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Yagita's counter-examples and beyond

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Abstract. A conjecture on a relationship between the Chow and Grothendieck rings for the generically twisted variety of Borel subgroups in a split semisimple group G, stated by the second author, has been disproved by Nobuaki Yagita in characteristic 0 for $G=\mathrm{Spin}(2n+1)$ with $n=8$ and $n=9$. For $n=8$, the second author provided an alternative simpler proof of Yagita's result, working in any characteristic, but failed to do so for $n=9$. This gap is filled here by involving a new ingredient – Pieri type K-theoretic formulas for highest orthogonal grassmannians. Besides, a similar counter-example for $n=10$ is produced. Note that the conjecture on $\mathrm{Spin}(2n+1)$ holds for n up to 5; it remains open for $n=6$, $n=7$, and every $n\geq 11$.

Contents

1. Introduction

Let X be a smooth variety (over a field of arbitrary characteristic). Consider the Grothendieck ring $K(X)$ of X and its filtration by codimension of support of coherent sheaves:

$$0=K(X)^{(\dim X+1)}\subset K(X)^{(\dim X)}\subset \ldots \subset K(X)^{(1)}\subset K(X)^{(0)}=K(X),$$

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called the topological filtration or the coniveau filtration. We consider the associated graded ring

$$GK(X) = \bigoplus_{i=0}^{\dim X} K(X)^{(i/i+1)}, \quad \text{where} \quad K(X)^{(i/i+1)} := K(X)^{(i)}/K(X)^{(i+1)}.$$

Let $\mathrm{CH}(X)$ be the Chow ring of X. There is a canonical surjective graded ring homomorphism

$$\varphi\colon\ \mathrm{CH}(X) \longrightarrow GK(X), \tag{1.1}$$

mapping the class in $\mathrm{CH}^i(X)$ of a closed subvariety in X of codimension i to the class of its structure sheaf in the quotient $G^iK(X) = K(X)^{(i/i+1)}$. The morphism φ commutes with pull-backs, push-forwards, and Chern classes of the respective cohomology theories. Moreover, by the Riemann-Roch theorem, the kernel of the ith homogeneous component

$$\varphi^i\colon\ \mathrm{CH}^i(X) \longrightarrow G^iK(X)$$

is annihilated by $(i-1)!$.

Let G be a split semisimple group and let E be a *generic G-torsor*, i.e., the generic fiber of the quotient map $\mathrm{GL}(N) \to \mathrm{GL}(N)/G$ induced by an embedding $G \hookrightarrow \mathrm{GL}(N)$ for some $N \geq 1$. For the twisted by E variety X of Borel subgroups in G, the second author conjectured:

Conjecture 1.2. ([7, Conjecture 1.1]) *For X as above, the homomorphism* (1.1) *is an isomorphism.*

Note that since the group G is split, it contains a Borel subgroup B. For any choice of Borel B, the variety of all Borel subgroups in G is isomorphic to the quotient G/B and the variety X is isomorphic to E/B.

By [12, Theorem 3.1], the statement of Conjecture 1.2 for a given G is equivalent to absence of torsion in the connective K-theory of X. Also note that by [8, Lemma 4.2], Conjecture 1.2 is equivalent to the same statement with the Borel subgroups replaced by any conjugacy class of *special* parabolic subgroups in G, where an algebraic group P is called special if any P-torsor over any base field extension is trivial.

In [8], Conjecture 1.2 has been confirmed for simple groups G of type A and C. Moreover, by [7, Theorem 1.2], Conjecture 1.2 holds for a wider class of groups G including special orthogonal groups as well as the exceptional groups of types G_2, F_4, and simply connected E_6. Finally, by [9, Theorem 3.1], Conjecture 1.2 holds for $G = \mathrm{Spin}(2n+1)$ with $n \leq 5$. Note that for any $n \geq 1$, the statement of

Conjecture 1.2 on $\mathrm{Spin}(2n{+}1)$ (which is a simply connected group of type B_n) is equivalent to its statement on $\mathrm{Spin}(2n{+}2)$ (a simply connected group of type D_{n+1}), see Proposition 2.16.

