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# Regularity of symbolic powers of square-free monomial ideals

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**Abstract.** We study the regularity of symbolic powers of square-free monomial ideals. We prove that if $I=I_\Delta$ is the Stanley-Reisner ideal of a simplicial complex $\Delta$, then $\operatorname{reg}(I^{(n)})\leqslant\delta(n-1)+b$ for all $n\geqslant 1$, where $\delta=\lim_{n\to\infty}\operatorname{reg}(I^{(n)})/n$, $b=\max\{\operatorname{reg}(I_\Gamma)\,|\,\Gamma$ is a subcomplex of $\Delta$ with $\mathcal{F}(\Gamma)\subseteq\mathcal{F}(\Delta)\}$, and $\mathcal{F}(\Gamma)$ and $\mathcal{F}(\Delta)$ are the set of facets of $\Gamma$ and $\Delta$, respectively. This bound is sharp for any $n$. When $I=I(G)$ is the edge ideal of a simple graph $G$, we obtain a general linear upper bound $\operatorname{reg}(I^{(n)})\leqslant 2n+\text{ord-match}(G)-1$, where ord-match$(G)$ is the ordered matching number of $G$.

## Introduction

Throughout the paper, let $K$ be a field and $R=K[x_1,\dots,x_r]$ the polynomial ring of $r$ variables $x_1,\dots,x_r$ with $r\geqslant 1$. Let $I$ be a homogeneous ideal of $R$. Then the $n$-th symbolic power of $I$ is defined by

$$I^{(n)}=\bigcap_{\mathfrak{p}\in\operatorname{Min}(I)} I^nR_{\mathfrak{p}}\cap R,$$

where $\operatorname{Min}(I)$ is as usual the set of minimal associated prime ideals of $I$.

Cutkosky, Herzog, Trung [5], and independently Kodiyalam [21], proved that the function $\operatorname{reg}(I^n)$ is a linear function in $n$ for $n\gg 0$. The similar result for symbolic powers is not true even when $I$ is a square-free monomial ideal (see e.g. [8, Theorem 5.15]) except for the case $\dim(R/I)\leqslant 2$ (see [19]).

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If $I$ is a square-free monomial ideal, Hoa and the second author (see [18, Theorem 4.9]) proved that the limit

$$\delta(I)=\lim_{n\to\infty}\frac{\operatorname{reg}(I^{(n)})}{n}, \tag{1}$$

does exist, in fact the limit exists for arbitrary monomial ideals (see [8]). Moreover, $\operatorname{reg}(I^{(n)})<\delta(I)n+\dim(R/I)+1$ for all $n\geqslant 1$. This bound is obviously not sharp for every $n$ (see Corollary 2.4). There have been many recent results which establish sharp bounds for $\operatorname{reg}(I^{(n)})$ in the case $I$ is the edge ideal of a simple graph (see e.g. [1], [13], [14] and [20]).

The aim of this paper is to find sharp bounds for $\operatorname{reg}(I^{(n)})$, for a square-free monomial ideal $I$, in terms of combinatorial data from its associated simplicial complexes and hypergraphs.

For a simplicial complex $\Delta$ on the set $V=\{1,...,r\}$, the Stanley-Reisner ideal of $\Delta$ is defined by

$$I_\Delta=\left(\prod_{i\in\tau}x_i \,|\, \tau\subseteq V \text{ and } \tau\notin\Delta\right)\subseteq R.$$

Let us denote by $\mathcal{F}(\Delta)$ the set of all facets of $\Delta$.

The first main result of the paper is the following theorem.

**Theorem 2.3** *Let $\Delta$ be a simplicial complex. Then,*

$$\operatorname{reg}(I_\Delta^{(n)})\leqslant\delta(I_\Delta)(n-1)+b,\quad \textit{for all } n\geqslant 1,$$

*where $b=\max\{\operatorname{reg}(I_\Gamma)\,|\,\Gamma$ is a subcomplex of $\Delta$ with $\mathcal{F}(\Gamma)\subseteq\mathcal{F}(\Delta)\}$.*

This bound is sharp for every $n$ (see Example 2.7). It is worth mentioning that the number $\delta(I_\Delta)$, which is determined by Equation (1), may be not an integer and even bigger than $\operatorname{reg}(I_\Delta)$ (see [8, Lemma 5.14 and Theorem 5.15]).

For a simple hypergraph $\mathcal{H}=(V,E)$ with vertex set $V=\{1,...,r\}$, the edge ideal of $\mathcal{H}$ is defined by

$$I(\mathcal{H})=\left(\prod_{i\in e}x_i\,|\,e\in E\right)\subseteq R.$$

Let $\mathcal{H}^*$ be the simple hypergraph corresponding to the Alexander duality $I(\mathcal{H})^*$ of $I(\mathcal{H})$. Let $\varepsilon(\mathcal{H}^*)$ be the minimum number of cardinality of edgewise dominant sets of $\mathcal{H}^*$, this concept was introduced by Dao and Schweig [7].

Then second main result of the paper is the following theorem.

**Theorem 2.6** *Let $\mathcal{H}$ be a simple hypergraph. Then,*

$$\operatorname{reg}(I(\mathcal{H})^{(n)})\leqslant\delta(I(\mathcal{H}))(n-1)+|V(\mathcal{H})|-\varepsilon(\mathcal{H}^*),\quad \textit{for all } n\geqslant 1.$$

A hypergraph is a graph if every edge has exactly two vertices. For a graph $G$, a linear lower bound for $\operatorname{reg}(I(G)^{(n)})$ is given in [14]:

$$\operatorname{reg}(I(G)^{(n)}) \geqslant 2n+\nu(G)-1,$$

where $\nu(G)$ is the induced matching number of $G$. Note that this lower bound is also valid for ordinary powers (see [2, Theorem 4.5]).

On the upper bounds, Fakhari (see [13, Conjecture 1.3]) conjectured that

$$\operatorname{reg}(I(G)^{(n)}) \leqslant 2n+\operatorname{reg}(I(G))-2,$$

This conjecture may be the best bound up to now of our knowledge.

By using Theorem 2.3, we obtain a general linear upper bound for $\operatorname{reg}(I(G)^{(n)})$ in terms of the ordered matching number of $G$, although it is weaker than the one in this conjecture, it provides us a sharp bound. Note that this result also settles the question (2) of Fakhari in [12].

**Theorem 3.5** *Let $G$ be a graph. Then,*

$$\operatorname{reg}(I(G)^{(n)}) \leqslant 2n+\text{ord-match}(G)-1, \text{ for all } n \geqslant 1,$$

*where* ord-match$(G)$ *is the ordered matching number of $G$.*

Let us explain the idea to prove Theorems 2.3 and 2.6 as follows. Let $i\geqslant 0$ such that $\operatorname{reg}(R/I^{(n)})=a_i(R/I^{(n)})+i$ (See Section 1.1 for more details).

The first key point is to prove that $a_i(R/I^{(n)})\leqslant\delta(I)(n-1)$. Assume that $\boldsymbol{\alpha}=(\alpha_1,...,\alpha_r)\in\mathbb{Z}^r$ such that

$$H^i_{\mathfrak{m}}(R/I^{(n)})_{\boldsymbol{\alpha}} \neq 0, \text{ and } a_i(R/I^{(n)}) = |\boldsymbol{\alpha}|,$$

where $\mathfrak{m}=(x_1,...,x_r)$ and $|\boldsymbol{\alpha}|=\alpha_1+...+\alpha_r$. We reduce to the case $\boldsymbol{\alpha}\in\mathbb{N}^r$. In order to bound $|\boldsymbol{\alpha}|$, we use Takayama's formula (see Lemma 1.4) to compute $H^i_{\mathfrak{m}}(R/I^{(n)})_{\boldsymbol{\alpha}}$, which allows us to search for $\boldsymbol{\alpha}$ in a polytope in $\mathbb{R}^r$, so that we can get the desired bound of $|\boldsymbol{\alpha}|$ via theory of convex polytopes (see Theorem 2.2).

The second key point is to bound the index $i$ by using the regularity of a Stanley-Reisner ideal in terms of the vanishing of reduced homology of simplicial complexes which derived from Hochster's formula about the Hilbert series of the local cohomology module of Stanley-Reisner ideals (see Lemma 1.2).

Our paper is structured as follows. In the next section, we collect notations and terminology used in the paper, and recall a few auxiliary results. In Section 2, we prove Theorems 2.3 and 2.6. In the last section, we prove Theorem 3.5.

# 1. Preliminaries

We shall follow standard notations and terminology from usual texts in the research area (cf. [9], [16] and [22]). For simplicity, we denote the set $\{1, \ldots, r\}$ by $[r]$.

