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Quantum Euler class and virtual Tevelev degrees of Fano complete intersections

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Abstract. We compute the quantum Euler class of Fano complete intersections X in a projective space. In particular, we prove a recent conjecture of A. Buch and R. Pandharipande, namely [7, Conjecture 5.14]. Finally we apply our result to obtain formulas for the virtual Tevelev degrees of X. An algorithm computing all genus 0 two-point Hyperplane Gromov-Witten invariants of X is illustrated along the way.

Contents

1. Introduction

1.1. Quantum Euler class of a variety

Let X be a nonsingular, projective, algebraic variety over $\mathbb{C}$ of dimension r and let $\{\gamma_j\}_{j=0}^N \subset H^*(X)$ be a homogeneous basis with $\gamma_0 = 1$ and $\gamma_N = \mathsf{P}$ the point class. The small quantum cohomology ring $QH^*(X)$(1) of X is defined via the 3-point genus 0 Gromov-Witten invariants:

$$\gamma_i \star \gamma_j = \sum_{\beta \in H_2(X,\mathbb{Z})} \sum_k \langle \gamma_i, \gamma_j, \gamma_k^\vee \rangle_{0,\beta}^X q^\beta \gamma_k$$

where $\gamma_k^\vee \in H^*(X)$ is the dual of γ_k with respect to the intersection form on X, defined by the conditions

$$\int_X \gamma_j \cup \gamma_k^\vee = \delta_{j,k} \quad \text{for } j = 0, ..., N.$$

Here we are following the notation of [14].

Let also

$$\Delta = \sum_j \gamma_j^\vee \otimes \gamma_j \in H^*(X) \otimes H^*(X)$$

be the Künneth decomposition of the diagonal class of $X \times X$.

The **quantum Euler class** of X is the image of Δ under the product map

$$H^*(X) \otimes H^*(X) \xrightarrow{\star} QH^*(X).$$

This is a canonically defined element of $QH^*(X)$, first introduced by Abrams in [2]. In terms of the basis $\{\gamma_j\}$, we have

$$\mathsf{E} = \sum_j \gamma_j^\vee \star \gamma_j.$$

Note that in particular

$$\mathsf{E} \equiv \chi(X)\mathsf{P} \bmod q$$

where $\chi(X)$ is the Euler characteristic of X.

In this paper we compute the quantum Euler class of all Fano nonsingular complete intersections of dimension at least 3 in a projective space (see Theorem 5 below). In particular, we prove a conjecture of Buch-Pandharipande, namely [7, Conjecture 5.14].

(1) Unless otherwise specified, (co)homology and quantum cohomology will always be taken with $\mathbb{Q}$-coefficients in this paper.

It is worth noting that although a priori the definition of E involves also the primitive cohomology of X, in our case of interest, this class actually lies in the restricted quantum cohomology ring $QH^*(X)^{\text{res}}$ of X, that is the quantum cohomology ring coming from the projective space (see Proposition 1 below for the exact definition of $QH^*(X)^{\text{res}}$). This is a key reason we were able to obtain so explicit a formula for E.

Finally, in [7] the quantum Euler class E of any variety X is related via a very simple formula (see [7, Theorem 1.4]) to the virtual Tevelev degrees of X, that is the virtual count of genus g maps of fixed complex structure in a given curve class β through n general points of X. Exploiting their formula and our explicit expression of E for X a Fano nonsingular complete intersection of dimension at least 3, we are able to compute all the virtual Tevelev degrees of such varieties X (see Theorem 10 below).

1.2. Preliminary results on complete intersections

We now specialize to smooth complete intersections of dimension at least 3. Let $X=V(f_1,...,f_L)\subset\mathbb{P}^{r+L}$ be a nonsingular complete intersection of dimension r. Assume for the rest of the paper that $r\geq 3$ and that for $i=1,...,L$,

$$f_i\in H^0(\mathbb{P}^{r+L},\mathcal{O}(m_i))$$

where $m_i\geq 2$.

Let $\mathsf{m}=(m_1,...,m_L)$ be the vector of degrees and, for $a,b\in\mathbb{Z}$ adopt the following notation:

$$|\mathsf{m}|=\sum_{i=1}^{L}m_i,\qquad \mathsf{m}^{a\mathsf{m}+b}=\prod_{i=1}^{L}m_i^{am_i+b}.$$

1.2.1. Cohomology of complete intersections

Consider the map

$$H^i(\mathbb{P}^{r+L})\longrightarrow H^i(X) \tag{1}$$

induced by the inclusion $X\subset\mathbb{P}^{r+L}$. By the Lefschetz Hyperplane Theorem, this map is an isomorphism for all $i\leq 2r, i\neq r$ and is injective for $i=r$. Also, for $i=r$, we have a canonical decomposition

$$H^r(X)=H^r(X)^{\text{prim}}\oplus H^r(X)^{\text{res}}$$

as a direct sum of the primitive cohomology and the restricted cohomology of degree r.

Explicitly

$$H^r(X)^{\text{res}} = \text{Im}(H^r(\mathbb{P}^{r+L}) \hookrightarrow H^r(X))$$

and

$$H^r(X)^{\text{prim}} = \text{Ker}(\mathsf{H} \cup - : H^r(X) \longrightarrow H^{r+2}(X))$$

where $\mathsf{H} \in H^*(X)$ is the hyperplane class.

Note that

$$\dim H^r(X)^{\text{prim}} = (-1)^r(\chi(X) - (r+1))$$

where $\chi(X)$ is the Euler characteristic of X.

1.2.2. Quantum cohomology of Fano complete intersections

From now on, we will further restrict our attention to the Fano case

$$|\mathsf{m}| \leq r + L. \tag{2}$$

Since $r \geq 3$, the map in Equation 1 is an isomorphism when $i = 2$ and thus

$$H_2(X) = \mathbb{Q}.\mathsf{L}$$

where $\mathsf{L} \in H^*(X)$ is the class of a line in X. It follows that $QH^*(X)$ is a graded algebra over the polynomial ring $\mathbb{Q}[q]$ in one variable q and as $\mathbb{Q}[q]$-modules we have

$$QH^*(X) = H^*(X) \otimes_{\mathbb{Q}} \mathbb{Q}[q].$$

The degree of q is equal to $2d$ where

$$d = r + L + 1 - |\mathsf{m}|.$$

which is greater than 0 by Equation 2.

Depending on the degree $|\mathsf{m}|$ of X, the ring $QH^*(X)$ satisfies the following magic relation (due to A. Givental):

- if $|\mathsf{m}| \leq r + L - 1$ we have:

$$\mathsf{H}^{\star(r+1)} = \mathsf{m}^{\mathsf{m}} q \mathsf{H}^{\star(|\mathsf{m}| - L)} \tag{3}$$

- if $|\mathsf{m}| = r + L$ we have:

$$(\mathsf{H} + \mathsf{m}! q)^{\star(r+1)} = \mathsf{m}^{\mathsf{m}} q (\mathsf{H} + \mathsf{m}! q)^{\star r} \tag{4}$$

Some cases of Equation 3 are proved in [3]. A complete proof of both relations can be found in Givental's paper [15]. There also are two very nice expositions of Givental's work, see [22, Section 3.2] and [6, Corollary 4.4 and Corollary 4.19]. Relations 3 and 4 will be essential for the object of this paper.