In this paper we work with the group $G{=}\mathrm{Spin}(2n{+}1)$ for larger n. A generic G-torsor E yields a generic quadratic form q of dimension $2n{+}1$ with trivial discriminant and Clifford invariant (defined as the Brauer class of the even Clifford algebra of q). The twisted by E variety X of an appropriate conjugacy class of special parabolic subgroups in G is identified with the highest orthogonal grassmannian of q.

Counter-examples to Conjecture 1.2 have been constructed with $G{=}\mathrm{Spin}(17)$ and $G{=}\mathrm{Spin}(19)$ by N. Yagita in [19]. Later, Yagita's counter-example for Spin(17) has been modified, simplified, and extended to the base field of arbitrary characteristic in [11]. However, an attempt to treat Spin(19) failed at that time.

In the present paper, we successfully treat Spin(19) by involving a new ingredient – a Pieri type formula for K-theory of highest orthogonal grassmannians. The Pieri formula (3.12) we need is formulated in [4, Theorem 1.2] in a combinatorial way. To avoid combinatorial computations, we reprove it using a technique of partially split generic forms, see the proof of Lemma 3.10. The Pieri formula is used in Lemma 3.24 as well.

We also do a similar treatment for Spin(21) thus showing the failure of Conjecture 1.2 for this group. (The corresponding Pieri formula (3.15) is involved in Lemmas 3.14 and 3.25.) In other terms, combined with the previously available results, we show

Theorem 1.3. *Let X be the highest orthogonal grassmannian of a generic quadratic form q of dimension* 17, 19 *or* 21 *with trivial discriminant and Clifford invariant. Then the canonical surjective homomorphism $\varphi{:}\mathrm{CH}(X){\to}GK(X)$ is not an isomorphism.*

Recall that Conjecture 1.2 on $\mathrm{Spin}(2n{+}1)$ holds for n up to 5. By Theorem 1.3 it fails for $8{\le}n{\le}10$. However, it remains widely open for every of the remaining values of n. One of the obstacles to extend the counter-examples to $n{=}11$ is the "drop" to 2^5 of the torsion index of Spin(23): the torsion indexes of Spin(19) and Spin(21) are 2^4 and 2^5, see [16]. (The similar drop for Spin(19) was also the origin of the difficulties with this case.) Generally speaking, it seems that every n needs an individual treatment. However, since the next drop occurs with Spin(35) only, all $11{\le}n{\le}16$ can probably be treated in a common way.

We do expect that the conjecture fails for every $n{\ge}11$. However, the situation with the pairs $n{=}6$ and $n{=}7$ looks completely misty.

For the proof of Theorem 1.3, following the approach of [11], we provide certain elements in the Chow groups of the highest orthogonal grassmannians, which are not divisible by 2, whereas their images under φ are.

2. Preliminaries

In this section we collect some basic results on the Chow and the Grothendieck rings of highest orthogonal grassmannians. For details and the general theory we refer the reader to [5], [11], and [17]. In the last part of the section, we discuss the equivalence between Conjecture 1.2 for $\mathrm{Spin}(2n+1)$ and for $\mathrm{Spin}(2n+2)$; in the course of this discussion it is also demonstrated how information on a generic object can serve to gain some information on a more general one – see Lemma 2.15.

2a. Chow ring of highest orthogonal grassmannians

For an integer $n\geq 1$, let q be a generic $(2n+1)$-dimensional quadratic form over a field F of trivial discriminant and Clifford invariant corresponding to a generic $\mathrm{Spin}(2n+1)$-torsor. The highest orthogonal grassmannian X_n of q is the variety of its n-dimensional totally isotropic subspaces.

We have $\dim X_n = n(n+1)/2$. The *index* $\operatorname{ind} X_n$ of X_n (defined as the greatest common divisor of the degrees of closed points on X_n) coincides with the torsion index of $\mathrm{Spin}(2n+1)$, determined by Totaro in [16]. In particular, we have

$$\operatorname{ind} X_8 = 2^4, \quad \operatorname{ind} X_9 = 2^4, \quad \operatorname{ind} X_{10} = 2^5. \tag{2.1}$$