## 1.1. Regularity and projective dimension

Through out this paper, let $K$ be a field, and let $R = K[x_1, \ldots, x_r]$ be a standard graded polynomial ring of $r$ variables over $K$. The object of our work is the Castelnuovo-Mumford regularity of graded modules and ideals over $R$. This invariant can be defined via either the minimal free resolutions or the local cohomology modules.

Let $M$ be a nonzero finitely generated graded $R$-module and let

$$0 \longrightarrow \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{p,j}(M)} \longrightarrow \ldots \longrightarrow \bigoplus_{j \in \mathbb{Z}} R(-j)^{\beta_{0,j}(M)} \longrightarrow 0$$

be the minimal free resolution of $M$. The *Castelnuovo–Mumford regularity* (or regularity for short) of $M$ is defined by

$$\operatorname{reg}(M) = \max\{j - i \mid \beta_{i,j}(M) \neq 0\},$$

and the *projective dimension* of $M$ is the length of this resolution

$$\operatorname{pd}(M) = p.$$

Let us denote by $d(M)$ the maximal degree of a minimal homogeneous generator of $M$. The definition of the regularity implies

$$d(M) \leqslant \operatorname{reg}(M).$$

For any nonzero proper homogeneous ideal $I$ of $R$, by looking at the minimal free resolution, it is easy to see that $\operatorname{reg}(I) = \operatorname{reg}(R/I) + 1$, so we shall work with $\operatorname{reg}(I)$ and $\operatorname{reg}(R/I)$ interchangeably.

The regularity of $M$ can also be computed via the local cohomology modules of $M$. For $i = 0, \ldots, \dim(M)$, we define the $a_i$-invariant of $M$ as follows

$$a_i(M) = \max\{t \mid H^i_{\mathfrak{m}}(M)_t \neq 0\}$$

where $H^i_{\mathfrak{m}}(M)$ is the $i$-th local cohomology module of $M$ with the support $\mathfrak{m} = (x_1, \ldots, x_r)$ (with the convention $\max \varnothing = -\infty$). Then,

$$\operatorname{reg}(M) = \max\{a_i(M) + i \mid i = 0, \ldots, \dim(M)\},$$

and

$$\operatorname{pd}(M) = r - \min\{i \mid H^i_{\mathfrak{m}}(M) \neq 0\}.$$

For example, since $\dim(R/\mathfrak{m})=0$ and $H^0_{\mathfrak{m}}(R/\mathfrak{m})=R/\mathfrak{m}$, we have

$$\operatorname{reg}(\mathfrak{m}) = \operatorname{reg}(R/\mathfrak{m})+1 = a_0(R/\mathfrak{m})+1 = \max\{i \mid (R/\mathfrak{m})_i \neq 0\}+1 = 1.$$

**Remark 1.1.** As usual we shall make the convention that $\operatorname{reg}(M)=-\infty$ if $M=0$.

### 1.2. Simplicial complexes and Stanley-Reisner ideals

A simplicial complex $\Delta$ over a finite set $V$ is a collection of subsets of $V$ such that if $F\in\Delta$ and $G\subseteq F$ then $G\in\Delta$. Elements of $\Delta$ are called faces. Maximal faces (with respect to inclusion) are called facets. For $F\in\Delta$, the dimension of $F$ is defined to be $\dim F=|F|-1$. The empty set, $\varnothing$, is the unique face of dimension $-1$, as long as $\Delta$ is not the void complex $\{\}$ consisting of no subsets of $V$. If every facet of $\Delta$ has the same cardinality, then $\Delta$ is called a *pure* complex. The dimension of $\Delta$ is $\dim\Delta=\max\{\dim F \mid F\in\Delta\}$. The link of $F$ inside $\Delta$ is its subcomplex:

$$\operatorname{lk}_\Delta(F) = \{H\in\Delta \mid H\cup F\in\Delta \text{ and } H\cap F=\varnothing\}.$$

Every element in a face of $\Delta$ is called a *vertex* of $\Delta$. Let us denote $V(\Delta)$ to be the set of vertices of $\Delta$. If there is a vertex, say $j$, such that $\{j\}\cup F\in\Delta$ for every $F\in\Delta$, then $\Delta$ is called a *cone* over $j$. It is well-known that if $\Delta$ is a cone, then it is an acyclic complex. Recall that a chain complex is called an *acyclic* complex if all of whose homology groups are zero. A complex is called a *simplex* if it contains all subsets of its vertices, and thus a simplex is a cone over every its vertex.

For a subset $\tau=\{j_1,...,j_i\}$ of $[r]$, denote $\mathbf{x}^\tau=x_{j_1}...x_{j_i}$. Let $\Delta$ be a simplicial complex over the set $V=\{1,...,r\}$. The Stanley-Reisner ideal of $\Delta$ is defined to be the squarefree monomial ideal

$$I_\Delta = (\mathbf{x}^\tau \mid \tau\subseteq[r] \text{ and } \tau\notin\Delta) \quad \text{in } R=K[x_1,...,x_r]$$

and the *Stanley-Reisner* ring of $\Delta$ to be the quotient ring $k[\Delta]=R/I_\Delta$. This provides a bridge between combinatorics and commutative algebra (see [22], [26]).

Note that if $I$ is a square-free monomial ideal, then it is a Stanley-Reisner ideal of the simplicial complex $\Delta(I)=\{\tau\subseteq[r] \mid \mathbf{x}^\tau\notin I\}$. When $I$ is a monomial ideal (maybe not square-free) we also use $\Delta(I)$ to denote the simplicial complex corresponding to the square-free monomial ideal $\sqrt{I}$.

The regularity of a square-free monomial ideal can compute via the vanishing of reduced homology of simplicial complexes. From Hochster's formula on the Hilbert series of the local cohomology module $H^i_{\mathfrak{m}}(R/I_\Delta)$ (see [22, Theorem 13.13]), one has

**Lemma 1.2.** *For a simplicial complex* $\Delta$, *we have*

$$\mathrm{reg}(I_\Delta)=\mathrm{reg}(R/I_\Delta)+1=\max\{d \mid \widetilde{H}_{d-1}(\mathrm{lk}_\Delta(\sigma);K)\neq 0, \text{ for some } \sigma\in\Delta\}+1.$$

The *Alexander dual* of $\Delta$, denoted by $\Delta^*$, is the simplicial complex over $V$ with faces

$$\Delta^*=\{V\setminus\tau \mid \ \tau\notin\Delta\}.$$

Notice that $(\Delta^*)^*=\Delta$. If $I=I_\Delta$ then we shall denote the Stanley-Reisner ideal of the Alexander dual $\Delta^*$ by $I^*$. It is a well-known result of Terai [28] (or see [22, Theorem 5.59]) that the regularity of a squarefree monomial ideal can be related to the projective dimension of its Alexander dual.

**Lemma 1.3.** *Let* $I\subseteq R$ *be a square-free monomial ideal. Then,*

$$\mathrm{reg}(I)=\mathrm{pd}(R/I^*).$$

Let $\mathcal{F}(\Delta)$ denote the set of all facets of $\Delta$. We say that $\Delta$ is generated by $\mathcal{F}(\Delta)$ and write $\Delta=\langle\mathcal{F}(\Delta)\rangle$. Note that $I_\Delta$ has the minimal primary decomposition (see [22, Theorem 1.7]):

$$I_\Delta=\bigcap_{F\in\mathcal{F}(\Delta)}(x_i \mid i\notin F),$$

and therefore the $n$-th symbolic power of $I_\Delta$ is

$$I_\Delta^{(n)}=\bigcap_{F\in\mathcal{F}(\Delta)}(x_i \mid i\notin F)^n.$$

We next describe a formula to compute the local cohomology modules of monomial ideals. Let $I$ be a non-zero monomial ideal. Since $R/I$ is an $\mathbb{N}^r$-graded algebra, $H^i_{\mathfrak{m}}(R/I)$ is an $\mathbb{Z}^r$-graded module over $R/I$ for every $i$. For each degree $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_r)\in\mathbb{Z}^r$, in order to compute $\dim_K H^i_{\mathfrak{m}}(R/I)_{\boldsymbol{\alpha}}$ we use a formula given by Takayama [27, Theorem 2.2] which is a generalization of Hochster's formula for the case $I$ is square-free [26, Theorem 4.1].

