z}} f(z) = \frac{1}{z^2} \int_0^z w f'(w)\, dw.$$

The next lemma is about the Fredholm theory of Toeplitz operators with continuous symbols on the Bergman space [22] and [30].

Lemma 2.2. *Suppose that $\varphi \in C(\overline{\mathbb{D}})$. Then the essential spectrum of the Toeplitz operator T_φ is given by*

$$\sigma_e(T_\varphi) = \varphi(\partial\mathbb{D}).$$

Moreover, if T_φ is a Fredholm operator, then the Fredholm index of T_φ is given by

$$\operatorname{index}(T_\varphi) = \dim\,\ker(T_\varphi) - \dim\,\ker(T_\varphi^*) = -\operatorname{wind}\big(\varphi(\partial\mathbb{D}), 0\big),$$

where $\operatorname{wind}\big(\varphi(\partial\mathbb{D}),0\big)$ *is the winding number of the closed oriented curve* $\varphi(\partial\mathbb{D})$ *with respect to the origin, which is defined by*

$$\operatorname{wind}\big(\varphi(\partial\mathbb{D}),0\big)=\frac{1}{2\pi\mathrm{i}}\int_{\varphi(\partial\mathbb{D})}\frac{dz}{z}.$$

We will use the following spectral picture theorem [18, Proposition 1.27] to analyze isolated points in the spectrum of a Toeplitz operator on the Bergman space.

Theorem 2.3. (Pearcy) *Let* T *be a bounded linear operator on a Hilbert space* $\mathscr{H}$ *and* H *be "a hole in* $\sigma_e(T)$*" (which is a bounded component of* $\mathbb{C}\backslash\sigma_e(T)$*) such that*

$$\operatorname{index}(T-\lambda I)=0,\qquad \lambda\in H,$$

then either

(a) $H\cap\sigma(T)=\varnothing$,

(b) $H\subset\sigma(T)$, *or*

(c) $H\cap\sigma(T)$ *is a countable set of isolated eigenvalues of* T*, each having finite multiplicity.*

Furthermore, the intersection of $\sigma(T)$ *with the unbounded component of* $\mathbb{C}\backslash\sigma_e(T)$ *is a countable set of isolated eigenvalues of* T*, each of which has finite multiplicity.*

The following theorem gives a characterization for the eigenvalues of a class of Toeplitz operators with harmonic symbols on the Bergman space, which is useful for us to study the isolated points in the spectra of Toeplitz operators with some bounded harmonic symbols.

Theorem 2.4. *Let* p *be a function in* $H^\infty\cap C(\overline{\mathbb{D}})$*. Suppose that* λ *is a complex number not in the essential spectrum of the Toeplitz operator* $T_{\overline{z}+p}$*. Then* λ *is an eigenvalue of* $T_{\overline{z}+p}$ *if and only if either* $1+z[p(z)-\lambda]$ *does not vanish on the unit disk or* $1+z[p(z)-\lambda]$ *has finitely many simple zeros* $\big\{z_1,\dots,z_k\big\}$ *in the unit disk which satisfy*

$$z_j^2p'(z_j)=\frac{n_j+2}{n_j+1}\tag{1}$$

for some integer $n_j\in\big\{0,1,2,\dots\big\}$ *with* $j=1,2,\dots,k$*.*

Proof. Let λ be a complex number which is not contained in the essential spectrum of the Toeplitz operator $T_{\overline{z}+p}$. We have by Lemma 2.2 that $1+z[p(z)-\lambda]$ does not vanish on the unit circle $\partial\mathbb{D}$, which yields that $1+z[p(z)-\lambda]$ has at most finitely many zeros in the open unit disk $\mathbb{D}$.

Clearly, λ is an eigenvalue of $T_{\overline{z}+p}$ if and only if

$$(T_{\overline{z}+p}-\lambda)f=0$$

for some nonzero function f in the Bergman space L_a^2. By Lemma 2.1, we have the following integral equation:

$$\frac{1}{z^2}\int_0^z wf'(w)\,dw=[\lambda-p(z)]f(z).$$

Multiplying both sides of the above equation by z^2 and then taking derivatives give the following first order differential equation:

$$\big(1+z[p(z)-\lambda]\big)f'(z)=-\big(2[p(z)-\lambda]+zp'(z)\big)f(z). \tag{2}$$

Therefore, the above arguments show that $\lambda\in\sigma_p(T_{\overline{z}+p})$ if and only if λ satisfies Equation (2) for some $f\neq 0$ in the Bergman space.

Let us first show the necessity. Suppose that λ is an eigenvalue of the Toeplitz operator $T_{\overline{z}+p}$. If $1+z[p(z)-\lambda]$ has no zeros in the unit disk, then we obtain the desired conclusion. Otherwise, we may assume that the zeros of $1+z[p(z)-\lambda]$ in $\mathbb{D}$ are given by $\{z_1,z_2,...,z_k\}$.

We first show that $z_1,z_2,...,z_k$ are all simple zeros of $1+z[p(z)-\lambda]$. If some of them, e.g., z_j $(j\in\{1,2,...,k\})$ is a multiple zero, using

$$\begin{cases} 1+z_j[p(z_j)-\lambda]=0,\\ p(z_j)-\lambda+z_jp'(z_j)=0, \end{cases}$$

we get

$$z_j^2p'(z_j)=1,$$

to obtain

$$2z_j[p(z_j)-\lambda]+z_j^2p'(z_j)=-2+1=-1.$$

Combining this with (2) gives us $f(z_j)=0$. Taking derivatives on both sides of (2) gives

$$(p-\lambda+zp')f'+\big[1+z(p-\lambda)\big]f''=-\big(3p'+zp''\big)f-\big[2(p-\lambda)+zp'\big]f',$$

to get $f'(z_j)=0$, since we have

$$\begin{cases} f(z_j)=0,\\ 1+z_j[p(z_j)-\lambda]=0,\\ p(z_j)-\lambda+z_jp'(z_j)=0,\\ 2[p(z_j)-\lambda]+z_jp'(z_j)=-\dfrac{1}{z_j}. \end{cases}$$

Thus we can write

$$f(z) = (z - z_j)^{n_j} g_j(z)$$

for some integer $n_j > 1$ and $g_j \in L_a^2$ with $g(z_j) \neq 0$. According to the above expression of f, we get

$$-\frac{f'(z)}{f(z)} = \frac{-n_j}{z - z_j} + \frac{-g_j'(z)}{g_j(z)},$$

which gives that $-f'/f$ has a pole z_j with order at most 1. On the other hand, note that the function

$$\frac{2[p(z) - \lambda] + zp'(z)}{1 + z[p(z) - \lambda]}$$

has a pole z_j with order greater than 1, since z_j is a multiple zero of $1 + z[p(z) - \lambda]$ (by the assumption) but not a zero of $2[p(z) - \lambda] + zp'(z)$. This implies that λ can not satisfy the following equation:

$$-\frac{f'(z)}{f(z)} = \frac{2[p(z) - \lambda] + zp'(z)}{1 + z[p(z) - \lambda]},$$

which contradicts that λ is an eigenvalue of $T_{\overline{z} + p}$. Thus $z_1, z_2, ..., z_k$ are all simple zeros of $1 + z[p(z) - \lambda]$ in the unit disk.

In order to derive the condition for z_j in (1), we first consider the case of

$$2[p(z_j) - \lambda] + z_j p'(z_j) \neq 0,$$

where $z_j \in \{z_1, z_2, ..., z_k\}$. Since the function f is analytic on $\mathbb{D}$, we can choose a small circle $\gamma \subset \mathbb{D}$ centered at z_j such that there is no zero of f on γ and all other zeros of $1 + z[p(z) - \lambda]$ are outside γ. Then the Cauchy residue theorem implies that there exists some $n_j \in \{0, 1, 2, ...\}$ such that

$$\begin{aligned}
-n_j &= -\frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{f'(z)}{f(z)}\, dz \\
&= \frac{1}{2\pi \mathrm{i}} \int_\gamma \frac{2[p(z) - \lambda] + zp'(z)}{1 + z[p(z) - \lambda]}\, dz \quad \big(\text{by Equation (2)}\big) \\
&= \mathrm{Res}\left(\frac{2[p(z) - \lambda] + zp'(z)}{1 + z[p(z) - \lambda]};\ z_j \right) \\
&= \frac{2p(z_j) - 2\lambda + z_j p'(z_j)}{p(z_j) - \lambda + z_j p'(z_j)} \\
&= 1 + \frac{1}{1 + \frac{z_j}{p(z_j) - \lambda} p'(z_j)} \\
&= 1 + \frac{1}{1 - z_j^2 p'(z_j)},
\end{aligned}$$

where the fourth equality follows from that $2[p(z_j)-\lambda]+z_jp'(z_j)\neq 0$ and z_j is a simple zero of $1+z[p(z)-\lambda]$, and the last equality comes from that

$$p(z_j)-\lambda=-\frac{1}{z_j}.$$

Solving the above equation gives

$$z_j^2p'(z_j)=\frac{n_j+2}{n_j+1},$$

to obtain (1).