1.3. Statement of the main theorem

In Theorem 5 below we will give explicit formulas for the quantum Euler class of any smooth Fano complete intersection $X \subseteq \mathbb{P}^{r+L}$ as above. Much of our work starts with the results in [7], some of which we now recall for the reader's convenience.

Proposition 1. (due to T. Graber) *Let* $\mathsf{R} = \mathrm{Span}\{1, \mathsf{H}, ..., \mathsf{H}^r\} \subset H^*(X)$. *Then,* $(\mathsf{R} \otimes_{\mathbb{Q}} \mathbb{Q}[q], \star)$ *is a subring of* $QH^*(X)$.

Proof. This is [7, Proposition 5.1]. □

This ring is denoted by $QH^*(X)^{\mathrm{res}}$.

Remark 2. The elements $1, \mathsf{H}, ..., \mathsf{H}^{\star r}$ form a basis of $QH^*(X)^{\mathrm{res}}$ as $\mathbb{Q}[q]$-module. This follows from the fact that $\mathsf{H}^i = \mathsf{H}^{\star i}$ mod q and $\mathsf{H}^{\star i} \in QH^*(X)^{\mathrm{res}}$ for $i = 0, ..., r$.

Let E be the quantum Euler class of X.

Lemma 3. *We have* $\mathsf{E} \in QH^*(X)^{\mathrm{res}}$.

Proof. See [7, Proof of Proposition 5.5]. □

By Remark 2 and Lemma 3, we can uniquely write

$$\mathsf{E} = \sum_{i=0}^{\lfloor \frac{r}{d} \rfloor} \mathrm{Coeff}(\mathsf{E}, q^i \mathsf{H}^{\star(r-id)}) q^i \mathsf{H}^{\star(r-id)}.$$

where $\mathrm{Coeff}(\mathsf{E}, q^i \mathsf{H}^{\star(r-id)}) \in \mathbb{Q}$. Our goal is to make this coefficients explicit.

Remark 4. We have

$$\mathrm{Coeff}(\mathsf{E}, \mathsf{H}^{\star r}) = \mathsf{m}^{-1} \sum_j \int_X \gamma_j^\vee \cup \gamma_j = \mathsf{m}^{-1} \sum_j (-1)^{\deg(\gamma_j)} = \mathsf{m}^{-1} \chi(X).$$

The main result of the paper is the following:

Theorem 5. (Main Theorem) *The following equalities hold*:

- *if* $|\mathsf{m}| \leq r + L - 1$ *then*

$$\mathsf{E} = \mathsf{m}^{-1} \chi(X) \mathsf{H}^{\star r} + (r + L + 1 - |\mathsf{m}| - \chi(X)) \mathsf{m}^{\mathsf{m}-1} q \mathsf{H}^{\star |\mathsf{m}| - L - 1},$$

- *if* $|\mathsf{m}| = r + L$ *then*

$$\mathsf{E} = \mathsf{m}^{-1} \chi(X) \mathsf{H}^{\star r} + \sum_{j=1}^{r} \mathsf{m}^{-1} (j - \chi(X)) \binom{r}{j-1} (\mathsf{m}!)^{j-1} \left[\mathsf{m}^{\mathsf{m}} - \frac{\mathsf{m}!}{j}(r+1) \right] q^j \mathsf{H}^{\star r - j}.$$

The case $|\mathsf{m}| \leq r + L - 1$ in the theorem is exactly [7, Conjecture 5.14] and is already shown to be true mod q^2 in [7, Corollary 5.13]. The proof of this theorem is given in Section 3.

1.4. Application: virtual Tevelev degrees of Fano complete intersections

1.4.1. Virtual Tevelev degrees

Let X be a nonsingular, projective, algebraic variety over $\mathbb{C}$ of dimension r. Fix integers $g,n\geq 0$ satisfying the stability condition $2g-2+n>0$ and fix $\beta\in H_2(X,\mathbb{Z})$ an effective curve class satisfying the condition

$$\int_\beta c_1(T_X)>0.$$

Let $\overline{\mathcal{M}}_{g,n}(X,\beta)$ be the moduli space of genus g, n-pointed stable maps to X in class β and assume that the dimensional constraint

$$\mathrm{vdim}(\overline{\mathcal{M}}_{g,n}(X,\beta))=\dim(\overline{\mathcal{M}}_{g,n}\times X^n)$$

holds. This is equivalent to

$$\int_\beta c_1(T_X)=r(n+g-1). \tag{5}$$

Let

$$\tau:\overline{\mathcal{M}}_{g,n}(X,\beta)\longrightarrow\overline{\mathcal{M}}_{g,n}\times X^n$$

be the canonical morphism obtained from the domain curve and the evaluation maps:

$$\pi:\overline{\mathcal{M}}_{g,n}(X,\beta)\longrightarrow\overline{\mathcal{M}}_{g,n},\quad \mathrm{ev}:\overline{\mathcal{M}}_{g,n}(X,\beta)\longrightarrow X^n.$$

Then the **virtual Tevelev degree** $\mathrm{vTev}^X_{g,n,\beta}\in\mathbb{Q}$ of X is defined by the equality

$$\tau_*[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}=\mathrm{vTev}^X_{g,n,\beta}[\overline{\mathcal{M}}_{g,n}\times X^n]\in A^0(\overline{\mathcal{M}}_{g,n}\times X^n).$$

Alternatively, denoting by $\Omega^X_{g,n,\beta}:H^*(X)^{\otimes n}\to H^*(\overline{\mathcal{M}}_{g,n})$ the Gromov-Witten class

$$\Omega^X_{g,n,\beta}(\alpha):=\pi_*(\mathrm{ev}^*(\alpha)\cap[\overline{\mathcal{M}}_{g,n}(X,\beta)]^{\mathrm{vir}}),$$

we have

$$\mathrm{vTev}^X_{g,n,\beta}[\overline{\mathcal{M}}_{g,n}]=\Omega^X_{g,n,\beta}(\mathsf{P}^{\otimes n}).$$

Fixed-domain curve counts for Grassmanians have a beautiful story at the intersection between algebraic geometry and physics. They are computed by the celebrated Vafa-Intriligator formula, conjectured by the physicists Vafa and Intriligator [16] and partially proved by Siebert-Tian [23] and by Bertram-Daskalopoulos-Wentworth in [4] and [5], and fully proven by Marian-Oprea in [20] using Quot-schemes. The equivalence with the formulation in terms of stable maps was then proven by Marian-Oprea-Pandharipande in [21]. The systematic study of Tevelev degrees for general targets started with [11], motivated by work of Tevelev [24] on scattering amplitudes in mathematical physics. The paper [11] then stimulated a series of subsequent studies. In [11] these degrees were formally defined and computed via Hurwitz theory for the case of $\mathbb{P}^1$; then, in [13], using Schubert calculus the problem was posed and solved for the case of $\mathbb{P}^n$; in [10] a generalization of these degrees is presented for $\mathbb{P}^1$. In [7] a virtual perspective is adopted via Gromov-Witten theory and in [19] an equality between virtual and geometric Tevelev degrees is proven for certain Fano varieties and large degree curve classes. The relationship between virtual and geometric degrees is studied for point blow-up of projective spaces in [9], where the authors also provided simpled closed formulas for the Tevelev degrees of such varieties. In [18] geometric Tevelev degrees are computed for low degree hypersurfaces and large degree curve classes via projective geometry. Finally, a tropical perspective is presented in [8] where, after proving a generalization of Mikhalkin's correspondence theorem for logarithmic and tropical Tevelev degrees, the authors computed these degrees for Hirzebruch surfaces in genus 0.