Set $G_{\boldsymbol{\alpha}}=\{i|\alpha_i<0\}$. For a subset $F\subseteq[r]$, we set $R_F=R[x_i^{-1}|i\in F\cup G_{\boldsymbol{\alpha}}]$. Define the simplicial complex $\Delta_{\boldsymbol{\alpha}}(I)$ by

$$\Delta_\alpha(I)=\{F\subseteq[r]\setminus G_{\boldsymbol{\alpha}} \mid x^\alpha\notin IR_F\}. \tag{2}$$

**Lemma 1.4.** [27, *Theorem* 2.2] $\dim_K H^i_{\mathfrak{m}}(R/I)_{\boldsymbol{\alpha}}=\dim_K \widetilde{H}_{i-|G_\alpha|-1}(\Delta_{\boldsymbol{\alpha}}(I);K)$.

The following result of Minh and Trung is very useful to compute $\Delta_{\boldsymbol{\alpha}}(I_\Delta^{(n)})$, which allows us to investigate $\mathrm{reg}(I_\Delta^{(n)})$ by using the theory of convex polyhedra.

**Lemma 1.5.** [23, *Lemma* 1.3] *Let* $\Delta$ *be a simplicial complex and* $\boldsymbol{\alpha}\in\mathbb{N}^r$. *Then,*

$$\mathcal{F}(\Delta_{\boldsymbol{\alpha}}(I_\Delta^{(n)}))=\left\{F\in\mathcal{F}(\Delta)\,|\sum_{i\notin F}\alpha_i\leqslant n-1\right\}.$$

This lemma can be generalized a little bit as follows.

**Lemma 1.6.** [19, *Lemma* 1.3] *Let* $\Delta$ *be a simplicial complex and* $\boldsymbol{\alpha}\in\mathbb{Z}^r$. *Then,*

$$\mathcal{F}(\Delta_{\boldsymbol{\alpha}}(I_\Delta^{(n)}))=\left\{F\in\mathcal{F}(\mathrm{lk}_\Delta(G_{\boldsymbol{\alpha}}))\,|\sum_{i\notin F\cup G_{\boldsymbol{\alpha}}}\alpha_i\leqslant n-1\right\}.$$

### 1.3. Hypergraphs

Let $V$ be a finite set. A simple hypergraph $\mathcal{H}$ with vertex set $V$ consists of a set of subsets of $V$, called the edges of $\mathcal{H}$, with the property that no edge contains another. We use the symbols $V(\mathcal{H})$ and $E(\mathcal{H})$ to denote the vertex set and the edge set of $\mathcal{H}$, respectively.

In this paper we assume that all hypergraphs are simple unless otherwise specified.

In the hypergraph $\mathcal{H}$, an edge is *trivial* if it contains only one element, a vertex is *isolated* if it does not appear in any edge, a vertex is a *neighbor* of another one if they are in some edge.

A hypergraph $\mathcal{H}'$ is a *subhypergraph* of $\mathcal{H}$ if $V(\mathcal{H}')\subseteq V(\mathcal{H})$ and $E(\mathcal{H}')\subseteq E(\mathcal{H})$. For an edge $e$ of $\mathcal{H}$, we define $\mathcal{H}\setminus e$ to be the hypergraph obtained by deleting $e$ from the edge set of $\mathcal{H}$. For a subset $S\subseteq V(\mathcal{H})$, we define $\mathcal{H}\setminus S$ to be the hypergraph obtained from $\mathcal{H}$ by deleting the vertices in $S$ and all edges containing any of those vertices.

A set $S\subseteq E(\mathcal{H})$ is called an *edgewise dominant set* of $\mathcal{H}$ if every non-isolated vertex of $\mathcal{H}$ is either contained in a non-trivial edge of $S$ or has a neighbor contained in an edge of $S$. Define,

$$\varepsilon(\mathcal{H})=\min\{|S|\mid S\text{ is edgewise dominant}\}.$$

For a hypergraph $\mathcal{H}$ with $V(\mathcal{H})\subseteq[r]$, we associate to the hypergraph $\mathcal{H}$ a square-free monomial ideal

$$I(\mathcal{H})=(\mathbf{x}^e\mid e\in E(\mathcal{H}))\subseteq R,$$

which is called the *edge ideal* of $\mathcal{H}$.

Notice that if $I$ is a square-free monomial ideal, then $I$ is an edge ideal of a hypergraph with the edge set uniquely determined by the generators of $I$.

Let $\mathcal{H}^*$ be the simple hypergraph corresponding to the Alexander duality $I(\mathcal{H})^*$ of $I(\mathcal{H})$. We will determine the edge set of $\mathcal{H}^*$, it turns out that $E(\mathcal{H}^*)$ is the set of all minimal vertex covers of $\mathcal{H}$. A *vertex cover* in a hypergraph is a set of vertices, such that every edge of the hypergraph contains at least one vertex of that set. It is an extension of the notion of vertex cover in a graph. A vertex cover $S$ is called minimal if no proper subset of $S$ is a vertex cover. From the minimal primary decomposition (see [22, Definition 1.35 and Proposition 1.37]):

$$I(\mathcal{H}^*) = \bigcap_{e \in E(\mathcal{H})} (x_i \mid i \in e),$$

it follows that $E(\mathcal{H}^*)$ is just the set of minimal vertex covers of $\mathcal{H}$. Thus,

$$I(\mathcal{H}^*) = (\mathbf{x}^\tau \mid \tau \text{ is a minimal vertex cover of } \mathcal{H}).$$

In the sequel, we need the following result of Dao and Schweig [7, Theorem 3.2].

**Lemma 1.7.** *Let $\mathcal{H}$ be a hypergraph. Then,* $\operatorname{pd}(R/I(\mathcal{H})) \leqslant |V(\mathcal{H})| - \varepsilon(\mathcal{H})$.

### 1.4. Matchings in a graph

Let $G$ be a graph. A *matching* in $G$ is a subgraph consisting of pairwise disjoint edges. If this subgraph is an induced subgraph, then the matching is called an *induced matching*. A matching of $G$ is maximal if it is maximal with respect to inclusion. The *matching number* of G, denoted by $\operatorname{match}(G)$, is the maximum size of a matching in $G$; and the *induced matching number* of $G$, denoted by $\nu(G)$, is the maximum size of an induced matching in $G$.

An *independent set* in $G$ is a set of vertices no two of which are adjacent to each other. An independent set in $G$ is maximal (with respect to set inclusion) if the set cannot be extended to a larger independent set. Let $\Delta(G)$ denote the set of all independent sets of $G$. Then, $\Delta(G)$ is a simplicial complex, called the *independence complex* of $G$. It is well-known that $I(G) = I_{\Delta(G)}$.

According to Constantinescu and Varbaro [3], we say that a matching $M = \{\{u_i, v_i\} \mid i = 1, ..., s\}$ is an *ordered matching* if:

(1) $\{u_1, ..., u_s\} \in \Delta(G)$,

(2) $\{u_i, v_j\} \in E(G)$ implies $i \leqslant j$.

The *ordered matching number* of $G$, denoted by $\operatorname{ord-match}(G)$ is the maximum size of an ordered matching in $G$.

The following result gives a lower bound for $\operatorname{reg}(I(G)^{(n)})$ in terms of the induced matching number $\nu(G)$

**Lemma 1.8.** [14, *Theorem* 4.6] *Let $G$ be a graph. Then,*

$$\operatorname{reg}(I(G)^{(n)}) \geqslant 2n+\nu(G)-1, \text{ for all } n \geqslant 1.$$

## 1.5. Convex polyhedra

The theory of convex polyhedra plays a key role in our study.

For a vector $\boldsymbol{\alpha}=(\alpha_1,...,\alpha_r)\in\mathbb{R}^r$, we set $|\boldsymbol{\alpha}|:=\alpha_1+...+\alpha_r$ and for a nonempty bounded closed subset $S$ of $\mathbb{R}^r$ we set

$$\delta(S) := \max\{|\boldsymbol{\alpha}| \mid \boldsymbol{\alpha} \in S\}.$$

Let $\Delta$ be a simplicial complex over $[r]$. In general, $\operatorname{reg}(I_\Delta^{(n)})$ is not a linear function in $n$ for $n\gg 0$ (see e.g. [8, Theorem 5.15]), but a quasi-linear function as in the following result.

**Lemma 1.9.** [18, *Theorem* 4.9] *There exist positive integers $N, n_0$ and rational numbers $a, b_0, ..., b_{N-1} < \dim(R/I_\Delta)+1$ such that*

$$\operatorname{reg}(I_\Delta^{(n)}) = an+b_k, \text{ for all } n\geqslant n_0 \text{ and } n\equiv k \mod N, \text{ where } 0\leqslant k\leqslant N-1.$$

*Moreover,* $\operatorname{reg}(I_\Delta^{(n)}) < an+\dim(R/I_\Delta)+1$ *for all* $n\geqslant 1$.