For the case of

$$2[p(z_j)-\lambda]+z_jp'(z_j)=0,$$

we have following system:

$$\begin{cases}2[p(z_j)-\lambda]+z_jp'(z_j)=0,\\ 1+z_j[p(z_j)-\lambda]=0,\end{cases}$$

which implies that

$$z_j^2p'(z_j)=2.$$

In conclusion, we obtain (1) for $n_j=0$, to finish the proof of the necessity.

Now we turn to the proof of the sufficiency, we need to show $\lambda\in\sigma_p(T_{\overline{z}+p})$. By the assumption, we need to consider two cases. First, let us discuss the case that $1+z[p(z)-\lambda]$ has no zeros on the closed unit disk $\overline{\mathbb{D}}$. In this case, we define a nonzero function f by

$$f(z)=\exp\left\{-\int_0^z\frac{2[p(w)-\lambda]+wp'(w)}{1+w[p(w)-\lambda]}\,dw\right\},$$

where the integrand

$$\frac{2[p(w)-\lambda]+wp'(w)}{1+w[p(w)-\lambda]}$$

is a bounded and analytic function on $\mathbb{D}$. Thus f belongs to L_a^2. Moreover, simple calculations give us that

$$\frac{f'(z)}{f(z)}=-\frac{2[p(z)-\lambda]+zp'(z)}{1+z[p(z)-\lambda]},$$

which means that the function f defined above satisfies Equation (2), and so λ is an eigenvalue of the Toeplitz operator $T_{\overline{z}+p}$.

To finish the proof, it remains to consider the case that $1+z[p(z)-\lambda]$ has simple zeros $\{z_1, z_2, ..., z_k\}$ in $\mathbb{D}$ such that

$$z_j^2 p'(z_j) = \frac{n_j+2}{n_j+1}$$

for some integer $n_j \in \{0, 1, 2, ...\}$ with $j=1, 2, ..., k$. In order to show that λ is an eigenvalue of $T_{\overline{z}+p}$ in this case, we need to solve Equation (2). Let us consider the following function:

$$f(z) = g(z) \prod_{j=1}^{k} (z-z_j)^{n_j},$$

where

$$g(z) = \exp\left\{ -\int_0^z \left(\frac{2[p(w)-\lambda]+wp'(w)}{1+w[p(w)-\lambda]} + \sum_{j=1}^{k} \frac{n_j}{w-z_j} \right) dw \right\}.$$

We are going to verify that f is an eigenvector of $T_{\overline{z}+p}$.

Since the zeros $\{z_1, z_2, ..., z_k\} \subset \mathbb{D}$ of $1+w[p(w)-\lambda]$ are all simple, we can write

$$\frac{2[p(w)-\lambda]+wp'(w)}{1+w[p(w)-\lambda]} = \sum_{j=1}^{k} \frac{a_j}{w-z_j} + \sum_{l=1}^{N} \frac{b_l}{(w-w_l)^{m_l}}$$

for some w_l outside the closure of the unit disk, $m_l \in \mathbb{N}$ and a_j, b_l are all complex constants. Recall that

$$z_j^2 p'(z_j) = \frac{n_j+2}{n_j+1} \quad (j = 1, 2, ..., k)$$

and each $z_j \in \mathbb{D}$ is the zero of $1+z[p(z)-\lambda]$ with multiplicity 1. Repeating the same arguments as the one in the proof of the necessity, we obtain by the Cauchy residue theorem that $a_j = -n_j$ for every $1 \leqslant j \leqslant k$, so

$$\frac{2[p(w)-\lambda]+wp'(w)}{1+w[p(w)-\lambda]} + \sum_{j=1}^{k} \frac{n_j}{w-z_j} = \sum_{l=1}^{N} \frac{b_l}{(w-w_l)^{m_l}}$$

is in H^∞ and hence $g \in H^\infty$. Moreover, from the definitions of f and g we have

$$\begin{aligned} \frac{f'(z)}{f(z)} &= \sum_{j=1}^{k} \frac{n_j}{z-z_j} + \frac{g'(z)}{g(z)} \\ &= \sum_{j=1}^{k} \frac{n_j}{z-z_j} - \left(\frac{2[p(z)-\lambda]+zp'(z)}{1+z[p(z)-\lambda]} + \sum_{j=1}^{k} \frac{n_j}{z-z_j} \right) \end{aligned}$$

$$= -\frac{2[p(z)-\lambda]+zp'(z)}{1+z[p(z)-\lambda]}.$$

Noting that f is analytic on $\mathbb{D}$ and bounded on $\overline{\mathbb{D}}$, we conclude that f is a nonzero solution of Equation (2) in the Bergman space L_a^2. Hence f is an eigenvector of $T_{\overline{z}+p}$ corresponding to the eigenvalue λ. This completes the proof of Theorem 2.4. $\square$

The above theorem leads to the following complete characterization on the invertibility of the Toeplitz operator $T_{\overline{z}+p}$ with $p\in H^\infty\cap C(\overline{\mathbb{D}})$ immediately.

Theorem 2.5. *Let p be a function in $H^\infty\cap C(\overline{\mathbb{D}})$. Then the Toeplitz operator $T_{\overline{z}+p}$ is invertible on the Bergman space L_a^2 if and only if the following two conditions hold*:

(i) *$1+zp$ has no zeros on the unit circle $\partial\mathbb{D}$*;

(ii) *$1+zp$ has exactly one simple zero z_0 in the open disk $\mathbb{D}$ which satisfies that*

$$z_0^2 p'(z_0) - \frac{n+2}{n+1} \neq 0$$

for any nonnegative integer n.

3. Quadratic polynomials

In this section, we show that the spectrum of the Topelitz operator $T_{\overline{z}+p}$ is connected for every quadratic polynomial p. More precisely, we obtain the following theorem which is analogous to [2, Theorem 4.6.1] or [11, Corollary 7.28] about the spectra of Toeplitz operators on the Hardy space.

Theorem 3.1. *Let $\varphi(z)=\overline{z}+p(z)$, where p is a quadratic polynomial. The spectrum of the Toeplitz operator operator T_φ is given by*

$$\sigma(T_\varphi) = \varphi(\partial\mathbb{D}) \bigcup \Big\{\lambda\in\mathbb{C} : \lambda\notin\varphi(\partial\mathbb{D}) \text{ and } \operatorname{wind}\big(\varphi(\partial\mathbb{D}),\lambda\big)\neq 0\Big\},$$

which coincides with the spectrum of the corresponding Hardy-Toeplitz operator with symbol $e^{-\mathrm{i}\theta}+p(e^{\mathrm{i}\theta})$. Hence the spectrum of $T_{\overline{z}+p}$ is connected for every $p(z)=az^2+bz+c$ with $a,b,c\in\mathbb{C}$.

Proof. Let p be a quadratic polynomial az^2+bz+c and let $\varphi(z)=\overline{z}+p(z)$. By Lemma 2.2, we have

$$\sigma(T_\varphi) = \varphi(\partial\mathbb{D}) \bigcup \Big\{\lambda\in\mathbb{C} : \lambda\notin\varphi(\partial\mathbb{D}) \text{ and } \operatorname{wind}\big(\varphi(\partial\mathbb{D}),\lambda\big)\neq 0\Big\}$$

$$\bigcup\left(\sigma_p(T_\varphi)\bigcap\left\{\lambda\in\mathbb{C}:\lambda\notin\sigma_e(T_\varphi)\text{ and index}(T_\varphi-\lambda I)=0\right\}\right).$$

To prove the theorem we need only to show that

$$\sigma_p(T_\varphi)\bigcap\left\{\lambda\in\mathbb{C}:\lambda\notin\sigma_e(T_\varphi)\text{ and index}(T_\varphi-\lambda I)=0\right\}=\varnothing.$$

Since $\sigma_e(T_\varphi)=\varphi(\partial\mathbb{D})$, and for each complex number μ,

$$T_\varphi-\mu I=T_{\varphi-\mu}=T_{\overline{z}+az^2+bz+(c-\mu)},$$

it is sufficient to show that 0 does not belong to

$$\sigma_p(T_\varphi)\bigcap\left\{\lambda\in\mathbb{C}:\lambda\notin\varphi(\partial\mathbb{D})\text{ and wind}\big(\varphi(\partial\mathbb{D}),\lambda\big)=0\right\}$$

for any $\varphi(z)=\overline{z}+(az^2+bz+c)$.