1.4.2. Virtual Tevelev degrees of Fano complete intersections

In this paper, we are concerned with exact computations of virtual Tevelev degrees of Fano complete intersections following the perspective presented in [7] and described above in Section 1.4.1.

Let X be a smooth Fano complete intersection in $\mathbb{P}^{r+L}$ of dimension $r \geq 3$ and vector of degrees m. Writing $\beta = k\mathsf{L}$ with $k > 0$, condition 5 becomes

$$k = k[g,n] := \frac{n+g-1}{d}r.$$

For us, the main ingredient to compute $\mathrm{vTev}^X_{g,n,k}$ will be the following result:

Theorem 6. *Suppose $k = k[g,n]$. Then*

$$\mathrm{vTev}^X_{g,n,k} = \mathsf{m}^1 \mathrm{Coeff}(\mathsf{P}^{\star n} \star \mathsf{E}^{\star g}, q^k \mathsf{H}^{\star r}).$$

Proof. This is [7, Theorem 1.4]. □

Before stating our theorem, we require a remark and some additional notation.

Remark 7. Given the form of Equation 4, when $|\mathsf{m}| = r+L$ it will be more convenient to use $1, (\mathsf{H}+\mathsf{m}!q), \ldots, (\mathsf{H}+\mathsf{m}!q)^{\star r}$ instead of $1, \mathsf{H}, \ldots, \mathsf{H}^{\star r}$ as a basis of $QH^*(X)$ as $\mathbb{Q}[q]$-module.

Definition 8. Define

$$\mathbb{Q} \ni P_i = \begin{cases} \operatorname{Coeff}(\mathsf{P}, q^i \mathsf{H}^{\star r-id}) & \text{when } |\mathsf{m}| \leq r+L-1, \\ \operatorname{Coeff}(\mathsf{P}, q^i (\mathsf{H}+\mathsf{m}!q)^{\star r-i}) & \text{when } |\mathsf{m}| = r+L, \end{cases}$$

for $i=0, ..., \lfloor \frac{r}{d} \rfloor$ and

$$\mathbb{Q} \ni b_i = \begin{cases} \operatorname{Coeff}(\mathsf{P}^{\star n} \star \mathsf{E}^{\star g}, q^{i+k} \mathsf{H}^{\star r-id}) & \text{when } |\mathsf{m}| \leq r+L-1, \\ \operatorname{Coeff}(\mathsf{P}^{\star n} \star \mathsf{E}^{\star g}, q^{i+k} (\mathsf{H}+\mathsf{m}!q)^{\star r-i}) & \text{when } |\mathsf{m}| = r+L, \end{cases}$$

for $i=0, ..., \lfloor \frac{r}{d} \rfloor$.

Note that, by Theorem 6

$$\mathrm{vTev}^X_{g,n,k} = \mathsf{m}^1 b_0$$

and that by Theorem 5 the b_i's are determined by the P_i's.

Definition 9. Following [7, Definition 5.15], we define the **discrepancy** of $\mathsf{P}^{\star n} \star \mathsf{E}^{\star g}$ to be

$$\operatorname{Disc}(\mathsf{P}^{\star n} \star \mathsf{E}^{\star g}) = \sum_{i=1}^{\lfloor \frac{r}{d} \rfloor} b_i \mathsf{m}^{-i\mathsf{m}+1}.$$

Putting everything together we obtain explicit formulas for all virtual Tevelev degrees of X (once all the coefficients P_i are known):

Theorem 10. *Suppose $k=k[g,n]$. Then, the virtual Tevelev degrees of X are as follows*:

- *if* $|\mathsf{m}| \leq r+L-1$ *then*

$$\mathrm{vTev}^X_{g,n,k} = \left(\sum_{i=0}^{\lfloor \frac{r}{d} \rfloor} P_i \mathsf{m}^{-i\mathsf{m}} \right)^n (r+L+1-|\mathsf{m}|)^g \mathsf{m}^{k\mathsf{m}-g+1} - \operatorname{Disc}(\mathsf{P}^{\star n} \star \mathsf{E}^{\star g}),$$

- *if* $|\mathsf{m}| = r+L$ *then*

$$\mathrm{vTev}^X_{g,n,k} = \left(\sum_{i=0}^{r} P_i \mathsf{m}^{-i\mathsf{m}} \right)^n \left(1 - \mathsf{m}^{-r\mathsf{m}} (\mathsf{m}!)^r (r+1-\chi(X)) \right)^g \mathsf{m}^{k\mathsf{m}-g+1} - \operatorname{Disc}(\mathsf{P}^{\star n} \star \mathsf{E}^{\star g}).$$

The case $|\mathsf{m}| \le r+L-1$ already appears in [7, Proposition 5.16] (where they assume that [7, conjecture 5.14] holds for X), the case $|\mathsf{m}|=r+L$ is instead completely new.

The last question would be if we can actually express the coefficients P_i appearing in Theorem 10 in a closed formula, obtaining in this way a closed formula for the virtual Tevelev degrees.

Partial results have been obtained in [7], where they gave a complete answer in the following cases:

- for quadric hypersurfaces (see [7, Theorem 1.5 and Example 2.4];
- for low degree complete complete intersections $r>2|\mathsf{m}|-2L-2$ which are not quadrics (see [7, Corollary 5.11 and Theorem 5.19]);
- for the border case $r=2|\mathsf{m}|-2L-2$ (see [7, Lemma 5.21 and Corollary 5.23]).

Here we will content ourselves with illustrating in Section 5 an algorithm that calculates all the coefficients P_i. It should be noted here that the method we will describe is more effective than the general result in [1], where they deal with Gromov-Witten invariants in all genera with arbitrary insertions.

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2. A preliminary computation

We start with expressing H^i for $i=1,...,r$ as a linear combination of $1,...,\mathsf{H}^{\star r}$ with coefficients in $\mathbb{Q}[q]$.

The following notation will be convenient. For $k\ge 0$ and $0\le j\le r$ let

$$\alpha^k_{r-j} := \mathsf{m}^{-1}\langle \mathsf{H}^{kd+j-1},\mathsf{H}^{r-j}\rangle^X_{0,k} = \mathsf{m}^{-1}\int_{[\overline{\mathcal{M}}_{0,2}(X,k\mathsf{L})]^{\mathrm{vir}}} \mathrm{ev}_1{}^*\mathsf{H}^{kd+j-1}\cup \mathrm{ev}_2^*\mathsf{H}^{r-j}$$

where L is the class of a line in X and $\mathrm{ev}_i : \overline{\mathcal{M}}_{0,2}(X,k\mathsf{L})\to X$ are the evaluation maps for $i=1,2$. Note the following symmetry:

$$\alpha^k_{r-j} = \alpha^k_{kd+j-1}.$$

Proposition 11. *Let* $0 \le i \le r$*. Then, for* $1 \le j \le \lfloor \frac{i}{d} \rfloor$*, we have*

$$\mathrm{Coeff}(\mathsf{H}^i, q^j \mathsf{H}^{\star(i-jd)}) =$$

$$= \sum_{\ell:1\le\ell\le j} (-1)^\ell \sum_{\substack{(i_1,\dots,i_\ell)\in\mathbb{Z}_{\ge1}:\\ i_1+\dots+i_\ell=j}} \sum_{\substack{(u_1,\dots,u_\ell)\in(\mathbb{Z}_{\ge0})^{\times\ell}:\\ 0\le u_\ell\le\dots\le u_1\le i-jd}} \prod_{a=1}^{\ell} i_a \alpha^{i_a}_{r-(j-i_1-\dots-i_a)d-u_a}.$$

Proof. We proceed by induction on i.