By virtue of this result, we define

$$\delta(I_\Delta) = a = \lim_{n\to\infty} \frac{\operatorname{reg}(I_\Delta^{(n)})}{n}.$$

In order to compute this invariant we can use the geometric interpretation of it by means of symbolic polyhedra defined in [4], [8]. Let $\mathcal{SP}(I_\Delta)$ be the convex polyhedron in $\mathbb{R}^r$ defined by the following system of linear inequalities:

$$(3) \qquad \begin{cases} \sum_{i\notin F} x_i \geqslant 1 & \text{for } F\in\mathcal{F}(\Delta), \\ x_1\geqslant 0, ..., x_r\geqslant 0, \end{cases}$$

which is called the *symbolic polyhedron* of $I_\Delta$. Then, $\mathcal{SP}(I_\Delta)$ is a convex polyhedron in $\mathbb{R}^r$. By [8, Theorem 3.6] we have

$$(4) \qquad \delta(I_\Delta) = \max\{|\mathbf{v}| \mid \mathbf{v} \text{ is a vertex of } \mathcal{SP}(I_\Delta)\}.$$

Now assume that

$$H^i_{\mathfrak{m}}(R/I_\Delta^{(n)})_{\boldsymbol{\alpha}} \neq 0$$

for some $0\leqslant i\leqslant \dim(R/I_\Delta)$ and $\boldsymbol{\alpha}=(\alpha_1,...,\alpha_r)\in\mathbb{N}^r$.

By Lemma 1.4 we have

$$\dim_K \widetilde{H}_{i-1}(\Delta_{\boldsymbol{\alpha}}(I_\Delta^{(n)}); K) = \dim_K H_{\mathfrak{m}}^i(R/I_\Delta^{(n)})_{\boldsymbol{\alpha}} \neq 0. \tag{5}$$

In particular, $\Delta_{\boldsymbol{\alpha}}(I_\Delta^{(n)})$ is not acyclic.

Suppose that $\mathcal{F}(\Delta) = \{F_1, ..., F_t\}$ for $t \geqslant 1$. By Lemma 1.5 we may assume that

$$\mathcal{F}(\Delta_{\boldsymbol{\alpha}}(I_\Delta^{(n)})) = \{F_1, ..., F_s\}, \text{ where } 1 \leqslant s \leqslant t.$$

For each integer $m \geqslant 1$, let $\mathcal{P}_m$ be the convex polyhedron of $\mathbb{R}^r$ defined by:

$$\begin{cases} \sum\limits_{i \notin F_j} x_i \leqslant m-1 & \text{for } j = 1, ..., s, \\ \sum\limits_{i \notin F_j} x_i \geqslant m & \text{for } j = s+1, ..., t, \\ x_1 \geqslant 0, ..., x_r \geqslant 0. \end{cases} \tag{6}$$

Then, $\boldsymbol{\alpha} \in \mathcal{P}_n$. Moreover, by Lemma 1.5 one has

$$\Delta_{\boldsymbol{\beta}}(I_\Delta^{(m)}) = \langle F_1, ..., F_s \rangle = \Delta_{\boldsymbol{\alpha}}(I_\Delta^{(n)}) \ \text{ whenever } \boldsymbol{\beta} \in \mathcal{P}_m \cap \mathbb{N}^r. \tag{7}$$

Note also that for such a vector $\boldsymbol{\beta}$, by Formula (7) we have

$$\dim_K \widetilde{H}_{i-1}(\Delta_{\boldsymbol{\beta}}(I_\Delta^{(m)}); K) = \dim_K \widetilde{H}_{i-1}(\Delta_{\boldsymbol{\alpha}}(I_\Delta^{(n)}); K) \neq 0.$$

Together with Lemma 1.4, this fact yields

$$H_{\mathfrak{m}}^i(R/I_\Delta^{(m)})_{\boldsymbol{\beta}} \neq 0. \tag{8}$$

In order to investigate the convex polyhedron $\mathcal{P}_m$ we also consider the convex polyhedron $\mathcal{C}_m$ in $\mathbb{R}^r$ defined by:

$$\begin{cases} \sum\limits_{i \notin F_j} x_i \leqslant m & \text{for } j = 1, ..., s, \\ \sum\limits_{i \notin F_j} x_i \geqslant m & \text{for } j = s+1, ..., t, \\ x_1 \geqslant 0, ..., x_r \geqslant 0. \end{cases} \tag{9}$$

Note that $\mathcal{C}_m = m\mathcal{C}_1$ for all $m \geqslant 1$, where $m\mathcal{C}_1 = \{m\mathbf{y} \mid \mathbf{y} \in \mathcal{C}_1\}$.

By the same way as in the proof of [15, Lemma 2.1] we obtain the following lemma.

**Lemma 1.10.** *$\mathcal{C}_1$ is a polytope with* $\dim \mathcal{C}_1 = r$.

The next lemma gives an upper bound for $\delta(\mathcal{C}_1)$.

**Lemma 1.11.** $\delta(C_1) \leqslant \delta(I_\Delta)$.

*Proof.* Since $\mathcal{C}_1$ is a polytope with $\dim \mathcal{C}_1 = r$ by Lemma 1.10, $\delta(\mathcal{C}_1) = |\boldsymbol{\gamma}|$ for some vertex $\boldsymbol{\gamma}$ of $\mathcal{C}_1$. By [25, Formula (23) on Page 104] we obtain that $\boldsymbol{\gamma}$ is the unique solution of a system of linear equations of the form

$$\begin{cases} \sum_{i \notin F_j} x_i = 1 & \text{for } j \in S_1, \\ x_j = 0 & \text{for } \ j \in S_2, \end{cases} \tag{10}$$

where $S_1 \subseteq [t]$ and $S_2 \subseteq [r]$ such that $|S_1| + |S_2| = r$. By using Cramer's rule to get $\boldsymbol{\gamma}$, we conclude that $\boldsymbol{\gamma}$ is a rational vector. In particular, there is a positive integer, say $p$, such that $p\boldsymbol{\gamma} \in \mathbb{N}^r$. Note that $\mathcal{C}_p = p\mathcal{C}_1$, so $p\boldsymbol{\gamma} \in \mathcal{C}_p \cap \mathbb{N}^r$.

For every $j \geqslant 1$, let $\mathbf{y} = jp\boldsymbol{\gamma} + \boldsymbol{\alpha}$. Then, $\mathbf{y} \in \mathbb{N}^r$ and $|\mathbf{y}| = \delta(\mathcal{C}_1)jp + |\boldsymbol{\alpha}|$. On the other hand, by using the fact that $jp\boldsymbol{\gamma} \in \mathcal{C}_{jp}$ and $\boldsymbol{\alpha} \in \mathcal{P}_n$, we can check that

$$\begin{cases} \sum_{i \notin F_j} y_i \leqslant jp + n - 1 & \text{for } j = 1, \ldots, s, \\ \sum_{i \notin F_j} y_i \geqslant jp + n & \text{for } j = s+1, \ldots, t, \end{cases}$$

and so $\mathbf{y} \in \mathcal{P}_{jp+n} \cap \mathbb{N}^r$.

Together with Equation (8), we deduce that $H^i_{\mathfrak{m}}(R/I_\Delta^{(jp+n)})_{\mathbf{y}} \neq 0$, and therefore

$$\operatorname{reg}(R/I_\Delta^{(jp+n)}) \geqslant |\mathbf{y}| + i = \delta(\mathcal{C}_1)jp + |\boldsymbol{\alpha}| + i.$$

Combining with Lemma 1.9, this inequality yields

$$\delta(\mathcal{C}_1)jp + |\boldsymbol{\alpha}| + i < \delta(I_\Delta)(jp+n) + \dim(R/I_\Delta).$$

Since this inequality valid for any positive integer $j$, it forces $\delta(\mathcal{C}_1) \leqslant \delta(I_\Delta)$. $\square$

## 2. Regularity of symbolic powers of ideals

In this section we will prove the upper bound for $\operatorname{reg}(I_\Delta^{(n)})$. First, we start with the following fact.

**Lemma 2.1.** *Let $\sigma \subseteq [r]$ with $\sigma \neq [r]$, $S = K[x_i \mid i \notin \sigma]$ and $J = IR_\sigma \cap S$. Then,*

$$\operatorname{reg}(J^{(n)}) \leqslant \operatorname{reg}(I^{(n)}) \quad \textit{for all } n \geqslant 1.$$

*In particular, $\delta(J) \leqslant \delta(I)$.*

*Proof.* We may assume that $S=K[x_1,\dots,x_s]$ for some $1\leqslant s\leqslant r$. Let $i$ be an index and $\boldsymbol{\alpha}$ a vector in $\mathbb{Z}^s$ such that

$$H^i_{\mathfrak{n}}(S/J^{(n)})_{\boldsymbol{\alpha}}\neq 0 \text{ and } \operatorname{reg}(S/J^{(n)})=|\boldsymbol{\alpha}|+i,$$

where $\mathfrak{n}=(x_1,\dots,x_s)$ is the homogeneous maximal ideal of $S$.