Let $\overline{X}_n$ be the base change of X_n to an algebraic closure $\overline{F}$ of F and let $\overline{Y}_n$ be the base change of the quadric Y_n of q to $\overline{F}$. Consider the projective bundle $\pi\colon \mathcal{P}\to\overline{X}_n$ associated with the tautological vector bundle on $\overline{X}_n$ and the projection $\pi'\colon \mathcal{P}\to\overline{Y}_n$. For $i=0,\dots,n$, let l_i be the class in $\mathrm{CH}(\overline{Y}_n)$ of a projective i-dimensional subspace on $\overline{Y}_n$ and let $e_i\in\mathrm{CH}^i(\overline{X}_n)$ be the image of l_{n-i} under the composition of the pullback of π' and the push-forward of π. Then, the Chow group $\mathrm{CH}(\overline{X}_n)$ is free with basis given by the products $\prod_{i\in I} e_i$, where I runs over the subsets of the set $\{1,\dots,n\}$. In particular, both groups $\mathrm{CH}^{\dim \overline{X}_n}(\overline{X}_n)$ and $\mathrm{CH}^{\dim \overline{X}_n-1}(\overline{X}_n)$ are cyclic generated by

$$p := \prod_{i=1}^{n} e_i \quad \text{and} \quad l := \prod_{i=2}^{n} e_i \tag{2.2}$$

respectively.

The Chow ring $\mathrm{CH}(\overline{X}_n)$ is generated by $e_1,\dots,e_n$ subject to the relations

$$e_i^2-2e_{i-1}e_{i+1}+2e_{i-2}e_{i+2}-\dots+(-1)^{i-1}2e_1e_{2i-1}+(-1)^ie_{2i}=0 \tag{2.3}$$

for all $i\geq 1$, where $e_i:=0$ for $i>n$. In particular, for $n\leq 10$ we have the following relations:

$$\begin{aligned} &e_1^2=e_2, && e_1^4=2e_1e_3-e_4,\\ &e_1^8\equiv 2(e_3e_5-e_2e_6+e_1e_7)-e_8 \pmod{2^2}, && e_1^{16}\equiv 2(e_7e_9-e_6e_{10}) \pmod{2^2}. \end{aligned} \tag{2.4}$$

Let c_i be the Chern class of the dual of the (rank n) tautological vector bundle on X_n. As the Clifford invariant of q is trivial, by [5, Exercise 88.14] there is an element $e\in\mathrm{CH}^1(X_n)$ with $c_1=2e$. (As the group $\mathrm{CH}^1(X_n)$ is torsion free, the element e is uniquely defined.) Consider the restriction map

$$\mathrm{res}:\mathrm{CH}(X_n)\longrightarrow\mathrm{CH}(\overline{X}_n).$$

Since the map res commutes with Chern classes, by [5, Proposition 86.13] we have $\mathrm{res}(c_i)=2e_i$ for all $1\leq i\leq n$ and $\mathrm{res}(e)=e_1$.

For a smooth quasi-projective variety X, we consider the total cohomological Steenrod operation $S\colon \mathrm{Ch}(X)\to\mathrm{Ch}(X)$, where $\mathrm{Ch}(X):=\mathrm{CH}(X)/2\,\mathrm{CH}(X)$ denotes the modulo 2 Chow ring of X. (The Steenrod operation in characteristic 2 has recently been constructed in [15].) For any $i\geq 0$, we write $S^i\colon \mathrm{Ch}^*(X)\to\mathrm{Ch}^{*+i}(X)$ for the ith component of S, which corresponds to the Steenrod operation Sq^{2i} on mod 2 cohomology. The image of an element $x\in\mathrm{CH}(X)$ under the map $\mathrm{CH}(X)\to\mathrm{Ch}(X)$ will be denoted by $\bar{x}$.

The values of Steenrod operations on Chern classes have been computed in [18] (see also [1, Théorème 7.1]). In [11, Proposition 3.1] only the linear part of the formula is indicated. In fact, there is also a quadratic part, but it is irrelevant for our purposes.

Lemma 2.5. ([18], [1, Théorème 7.1]) *For $i\geq 0$, let $\bar{c}_i$ be the image in $\mathrm{Ch}^i(X_n)$ of the ith Chern class $c_i\in\mathrm{CH}^i(X_n)$ of the dual of the (rank n) tautological vector bundle on X_n. Then, for any $j\geq 0$,*

$$S^j(\bar{c}_i)=\binom{i-1}{j}\bar{c}_{i+j}+\dots,$$

where … stands for a linear combination of $c_1c_{i+j-1},\dots,c_ic_j$.