When $i<d$ there is nothing to prove. When $i=d$, it follows that $j=1$ and the right-hand side of the equation is just α^1_r. To compute the left-hand side note that $\mathsf{H}^i = \mathsf{H}^{\star i}$ for $i<d$ and thus

$$\mathsf{H}^{\star d} = \mathsf{H}\star\mathsf{H}^{d-1} = \mathsf{H}^d + q\langle \mathsf{H}, \mathsf{H}^{d-1}, \mathsf{H}^r\rangle^X_{0,1}\mathsf{m}^{-1}.$$

Note that, since $\mathsf{H}\star\mathsf{H}^{d-1}\in QH^*(X)^{\mathrm{res}}$, the primitive cohomology contribution in $\mathsf{H}\star\mathsf{H}^{d-1}$ is 0.

Use now the divisor equation in Gromov-Witten theory to obtain

$$\langle \mathsf{H}, \mathsf{H}^{d-1}, \mathsf{H}^r\rangle^X_{0,1}\mathsf{m}^{-1} = \alpha^1_r.$$

Assume now that the Theorem is true for $i=d,\dots,t-1<r$.

Write

$$\mathsf{H}\star\mathsf{H}^{t-1} = \mathsf{H}^t + \sum_{k=1}^{\lfloor \frac{t}{d}\rfloor} \langle \mathsf{H}, \mathsf{H}^{t-1}, \mathsf{H}^{kd+r-t}\rangle^X_{0,k}\mathsf{m}^{-1} q^k \mathsf{H}^{t-kd}.$$

where again the primitive cohomology contributions in $\mathsf{H}\star\mathsf{H}^{t-1}$ is 0 and by the divisor equation

$$\mathsf{m}^{-1}\langle \mathsf{H}, \mathsf{H}^{t-1}, \mathsf{H}^{kd+r-t}\rangle^X_{0,k} = k\alpha^k_{r-(t-kd)}.$$

Note now that, by induction, we know how to write the H^{t-kd} and H^{t-1} in terms of $1, \mathsf{H}, \dots, \mathsf{H}^{\star r}$. Putting everything together we obtain for $1\le j\le \lfloor \frac{t}{d}\rfloor$

$$\begin{aligned}
&\mathrm{Coeff}(\mathsf{H}^t, q^j\mathsf{H}^{\star(t-jd)})\\
&= \mathrm{Coeff}(\mathsf{H}^{t-1}, q^j\mathsf{H}^{\star(t-1-jd)}) - \sum_{k=1}^{j} k\alpha^k_{r-(t-kd)}\mathrm{Coeff}(\mathsf{H}^{t-kd}, q^{j-k}\mathsf{H}^{\star(t-jd)})\\
&= \sum_{\ell=1}^{j}(-1)^\ell \sum_{\substack{i_1+\dots+i_\ell=j,\\ 0\le u_\ell\le\dots\le u_1\le t-1-jd}} \prod_{a=1}^{\ell} i_a\alpha^{i_a}_{r-(j-i_1-\dots-i_a)d-u_a}\\
&\quad - \sum_{k=1}^{j} k\alpha^k_{r-(t-kd)}\left(\sum_{\ell=0}^{j-k}(-1)^\ell \sum_{\substack{i_1+\dots+i_\ell=j-k,\\ 0\le u_\ell\le\dots\le u_1\le t-jd}} \prod_{a=1}^{\ell} i_a\alpha^{i_a}_{r-(j-k-i_1-\dots-i_a)d-u_a}\right)
\end{aligned}$$

and, since

$$r-(t-kd)=r-(j-k)d-(t-jd),$$

this is exactly the right-hand side appearing in the statement above with $i=t$. □

Corollary 12. *For* $1\le j\le\lfloor\frac{r}{d}\rfloor$ *we have*

$$\begin{aligned}&\mathrm{Coeff}(\mathsf{H}^{\star r+1},q^j\mathsf{H}^{\star r+1-jd})\\ (6)\qquad &=\sum_{\ell:1\le\ell\le j}(-1)^{\ell+1}\sum_{\substack{(i_1,\dots,i_\ell)\in\mathbb{Z}_{\ge1}:\\ i_1+\dots+i_\ell=j}}\ \sum_{\substack{(u_1,\dots,u_\ell)\in(\mathbb{Z}_{\ge0})^{\times\ell}:\\ 0\le u_\ell\le\dots\le u_1\le r+1-jd}}\prod_{a=1}^{\ell}i_a\alpha^{i_a}_{r-(j-i_1-\dots-i_a)d-u_a}.\end{aligned}$$

Proof. Proceeding as in the proof of Proposition 11, we write $\mathsf{H}^{\star r+1}=\mathsf{H}\star\mathsf{H}^{\star r}$ and

$$\mathsf{H}^{\star r}=\mathsf{H}^r-\sum_{j=1}^{\lfloor\frac{r}{d}\rfloor}\mathrm{Coeff}(\mathsf{H}^r,q^j\mathsf{H}^{\star r-jd})q^j\mathsf{H}^{\star r-jd}.$$

Since

$$\mathsf{H}\star\mathsf{H}^r=\sum_{k=1}^{\lfloor\frac{r+1}{d}\rfloor}k\alpha^k_{kd-1}q^k\mathsf{H}^{r-kd+1},$$

we have

$$\begin{aligned}&\mathrm{Coeff}(\mathsf{H}^{\star(r+1)},q^j\mathsf{H}^{\star(r+1-jd)})\\ &\quad=-\mathrm{Coeff}(\mathsf{H}^r,q^j\mathsf{H}^{\star(r-jd)})\\ &\qquad+\sum_{k=1}^{j}k\alpha^k_{r-(r+1-kd)}\mathrm{Coeff}(\mathsf{H}^{r+1-kd},q^{j-k}\mathsf{H}^{\star r+1-jd})\\ &\quad=\sum_{1\le\ell\le j}(-1)^{\ell+1}\sum_{\substack{i_1+\dots+i_\ell=j,\\ 0\le u_\ell\le\dots\le u_1\le r-jd}}\prod_{a=1}^{\ell}i_a\alpha^{i_a}_{r-(j-i_1-\dots-i_a)d-u_a}\\ &\qquad-\sum_{k=1}^{j}k\alpha^k_{kd-1}\sum_{1\le\ell\le j-k}(-1)^{\ell+1}\sum_{\substack{i_1+\dots+i_\ell=j-k,\\ 0\le u_\ell\le\dots\le u_1\le r+1-jd}}\prod_{a=1}^{\ell}i_a\alpha^{i_a}_{r-(j-k-i_1-\dots-i_a)d-u_a}\end{aligned}$$

for $j=1,\dots,\lfloor\frac{r}{d}\rfloor$. Finally, since $kd-1=r-(j-k)d-(r+1-jd)$, we are done. □

3. Proof of the main theorem

3.1. Plan of the proof

In this subsection we explain how the proof of Theorem 5 goes.