Let $\boldsymbol{\beta}=(\alpha_1,\dots,\alpha_s,-1,\dots,-1)\in\mathbb{Z}^r$ so that $G_{\boldsymbol{\beta}}=G_{\boldsymbol{\alpha}}\cup\{s+1,\dots,r\}$. By Formula (2) we deduce that

$$\Delta_{\boldsymbol{\alpha}}(J^{(n)})=\Delta_{\boldsymbol{\beta}}(I^{(n)}). \tag{11}$$

By Lemma 1.4,

$$\dim_K H^i_{\mathfrak{n}}(S/J^{(n)})_{\boldsymbol{\alpha}}=\dim_K \widetilde{H}_{i-|G_{\boldsymbol{\alpha}}|-1}(\Delta_{\boldsymbol{\alpha}}(J^{(n)});K),$$

and thus $\widetilde{H}_{i-|G_{\boldsymbol{\alpha}}|-1}(\Delta_{\boldsymbol{\alpha}}(J^{(n)});K)\neq 0$. Together with Equation (11), it yields

$$\widetilde{H}_{i-|G_{\boldsymbol{\alpha}}|-1}(\Delta_{\boldsymbol{\beta}}(I^{(n)});K)\neq 0.$$

By Lemma 1.4 again, it gives $H^{i+(r-s)}_{\mathfrak{m}}(R/I^{(n)})_{\boldsymbol{\beta}}\neq 0$ since $|G_{\boldsymbol{\beta}}|=|G_{\boldsymbol{\alpha}}|+(r-s)$. Therefore,

$$\operatorname{reg}(R/I^{(n)})\geqslant|\boldsymbol{\beta}|+i+(r-s)=|\boldsymbol{\alpha}|+i=\operatorname{reg}(S/J^{(n)}),$$

it follows that $\operatorname{reg}(J^{(n)})\leqslant\operatorname{reg}(I^{(n)})$.

Finally, together this inequality with Lemma 1.9 we have

$$\delta(J)=\lim_{n\to\infty}\frac{\operatorname{reg}(J^{(n)})}{n}\leqslant\lim_{n\to\infty}\frac{\operatorname{reg}(I^{(n)})}{n}=\delta(I),$$

and the lemma follows. □

**Theorem 2.2.** *Let $I$ be a square-free monomial ideal. Then, for all $i\geqslant 0$ we have*

$$a_i(R/I^{(n)})\leqslant\delta(I)(n-1).$$

*Proof.* If $n=1$, the theorem follows from Hochster's formula on the Hilbert series of the local cohomology module $H^i_{\mathfrak{m}}(R/I_\Delta)$ (see [26, Theorem 4.1]).

We may assume that $n\geqslant 2$. If $a_i(R/I^{(n)})=-\infty$, the theorem is obvious, so that we also assume that $a_i(R/I^{(n)})\neq-\infty$.

Suppose $\boldsymbol{\alpha}\in\mathbb{Z}^r$ such that

$$H^i_{\mathfrak{m}}(R/I^{(n)})_{\boldsymbol{\alpha}}\neq 0 \quad\text{and}\quad a_i(R/I^{(n)})=|\boldsymbol{\alpha}|.$$

By Lemma 1.4 we have

$$\dim_K \widetilde{H}_{i-|G_{\boldsymbol{\alpha}}|-1}(\Delta_{\boldsymbol{\alpha}}(I^{(n)});K)=\dim_K H^i_{\mathfrak{m}}(R/I^{(n)})_{\boldsymbol{\alpha}}\neq 0. \tag{12}$$

In particular, $\Delta_{\boldsymbol{\alpha}}(I^{(n)})$ is not acyclic.

If $G_{\boldsymbol{\alpha}}=[r]$, then $a_i(R/I^{(n)})=|\boldsymbol{\alpha}|\leqslant 0$, and hence the theorem holds in this case.

We therefore assume that $G_{\boldsymbol{\alpha}}=\{m+1,\dots,r\}$ for $1\leqslant m\leqslant r$. Let $S=K[x_1,\dots,x_m]$ and $J=IR_{G_{\boldsymbol{\alpha}}}\cap S$.

Let $\boldsymbol{\alpha}'=(\alpha_1,\dots,\alpha_m)\in\mathbb{N}^m$. By using Formula (2), we have

$$\Delta_{\boldsymbol{\alpha}'}(J^{(n)})=\Delta_{\boldsymbol{\alpha}}(I^{(n)}). \tag{13}$$

Together with (12), it gives $\widetilde{H}_{i-|G_{\boldsymbol{\alpha}}|-1}(\Delta_{\boldsymbol{\alpha}'}(J^{(n)});K)\neq 0$. By Lemma 1.4 we get

$$H_{\mathfrak{n}}^{i-|G_{\boldsymbol{\alpha}}|}(S/J^{(n)})_{\boldsymbol{\alpha}'}\neq 0,$$

where $\mathfrak{n}=(x_1,\dots,x_m)$ is the homogeneous maximal ideal of $S$.

Let $\Delta$ be the simplicial complex over $[m]$ corresponding to the square-free monomial ideal $J$. Assume that $\mathcal{F}(\Delta)=\{F_1,\dots,F_t\}$.

By Lemma 1.5 we may assume that $\mathcal{F}(\Delta_{\boldsymbol{\alpha}'}(J^{(n)}))=\{F_1,\dots,F_s\}$ for $1\leqslant s\leqslant t$. Let

$$\boldsymbol{\beta}=(\beta_1,\dots,\beta_m)=\frac{1}{n-1}\boldsymbol{\alpha}'\in\mathbb{R}^m.$$

By Lemma 1.5 again, we deduce that

$$\begin{cases} \sum\limits_{i\notin F_j}\beta_i=\dfrac{1}{n-1}\sum\limits_{i\notin F_j}\alpha_i\leqslant 1 & \text{for } j=1,\dots,s,\\ \sum\limits_{i\notin F_j}\beta_i=\dfrac{1}{n-1}\sum\limits_{i\notin F_j}\alpha_i\geqslant\dfrac{n}{n-1}>1 & \text{for } j=s+1,\dots,t.\end{cases}$$

It follows that $\boldsymbol{\beta}\in C_1$, where $C_1$ is a polyhedron in $\mathbb{R}^m$ defined by

$$\begin{cases} \sum\limits_{i\notin F_j}x_i\leqslant 1 & \text{for } j=1,\dots,s,\\ \sum\limits_{i\notin F_j}x_i\geqslant 1 & \text{for } j=s+1,\dots,t,\\ x_1\geqslant 0,\dots,x_m\geqslant 0. \end{cases}$$

By Lemma 1.10, $C_1$ is a polytope in $\mathbb{R}^m$.

Hence $|\boldsymbol{\beta}|\leqslant\delta(C_1)$, and hence $|\boldsymbol{\alpha}'|=(n-1)|\boldsymbol{\beta}|\leqslant\delta(C_1)(n-1)$. Observe that $\alpha_j<0$ for all $j\in G_{\boldsymbol{\alpha}}=\{m+1,\dots,r\}$, so

$$a_i(R/I^{(n)})=|\boldsymbol{\alpha}|=|\boldsymbol{\alpha}'|+(\alpha_{m+1}+\dots+\alpha_r)\leqslant|\boldsymbol{\alpha}'|\leqslant\delta(C_1)(n-1). \tag{14}$$

On the other hand, by Lemmas 1.11 and 2.1 we deduce that

$$\delta(C_1)\leqslant\delta(J)\leqslant\delta(I).$$

Together with Formula (14), it yields $a_i(R/I^{(n)})\leqslant\delta(I)(n-1)$, and the proof of the theorem is complete. □

We are now in position to prove the main result of the paper.

**Theorem 2.3.** *Let $\Delta$ be a simplicial complex. Then,*

$$\operatorname{reg}(I_\Delta^{(n)}) \leqslant \delta(I_\Delta)(n-1)+b, \quad \textit{for all } n\geqslant 1,$$

*where* $b=\max\{\operatorname{reg}(I_\Gamma) \mid \Gamma$ *is a subcomplex of* $\Delta$ *with* $\mathcal{F}(\Gamma)\subseteq\mathcal{F}(\Delta)\}$.