In particular, by Lemma 2.5, we have

$$\begin{aligned} &S(\bar{c}_2)=\bar{c}_2+\bar{c}_3+\dots, && S(\bar{c}_3)=\bar{c}_3+\bar{c}_5+\dots,\\ &S(\bar{c}_6)=\bar{c}_6+\bar{c}_7+\bar{c}_{10}+\bar{c}_{11}+\dots, && S(\bar{c}_7)=\bar{c}_7+\bar{c}_9+\bar{c}_{11}+\bar{c}_{13}+\dots, \end{aligned} \tag{2.6}$$

where $\bar{c}_i=0$ for $i>n$. Besides, since $\bar{e}\in\mathrm{Ch}^1(X_n)$, we have

$$S(\bar{e})=\bar{e}+\bar{e}^2. \tag{2.7}$$

2b. Grothendieck ring of highest orthogonal grassmannians

For a smooth variety X, let $\widetilde{K}(X)$ denote the (extended) Rees ring of the Grothendieck ring $K(X)$ with respect to the topological filtration on $K(X)$, i.e.,

$$\widetilde{K}(X)=\bigoplus_{i\in\mathbb{Z}}\widetilde{K}^i(X),\ \ \text{where}\ \ \widetilde{K}^i(X)=K(X)^{(i)}t^{-i}$$

for a variable t, where $K(X)^{(i)}:=K(X)$ for $i<0$ (see [2, §4.5] for the definition of the extended Rees ring). We view $\widetilde{K}(X)$ as a subring of the Laurent polynomial ring $K(X)[t,t^{-1}]$. Note that $\widetilde{K}(X)$ is a graded ring, $\widetilde{K}^i(X)$ is its degree i component. The degree of $t\in\widetilde{K}^{-1}(X)$ is -1 and for any $i\in\mathbb{Z}$ the degree of any element of

$$K(X)^{(i)}\subset K(X)=\widetilde{K}^0(X)$$

is 0.

We have $GK(X)=\widetilde{K}(X)/t\widetilde{K}(X)$. We also have $\widetilde{K}(X)/I(X)=GK(X)/2GK(X)$, where $I(X)$ is the ideal of $\widetilde{K}(X)$ generated by t and 2.

To avoid the minus sign, we sometimes write u instead of t^{-1}. Note that $u\notin\widetilde{K}(X)$.

As in §2a, let X_n be the highest orthogonal grassmannian of a generic quadratic form of dimension $2n+1$ of trivial discriminant and Clifford invariant. Given $i\geq 0$, similarly to §2a, we now write $c_i\in K(X_n)^{(i)}$ for the K-theoretic Chern class of the dual of the tautological vector bundle on X_n and we also write $e_i\in K(\overline{X}_n)^{(i)}$ for the image of $l_{n-i}\in K(\overline{Y}_n)^{(n+i-1)}$ under the composition

$$\pi_*\circ(\pi')^*\colon K(\overline{Y}_n)^{(n+i-1)}\longrightarrow K(\mathcal{P})^{(n+i-1)}\longrightarrow K(\overline{X}_n)^{(i)}.$$

As the homomorphism φ in (1.1) commutes with Chern classes, we have $\varphi(c_i)=c_i$ and $\varphi(e_i)=e_i$ modulo $K(X_n)^{(i+1)}$ and $K(\overline{X}_n)^{(i+1)}$, respectively. We write p and l for the classes of $\prod_{i=1}^n e_i$ and $\prod_{i=2}^n e_i$ in $K(\overline{X}_n)^{(\dim\overline{X}_n)}$ and $K(\overline{X}_n)^{(\dim\overline{X}_n-1)}$, respectively. In particular, we have

$$K(\overline{X}_n)^{(\frac{n^2+n}{2})}=\mathbb{Z}\cdot p\quad\text{and}\quad K(\overline{X}_n)^{(\frac{n^2+n-2}{2})}=\mathbb{Z}\cdot p\oplus\mathbb{Z}\cdot l. \tag{2.8}$$