To do so, suppose that 0 belongs to

$$\sigma_p(T_\varphi)\bigcap\left\{\lambda\in\mathbb{C}:\lambda\notin\varphi(\partial\mathbb{D})\text{ and wind}\big(\varphi(\partial\mathbb{D}),\lambda\big)=0\right\}$$

for some $\varphi(z)=\overline{z}+(az^2+bz+c)$. Noting that $\text{index}(T_\varphi)=0$, Lemma 2.2 gives us

$$0=\text{index}(T_\varphi)=-\text{wind}\big(\varphi(\partial\mathbb{D}),0\big)=-\text{wind}\left(\frac{az^3+bz^2+cz+1}{z}\bigg|_{\partial\mathbb{D}},\ 0\right).$$

By the argument principle, the cubic polynomial az^3+bz^2+cz+1 has exactly one zero in the disk $\mathbb{D}$ with multiplicity 1 and hence has the following factorization:

$$az^3+bz^2+cz+1=a(z-\alpha)(z+\beta)(z+\gamma)$$

with $|\alpha|<1$, $|\beta|>1$ and $|\gamma|>1$. Evaluating both sides of the above equality at 0 gives

$$a\alpha\beta\gamma=-1. \tag{3}$$

Since 0 is an eigenvalue of the Toeplitz operator T_φ, T_φ has a nonzero eigenvector f in L^2_a such that

$$T_\varphi f=T_{\overline{z}+p}f=0.$$

By Theorem 2.4, we get

$$\alpha^2p'(\alpha)=\frac{n+2}{n+1} \tag{4}$$

for some $n\in\mathbb{N}$. Since

$$zp(z)+1=az^3+bz^2+cz+1=a(z-\alpha)(z+\beta)(z+\gamma),$$

taking derivatives both sides of the above equality and evaluating at α give

$$\alpha p(\alpha)+\alpha^2 p'(\alpha)=a\alpha(\alpha+\beta)(\alpha+\gamma)=a\alpha\beta\gamma\Big(1+\frac{\alpha}{\beta}\Big)\Big(1+\frac{\alpha}{\gamma}\Big).$$

Note that

$$\alpha p(\alpha)+1=0,$$

thus combining (3) with (4) gives

$$\Big(1+\frac{\alpha}{\beta}\Big)\Big(1+\frac{\alpha}{\gamma}\Big)=-\frac{1}{n+1}<0 \tag{5}$$

for some $n\in\mathbb{N}$. This will lead a contradiction by the following simple observation:

$$zw<0 \Longrightarrow \mathrm{Re}(z)\cdot\mathrm{Re}(w)\leqslant 0. \tag{6}$$

To derive a contradiction, using the above observation and Inequality (5) we have

$$\mathrm{Re}\Big(1+\frac{\alpha}{\beta}\Big)\cdot\mathrm{Re}\Big(1+\frac{\alpha}{\gamma}\Big)\leqslant 0. \tag{7}$$

On the other hand, we have that $\frac{\alpha}{\beta}$ and $\frac{\alpha}{\gamma}$ are in the unit disk, to get

$$\mathrm{Re}\Big(1+\frac{\alpha}{\beta}\Big)>0 \quad \text{and} \quad \mathrm{Re}\Big(1+\frac{\alpha}{\gamma}\Big)>0.$$

This contradicts with (7) and yields that 0 does not belong to

$$\sigma_p(T_\varphi)\bigcap\Big\{\lambda\in\mathbb{C}:\lambda\notin\varphi(\partial\mathbb{D}) \text{ and } \mathrm{wind}\big(\varphi(\partial\mathbb{D}),\lambda\big)=0\Big\}$$

for any $\varphi(z)=\overline{z}+(az^2+bz+c)$.

To complete the proof, we need to prove Observation (6). Writing

$$z=x_1+\mathrm{i}y_1 \quad \text{and} \quad w=x_2+\mathrm{i}y_2 \quad \big(x_1,x_2,y_1,y_2\in\mathbb{R}\big),$$

we have that $zw<0$ if and only if

$$\begin{cases} x_1x_2-y_1y_2<0, \\ x_1y_2+x_2y_1=0. \end{cases}$$

If $zw<0$ and neither x_1 nor x_2 equals 0, then x_1 and x_2 have opposite signs. Otherwise, if both x_1 and x_2 have the same signs, then the product x_1x_2 is positive. Thus the first inequality above gives

$$y_1y_2>x_1x_2>0,$$

so y_1 and y_2 also have the same signs. But the second equality above gives

$$\frac{x_1}{x_2} = -\frac{y_1}{y_2}.$$

The left-hand side of the above equality is positive but the right-hand side is negative. We get a contradiction to complete the proof of Theorem 3.1. $\square$

It was shown in [28, Theorem 3.1] that $\sigma(T_h)=\mathrm{clos}[h(\mathbb{D})]$ is a solid ellipse if $h(z)=\overline{z}+(az+b)$ $(a,b\in\mathbb{C})$. Thus, Theorem 3.1 is true for any polynomial p with degree at most 2. From the proof of Theorem 3.1, we get the following two corollaries immediately.

Corollary 3.2. *Let $\varphi(z)=\overline{z}+(az^2+bz+c)$, where a,b and c are all complex constants. Then the Toeplitz operator T_φ is invertible if and only if the cubic polynomial az^3+bz^2+cz+1 has a unique zero in the unit disk $\mathbb{D}$ with multiplicity 1, but does not have any zero on the unit circle $\partial\mathbb{D}$.*

Proof. If T_φ is invertible, then T_φ is a Fredholm operator with Fredholm index 0. By Lemma 2.2, since 0 is not in $\varphi(\partial\mathbb{D})$ and

$$0 = -\mathrm{wind}\big(\varphi(\partial\mathbb{D}),0\big) = -\mathrm{wind}\left(\frac{az^3+bz^2+cz+1}{z}\bigg|_{\partial\mathbb{D}},\ 0\right),$$

the argument principle implies that az^3+bz^2+cz+1 has a unique zero with multiplicity 1 in the unit disk $\mathbb{D}$.

Conversely, we assume that the cubic polynomial satisfies the conditions in the corollary, then T_φ is a Fredholm operator with $\mathrm{index}(T_\varphi)=0$. We are going to show that T_φ is invertible on the Bergman space. If it is not invertible, then there exists a nonzero function $f\in L_a^2$ such that $T_\varphi f=0$.

Write

$$az^3+bz^2+cz+1 = a(z-\alpha)(z+\beta)(z+\gamma),$$

where $|\alpha|<1$, $|\beta|>1$ and $|\gamma|>1$. Since f is nonzero, the proof of Theorem 3.1 shows

$$\mathrm{Re}\left(1+\frac{\alpha}{\beta}\right)\cdot\mathrm{Re}\left(1+\frac{\alpha}{\gamma}\right) \leqslant 0.$$

However, Observation (6) in the proof of Theorem 3.1 tells us that one of $\frac{\alpha}{\beta}$ and $\frac{\alpha}{\gamma}$ lies outside the unit disk. This implies that either β or γ is inside the unit disk, since $|\alpha|<1$. It is a contradiction. This finishes the proof of Corollary 3.2. $\square$

Corollary 3.3. *Let $\varphi(z)=\overline{z}+(az^2+bz+c)$ with $a,b,c\in\mathbb{C}$. Then we have*

$$\sigma(T_\varphi)\subset\mathrm{clos}[\varphi(\mathbb{D})].$$

Proof. Let λ be any complex number not in $\operatorname{clos}[\varphi(\mathbb{D})]$. Thus we obtain

$$|\varphi(z)-\lambda| \geqslant \delta > 0$$

for some constant δ and all z in the unit disk.

Let

$$F(e^{\mathrm{i}\theta}, t) := \varphi(te^{\mathrm{i}\theta})-\lambda, \quad \theta \in [0, 2\pi], \ t \in [0, 1].$$

Then we have $F(e^{\mathrm{i}\theta}, t) \in C\big(\partial\mathbb{D} \times [0,1], \mathbb{C}\setminus\{0\}\big)$ and

$$F(e^{\mathrm{i}\theta}, 0) = \varphi(0)-\lambda, \quad F(e^{\mathrm{i}\theta}, 1) = \varphi(e^{\mathrm{i}\theta})-\lambda, \quad e^{\mathrm{i}\theta} \in \partial\mathbb{D}.$$

This shows that the curve $\varphi(\partial\mathbb{D})-\lambda$ is homotopy equivalent to a nonzero point $\varphi(0)-\lambda$ $\big($in $\mathbb{C}\setminus\{0\}\big)$. Thus we have

$$0 = \operatorname{wind}\big(\varphi(\partial\mathbb{D})-\lambda, 0\big) = \operatorname{wind}\left(\frac{az^3+bz^2+(c-\lambda)z+1}{z}\bigg|_{\partial\mathbb{D}}, \ 0\right).$$

The argument principle gives that the polynomial

$$az^3+bz^2+(c-\lambda)z+1$$

has exactly one zero with multiplicity 1 in $\mathbb{D}$. By Corollary 3.2, we have that $T_{\varphi-\lambda}$ is invertible, and hence λ is not in the spectrum $\sigma(T_\varphi)$. So we obtain

$$\sigma(T_\varphi) \subset \operatorname{clos}[\varphi(\mathbb{D})],$$

to complete the proof of this corollary. $\square$

Remark 3.4. Athough it is true [28, Theorem 3.1] that for every linear polynomial p and $\varphi(z)=\overline{z}+p(z)$, we have

$$\sigma(T_\varphi) = \operatorname{clos}[\varphi(\mathbb{D})],$$

the inclusion

$$\operatorname{clos}[\varphi(\mathbb{D})] \subset \sigma(T_\varphi)$$

does not hold for general harmonic polynomials

$$\varphi(z) = \overline{z}+(az^2+bz+c).$$

Indeed, it was shown in [28, Theorem 4.1] that the Toeplitz operator $T_{\overline{z}+(z^2-z)}$ is invertible on the Bergman space L_a^2, but clearly the harmonic function $\overline{z}+(z^2-z)$ is not invertible in $L^\infty(\mathbb{D})$.