*Proof.* For simplicity, we put $I=I_\Delta$. Let $i\in\{0,\dots,\dim(R/I)\}$ and $\boldsymbol{\alpha}\in\mathbb{Z}^r$ such that

$$H_{\mathfrak{m}}^i(R/I^{(n)})_{\boldsymbol{\alpha}} \neq 0, \text{ and } \operatorname{reg}(R/I^{(n)}) = a_i(R/I^{(n)})+i = |\boldsymbol{\alpha}|+i.$$

By Lemma 1.4, we have

$$\dim_K \widetilde{H}_{i-|G_{\boldsymbol{\alpha}}|-1}(\Delta_{\boldsymbol{\alpha}}(I^{(n)});K) = \dim_K H_{\mathfrak{m}}^i(R/I^{(n)})_{\boldsymbol{\alpha}} \neq 0. \tag{15}$$

In particular, $\Delta_{\boldsymbol{\alpha}}(I^{(n)})$ is not acyclic.

If $G_{\boldsymbol{\alpha}}=[r]$, then $\Delta_{\boldsymbol{\alpha}}(I^{(n)})$ is either $\{\varnothing\}$ or a void complex. Because it is not acyclic, $\Delta_{\boldsymbol{\alpha}}(I^{(n)})=\{\varnothing\}$. By Formula (15) we deduce that $i=|G_{\boldsymbol{\alpha}}|=r$, and hence $\dim R/I=r$. It means that $I=0$, so $I^{(n)}=0$ as well. Therefore, $\operatorname{reg}(I^{(n)})=-\infty$, and the theorem holds in this case.

We may assume that $G_{\boldsymbol{\alpha}}=\{m+1,\dots,r\}$ for some $1\leqslant m\leqslant r$. Let $S=K[x_1,\dots,x_m]$ and $J=IR_{G_{\boldsymbol{\alpha}}}\cap S$.

Let $\boldsymbol{\alpha}'=(\alpha_1,\dots,\alpha_m)\in\mathbb{N}^m$. By using Formula (2), we have

$$\Delta_{\boldsymbol{\alpha}'}(J^{(n)}) = \Delta_{\boldsymbol{\alpha}}(I^{(n)}). \tag{16}$$

Together with (15), it gives $\widetilde{H}_{i-|G_{\boldsymbol{\alpha}}|-1}(\Delta_{\boldsymbol{\alpha}'}(J^{(n)});K)\neq 0$. By Lemma 1.4 we get

$$H_{\mathfrak{n}}^{i-|G_{\boldsymbol{\alpha}}|}(S/J^{(n)})_{\boldsymbol{\alpha}'} \neq 0,$$

where $\mathfrak{n}=(x_1,\dots,x_m)$ is the homogeneous maximal ideal of $S$. In particular,

$$|\boldsymbol{\alpha}'| \leqslant a_{i-|G_{\boldsymbol{\alpha}}|}(S/J^{(n)}).$$

Together with Lemma 2.1 and Theorem 2.2, it yields

$$|\boldsymbol{\alpha}'| \leqslant \delta(J)(n-1) \leqslant \delta(I)(n-1).$$

Therefore,

$$\begin{aligned}\operatorname{reg}(I^{(n)}) = \operatorname{reg}(R/I^{(n)})+1 = |\boldsymbol{\alpha}|+i+1 = |\boldsymbol{\alpha}'| + \sum_{j=m+1}^{r} \alpha_j + i+1 \\ \leqslant |\boldsymbol{\alpha}'|+i-|G_{\boldsymbol{\alpha}}|+1 \leqslant \delta(I)(n-1)+i-|G_{\boldsymbol{\alpha}}|+1.\end{aligned}$$

It remains to prove that $i-|G_{\boldsymbol{\alpha}}|+1\leqslant b$. By Lemma 1.6, we have

$$\mathcal{F}(\Delta_{\boldsymbol{\alpha}'}(J^{(n)}))=\mathcal{F}(\Delta_{\boldsymbol{\alpha}}(I^{(n)}))=\left\{F\in\mathcal{F}(\mathrm{lk}_{\Delta}(G_{\boldsymbol{\alpha}}))\,|\sum_{j\notin F\cup G_{\boldsymbol{\alpha}}}\boldsymbol{\alpha}_j\leqslant n-1\right\}.$$

It follows that there is a simplicial complex $\Gamma$ with $\mathcal{F}(\Gamma)\subseteq\mathcal{F}(\Delta)$ such that

$$\Delta_{\boldsymbol{\alpha}'}(J^{(n)})=\mathrm{lk}_{\Gamma}(G_{\boldsymbol{\alpha}}).$$

Since $\widetilde{H}_{i-|G_{\boldsymbol{\alpha}}|-1}(\mathrm{lk}_{\Gamma}(G_{\boldsymbol{\alpha}});K)\neq 0$, by Lemma 1.2 we have $i-|G_{\boldsymbol{\alpha}}|+1\leqslant\mathrm{reg}(I_{\Gamma})\leqslant b$, and then proof of the theorem is complete. □

As a direct consequence of Theorem 2.3, we have a simple bound. Namely,

**Corollary 2.4.** *Let $I$ be a square-free monomial ideal. Then,*

$$\mathrm{reg}(I^{(n)})\leqslant\delta(I)(n-1)+\dim(R/I)+1,\quad \textit{for all } n\geqslant 1.$$

*Proof.* Let $\Delta$ be the simplicial complex corresponding to the square-free ideal $I$. For every subcomplex $\Gamma$ of $\Delta$ we have $\dim\Gamma\leqslant\dim\Delta$. It follows from Lemma 1.2 that

$$\mathrm{reg}(I_{\Gamma})\leqslant\dim(R/I_{\Gamma})+1\leqslant\dim(R/I_{\Delta})+1.$$

Therefore, $b=\max\{\mathrm{reg}(I_{\Gamma})\,|\,\mathcal{F}(\Gamma)\subseteq\mathcal{F}(\Delta)\}\leqslant\dim(R/I_{\Delta})+1$. Now the corollary follows from Theorem 2.3. □

We next reformulate Theorem 2.3 for a square-free monomial ideal arising from a hypergraph.

**Theorem 2.5.** *Let $\mathcal{H}$ be a hypergraph. Then, for all $n\geqslant 1$, we have*

$$\mathrm{reg}(I(\mathcal{H})^{(n)})\leqslant\delta(I(\mathcal{H}))(n-1)+b,$$

*where $b=\max\{\mathrm{pd}(R/I(\mathcal{H}'))\,|\,\mathcal{H}'$ is a subhypergraph of $\mathcal{H}^*$ with $E(\mathcal{H}')\subseteq E(\mathcal{H}^*)\}$.*

*Proof.* Let $\Delta$ be the corresponding simplicial complex of the square-free monomial ideal $I(\mathcal{H})$. Assume that $\mathcal{F}(\Delta)=\{F_1,\dots,F_p\}$. Since

$$I(\mathcal{H})=\bigcap_{j=1}^{p}(x_i\,|\,i\notin F_j),$$

so that $E(\mathcal{H}^*)=\{C_1,\dots,C_p\}$, where $C_j=[r]\setminus F_j$ for all $j=1,\dots,p$.

Let $\Gamma$ be a subcomplex of $\Delta$ with $\mathcal{F}(\Gamma)\subseteq\mathcal{F}(\Delta)$. We may assume that $\mathcal{F}(\Gamma)=\{F_1,\dots,F_k\}$ for $1\leqslant k\leqslant p$. Then, we have $I_{\Gamma}^*=I(\mathcal{H}')$ where $\mathcal{H}'$ is the subhypergraph of $\mathcal{H}^*$ with $E(\mathcal{H}')=\{C_1,\dots,C_k\}$.

By Lemma 1.3 we have $\mathrm{reg}(I_{\Gamma})=\mathrm{pd}(R/I_{\Gamma}^*)=\mathrm{pd}(R/I(\mathcal{H}'))$, and therefore the theorem follows from Theorem 2.3. □

The next theorem is the second main result of the paper. It bounds the regularity of symbolic powers of a square-free monomial ideal via the combinatorial properties of the associated hypergraph.