Since the Clifford invariant of q is trivial, we have

$$K(X_n)=K(\overline{X}_n) \tag{2.9}$$

by [14]. In addition, the following relations hold:

$$K(X_n)^{(1)}=K(\overline{X}_n)^{(1)}\quad\text{and}\quad K(X_n)^{(i)}\subset K(\overline{X}_n)^{(i)} \tag{2.10}$$

for any $i \geq 2$. We shall still write c_i for the image of $c_i \in K(X_n)^{(i)}$ under the restriction map in (2.10) and write $e \in K(X_n)^{(1)}$ for the element $e_1 \in K(\overline{X}_n)^{(1)}$. Note that in general, the (injective) restriction map $K(X_n)^{(i)} \to K(\overline{X}_n)^{(i)}$ is not an isomorphism. However, a restriction-corestriction argument shows that $(\operatorname{ind} X_n) \cdot K(\overline{X}_n)^{(i)} \subset K(X_n)^{(i)}$, so that

$$(\operatorname{ind} X_n) \cdot \widetilde{K}(\overline{X}_n) \subset \widetilde{K}(X_n). \tag{2.11}$$

We shall need [11, Lemma 4.1]. A typo (some plus sign in place of minus) made there is corrected here:

Lemma 2.12. ([11, Lemma 4.1]) *For any $i \geq 0$, the difference*

$$(2e_i - e_{i+1}) - c_i$$

is a sum of monomials in $c_1, \dots, c_n$ of degrees greater than or equal to $i+1$, where the degree of c_j for any $j \geq 0$ is defined to be j. In particular, the difference $2e_i - e_{i+1}$, lying a priori in $K(\overline{X}_n)^{(i)}$, actually lies in $K(X_n)^{(i)}$ and $2e_i - e_{i+1} = c_i$ in $K(X_n)^{(i/i+1)}$.

2c. Relation between Spin(2n+1) and Spin(2n+2)

Given any $n \geq 1$, we are going to show that Conjecture 1.2 with $G = \mathrm{Spin}(2n+1)$ is equivalent to the same conjecture with $G = \mathrm{Spin}(2n+2)$. This statement has already been mentioned in [11, §1] but no proof was provided.

Before proving the equivalence, let us mention that the genericity of q in Theorem 1.3 is only used for determination of the index of the variety X. So, the assumption that q is generic can be replaced by the assumption on the value of the index. Then it is also not needed to require the triviality of the discriminant because any quadratic form of odd dimension is similar to a quadratic form of trivial discriminant which has the same highest orthogonal grassmannians. So, we will actually prove the following stronger result:

Theorem 2.13. *For $n = 8, 9, 10$, let $X = X_n$ be the highest orthogonal grassmannian of a non-degenerate quadratic form q of dimension $2n+1$ with trivial Clifford invariant. If $\operatorname{ind} X$ is as in (2.1), then $\varphi\colon \mathrm{CH}(X) \to GK(X)$ is not an isomorphism.*

This stronger result is easier to use for producing the counter-examples to Conjecture 1.2 with $G = \mathrm{Spin}(2n+2)$.

Indeed, any $\mathrm{Spin}(2n+2)$-torsor E (over a field) yields a non-degenerate quadratic form q of dimension $2n+2$ with trivial discriminant and Clifford invariant. The highest orthogonal grassmannian of q consists of two connected components each of which is isomorphic to $X:=E/P$ for an appropriately chosen special parabolic subgroup $P\subset G$. Besides, X is isomorphic to the highest orthogonal grassmannian X' of any non-degenerate $(2n+1)$-dimensional subform q' of q, [5, Proposition 85.2]. The Clifford invariant of q' coincides with the Clifford invariant of q which is trivial. We also have $\operatorname{ind} X' = \operatorname{ind} X$ and, if E is generic, this is the torsion index of $\mathrm{Spin}(2n+2)$. By [16], the torsion index of $\mathrm{Spin}(2n+2)$ coincides with the torsion index of $\mathrm{Spin}(2n+1)$. It follows that Theorem 2.13 applies to X' and we get

Theorem 2.14. *For $n=8,9,10$, let X be a connected component of the highest orthogonal grassmannian of a generic quadratic form q of dimension $2n+2$ with trivial Clifford invariant. Then $\varphi\colon \mathrm{CH}(X)\to GK(X)$ is not an isomorphism. In particular, Conjecture 1.2 fails for $G=\mathrm{Spin}(2n+2)$.*