Define

$$\Gamma := \sum_{j:\gamma_j \in H^r(X)^{\text{prim}}} \gamma_j^\vee \star \gamma_j$$

and

$$\mathsf{E}' := \mathsf{m}^{-1} \sum_{i=0}^{r} \mathsf{H}^i \star \mathsf{H}^{r-i}$$

Then

$$\mathsf{E} = \Gamma + \mathsf{E}'.$$

So by Lemma 3, we see that $\Gamma \in QH^*(X)^{\text{res}}$.

Using relations 3 and 4, the proof of Theorem 5 becomes an easy algebraic count (done in Section 3.4) once we know the following two propositions.

Proposition 13. *For* $j=1,\dots,\lfloor \frac{r}{d} \rfloor$ *we have*

$$\text{Coeff}(\Gamma, q^j \mathsf{H}^{\star r-jd}) = \mathsf{m}^{-1}(r+1-\chi(X))\text{Coeff}(\mathsf{H}^{\star r+1}, q^j \mathsf{H}^{\star r+1-jd}).$$

The proof is presented in Section 3.2.

Proposition 14. *For* $j=1,\dots,\lfloor \frac{r}{d} \rfloor$ *we have*

$$\text{Coeff}(\mathsf{E}', q^j \mathsf{H}^{\star r-jd}) = -\mathsf{m}^{-1}(r-jd+1)\text{Coeff}(\mathsf{H}^{\star r+1}, q^j \mathsf{H}^{\star r+1-jd}).$$

The proof is presented in Section 3.3.

We remark here that the way we prove Proposition 14 is purely algebraic. It would be interesting to find a more conceptual explanation for this equality.

3.2. Computation of Γ

The proof of Proposition 13 relies on the following preliminary lemma which is very similar to [7, Lemma 5.2].

Lemma 15. *Let* $\Lambda \in QH^*(X)^{\text{res}}$ *be a degree* $2r$ *class such that*

$$\Lambda = a\mathsf{H}^r \text{ mod } q \text{ and } \mathsf{H}\star\Lambda = 0$$

where $a \in \mathbb{Q}$. *Then*

$$\text{Coeff}(\Lambda, q^j \mathsf{H}^{\star r-jd}) = -a\text{Coeff}(\mathsf{H}^{\star r+1}, q^j \mathsf{H}^{r+1-jd})$$

for $i=1,\dots,\lfloor \frac{r}{d} \rfloor$.

Proof. Write

$$\Lambda = a\mathsf{H}^{\star r} + \sum_{i=1}^{\lfloor \frac{r}{d} \rfloor} \mathrm{Coeff}(\Lambda, q^i \mathsf{H}^{\star r-id}) q^i \mathsf{H}^{\star r-id}.$$

Then we have

$$0 = \mathsf{H} \star \Lambda = a\mathsf{H}^{\star r+1} + \sum_{i=1}^{\lfloor \frac{r}{d} \rfloor} \mathrm{Coeff}(\Lambda, q^i \mathsf{H}^{\star r-id}) q^i \mathsf{H}^{\star r+1-id}$$

from which we obtain the lemma. ☐

Proof of Proposition 13. By [7, Corollary 5.3] (the proof of that corollary also applies when $|\mathsf{m}| = r + L$), we have $\mathsf{H} \star \Gamma = 0$. Moreover

$$\Gamma = \sum_{j : \gamma_j \in H^r(X)^{\mathrm{Prim}}} \gamma_j^\vee \cup \gamma_j = \dim(H^r(X)^{\mathrm{prim}})(-1)^r \mathsf{m}^{-1} \mathsf{H}^r \mod q.$$

Note that we have $\dim(H^r(X)^{\mathrm{prim}}) = (-1)^r(\chi(X) - r - 1)$. The proposition now follows from an application of Lemma 15. ☐

3.3. Computation of E'

In this subsection we prove Proposition 14 by showing the following equivalent result:

Lemma 16. *For $j = 1, \ldots, \lfloor \frac{r}{d} \rfloor$ we have*

$$\sum_{i=0}^{r} \mathrm{Coeff}(\mathsf{H}^i \star \mathsf{H}^{r-i}, q^j \mathsf{H}^{\star(r-jd)}) = -(r - jd + 1) \text{ times the RHS of Equation 6.}$$

Proof. For $0 \le i \le r$ and $1 \le j \le \lfloor \frac{r}{d} \rfloor$ we have

$$\begin{aligned} &\mathrm{Coeff}(\mathsf{H}^i \star \mathsf{H}^{r-i}, q^j \mathsf{H}^{\star(r-jd)}) \\ &\quad = \sum_{(h,s) \in \mathbb{Z}_{\ge 0}^{\times 2} : h+s=j} \mathrm{Coeff}(\mathsf{H}^i, q^h \mathsf{H}^{\star(i-hd)}) \mathrm{Coeff}(\mathsf{H}^{r-i}, q^s \mathsf{H}^{\star(r-i-sd)}) \end{aligned}$$

and for each (h, s) as above such that $hd \le i$ and $r - i \ge sd$, the product

$$\mathrm{Coeff}(\mathsf{H}^i, q^h \mathsf{H}^{\star(i-hd)}) \mathrm{Coeff}(\mathsf{H}^{r-i}, q^s \mathsf{H}^{\star(r-i-sd)})$$

is equal to

$$\sum_{\substack{1\le w\le h\\1\le z\le s}}(-1)^{w+z}\sum_{\substack{y_1+\ldots+y_w=h\\x_1+\ldots+x_z=s}}\sum_{\substack{0\le p_w\le\ldots\le p_1\le i-hd\\0\le v_z\le\ldots\le v_1\le r-i-sd}}\prod_{a=1}^{w} i_a\alpha^{i_a}_{r-(h-y_1-\ldots-y_a)d-p_a}$$

$$\times\prod_{b=1}^{z}x_b\alpha^{x_b}_{r-(s-x_1-\ldots-x_b)d-v_b}. \tag{7}$$

Note that it could be that $h{=}0$ or $s{=}0$ (but not $h{=}s{=}0$, since $h{+}s{=}j{>}0$). Observe the symmetry

$$\alpha^{x_b}_{r-(s-x_1-\ldots-x_b)d-v_b}=\alpha^{x_b}_{(s-x_1-\ldots-x_{b-1})d+v_b-1},$$

and that

$$(s-x_1-\ldots-x_{b-1})d+v_b-1=r-[(j-x_b-\ldots-x_z)d+r-v_b+1-jd]$$

where $r-v_b+1-jd$ varies in $[i-hd+1,r+1-jd]$ for $0\le v_b\le r-i-sd$. Therefore we can rewrite the quantity in Equation 7 as

$$\sum_{\substack{1\le w\le h\\1\le z\le s}}(-1)^{w+z}\sum_{\substack{y_1+\ldots+y_w=h\\x_1+\ldots+x_z=s}}\sum_{\substack{0\le p_w\le\ldots\le p_1\le i-hd\\i-hd+1\le v_1\le\ldots\le v_z\le r+1-jd}}\prod_{a=1}^{w} i_a\alpha^{i_a}_{r-(h-y_1-\ldots-y_a)d-p_a}$$

$$\times\prod_{b=1}^{z}x_b\alpha^{x_b}_{r-(j-x_b-\ldots-x_z)d-v_b}. \tag{8}$$

Note that the quantity $i-hd$ appearing in Equation 8 under the third summation symbol varies in $[0,r-jd]$ and not in $[0,r]$ (if $i-hd>r-jd$, then $r-i<(j-h)d=sd$ and so $\mathrm{Coeff}(\mathsf{H}^{r-i},q^s\mathsf{H}^{\star(r-i-sd)}){=}0$).