**Theorem 2.6.** *Let $\mathcal{H}$ be a simple hypergraph. Then,*

$$\operatorname{reg}(I(\mathcal{H})^{(n)}) \leqslant \delta(I(\mathcal{H}))(n-1)+|V(\mathcal{H})|-\varepsilon(\mathcal{H}^*), \text{ for all } n \geqslant 1.$$

*Proof.* By Theorem 2.5, it suffices to show that

$$\operatorname{pd}(R/I(\mathcal{G})) \leqslant |V(\mathcal{H})|-\varepsilon(\mathcal{H}^*)$$

for every hypergraph $\mathcal{G}$ with $E(\mathcal{G}) \subseteq E(\mathcal{H}^*)$. By Lemma 1.7, it suffices to prove that

$$|V(\mathcal{G})|-\varepsilon(\mathcal{G}) \leqslant |V(\mathcal{H}^*)|-\varepsilon(\mathcal{H}^*).$$

In order to prove this inequality, without loss of generality we may assume that $\mathcal{H}^*$ has no trivial edges or isolated vertices.

Let $S$ be an edgewise-dominant set of $\mathcal{G}$ such that $|S|=\varepsilon(\mathcal{G})$. For each vertex $v \in V(\mathcal{H}^*) \setminus V(\mathcal{G})$, we take an edge of $\mathcal{H}^*$ containing $v$, and denote this edge by $F(v)$. Then,

$$S' = S \cup \{F(v) \mid v \in V(\mathcal{H}^*) \setminus V(\mathcal{G})\}$$

is an edgewise-dominant set of $\mathcal{H}^*$. It follows that

$$\varepsilon(\mathcal{H}^*) \leqslant |S'| \leqslant |S|+|V(\mathcal{H}^*) \setminus V(\mathcal{G})| = |S|+|V(\mathcal{H}^*)|-|V(\mathcal{G})|,$$

and therefore $|V(\mathcal{G})|-\varepsilon(\mathcal{G}) \leqslant |V(\mathcal{H}^*)|-\varepsilon(\mathcal{H}^*)$, as required. $\square$

The following example shows that the bound in Theorem 2.3 is sharp at every $n$ for the class of matroid complexes. Recall that a simplicial complex $\Delta$ is called a *matroid complex* if for every subset $\sigma$ of $V(\Delta)$, the simplicial complex $\Delta[\sigma]$ is pure (see e.g. [26, Chapter 3]). Here, $\Delta[\sigma]$ is the restriction of $\Delta$ to $\sigma$ and defined by $\Delta[\sigma] = \{\tau \mid \tau \in \Delta \quad \text{and} \quad \tau \subseteq \sigma\}$.

**Example 2.7.** Let $\Delta$ be a matroid complex that is not a cone. Then,

$$\operatorname{reg}(I_\Delta^{(n)}) = \delta(I_\Delta)(n-1)+b, \quad \text{for all } n \geqslant 1,$$

where $b = \max\{\operatorname{reg}(I_\Gamma) \mid \Gamma \text{ is a subcomplex of } \Delta \text{ with } \mathcal{F}(\Gamma) \subseteq \mathcal{F}(\Delta)\}$.

*Proof.* Let $I=I_\Delta$ and $s=\dim(R/I_\Delta)$. By [24, Theorem 4.5], for all $n\geqslant 1$ we have:

$$\operatorname{reg}(I^{(n)})=d(I)(n-1)+s+1.$$

It implies that

$$\lim_{n\to\infty}\frac{\operatorname{reg}(I^{(n)})}{n}=d(I),$$

so $\delta(I)=d(I)$. It remains to show that $b=s+1$.

Together the fact $\delta(I)=d(I)$ with Theorem 2.3, we get $s+1\leqslant b$. On the other hand, by the same argument as in the proof of Corollary 2.4, we obtain $b\leqslant s+1$. Hence, $b=s+1$, as required. □

We conclude this section with a remark on lower bounds.

**Remark 2.8.** Let $I$ be a square-free monomial ideal. By [8, Lemma 4.2(ii)] we deduce that $d(I)n\leqslant d(I^{(n)})$, and therefore

$$\operatorname{reg}(I^{(n)})\geqslant d(I)n,\quad \text{for all } n\geqslant 1.$$

In general, $d(I)<\delta(I)$ (see e.g. [8, Lemma 5.14]), so that the bound is not optimal.

On the other hand, by Lemma 1.9, there is a number $b$ such that

$$\operatorname{reg}(I^{(n)})\geqslant \delta(I)n+b,\quad \text{for all}\quad n\geqslant 1.$$

The natural question is to find a good bound for $b$.

## 3. Applications

In this section we will apply Theorem 2.3 to the regularity of symbolic powers of the edge ideal of a graph. We start with a result which allows us to bound the number $b$ in Theorem 2.3 by choosing a suitable numerical function, which is of independent interest.

**Theorem 3.1.** *Let $\Delta$ be a simplicial complex over $[r]$ and let*

$$\operatorname{Simp}(\Delta)=\{\operatorname{lk}_\Delta(\sigma)\mid \sigma\in\Delta\}.$$

*Assume that $f\colon \operatorname{Simp}(\Delta)\to\mathbb{N}$ is a function which satisfies the following properties:*

(1) *If $\Lambda\in\operatorname{Simp}(\Delta)$ is a simplex, then $f(\Lambda)=0$.*

(2) *For every $\Lambda\in\operatorname{Simp}(\Delta)$ and every $v\in V(\Lambda)$ such that $\Lambda$ is not a cone over $v$, $f(\operatorname{lk}_\Lambda(v))+1\leqslant f(\Lambda)$.*

*Then, for every subcomplex $\Gamma$ of $\Delta$ with $\mathcal{F}(\Gamma)\subseteq\mathcal{F}(\Delta)$ we have $\operatorname{reg}(I_\Gamma)\leqslant f(\Delta)+1$.*

*Proof.* For a subset $S$ of $[r]$ we set $\mathfrak{p}_S=(x_i|i\in S)\subseteq R$. In order to facilitate an induction argument on the number of vertices of $\Delta$ we prove the following assertion:

$$\operatorname{reg}(\mathfrak{p}_S+I_\Gamma)\leqslant f(\Delta)+1, \quad \text{for every } S\subseteq[r], \tag{17}$$

where all simplicial complexes are considered over $[r]$.

Indeed, if $|V(\Delta)|\leqslant 1$, then $\Delta$ is a simplex. In this case, the assertion is obvious.

Assume that $|V(\Delta)|\geqslant 2$. If $\Delta$ is a simplex, the assertion holds, so we assume that $\Delta$ is not a simplex. We now proceed by backward induction on $|S|$. If $|S|=r$, then

$$\mathfrak{p}_S+I_\Gamma=(x_1,\dots,x_r).$$

In this case $\operatorname{reg}(\mathfrak{p}_S+I_\Gamma)=1$, and so the assertion holds.

Assume that $|S|<r$. If $\mathfrak{p}_S+I_\Gamma$ is a prime, i.e. it is generated by variables, then $\operatorname{reg}(\mathfrak{p}_S+I_\Gamma)=1$, and then the assertion holds.

Assume that $\mathfrak{p}_S+I_\Gamma$ is not a prime. Then, there is a variable, say $x_v$ with $v\in[r]$, such that $x_v$ appears in some monomial generator of $\mathfrak{p}_S+I_\Gamma$ of degree at least 2 and $v\notin S$. Note that if $u$ is not a vertex of $\Gamma$ then $x_u$ is a monomial generator of $I_\Gamma$, and if $\Gamma$ is a cone over some vertex $w$ then $x_w$ does not appear in any monomial generator of $I_\Gamma$. It implies that $v$ is a vertex of $\Gamma$ and $\Gamma$ is not a cone over $v$. In particular, $\Delta$ is not a cone over $v$ since $\mathcal{F}(\Gamma)\subseteq\mathcal{F}(\Delta)$.

Since

$$(\mathfrak{p}_S+I_\Gamma)+(x_v)=\mathfrak{p}_{S\cup\{v\}}+I_\Gamma, \text{ and } (\mathfrak{p}_S+I_\Gamma):(x_v)=\mathfrak{p}_S+I_{\Gamma'},$$

where $\Gamma'$ is a subcomplex of $\Gamma$ with $\mathcal{F}(\Gamma')=\{F\in\mathcal{F}(\Gamma)|v\in F\}$, by [6, Lemma 2.10] we have

$$\operatorname{reg}(\mathfrak{p}_S+I_\Gamma)\leqslant\max\{\operatorname{reg}(\mathfrak{p}_{S\cup\{v\}}+I_\Gamma),\operatorname{reg}(\mathfrak{p}_S+I_{\Gamma'})+1\}. \tag{18}$$

By the backward induction hypothesis, we have

$$\operatorname{reg}(\mathfrak{p}_{S\cup\{v\}}+I_\Gamma)\leqslant f(\Delta)+1. \tag{19}$$

We now claim that

$$\operatorname{reg}(\mathfrak{p}_S+I_{\Gamma'})\leqslant f(\Delta). \tag{20}$$

Indeed, if $\mathfrak{p}_S+I_{\Gamma'}$ is prime, then $\operatorname{reg}(\mathfrak{p}_S+I_{\Gamma'})=1$. As $\Delta$ is not a cone over $v$, by the definition of $f$ we have $f(\Delta)\geqslant f(\operatorname{lk}_\Delta(v))+1\geqslant 1$, and the claim holds in this case.