4. Polynomials with higher degree

In this section, we mainly study the spectral structure of the Toeplitz operator $T_{\overline{z}+p}$, where p is a polynomial with degree greater than 2. For each integer $k>2$, we will construct a polynomial p of degree k such that the spectrum of the Toeplitz operator $T_{\overline{z}+p}$ has at least one isolated point but has at most finitely many isolated points.

The first main result of this section is contained in the following theorem, which provides a general method to construct Toeplitz operators with harmonic symbols on the Bergman space having disconnected spectra.

Theorem 4.1. *For each integer k greater than 2, there is a polynomial p with degree k such that the spectrum $\sigma(T_{\overline{z}+p})$ has an isolated point and hence is disconnected.*

Proof. Let k and n be two integers such that $k\geqslant 3$ and $n\geqslant 1$. Let β be the complex number defined by:

$$\beta=\left[1-\left(\frac{1}{n+1}\right)^{\frac{1}{k}}e^{\frac{\pi}{k}\mathrm{i}}\right]^{-\frac{1}{k+1}}.$$

First we show $|\beta|>1$. To do so, using a simple computation we have

$$\begin{aligned}\left|1-\left(\frac{1}{n+1}\right)^{\frac{1}{k}}e^{\frac{\pi}{k}\mathrm{i}}\right|^2&=1-2\left(\frac{1}{n+1}\right)^{\frac{1}{k}}\cos\left(\frac{\pi}{k}\right)+\left(\frac{1}{n+1}\right)^{\frac{2}{k}}\\&=1-\left[2^k\left(\frac{1}{n+1}\right)\cos^k\left(\frac{\pi}{k}\right)\right]^{\frac{1}{k}}+\left(\frac{1}{n+1}\right)^{\frac{2}{k}}\\&\leqslant 1-\left[2^k\left(\frac{1}{n+1}\right)\cos^k\left(\frac{\pi}{3}\right)\right]^{\frac{1}{k}}+\left(\frac{1}{n+1}\right)^{\frac{2}{k}}\\&=1-\left(\frac{1}{n+1}\right)^{\frac{1}{k}}+\left(\frac{1}{n+1}\right)^{\frac{2}{k}}\\&<1-\left(\frac{1}{n+1}\right)^{\frac{1}{k}}+\left(\frac{1}{n+1}\right)^{\frac{1}{k}}=1\end{aligned}$$

for $k\geqslant 3$ and $n\geqslant 1$. The first inequality follows from that the function $\cos(t)$ is decreasing for positive t nearby 0.

Letting $\alpha=-\beta^{-k}$, then we have $|\alpha|<1$. Then we can define a class of polynomials p such that $\sigma(T_{\overline{z}+p})$ is disconnected as follows:

$$p(z)=\frac{(z-\alpha)(z+\beta)^k-1}{z}.$$

Clearly, the degree of the polynomial p is k. We will show that 0 is an isolated point in $\sigma(T_{\overline{z}+p})$. To do so, we first show that 0 is an eigenvalue of $T_{\overline{z}+p}$. As in the proof of Theorem 2.4, simple calculations yield

$$\alpha^2 p'(\alpha) = \frac{n+2}{n+1}.$$

By Theorem 2.4, we have that 0 is an eigenvalue of $T_{\overline{z}+p}$.

Next we will show that 0 is an isolated point of $\sigma(T_{\overline{z}+p})$. Noting that

$$\overline{z}+p(z) = \frac{1+zp(z)}{z} = \frac{(z-\alpha)(z+\beta)^k}{z}$$

on the unit circle, we have that 0 is not in $\sigma_e(T_{\overline{z}+p})$. Since α is the unique zero of $1+zp(z)$ in the unit disk, we have

$$\operatorname{index}(T_{\overline{z}+p}) = 0,$$

and hence 0 is in the intersection

$$\Lambda := \sigma_p(T_{\overline{z}+p}) \bigcap \Big\{\lambda \in \mathbb{C} : \lambda \notin \sigma_e(T_{\overline{z}+p}) \text{ and } \operatorname{index}(T_{\overline{z}+p} - \lambda I) = 0\Big\}.$$

Suppose that λ belongs to Λ, then we have by the argument principle that there exists a $z_\lambda \in \mathbb{D}$ such that

$$1 + z_\lambda [p(z_\lambda) - \lambda] = 0.$$

Moreover, since λ is an eigenvalue of $T_{\overline{z}+p}$, the proof of Theorem 2.4 implies that

$$z_\lambda^2 p'(z_\lambda) = \frac{n+2}{n+1}$$

for some integer $n \geqslant 0$. Therefore, Λ is contained in the following countable set:

$$\left\{\lambda \in \mathbb{C} : z_\lambda \text{ is the root of } \lambda = \frac{1}{z} + p(z) \text{ in } \mathbb{D} \text{ and } z_\lambda^2 p'(z_\lambda) = \frac{n+2}{n+1} \text{ for some } n \in \mathbb{N}\right\},$$

which is a discrete set and hence is not connected. By Theorem 2.3, we conclude that the eigenvalue 0 is an isolated point in $\sigma(T_{\overline{z}+p})$. Consequently, the spectrum of the Toeplitz operator $T_{\overline{z}+p}$ is disconnected, to complete the proof of Theorem 4.1. $\square$

An interesting problem in the study of the Topelitz operator theory is to characterize the spectral structure of such operators. However, as mentioned in Section 1, people know very little about this problem. Although we have shown in the previous theorem that there exists a class of Toeliptz operators with harmonic polynomial symbols such that they have disconnected spectra, it is natural to study more about the topological structure for the spectra of this class of Toeplitz operators. As the isolated points in the spectrum of an operator can be used for producing hyperinvariant subspaces (which was mentioned in Section 1) and for studying Weyl type theorems, we hope to further discuss the isolated points in the spectrum of the Toeplitz operator constructed in Theorem 4.1.

In fact, we will show in the next theorem that there are at most finitely many isolated points in $\sigma(T_{\overline{z}+p})$ for each polynomial p defined in the above theorem, which suggests that there can not be too many isolated points in the spectra of such class of operators. For this purpose, we need to estimate the number of zeros of certain high-degree polynomials. In order to analyze the zero distribution of complex polynomials, we need the following lemma, which is known as "the continuous dependence of the zeros of a polynomial on its coefficients", see [17, appendix A] or [19, Theorem 1.3.1] if necessary.

Lemma 4.2. *Let*

$$p(z)=z^n+a_{n-1}z^{n-1}+\ldots+a_1z+a_0$$

be a monic polynomial with complex coefficients. Then, for every small $\varepsilon>0$, *there exists a* $\tau>0$ *such that for any polynomial*

$$q(z)=z^n+b_{n-1}z^{n-1}+\ldots+b_1z+b_0$$

satisfying

$$\max_{0\leqslant k\leqslant n-1}|a_k-b_k|<\tau,$$

the zeros z_j *and* w_j *of* p *and* q, *respectively, can be ordered in such a way that we have*

$$|z_j-w_j|<\varepsilon\quad(j=1,2,\ldots,n).$$

Another main result of this section is contained in the next theorem. For convenience, we only consider the case of $n=1$ in Theorem 4.1.

Theorem 4.3. *Fix an integer* $k\geqslant 3$. *Suppose that* β *is such that*

$$\beta^{k+1}=\frac{1}{1-(\frac{1}{2})^{\frac{1}{k}}e^{\frac{\pi}{k}\mathrm{i}}},$$

and let $\alpha=-\beta^{-k}$*. Let* p *be the polynomial with degree* k *as the following*:

$$p(z)=\frac{(z-\alpha)(z+\beta)^k-1}{z}.$$

Then 0 *is an isolated point in* $\sigma(T_{\overline{z}+p})$ *and* $\sigma(T_{\overline{z}+p})$ *has at most finitely many isolated points.*

In order to prove Theorem 4.3, we need one more lemma.