In order to prove the equivalence between $\mathrm{Spin}(2n+1)$ and $\mathrm{Spin}(2n+2)$ cases for arbitrary $n\geq 1$, we first deduce another consequence of Conjecture 1.2 with $G=\mathrm{Spin}(2n+1)$:

Lemma 2.15. *Assume that for some $n\geq 1$, Conjecture 1.2 holds for $\mathrm{Spin}(2n+1)$. Let X be the highest orthogonal grassmannian of a non-degenerate $(2n+1)$-dimensional quadratic form q and let $c_i\in \mathrm{CH}^i(X)$ for $i=1,\dots,n$ be the ith Chern class of the tautological vector bundle on X. Assume that c_1 is divisible by 2 and that the ring $\mathrm{CH}(X)$ is generated by $e:=c_1/2$ along with $c_2,\dots,c_n$. Then $\varphi\colon \mathrm{CH}(X)\to GK(X)$ is an isomorphism.*

Proof. Since c_1 is divisible by 2, the Clifford invariant of q is trivial, [5, Exercise 88.14(1)]. We consider the group $G=\mathrm{Spin}(2n+1)$ over the field of definition of q. We choose an embedding $G\hookrightarrow \mathrm{GL}(N)$ with some $N\geq 1$. Let $\tilde q$ be the quadratic form, given by the generic fiber of the quotient map

$$f\colon\ \mathrm{GL}(N)\longrightarrow Q:=\mathrm{GL}(N)/G,$$

and let $\widetilde X$ be the highest orthogonal grassmannian of $\tilde q$. The smooth variety Q has a rational point x such that the fiber of f over x is a $\mathrm{Spin}(2n+1)$-torsor that yields q, see [13, §3]. Therefore we have a specialization homomorphism $\mathrm{CH}(\widetilde X)\to \mathrm{CH}(X)$ which is a homomorphism of graded rings mapping for every i the ith Chern class of the tautological vector bundle on $\widetilde X$ to c_i. The specialization map $K(\widetilde X)\to K(X)$ is an isomorphism. Moreover, since the rings $\mathrm{CH}(\widetilde X)$ and $\mathrm{CH}(X)$ are generated by Chern classes (the element e is also the Chern class of a line bundle), the topological filtrations on both $K(\widetilde{X})$ and $K(X)$ coincide with the gamma-filtrations ([6, Proof of Theorem 3.7]) implying that the specialization map is an isomorphism of rings with filtrations. It follows that the specialization map $GK(\widetilde{X})\to GK(X)$ is an isomorphism. From the commutative square

$$\begin{array}{ccc} \mathrm{CH}(\widetilde{X}) & \xrightarrow{\text{iso}} & GK(\widetilde{X}) \\ \Big\downarrow{\scriptstyle \text{onto}} & & \Big\downarrow{\scriptstyle \text{iso}} \\ \mathrm{CH}(X) & \xrightarrow{\varphi} & GK(X) \end{array}$$

we conclude that the bottom map is an isomorphism. $\square$

Proposition 2.16. *For any* $n\geq 1$, *Conjecture* 1.2 *with* $G=\mathrm{Spin}(2n+1)$ *is equivalent to the same conjecture with* $G=\mathrm{Spin}(2n+2)$.

Proof. Assume that Conjecture 1.2 with $G=\mathrm{Spin}(2n+1)$ holds. To prove Conjecture 1.2 with $G=\mathrm{Spin}(2n+2)$, it suffices to show that φ: $\mathrm{CH}(X)\to GK(X)$ is an isomorphism, where X is a connected component of the highest orthogonal grassmannian given by a generic $(2n+2)$-dimensional quadratic form q of trivial discriminant and Clifford invariant. Since the variety X also is the highest orthogonal grassmannian of a non-degenerate $(2n+1)$-dimensional subform of q, φ is an isomorphism by Lemma 2.15.

The proof of the inverse implication is similar (with Lemma 2.15 replaced by its $\mathrm{Spin}(2n+2)$-analogue). $\square$

3. Proof of Theorem 1.3

Theorem 1.3 for q of dimension 17 is [11, Theorem 1.1]. So, here we assume that $\dim q=2n+1$ for $n=9,10$ and consider the highest orthogonal grassmannian X_n of q.