Fix $j\in\{1,\ldots,\lfloor\frac{r}{d}\rfloor\}$ and let $\ell,(i_1,\ldots,i_\ell)$ and $(u_1,\ldots,u_\ell)$ be such that

$$0\le\ell\le j,\ i_1+\ldots+i_\ell=j\quad\text{and}\quad 0\le u_\ell\le\ldots\le u_1\le r+1-jd.$$

We want to count how many times the term

$$(-1)^\ell\prod_{a=1}^{\ell}i_a\alpha^{i_a}_{r-(j-i_1-\ldots-i_a)d-u_a} \tag{9}$$

appears in

$$\begin{aligned}b_j&:=\sum_{i=0}^{r}\mathrm{Coeff}(\mathsf{H}^i\star\mathsf{H}^{r-i},q^j\mathsf{H}^{\star(r-jd)})\\&=\sum_{i=0}^{r}\sum_{h+s=j}\mathrm{Coeff}(\mathsf{H}^i,q^h\mathsf{H}^{\star(i-hd)})\mathrm{Coeff}(\mathsf{H}^{r-i},q^s\mathsf{H}^{\star(r-i-sd)}).\end{aligned}$$

We observe that $w+z=\ell$ and $(x_z,...,x_1,y_1,...,y_w)=(i_1,...,i_\ell)$ must hold. Moreover, given any integer $g\in[0,r-jd]$, if $i-hd=g$ then in Equation 8 we must have

$$z=\min\{f:u_f\leq g\}-1 \quad \text{and} \quad w=\ell-z$$

where if $\{f:u_f\leq g\}=\varnothing$, we set $z=\ell$ and $w=\ell$.

Therefore

$$v_z=u_1,...,v_1=u_z,p_1=u_{z+1},...,p_w=u_\ell$$

and

$$x_z=i_1,...,x_1=i_z,y_1=i_{z+1},...,y_w=i_\ell$$

and finally

$$h=y_1+...+y_w \quad \text{and} \quad s=x_1+...+x_z.$$

This means that the term in Equation 9 appears in b_j once for every $g\in[0,r-jd]$, and thus a total of $r-jd+1$ times. This concludes the proof of the lemma. □

3.4. Computation of E

We finally prove Theorem 5. We will distinguish two cases.

- **Case $|\mathsf{m}|\leq r+L-1$.**

By Relation 3, in this case we have

$$\mathrm{Coeff}(\mathsf{H}^{\star r+1},q^j\mathsf{H}^{\star r+1-jd})=0 \text{ for } j>1 \quad \text{and} \quad \mathrm{Coeff}(\mathsf{H}^{\star r+1},q\mathsf{H}^{\star r+1-d})=\mathsf{m}^{\mathsf{m}}.$$

Therefore, by Propositions 13 and 14, we have

$$\mathrm{Coeff}(\mathsf{E},q^j\mathsf{H}^{\star r-jd})=0 \text{ for } j>1 \quad \text{and}$$
$$\mathrm{Coeff}(\mathsf{E},q\mathsf{H}^{\star r-d})=\mathsf{m}^{m-1}(r+L+1-|\mathsf{m}|-\chi(X)).$$

This is what we wanted to prove.

- **Case $|\mathsf{m}|=r+L$.**

Note that in this case $d=1$. Relation 4, can be rewritten as

$$\mathrm{Coeff}(\mathsf{H}^{\star r+1},q^j\mathsf{H}^{\star r+1-j})=\binom{r}{j-1}(\mathsf{m}!)^{j-1}\left[\mathsf{m}^{\mathsf{m}}-\frac{\mathsf{m}!}{j}(r+1)\right]$$

for $j=1,...,r+1$. Therefore for $j=1,...,r$ we have

$$\begin{aligned}\mathrm{Coeff}(\mathsf{E},q^j\mathsf{H}^{\star r-j})&=\mathsf{m}^{-1}(j-\chi(X))\mathrm{Coeff}(\mathsf{H}^{\star r+1},q^j\mathsf{H}^{\star r+1-j})\\&=\mathsf{m}^{-1}(j-\chi(X))\binom{r}{j-1}(\mathsf{m}!)^{j-1}\left[\mathsf{m}^{\mathsf{m}}-\frac{\mathsf{m}!}{j}(r+1)\right].\end{aligned}$$

This concludes the proof.

4. Virtual Tevelev degrees

We now apply our computations to prove Theorem 10. We distinguish two cases again.

- **Case** $|\mathsf{m}| \le r + L - 1$.

This case follows from [7, Proposition 5.16] and Theorem 5 above.

- **Case** $|\mathsf{m}| = r + L$.

The first step is to express E in terms of the basis $1, \mathsf{H} + \mathsf{m}!q, \ldots, (\mathsf{H} + \mathsf{m}!q)^{\star r}$. This will use the following simple combinatorial lemma.

Lemma 17. *For $j = 2, \ldots, r$ the following two equalities hold:*

$$\sum_{i=1}^{j} \binom{r}{i-1} \binom{r-i}{j-i} (-1)^{j-i} = 1 \tag{10}$$

and

$$\sum_{i=1}^{j} i \binom{r}{i-1} \binom{r-i}{j-i} (-1)^{j-i} = r + 1. \tag{11}$$

Proof. The proof is left to the reader. □

Lemma 18. *We have*

$$\begin{aligned}\mathsf{E} = {} & \mathsf{m}^{-1}\chi(X)(\mathsf{H} + \mathsf{m}!q)^{\star r} + [\mathsf{m}^{-1}(r + 1 - \chi(X))(\mathsf{m}^{\mathsf{m}} - \mathsf{m}!) - \mathsf{m}^{\mathsf{m}-1} r] q (\mathsf{H} + \mathsf{m}!q)^{\star r-1} \\ & + \sum_{j=2}^{r} [\mathsf{m}^{-1}(\mathsf{m}!)^{j-1}(r + 1 - \chi(X))(\mathsf{m}^{\mathsf{m}} - \mathsf{m}!)] q^j (\mathsf{H} + \mathsf{m}!q)^{\star r-j}.\end{aligned}$$

Proof. This is an algebraic check substituting

$$\mathsf{H} = (\mathsf{H} + \mathsf{m}!q) - \mathsf{m}!q$$

in the expression of E found in Theorem 5.

Here we will deal with $\mathrm{Coeff}(\mathsf{E}, q^j (\mathsf{H} + \mathsf{m}!q)^{\star r-j})$ for $j = 2, \ldots, r$. The cases $j = 0, 1$ are instead left to the reader.