Lemma 4.4. *Let* p *be the polynomial with degree* $k\geqslant 3$ *defined in Theorem* 4.3*. There is an* $N\in\mathbb{N}$ *such that for each integer* $n>N$ *and for each solution* w (*in the disk* $\mathbb{D}$) *of the following equation*:

$$z^2p'(z)=\frac{n+2}{n+1},$$

setting

$$\lambda=\frac{1}{w}+p(w),$$

the polynomial $1+z[p(z)-\lambda]$ *has at least two zeros* (*counting multiplicities*) *in the open unit disk.*

Proof. The proof of this lemma will be done in several steps. First we show that the equation $z^2p'(z)=1$ has exactly two distinct roots in the open unit disk $\mathbb{D}$. From the assumption, we have by the definition of p that

$$1+zp(z)=(z-\alpha)(z+\beta)^k.$$

Taking derivatives of both sides of the above equality and then multiplying both sides by z give us that

$$zp(z)+z^2p'(z)=z(z+\beta)^k+kz(z-\alpha)(z+\beta)^{k-1}.$$

This implies

$$\begin{aligned}
z^2p'(z)-1&=z^2p'(z)-(z-\alpha)(z+\beta)^k+zp(z)\\
&=z(z+\beta)^k+kz(z-\alpha)(z+\beta)^{k-1}-(z-\alpha)(z+\beta)^k\\
&=(z+\beta)^{k-1}\left[kz^2+\alpha(1-k)z+\alpha\beta\right]\\
&=(z+\beta)^{k-1}\left(kz^2+\frac{k-1}{\beta^k}z-\frac{1}{\beta^{k-1}}\right),
\end{aligned}$$

where the last equality follows from $\alpha=-\beta^{-k}$. Thus $z^2p'(z)=1$ has a multiple root $-\beta$ and other two complex roots are given by

$$z_\infty=-\frac{\sqrt{(k-1)^2+4k\beta^{k+1}}+(k-1)}{2k\beta^k}$$

and

$$w_\infty = \frac{\sqrt{(k-1)^2+4k\beta^{k+1}}-(k-1)}{2k\beta^k}.$$

Clearly, z_∞ is not equal to w_∞.

Before going further, we first show that z_∞ and w_∞ are both in $\mathbb{D}$. To do so, we need some routine computations. Noting that

$$\begin{aligned} \frac{1}{\beta^{k+1}} &= 1-\left(\frac{1}{2}\right)^{\frac{1}{k}} e^{\frac{\pi}{k}\mathrm{i}} \\ &= 1-\left(\frac{1}{2}\right)^{\frac{1}{k}} \cos\left(\frac{\pi}{k}\right) - \mathrm{i}\left(\frac{1}{2}\right)^{\frac{1}{k}} \sin\left(\frac{\pi}{k}\right), \end{aligned} \tag{8}$$

we obtain

$$\begin{aligned} \frac{1}{|\beta|^k} &= \left|1-\left(\frac{1}{2}\right)^{\frac{1}{k}} e^{\frac{\pi}{k}\mathrm{i}}\right|^{\frac{k}{k+1}} = \left(\left|1-\left(\frac{1}{2}\right)^{\frac{1}{k}} e^{\frac{\pi}{k}\mathrm{i}}\right|^2\right)^{\frac{k}{2(k+1)}} \\ &= \left[1-2^{1-k}\cos\left(\frac{\pi}{k}\right)+4^{-k}\right]^{\frac{k}{2(k+1)}} \end{aligned} \tag{9}$$

for $k\geqslant 3$. In order to estimate $|z_\infty|$, $|w_\infty|$ and $|\beta|^{-k}$, we need the following two real-valued functions on the interval $\left(0, \frac{1}{3}\right]$:

$$A(x) = \frac{1-2^{-x}\cos(\pi x)}{1-2^{1-x}\cos(\pi x)+4^{-x}} - \left(x+\frac{1}{20x}\right)$$

and

$$B(x) = \left[1-2^{1-x}\cos(\pi x)+4^{-x}\right]\left(2x+\frac{5}{16x}\right)^2.$$

Using Taylor's series for $\cos(t)$ with $t\in\left(0, \frac{\pi}{3}\right]$, we can show by standard calculus that

$$A(x) \geqslant \frac{1}{10} \quad \text{and} \quad 1 \leqslant B(x) \leqslant \frac{5}{2}$$

for all $0<x\leqslant\frac{1}{3}$. This yields

$$\frac{1-(\frac{1}{2})^{\frac{1}{k}}\cos(\frac{\pi}{k})}{\left|1-(\frac{1}{2})^{\frac{1}{k}} e^{\frac{\pi}{k}\mathrm{i}}\right|^2} \geqslant \frac{k}{20}+\frac{1}{k} \tag{10}$$

and

$$\frac{2}{5}\left(\frac{5k}{16}+\frac{2}{k}\right)^2 \leqslant \left|1-\left(\frac{1}{2}\right)^{\frac{1}{k}} e^{\frac{\pi}{k}\mathrm{i}}\right|^{-2} \leqslant \left(\frac{5k}{16}+\frac{2}{k}\right)^2 \tag{11}$$

for all $k\geqslant 3$. Combining (8) with (10) and (11) gives

$$
\begin{aligned}
\left|\left(\frac{k-1}{k}\right)^2+\frac{4\beta^{k+1}}{k}\right|^{\frac{1}{2}} &= \left|\left(\frac{k-1}{k}\right)^2+\frac{4}{k}\left(\frac{1-(\frac{1}{2})^{\frac{1}{k}}\cos(\frac{\pi}{k})+\mathrm{i}(\frac{1}{2})^{\frac{1}{k}}\sin(\frac{\pi}{k})}{\left|1-(\frac{1}{2})^{\frac{1}{k}}e^{\frac{\pi}{k}\mathrm{i}}\right|^2}\right)\right|^{\frac{1}{2}}\\
&\geqslant \sqrt{\left(\frac{k-1}{k}\right)^2+\frac{4}{k}\left(\frac{1-(\frac{1}{2})^{\frac{1}{k}}\cos(\frac{\pi}{k})}{\left|1-(\frac{1}{2})^{\frac{1}{k}}e^{\frac{\pi}{k}\mathrm{i}}\right|^2}\right)}\\
&\geqslant \sqrt{\left(\frac{k-1}{k}\right)^2+\frac{4}{k}\left(\frac{k}{20}+\frac{1}{k}\right)}\\
&= \sqrt{1+\frac{(k-5)^2}{5k^2}}\geqslant 1 \qquad (12)
\end{aligned}
$$

for all $k\geqslant 3$ and

$$
\begin{aligned}
\left|\left(\frac{k-1}{k}\right)^2+\frac{4\beta^{k+1}}{k}\right|^{\frac{1}{2}} &= \left|\left(\frac{k-1}{k}\right)^2+\frac{4}{k}\left(\frac{1}{1-(\frac{1}{2})^{\frac{1}{k}}e^{\frac{\pi}{k}\mathrm{i}}}\right)\right|^{\frac{1}{2}}\\
&\leqslant \sqrt{\left(\frac{k-1}{k}\right)^2+\frac{4}{k\left|1-(\frac{1}{2})^{\frac{1}{k}}e^{\frac{\pi}{k}\mathrm{i}}\right|}}\\
&\leqslant \sqrt{\left(\frac{k-1}{k}\right)^2+\frac{4}{k}\left(\frac{5k}{16}+\frac{2}{k}\right)}\\
&= \sqrt{\frac{9}{4}-\frac{2k-9}{k^2}}\leqslant\frac{3}{2} \qquad (13)
\end{aligned}
$$

for all $k\geqslant 5$, where the last inequality follows from $k\geqslant 5$.