Assume that $\mathfrak{p}_S+I_{\Gamma'}$ is not a prime. Observe that

$$I_{\Gamma''}=(x_v)+I_{\Gamma'},$$

where $\Gamma''=\mathrm{lk}_{\Gamma'}(v)$ and this simplicial complex is considered over $[r]$. Since variable $x_v$ does not appear in any generator of $I_{\Gamma'}$, hence $\mathrm{reg}(I_{\Gamma''})=\mathrm{reg}(I_{\Gamma'})$.

On the other hand, by the induction hypothesis, we have

$$\mathrm{reg}(I_{\Gamma''})=\mathrm{reg}(\mathrm{lk}_{\Gamma'}(v))\leqslant f(\mathrm{lk}_\Delta(v))+1.$$

It follows that

$$\mathrm{reg}(\mathfrak{p}_S+I_{\Gamma'})\leqslant \mathrm{reg}(I_{\Gamma'})=\mathrm{reg}(I_{\Gamma''})\leqslant f(\mathrm{lk}_\Delta(v))+1.$$

Together with the inequality $f(\mathrm{lk}_\Delta(v))+1\leqslant f(\Delta)$, it yields $\mathrm{reg}(\mathfrak{p}_S+I_{\Gamma'})\leqslant f(\Delta)$, as claimed.

By combining three Inequalities (18)-(20), we obtain $\mathrm{reg}(\mathfrak{p}_S+I_\Gamma)\leqslant f(\Delta)+1$, and so the inequality (17) is proved. The lemma now follows from the assertion by taking $S=\varnothing$, and the proof is complete. □

We now reformulate Theorem 3.1 for graphs. A graph $G$ is called *trivial* if it has no edges. For a subset $S$ of $V(G)$, the *closed neighborhood* of the set $S$ in $G$ is the set $N_G[S]=S\cup\{v\in V(G)\,|\,v$ is a neighbor of some vertex in $S\}$. For a vertex $v$ of $G$, we write $N_G[v]$ for $N_G[\{v\}]$. Recall that $\Delta(G)$ is the set of independent sets of $G$, which is a simplicial complex and $I(G)=I_{\Delta(G)}$.

**Corollary 3.2.** *Let $G$ be a graph and let $\mathcal{I}_G=\{G\setminus N_G[S]\,|\,S\in\Delta(G)\}$. Assume that $f\colon \mathcal{I}_G\to\mathbb{N}$ is a function which satisfies the following properties*:

(1) *$f(H)=0$ if $H$ is trivial.*

(2) *For every $H$ and every non-isolated vertex $v$ of $H$, $f(H\setminus N_H[v])+1\leqslant f(H)$. Then, for every subcomplex $\Gamma$ of $\Delta(G)$ with $\mathcal{F}(\Gamma)\subseteq\mathcal{F}(\Delta(G))$ we have*

$$\mathrm{reg}(I_\Gamma)\leqslant f(G)+1.$$

*Proof.* First we note that, for every graph $H$ and every $S\in\Delta(H)$ we have

$$\Delta(H\setminus N_H[S])=\mathrm{lk}_{\Delta(H)}(S).$$

It implies that

$$\mathrm{Simp}(\Delta(G))=\{\Delta(H)\,|\,H\in\mathcal{I}_G\}.$$

Therefore, we can define a function $g\colon\ \mathrm{Simp}(\Delta(G))\to\mathbb{N}$, by sending $\Delta(H)$ to $f(H)$ for all $H\in\mathcal{I}_G$.

Note that for every graph $H$, we have $\Delta(H)$ is a simplex if and only if $H$ is trivial; and $\Delta(H)$ is a cone over a vertex $v$ if and only if $v$ is an isolated vertex of $H$. Together with the definition of the function $g$, it shows that $g$ satisfies all conditions of Theorem 3.1, and therefore by this theorem we obtain $\mathrm{reg}(I_\Gamma)\leqslant g(\Delta(G))+1=f(G)+1$, as required. □

**Remark 3.3.** Suppose that $\mathcal{H}$ is a (simple) hypergraph and $S \subset V(\mathcal{H})$. Let $N_{\mathcal{H}}[S]$ be the closed neighborhood of $S$ in $\mathcal{H}$, this is the natural extension to hypergraphs from graphs of the notion just prior to Corollary 3.2. Furthermore, let $\Delta(\mathcal{H})$ be the independence complex of $\mathcal{H}$. The equalities $I(\mathcal{H}) = I_{\Delta(\mathcal{H})}$ and $\Delta(\mathcal{H} \setminus N_{\mathcal{H}}[S]) = lk_{\Delta(\mathcal{H})}(S)$ hold just as well for simple hypergraphs. Let

$$f : \{\mathcal{H} \setminus N_{\mathcal{H}}[S] : S \in \Delta(\mathcal{H})\} \longrightarrow \mathbb{N}$$

be the function defined by

$$f(\mathcal{H}') = \begin{cases} 0 & \text{if } \ \mathcal{H}' \text{ is trivial,} \\ |V(\mathcal{H}')| - \varepsilon(\mathcal{H}'^*) - 1 & \text{otherwise.} \end{cases}$$

By the same argument as in the proof of Theorem 2.6, we can verify $f$ satisfies conditions in Theorem 3.1. As a consequence, we recover a result of Dao and Schweig (see Lemma 1.7).

When applied to the edge ideal of a graph, Theorem 2.3 has the following form.

**Lemma 3.4.** *Let $G$ be a graph. Then,*

$$\operatorname{reg}(I(G)^{(n)}) \leqslant 2(n-1) + b, \ \text{for all } n \geqslant 1,$$

*where* $b = \max\{\operatorname{reg}(I_\Gamma) \mid \Gamma$ *is a subcomplex of* $\Delta(G)$ *with* $\mathcal{F}(\Gamma) \subseteq \mathcal{F}(\Delta(G))\}$.

*Proof.* Since $I(G) = I_{\Delta(G)}$ and $\delta(I(G)) = 2$ by [8, Example 4.4], therefore the lemma follows from Theorem 2.3. □

We are now in position to prove the main result of this section.

**Theorem 3.5.** *Let $G$ be a graph. Then,*

$$\operatorname{reg}(I(G)^{(n)}) \leqslant 2n + \text{ord-match}(G) - 1, \ \text{for all } n \geqslant 1.$$

*Proof.* By Lemma 3.4, it remains to show that $\operatorname{reg}(I_\Gamma) \leqslant \text{ord-match}(G) + 1$, for every subcomplex $\Gamma$ of $\Delta(G)$ with $\mathcal{F}(\Gamma) \subseteq \mathcal{F}(\Delta(G))$.

Consider the function $f : \mathcal{I}_G \to \mathbb{N}$ defined by

$$f(H) = \begin{cases} 0 & \text{if } H \text{ is trivial,} \\ \text{ord-match}(H) & \text{otherwise.} \end{cases}$$

For every non-isolated vertex $v$ of $H$, we have $f(H \setminus N_H[v]) + 1 \leqslant f(H)$ by [10, Lemma 2.1], hence $f$ satisfies all conditions of Corollary 3.2, so that by this corollary

$$\operatorname{reg}(I_\Gamma) \leqslant f(G) + 1 = \text{ord-match}(G) + 1,$$

and the theorem follows. □

**Remark 3.6.** Let $G$ be a graph with ord-match$(G)=\nu(G)$. Then,

$$\operatorname{reg}(I(G)^{(n)}) = 2n+\nu(G)-1, \quad \text{for all} \quad n \geqslant 1.$$

Indeed, for every positive integer $n$, the lower bound $\operatorname{reg}(I(G)^{(n)}) \geqslant 2n+\nu(G)-1$ comes from Lemma 1.8, and the upper bound follows from Theorem 3.5 because ord-match$(G)=\nu(G)$.

As a consequence, we quickly recover the main result of Fakhari in [12], which says that the equality holds when $G$ is a *Cameron-Walker* graph, where a graph $G$ is called Cameron-Walker if $\nu(G)=$match$(G)$ (see e.g. [17]). For such a graph $G$, ord-match$(G)=\nu(G)$ since $\nu(G)\leqslant$ord-match$(G)\leqslant$match$(G)$.

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