Using Theorem 5, we see that for $j = 2, \ldots, r$ the coefficient $\mathrm{Coeff}(\mathsf{E}, q^j (\mathsf{H} + \mathsf{m}!q)^{\star r-j})$ is equal to

$$\mathsf{m}^{-1}\chi(X)\binom{r}{j}(-1)^j(\mathsf{m}!)^j$$

$$+\sum_{i=1}^{j}\mathsf{m}^{-1}(i-\chi(X))\binom{r}{i-1}(\mathsf{m}!)^{i-1}\left[\mathsf{m}^{\mathsf{m}}-\mathsf{m}!\frac{(r+1)}{i}\right]\binom{r-i}{j-i}(-1)^{j-i}(\mathsf{m}!)^{j-i}$$

which we now rewrite as a sum of four terms. The first one is

$$\begin{aligned}&\mathsf{m}^{-1}\chi(X)(\mathsf{m}!)^{j-1}\mathsf{m}!\left[(-1)^j\binom{r}{j}+\sum_{i=1}^{j}(-1)^{j-i}\binom{r}{i-1}\binom{r-i}{j-i}\frac{r+1}{i}\right]\\&=\mathsf{m}^{-1}\chi(X)(\mathsf{m}!)^{j-1}\mathsf{m}!\end{aligned}$$

where we used Equation 10. The second one is

$$-\mathsf{m}^{-1}\chi(X)(\mathsf{m}!)^{j-1}\mathsf{m}^{\mathsf{m}}\sum_{i=1}^{j}\binom{r}{i-1}\binom{r-i}{j-i}(-1)^{j-i}=-\mathsf{m}^{-1}\chi(X)(\mathsf{m}!)^{j-1}\mathsf{m}^{\mathsf{m}}$$

where we used Equation 10. The third term is

$$-\mathsf{m}^{-1}(\mathsf{m}!)^{j-1}\mathsf{m}!\sum_{i=1}^{j}\binom{r}{i-1}\binom{r-i}{j-i}(-1)^{j-i}(r+1)=-\mathsf{m}^{-1}(\mathsf{m}!)^{j-1}\mathsf{m}!(r+1)$$

where we used again Equation 10. Finally the last term is

$$\mathsf{m}^{-1}(\mathsf{m}!)^{j-1}\mathsf{m}^{\mathsf{m}}\sum_{i=1}^{j}i\binom{r}{i-1}\binom{r-i}{j-i}(-1)^{j-i}=\mathsf{m}^{-1}(\mathsf{m}!)^{j-1}\mathsf{m}^{\mathsf{m}}(r+1)$$

where instead we used Equation 11.

Summing everything up we obtain the desired conclusion. □

Although the full expression of E might be a bit complicated, the product $(\mathsf{H}+\mathsf{m}!q)^{\star r}\star\mathsf{E}$ is quite simple.

Corollary 19. *We have*

$$(\mathsf{H}+\mathsf{m}!q)^{\star r}\star\mathsf{E}=[\mathsf{m}^{-1}-\mathsf{m}^{-r\mathsf{m}-1}(\mathsf{m}!)^r(r+1-\chi(X))](\mathsf{H}+\mathsf{m}!q)^{\star 2r}.$$

Proof. Use r times Equality 4. □

We can now finish the proof of Theorem 10.

Proof of Theorem 10 when $|\mathsf{m}|=r+L$. From Definition 8 and Equation 4, we see that

$$\mathsf{P}^{\star n}\star\mathsf{E}^{\star g}\star(\mathsf{H}+\mathsf{m}!q)^{\star r}=\left(\sum_{i=0}^{r}b_i\mathsf{m}^{-(k+i)\mathsf{m}}\right)(\mathsf{H}+q\mathsf{m}!)^{\star r+rg+nr}. \tag{12}$$

Using Definition 8 and Equation 4, we also have

$$\mathsf{P}^{\star n}\star(\mathsf{H}+\mathsf{m}!q)^{\star r}=\left(\sum_{i=0}^{r}P_i\mathsf{m}^{-i\mathsf{m}}\right)^n(\mathsf{H}+q\mathsf{m}!)^{\star nr+r},$$

and so by Corollary 19

$$\begin{aligned}&\mathsf{P}^{\star n}\star\mathsf{E}^{\star g}\star(\mathsf{H}+\mathsf{m}!q)^{\star r}\\ &=\left(\sum_{i=0}^{r}P_i\mathsf{m}^{-i\mathsf{m}}\right)^n\left(\mathsf{m}^{-1}-\mathsf{m}^{-r\mathsf{m}-1}(\mathsf{m}!)^r(r+1-\chi(X))\right)^g(\mathsf{H}+\mathsf{m}!q)^{\star r+gr+nr}.\end{aligned}\tag{13}$$

The theorem follows by comparing Equation 12 and Equation 13. □

5. An algorithm for the calculation of the coefficients P_i

In this final section we propose a method to compute the coefficients P_i appearing in Definition 8. In this way, up to implementing the algorithm with a computer, all the virtual Tevelev degrees of X can be explicitly calculated.

It is possible that our result is known to the experts, but we preferred to include it anyway for completeness.

5.1. Recursion for genus 0 two-pointed Hyperplane Gromov-Witten invariants

Proposition 11 reduces the computation of the P_i's to the computation of genus 0 two-pointed Hyperplane Gromov-Witten invariants of X. These invariants satisfies a recursion involving more general integrals which we now recall.

5.1.1. The recursion

For $g\geq 0$, $k>0$ and $n>0$ the **gravitational descendant invariants** of X are defined by:

$$\langle\tau_{a_1}(\gamma_1),\dots,\tau_{a_n}(\gamma_n)\rangle_{g,k}^X:=\int_{[\overline{\mathcal{M}}_{g,n}(X,k\mathsf{L})]^{\mathrm{vir}}}\mathrm{ev}_1^*(\gamma_1)\cup\psi_1^{a_1}\cup\dots\cup\mathrm{ev}_n^*(\gamma_n)\cup\psi_n^{a_n}$$

where $\gamma_1,\dots,\gamma_n\in H^*(X)$ and $\psi_i=c_1(\mathbb{L}_i)\in H^2(\overline{\mathcal{M}}_{g,n}(X,k\mathsf{L}))$ is the first Chern class of the cotangent line

$$\mathbb{L}_i\big|_{[f:(C,p_1,\dots,p_n)\to X]}=(T_{p_i}C)^\vee$$

for $i=1,\dots,n$.

We start with a monodromy result.

Lemma 20. *For any* $\gamma \in H^*(X)^{\mathrm{prim}}$, $\gamma_1, \ldots, \gamma_n \in H^*(X)^{\mathrm{res}}$ (*with* $n \geq 0$), $a_1, \ldots, a_n \in \mathbb{Z}_{\geq 0}$ *and* $k>0$ *we have*

$$\langle \tau_{a_1}(\gamma_1), \ldots, \tau_{a_n}(\gamma_n), \gamma \rangle_{0,k}^X = 0.$$

Proof. The proof is a monodromy argument. Let

$$U \subset \prod_{i=1}^{L} \mathbb{P}(H^0(\mathbb{P}^{r+L}, \mathcal{O}(m_i)))$$

be the open subscheme parametrizing smooth complete intersection in $\mathbb{P}^{r+L}$ of dimension r and degree m. Call $V = V^{\mathrm{prim}} \oplus V^{\mathrm{res}}$ where $V^{\mathrm{prim}} = H^*(X)^{\mathrm{prim}} \otimes_{\mathbb{Q}} \mathbb{R}$ and $V^{\mathrm{res}} = H^*(X)^{\mathrm{res}} \otimes_{\mathbb{Q}} \mathbb{R}$, and

$$\rho: \pi_1(U, u) \longrightarrow \mathrm{Aut}(V)$$

the monodromy homomorphism (here $u \in U$ is the point corresponding to X). The homomorphism ρ preserves the decomposition $V = V^{\mathrm{prim}} \oplus V^{\mathrm{res}}$ and actually its invariant subspace is exactly V^{res}. Let $G \subset \mathrm{GL}(V^{\mathrm{prim}})$ be the algebraic monodromy group defined as the Zariski closure of the image of $\pi_1(U, u) \to \mathrm{Aut}(V^{\mathrm{prim}})$. The lemma will follow from the following two standard facts:

- the invariance under deformations of X in Gromov-Witten theory tells us that for any $\alpha \in \pi_1(U, u)$ we have:

$$\langle \tau_{a_1}(\alpha.\gamma_1), \ldots, \tau_{a_n}(\alpha.\gamma_n), \alpha.\gamma \rangle_{0,k}^X = \langle \tau_{a_1}(\gamma_1), \ldots, \tau_{a_n}(\gamma_n), \gamma \rangle_{0,k}^X;$$

- the intersection form Q on V^{prim} is preserved by the monodromy action. When r is odd, Q is a non-degenerate skew-symmetric bilinear form, it follows that in this case $\dim(V^{\mathrm{prim}})$ is even and that $G \subseteq \mathrm{Sp}(V^{\mathrm{prim}})$. When instead r is even, Q is a non-degenerate symmetric bilinear form and we have $G \subseteq \mathrm{O}(V^{\mathrm{prim}})$. Since for us $r \geq 3$, by [12, Theorem 4.4.1] (see also [1, Proposition 4.2]), the previous inclusions are actually equalities except for the case when r is even and $\mathsf{m} = (2,2)$. In this latter case, $\dim(V^{\mathrm{prim}}) = r+3$ and G is the Weyl group $\mathcal{W}$ of D_{r+3}.

Since $-\mathrm{Id} \in \mathrm{Sp}(V^{\mathrm{prim}})$ and $-\mathrm{Id} \in \mathrm{O}(V^{\mathrm{prim}})$ the proof is complete in all cases except for the case r even and $\mathsf{m} = (2,2)$. In this case, note that if $L: V^{\mathrm{prim}} \to V^{\mathrm{prim}}$ is any $\mathbb{R}$-linear map invariant under $\mathcal{W}$ then $L = 0$ must hold (reason: if $\Phi \subset V^{\mathrm{prim}}$ is the root system corresponding to D_{r+3} then for all $v \in \Phi$ the reflection r_v along the hyperplane $v^{\perp}$ lies in $\mathcal{W}$ and sends v to $-v$, thus $L(v) = L(-v) = -L(v)$, from which $L(v) = 0$. Since $\mathrm{Span}_{\mathbb{R}}(\Phi) = V^{\mathrm{prim}}$ we are done). To conclude the proof of the lemma, apply this observation with $L = \langle \tau_{a_1}(\gamma_1), \ldots, \tau_{a_n}(\gamma_n), - \rangle_{0,k}^X$. $\square$

Proposition 21. *Let $i,a\geq 0$ and $j,k>0$ be integers satisfying*

$$i+j+a=\mathrm{vdim}(\overline{\mathcal{M}}_{0,2}(X,k\mathsf{L})).$$

Then we have

$$\langle\tau_a(\mathsf{H}^i),\mathsf{H}^j\ \rangle_{0,k}^X=\langle\tau_a(\mathsf{H}^{i+1}),\mathsf{H}^{j-1}\rangle_{0,k}^X+k\langle\tau_{a+1}(\mathsf{H}^i),\mathsf{H}^{j-1}\rangle_{0,k}^X \\ -\sum_{\ell=1}^{k-1}\mathsf{m}^{-1}\ell\langle\tau_a(\mathsf{H}^i),\mathsf{H}^{\ell d+r-1-i-a}\rangle_{0,\ell}^X\langle\mathsf{H}^{j-1},\mathsf{H}^{(k-\ell)d+r-j}\rangle_{0,k-\ell}^X.$$

Proof. An application of [17, Corollary 1] and the splitting principle in Gromov-Witten theory yields

$$\langle\tau_a(\mathsf{H}^i),\mathsf{H}^j\rangle_{0,k}^X=\langle\tau_a(\mathsf{H}^{i+1}),\mathsf{H}^{j-1}\rangle_{0,k}^X+k\langle\tau_{a+1}(\mathsf{H}^i),\mathsf{H}^{j-1}\rangle_{0,k}^X \\ -\sum_{\ell=1}^{k-1}\sum_{j=0}^{N}\ell\langle\tau_a(\mathsf{H}^i),\gamma_j^\vee\rangle_{0,\ell}^X\langle\mathsf{H}^{j-1},\gamma_j\rangle_{0,k-\ell}^X$$

where as always $\{\gamma_j\}_{j=0}^N$ is any homogeneous basis of $H^*(X)$ with $\gamma_0=1$ and $\gamma_N=\mathsf{P}$. Finally, apply Lemma 20 to conclude the proof. □

5.1.2. The base case

Consider the recursion of Proposition 21. In each two-pointed Gromov-Witten integral on the right-hand side, either the quantity j decreased or the quantity k decreased (when compared to those appearing in the left-hand side). Note also that when $k=1$, the recursion becomes simply

$$\langle\tau_a(\mathsf{H}^i),\mathsf{H}^j\rangle_{0,1}^X=\langle\tau_a(\mathsf{H}^{i+1}),\mathsf{H}^{j-1}\rangle_{0,1}^X+\langle\tau_{a+1}(\mathsf{H}^i),\mathsf{H}^{j-1}\rangle_{0,1}^X.$$

So, when $k=1$, k stabilizes while j continues to decrease. It follows that the recursion completely determines all the integrals

$$\langle\tau_a(\mathsf{H}^i),\mathsf{H}^j\rangle_{0,k}^X \text{ for } a,i,j\geq 0 \text{ such that } a+i+j=\mathrm{vdim}(\overline{\mathcal{M}}_{0,2}(X,k\mathsf{L}))$$

once the integrals $\langle\tau_a(\mathsf{H}^i),1\rangle_{0,k}^X$ are given for all $a,i\geq 0$ and $k>0$. These last invariants are indeed known as the next proposition shows.

Proposition 22. *Let $a,i\geq 0$ and $k>0$ be integers such that*

$$a+i=\mathrm{vdim}(\overline{\mathcal{M}}_{0,2}(X,k\mathsf{L})).$$

Then

- *for* $|\mathsf{m}|\le r+L-1$ *and* $i=0,\dots,r$ *we have*

$$\langle \tau_{r+kd-1-i}(\mathsf{H}^i),1\rangle^X_{0,k}=\mathrm{Coeff}\left(\frac{\prod_{j=1}^L\prod_{\ell=0}^{km_j}(m_jx+\ell)}{\prod_{\ell=1}^k(x+\ell)^{r+L+1}},x^{r+L-i}\right);$$

- *for* $|\mathsf{m}|=r+L$ *and* $i=0,\dots,r$ *we have*

$$\langle \tau_{r+k-1-i}(\mathsf{H}^i),1\rangle^X_{0,k}=\sum_{h=0}^k\frac{(-\mathsf{m}!)^{k-h}}{(k-h)!}\mathrm{Coeff}\left(\frac{\prod_{j=1}^L\prod_{\ell=0}^{hm_j}(m_jx+\ell)}{\prod_{\ell=1}^h(x+\ell)^{r+L+1}},x^{r+L-i}\right)$$

where in both cases the coefficient of x^{r+L-i} *is meant to be the coefficient of the Taylor expansion in* x *at* 0.

Proof. This is just a way of rephrasing [6, Theorem 4.2 and Theorem 4.17]. Note that in [6, Theorem 4.17] there is a typo: in their notation, their index m in the product appearing in the numerator should range from 0 to dl_j, instead of from 1 to d. □

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