Thus we derive by (12) that

$$
\begin{aligned}
|z_\infty| &\geqslant \frac{\left|\sqrt{(k-1)^2+4k\beta^{k+1}}\right|}{2k|\beta^k|}-\frac{k-1}{2k|\beta^k|}\\
&= \frac{1}{2|\beta|^k}\left|\left(\frac{k-1}{k}\right)^2+\frac{4\beta^{k+1}}{k}\right|^{\frac{1}{2}}-\frac{k-1}{2k|\beta|^k}\\
&\geqslant \frac{1}{2|\beta|^k}-\frac{k-1}{2k|\beta|^k}=\frac{1}{2k|\beta|^k}
\end{aligned}
$$

for all $k\geqslant 3$. On the other hand, using (8) and (13) we get

$$\begin{aligned}|z_\infty|&\leqslant \frac{1}{2|\beta|^k}\left|\left(\frac{k-1}{k}\right)^2+\frac{4\beta^{k+1}}{k}\right|^{\frac{1}{2}}+\frac{k-1}{2k|\beta|^k}\\&\leqslant \frac{3}{4|\beta|^k}+\frac{1}{2|\beta|^k}=\frac{5}{4|\beta|^k}\end{aligned}$$

for all $k\geqslant 5$. Moreover, by (9), (11) and some elementary calculations we obtain that

$$|\beta|^{-k}=\left(\left|1-\left(\frac{1}{2}\right)^{\frac{1}{k}}e^{\frac{\pi}{k}\mathrm{i}}\right|^2\right)^{\frac{k}{2(k+1)}}\leqslant\left[\frac{5}{2}\left(\frac{5k}{16}+\frac{2}{k}\right)^{-2}\right]^{\frac{k}{2(k+1)}}\leqslant\frac{39}{50}$$

if $k\geqslant 6$, which gives us that

$$|z_\infty|\leqslant\frac{5}{4|\beta|^k}\leqslant\frac{5}{4}\times\frac{39}{50}=\frac{39}{40}\quad(k\geqslant 6).$$

For $k=3$ or $k=4$ or $k=5$, by (9) we can compute directly that $|z_\infty|<\frac{49}{50}$ in these three cases. To summarize, we get the following estimate for $|z_\infty|$:

$$\frac{1}{2k|\beta|^k}\leqslant|z_\infty|<\frac{49}{50}$$

when $k\geqslant 3$. Observing that the above estimations are also valid for $|w_\infty|$, so we obtain

$$\frac{1}{2k|\beta|^k}\leqslant|w_\infty|<\frac{49}{50}\quad(k\geqslant 3).$$

For simplicity we denote the positive constant $\frac{1}{2k|\beta|^k}$ by c_k for $k\geqslant 3$, then we conclude by the above arguments that

$$0<c_k\leqslant|z_\infty|<\frac{49}{50}<1$$

and

$$0<c_k\leqslant|w_\infty|<\frac{49}{50}<1$$

for every integer $k\geqslant 3$. Since $z_\infty\neq w_\infty$, the polynomial $z^2p'(z)-1$ has exactly two different zeros in the unit disk $\mathbb{D}$ for each fixed integer $k\geqslant 3$.

Let

$$\lambda_\infty=\frac{1}{z_\infty}+p(z_\infty).$$

Next we show that z_∞ is a multiple zero of the polynomial

$$F(z):=1+z[p(z)-\lambda_\infty]$$

$$=(z-\alpha)(z+\beta)^k-\lambda_\infty z$$

in the unit disk $\mathbb{D}$. By the definition of p and the fact that

$$1=z_\infty^2 p'(z_\infty),$$

we get

$$\lambda_\infty = z_\infty p'(z_\infty)+p(z_\infty). \tag{14}$$

From the definition of λ_∞, we immediately obtain

$$F(z_\infty)=z_\infty\Big(\frac{1}{z_\infty}+p(z_\infty)-\lambda_\infty\Big)=0.$$

In order to show that z_∞ is a multiple zero of F, we calculate

$$F'(z)=p(z)+zp'(z)-\lambda_\infty,$$

to obtain

$$\begin{aligned} F'(z_\infty) &= p(z_\infty)+z_\infty p'(z_\infty)-\lambda_\infty \\ &= 0, \end{aligned}$$

where the last equality comes from (14). This gives us that

$$F(z_\infty)=F'(z_\infty)=0,$$

so $z_\infty \in \mathbb{D}$ is a multiple zero of F, as desired.

Using the same method as above, we can show that the polynomial

$$G(z):=1+z[p(z)-\mu_\infty]$$

also has a multiple zero w_∞ in the unit disk, where μ_∞ is defined by

$$\begin{aligned} \mu_\infty &:= \frac{1}{w_\infty}+p(w_\infty)=\frac{w_\infty^2 p'(w_\infty)}{w_\infty}+p(w_\infty) \\ &= w_\infty p'(w_\infty)+p(w_\infty). \end{aligned}$$

Last we will show that if w is a root of $z^2p'(z)=\frac{n+2}{n+1}$ in $\mathbb{D}$ and

$$\lambda=\frac{1}{w}+p(w),$$

then the equation

$$1+z[p(z)-\lambda]=0$$

has at least two roots (counting multiplicities) in $\mathbb{D}$ for all n large enough. To do this, we observe that the polynomials $z^2p'(z)-\frac{n+2}{n+1}$ and $z^2p'(z)-1$ have only one different coefficient, and the absolute value of the difference is given by $\frac{n+2}{n+1}-1$, which tends to 0 as $n\to\infty$. As shown above, $z^2p'(z)-1$ has exactly two distinct zeros z_∞ and w_∞ in the unit disk, Lemma 4.2 guarantees that there is a positive integer N_0 such that $z^2p'(z)-\frac{n+2}{n+1}$ has two distinct zeros z_n and w_n in $\mathbb{D}$ for each $n\geqslant N_0$, and moreover,

$$\lim_{n\to\infty} z_n = z_\infty \quad \text{and} \quad \lim_{n\to\infty} w_n = w_\infty. \tag{15}$$

Letting

$$\lambda_n = \frac{1}{z_n}+p(z_n)$$

and

$$F_n(z)=1+z[p(z)-\lambda_n]$$

for each integer $n\geqslant N_0$. Recall that

$$F(z)=1+z[p(z)-\lambda_\infty]$$

and z_∞ is a multiple zero of F in the unit disk $\mathbb{D}$. We will use the zeros of F to study the distribution of zeros of each polynomial F_n for n sufficiently large.

Observe that the only difference between F_n and F is the coefficient of the linear term. To be more precise, the absolute value of the difference is

$$|\lambda_n-\lambda_\infty| = \left|\left[\frac{1}{z_n}+p(z_n)\right]-\left[\frac{1}{z_\infty}+p(z_\infty)\right]\right|. \tag{16}$$

Applying Lemma 4.2 to F_n and F, there exists a small number $\tau>0$ (depending only on k) such that if $|\lambda_n-\lambda_\infty|<\tau$, then each polynomial F_n has two zeros ξ_n and $\tilde{\xi}_n$ in $\mathbb{D}$ which satisfy that

$$\begin{cases} |\xi_n - z_\infty| < \dfrac{1}{4^k}, \\ |\tilde{\xi}_n - z_\infty| < \dfrac{1}{4^k}, \end{cases} \tag{17}$$

because $z_\infty\in\mathbb{D}$ is a multiple zero of F. Notice that ξ_n does not necessarily equal $\tilde{\xi}_n$. (If $\xi_n=\tilde{\xi}_n$, then $\tilde{\xi}_n$ is a multiple root of F_n.)

To see that $|\lambda_n-\lambda_\infty|$ can be made as small as we need, recall that we have shown

$$c_k \leqslant |z_\infty| < \frac{49}{50}$$

and $z_n \to z_\infty$ as $n \to \infty$. Thus there is a positive integer $N_1 \geqslant N_0$ such that for each $n \geqslant N_1$, z_n belongs to the compact subset $\left\{z : \frac{c_k}{2} \leqslant |z| \leqslant \frac{49}{50}\right\}$. Noting that the function

$$Q(z) = \frac{1}{z} + p(z)$$

is uniformly continuous on $\left\{z : \frac{c_k}{2} \leqslant |z| \leqslant \frac{49}{50}\right\}$. For the constant $\tau > 0$ chosen above, there is a positive integer $N_2 \geqslant N_1$ such that if $n \geqslant N_2$, then (15) and (16) imply that

$$|\lambda_n - \lambda_\infty| < \tau,$$

as

$$\lambda_n = \frac{1}{z_n} + p(z_n) = Q(z_n)$$

and

$$\lambda_\infty = \frac{1}{z_\infty} + p(z_\infty) = Q(z_\infty),$$

as required.

Therefore, we obtain the following estimate for $|\xi_n|$:

$$\begin{aligned} |\xi_n| &\leqslant |z_\infty| + |\xi_n - z_\infty| \\ &\leqslant |z_\infty| + \frac{1}{4^k} < \frac{49}{50} + \frac{1}{4^k} \\ &\leqslant \frac{49}{50} + \frac{1}{4^3} < 1 \end{aligned}$$

for every $n \geqslant N_2$, where the second inequality follows from (17). Moreover, we also have that

$$|\tilde{\xi}_n| \leqslant |z_\infty| + |\tilde{\xi}_n - z_\infty| < 1$$

for $n \geqslant N_2$. Thus the polynomial $F_n(z) = 1 + z[p(z) - \lambda_n]$ has at least two zeros ξ_n and $\tilde{\xi}_n$ in the open unit disk $\mathbb{D}$ for each integer $n \geqslant N_2$.

In the case that

$$\mu_n = \frac{1}{w_n} + p(w_n),$$

similarly we can use the above arguments to prove that

$$1 + z[p(z) - \mu_n] = 0$$

has at least two roots η_n and $\tilde{\eta}_n$ in $\mathbb{D}$ for all n large enough (η_n does not necessarily equal $\tilde{\eta}_n$). Therefore, Lemma 4.4 is now proved. $\square$

Now we are ready to prove the second main theorem of this section.

Proof of Theorem 4.3. Recalling that

$$\left(1-\frac{1}{\beta^{k+1}}\right)^k = \left(1+\frac{\alpha}{\beta}\right)^k = -\frac{1}{2}$$

and

$$1+zp(z) = (z-\alpha)(z+\beta)^k,$$

we immediately obtain

$$\alpha^2 p'(\alpha) = \frac{3}{2}.$$

Theorem 4.1 gives that 0 is an isolated eigenvalue of the Toeplitz operator $T_{\overline{z}+p}$.

To complete the proof, we need to show that $\sigma(T_{\overline{z}+p})$ has at most finitely many isolated points. To this end, we denote

$$h(z) = \overline{z}+p(z).$$

From the proof of Theorem 3.1, we have that the isolated points of $\sigma(T_h)$ are contained in the subset

$$\begin{aligned}&\sigma_p(T_h)\bigcap\left\{\lambda\in\mathbb{C}:\lambda\notin\sigma_e(T_h)\text{ and index}(T_h-\lambda I)=0\right\}\\ &=\sigma_p(T_h)\bigcap\left\{\lambda\in\mathbb{C}:\lambda\notin h(\partial\mathbb{D})\text{ and wind}\left(\frac{1+z[p(z)-\lambda]}{z}\bigg|_{\partial\mathbb{D}},\ 0\right)=0\right\}.\end{aligned}$$

For simplicity, we let

$$\Lambda := \sigma_p(T_h)\bigcap\left\{\lambda\in\mathbb{C}:\lambda\notin h(\partial\mathbb{D})\text{ and wind}\left(\frac{1+z[p(z)-\lambda]}{z}\bigg|_{\partial\mathbb{D}},\ 0\right)=0\right\}.$$

Theorem 2.4 implies that Λ is a subset of $\bigcup_{n\geqslant 0}\Omega_n$, where for each nonnegative integer n,

$$\Omega_n := \left\{\lambda : z_\lambda \text{ is the root of } \lambda=\frac{1}{z}+p(z) \text{ in } \mathbb{D} \text{ and } z_\lambda^2 p'(z_\lambda)=\frac{n+2}{n+1} \text{ for some } n\in\mathbb{N}\right\}$$

is a finite set. On the other hand, Lemma 4.4 gives that $1+z[p(z)-\lambda]$ has at least two zeros in the unit disk for $\lambda\in\Omega_n$ if $n>N$. Hence, we deduce that

$$\text{wind}\left(\frac{1+z[p(z)-\lambda]}{z}\bigg|_{\partial\mathbb{D}},\ 0\right)\geqslant 1$$

for every $\lambda\in\Omega_n$ with $n>N$. It follows that Λ is contained in the finite union $\bigcup_{n=0}^{N}\Omega_n$ of finite sets Ω_n, and so Λ is a finite set. This completes the proof of Theorem 4.3. $\square$

Remark 4.5. Although Theorems 4.1 and 4.3 tell us that we can construct a polynomial p with $\deg(p)=k$ such that $\sigma(T_{\overline{z}+p})$ has isolated points for every $k\geqslant 3$, there exists a class of high-degree polynomials p such that the spectra of Toeplitz operators $T_{\overline{z}+p}$ are all connected. More specifically, let us consider

$$\varphi(z)=\overline{z}+(az^k+b)$$

with $k\geqslant 3$ and a,b are complex constants. Indeed, Guan and the second author showed in [13, Proposition 4.5] that

$$\sigma(T_\varphi)=\varphi(\partial\mathbb{D})\bigcup\Big\{\lambda\in\mathbb{C}:\lambda\notin\varphi(\partial\mathbb{D})\text{ and }\operatorname{wind}\big(\varphi(\partial\mathbb{D}),\lambda\big)\neq 0\Big\},$$

which is a connected set in the complex plane $\mathbb{C}$.

Let $h_k(z)=\overline{z}+p_k(z)$, where p_k is the polynomial with degree $k\geqslant 3$ constructed in Theorem 4.3. In view of Theorem 4.3, we end this section by discussing the topological structure of the spectra of these Toeplitz operators T_{h_k}.

For $\varphi\in C(\overline{\mathbb{D}})$, we denote its boundary function by φ^*, i.e.,

$$\varphi^*:=\varphi|_{\partial\mathbb{D}}.$$

Let $\mathbb{T}_{\varphi^*}$ be the Hardy-Toeplitz operator with symbol φ^*, then we always have $\sigma(\mathbb{T}_{\varphi^*})\subset\sigma(T_\varphi)$, see [28, Theorem 2.4] for the details. In addition, observe that the spectral structure of Hardy-Toeplitz operators with continuous symbols implies that $\sigma(T_\varphi)\backslash\sigma(\mathbb{T}_{\varphi^*})$ is at most countable. However, for the harmonic polynomial h_k, the following corollary tells us that the distinction between the spectra of the Bergman-Toeplitz operator T_{h_k} and the corresponding Hardy-Toeplitz operator $\mathbb{T}_{h_k^*}$ is just finitely many isolated points.

Corollary 4.6. *For each integer $k\geqslant 3$, let h_k be the harmonic polynomial mentioned above. Let T_{h_k} and $\mathbb{T}_{h_k^*}$ be the Bergman-Toeplitz operator and the Hardy-Toeplitz operator, respectively. Then the relationship between $\sigma(T_{h_k})$ and $\sigma(\mathbb{T}_{h_k^*})$ is given by*

$$\sigma(T_{h_k})=\sigma(\mathbb{T}_{h_k^*})\cup\Lambda_k,$$

where Λ_k is a finite subset of $\sigma_p(T_{h_k})$.

Proof. Since h_k^* is continuous on $\partial\mathbb{D}$, it follows from [11, Corollary 7.28] or [2, Theorem 4.6.1] that

$$\begin{aligned}\sigma(\mathbb{T}_{h_k^*})&=h_k^*(\partial\mathbb{D})\bigcup\Big\{\lambda\in\mathbb{C}:\lambda\notin h_k^*(\partial\mathbb{D})\text{ and }\operatorname{wind}\big(h_k^*(\partial\mathbb{D}),\lambda\big)\neq 0\Big\}\\&=h_k(\partial\mathbb{D})\bigcup\Big\{\lambda\in\mathbb{C}:\lambda\notin h_k(\partial\mathbb{D})\text{ and }\operatorname{wind}\big(h_k(\partial\mathbb{D}),\lambda\big)\neq 0\Big\}.\end{aligned}$$

Thus, from the proof of Theorem 3.1 we have that

$$\sigma(\mathbb{T}_{h_k})\backslash\sigma(\mathbb{T}_{h_k^*}) = \sigma_p(T_{h_k}) \bigcap \Big\{\lambda \in \mathbb{C} : \lambda \notin \sigma_e(T_{h_k}) \text{ and } \operatorname{index}(T_{h_k} - \lambda I) = 0\Big\}. \tag{18}$$

Denote the intersection on the right-hand side of (18) by Λ_k.

However, we have shown in Theorem 4.3 that all the isolated points of $\sigma(T_{h_k})$ are contained in Λ_k and Λ_k is a finite subset of $\sigma_p(T_{h_k})$. Therefore, we conclude that $\sigma(T_{h_k})$ is the union of $\sigma(\mathbb{T}_{h_k^*})$ and finitely many (isolated) eigenvalues of T_{h_k}, to finish the proof of Corollary 4.6. $\square$

5. Weyl's theorem for a class of Toeplitz operators

In this section we will show that Weyl's theorem holds for a class of Toeplitz operators on the Bergman space. More precisely, letting q be an arbitrary function in $H^\infty \cap C(\overline{\mathbb{D}})$, we will show that Weyl's theorem holds for the Bergman-Toeplitz operator $T_{\overline{z}+q}$. To do so, we begin with some standard notations related to the Weyl spectrum.

Suppose that T is a bounded linear operator on some Hilbert space. The Weyl spectrum $\omega(T)$ of T is defined by

$$\omega(T) := \bigcap_{K \text{ is compact}} \sigma(T+K).$$

Using the characterization in [21], the Weyl spectrum of T can be expressed as

$$\omega(T) = \sigma_e(T) \bigcup \Big\{\lambda \in \mathbb{C} : \lambda \notin \sigma_e(T) \text{ and } \operatorname{index}(T - \lambda I) \neq 0\Big\}.$$

Following the notation in [6] and [7], for simplicity we use $\pi_{00}(T)$ to denote the set of isolated points λ in the spectrum which are eigenvalues of finite geometric multiplicity, i.e.,

$$0 < \dim \ker(T - \lambda I) < \infty.$$

According to [8], we say that an operator T satisfies Weyl's theorem if

$$\omega(T) = \sigma(T)\backslash\pi_{00}(T).$$

For Bergman-Toeplitz operators with analytic and co-analytic symbols, it is clear that these operators satisfy Weyl's theorem, because their spectra are equal to the closure of the ranges of their symbols. Furthermore, Toeplitz operators with real-valued symbols and *radial symbols* (i.e., $\varphi(z) = \varphi(|z|)$ for all $z \in \mathbb{D}$) also satisfy Weyl's theorem, since they are normal. However, unlike Toeplitz operators on the Hardy space, there exist many Bergman-Toeplitz operators for which Weyl's theorem does not hold.

Example 5.1. Let

$$\varphi(z)=\chi_{\frac{1}{2}\mathbb{D}}(z)e^{-\mathrm{i}(\arg(z))},\quad z\in\mathbb{D},$$

where $\frac{1}{2}\mathbb{D}=\{z\in\mathbb{C}:|z|<\frac{1}{2}\}$. Considering the Toeplitz operator T_φ on the Bergman space L_a^2, we have

$$T_\varphi e_n(z)=\begin{cases}0, & n=0,\\ \dfrac{\sqrt{n(n+1)}}{2n+1}\Big(\dfrac{1}{2}\Big)^{2n}e_{n-1}(z), & n\geqslant 1,\end{cases}$$

where $\{e_n(z)\}_{n=0}^{\infty}=\{\sqrt{n+1}z^n\}_{n=0}^{\infty}$ is the orthonormal basis of L_a^2. Then the Toeplitz operator T_φ does not satisfy Weyl's theorem.

Indeed, noting that T_φ is a compact backward weighted shift, since

$$\lim_{n\to\infty}\frac{\sqrt{n(n+1)}}{2n+1}\Big(\frac{1}{2}\Big)^{2n}=0.$$

Using (a) of [10, Proposition 27.7], we conclude that the spectrum and essential spectrum of T_φ are both $\{0\}$, which implies that $\omega(T_\varphi)=\{0\}$. Moreover, we observe that

$$\ker(T_\varphi)=\operatorname{span}\{1\}.$$

It follows that 0 is an isolated eigenvalue with finite multiplicity and $\pi_{00}(T_\varphi)=\{0\}$. Thus we have

$$\sigma(T_\varphi)\backslash\pi_{00}(T_\varphi)=\varnothing$$

and

$$\omega(T_\varphi)\neq\sigma(T_\varphi)\backslash\pi_{00}(T_\varphi).$$

Therefore, the Bergman-Toeplitz operator with the symbol

$$\varphi(z)=\chi_{\frac{1}{2}\mathbb{D}}(z)e^{-\mathrm{i}(\arg(z))}$$

does not satisfy Weyl's theorem.

Nevertheless, in the rest of this section, we will use the characterizations for the point spectra of Toeplitz operators in Theorems 2.4 and 4.3 to obtain a class of Toeplitz operators with bounded harmonic symbols on the Bergman space for which Weyl's theorem holds.

Theorem 5.2. *Suppose that q is in the disk algebra $H^\infty\cap C(\overline{\mathbb{D}})$ and let $h(z)=\overline{z}+q(z)$. Then Weyl's theorem holds for the Bergman-Toeplitz operator T_h.*

Proof. Recall that the spectrum of the Toeplitz operator T_h can be decomposed as the following disjoint union:

$$\sigma(T_h)=h(\partial\mathbb{D})\bigcup\Big\{\lambda\in\mathbb{C}:\lambda\notin h(\partial\mathbb{D})\text{ and }\operatorname{wind}\big(h(\partial\mathbb{D}),\lambda\big)\neq 0\Big\}\bigcup\Lambda,$$

where

$$\Lambda:=\sigma_p(T_h)\bigcap\Big\{\lambda\in\mathbb{C}:\lambda\notin\sigma_e(T_h)\text{ and }\operatorname{index}(T_h-\lambda I)=0\Big\}.$$

By the definition of $\omega(T_h)$ and Lemma 2.2, we have

$$\omega(T_h)=h(\partial\mathbb{D})\bigcup\Big\{\lambda\in\mathbb{C}:\lambda\notin h(\partial\mathbb{D})\text{ and }\operatorname{wind}\big(h(\partial\mathbb{D}),\lambda\big)\neq 0\Big\},$$

to obtain

$$\sigma(T_h)=\omega(T_h)\cup\Lambda. \tag{19}$$

Since all the isolated points of $\sigma(T_h)$ are contained in Λ, we immediately have $\pi_{00}(T_h)\subset\Lambda$.

On the other hand, from the proofs of Theorems 2.4 and 4.1 we recall that Λ is a subset of the following countable set:

$$\Big\{\lambda\in\mathbb{C}:z_\lambda\text{ is the root of }\lambda=\frac{1}{z}+q(z)\text{ in }\mathbb{D}\text{ and }z_\lambda^2q'(z_\lambda)=\frac{n+2}{n+1}\text{ for some }n\in\mathbb{N}\Big\}.$$

It follows from Theorem 2.3 that each eigenvalue in Λ is an isolated point of $\sigma(T_h)$. Thus we have by the definition of $\pi_{00}(T_h)$ that $\Lambda\subset\pi_{00}(T_h)$, hence we get

$$\Lambda=\pi_{00}(T_h).$$

To show that T_h satisfies Weyl's theorem, we need to consider two cases. If T_h has no isolated eigenvalues with finite geometric multiplicity, then we have $\Lambda=\pi_{00}(T_h)=\varnothing$, it follows from (19) that

$$\omega(T_h)=\sigma(T_h).$$

For the case of $\pi_{00}(T_h)\neq\varnothing$, we actually have

$$\omega(T_h)=\sigma(T_h)\backslash\Lambda=\sigma(T_h)\backslash\pi_{00}(T_h),$$

as desired. This completes the proof of Theorem 5.2. $\square$

As every hyponormal operator satisfies Weyl's theorem, the following example shows that there are a lot of non-hyponormal Bergman-Toeplitz operators with harmonic symbols for which Weyl's theorem holds.

Example 5.3. In order to construct a non-hyponormal Toeplitz operator with harmonic polynomial symbol on the Bergman space, we first choose complex numbers $a_1, a_2, ..., a_n$ $(n \geqslant 2)$ such that $a_n \neq 0$ and

$$|a_1 + 2a_2 + ... + na_n| < 1. \tag{20}$$

Letting

$$q(z) = a_1 z + a_2 z^2 + ... + a_n z^n$$

and h be the harmonic polynomial $h(z) = \overline{z} + q(z)$. Then the Toeplitz operator T_h satisfies Weyl's theorem but it is not hyponormal on L_a^2.

In order to show that T_h defined above is not hyponormal, we shall recall a necessary condition for Toeplitz operators to be hyponormal on the Bergman space. Let f and g be analytic on the closed unit disk $\overline{\mathbb{D}}$. It was shown in [1] and [20] that

$$|f'(z)| \geqslant |g'(z)|$$

for all $z \in \partial\mathbb{D}$ if the Toeplitz operator $T_{f+\overline{g}}$ is hyponormal on L_a^2. But Condition (20) tells us that $|q'(1)| < 1$. Thus T_h is not a hyponormal operator. On the other hand, Theorem 5.2 implies that Weyl's theorem holds for the Bergman-Toeplitz operator T_h. So we obtain a class of non-hyponormal Toeplitz operators T_h on the Bergman space for which Weyl's theorem holds.

Acknowledgement. We would like to thank the referee for the constructive and valuable comments and suggestions that improved the content of this paper. This work was partially supported by NSFC (Grant Nos.: 11701052, 11871157, 12371125). The first author was supported by NNSF of China (12231005) and NSF of Shanghai (21ZR1404200). The second author was supported by the Fundamental Research Funds for the Central Universities (Grant Nos.: 2020CDJQY-A039, 2020CDJ-LHSS-003).

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Kunyu Guo
School of Mathematical Sciences
Fudan University
Shanghai 200433
PR China
kyguo@fudan.edu.cn

Xianfeng Zhao
College of Mathematics and Statistics
Chongqing University
Chongqing 401331
PR China
xianfengzhao@cqu.edu.cn

Dechao Zheng
Center of Mathematics
Chongqing University
Chongqing 401331
PR China
and
Department of Mathematics
Vanderbilt University
Nashville TN 37240
U.S.A.
dechao.zheng@vanderbilt.edu

Received October 28, 2021
in revised form January 6, 2023