3a. Non-divisibility in $\mathrm{CH}(X_n)$

We first show that certain elements in $\mathrm{CH}(X_n)$ are not divisible by 2:

Proposition 3.1. *The elements*

$$c_2c_3c_6e^{31}\in\mathrm{CH}(X_9)\quad \textit{and}\quad c_2c_3c_6c_{10}e^{31}\in\mathrm{CH}(X_{10})$$

are not divisible by 2.

Proof. Consider the elements

$$A_9=(c_2+c_3+\dots)(c_3+c_5+\dots)(c_6+c_7+\dots)(e+e^2)^{31}\in \mathrm{CH}(X_9) \text{ and}$$
$$A_{10}=(c_2+c_3+\dots)(c_3+c_5+\dots)(c_6+c_7+c_{10}+\dots)c_{10}(e+e^2)^{31}\in \mathrm{CH}(X_{10}),$$

where ... stand for certain sums of pairwise products of c_i with $i>0$. By (2.6) and (2.7), they are integral representatives of $S(\bar{c}_2\bar{c}_3\bar{c}_6\bar{e}^{31})$ and $S(\bar{c}_2\bar{c}_3\bar{c}_6\bar{c}_{10}\bar{e}^{31})$, respectively, where S is the total Steenrod operation. For $n=9,10$, the $(\dim X_n)$th degree homogeneous part $A_n[\dim X_n]$ of A_n is therefore the integral representative of $S^3(\bar{c}_2\bar{c}_3\bar{c}_6\bar{e}^{31})$ and $S^3(\bar{c}_2\bar{c}_3\bar{c}_6\bar{c}_{10}\bar{e}^{31})$, respectively. Let $\deg\colon \mathrm{CH}^{\dim X_n}(X_n)\to\mathbb{Z}$ be the degree homomorphism (induced by the structure morphism of the variety X_n). Then, by Lemma 3.2 below, the image of $A_n[\dim X_n]$ under the degree map is an odd multiple of $\operatorname{ind} X_n$. Hence, $A_n[\dim X_n]$ is not divisible by 2 in $\mathrm{CH}(X_n)$ and the statement follows. □

Lemma 3.2. *For $n=9,10$, with the above notation,* $\operatorname{res}\big(A_n[\dim X_n]\big)$ *is an odd multiple of* $(\operatorname{ind} X_n)\cdot p$*, where, as in* (2.2)*,* $p\in \mathrm{CH}^{\dim X_n}(\overline{X}_n)$ is the class of a rational point.

Proof. We will prove the statement case by case.

Case $n=9$: Since modulo 2 we have

$$\begin{aligned}A_9[45]\equiv{}&(e^{16})^2(c_3c_3c_6e+c_3c_3c_7+c_2c_3c_6e^2+c_2c_3c_7e+c_2c_5c_6+\dots)\\&+e^{31}(c_3c_5c_6+c_2c_5c_7+\dots),\end{aligned}$$

where ... stand for a sum of products of at least four c_i with $i>0$, and $(e_1^{16})\equiv 0 \pmod 2$ by (2.4), we obtain

$$\operatorname{res}(A_9[45])\equiv 2^3(e_3e_5e_6e_1^{31}+e_2e_5e_7\cdot e_1^{16}\cdot e_1^{15}) \pmod{2^5}. \tag{3.3}$$

As $e_7^2\equiv 0 \pmod 2$ and $e_1^{16}\equiv 2e_7e_9 \pmod{2^2}$ by (2.3) and (2.4), it follows from (3.3) that, modulo 2^5,

$$\operatorname{res}(A_9[45])\equiv 2^3e_3e_5e_6\cdot e_1^{1+2+4+8}\cdot e_1^{16}\equiv 2^3e_3e_5e_6\cdot e_1e_2e_4e_8\cdot 2e_7e_9=2^4\cdot p.$$

Since $\operatorname{ind} X_9=2^4$, we are done with this case.

Case $n=10$: As the formal expressions for $A_{10}[55]$ and $A_9[45]$ are related by the equality $A_{10}[55]=c_{10}A_9[45]$, using (3.3), we get that

$$\mathrm{res}(A_{10}[55]) \equiv 2^4 e_1^{31}(e_3e_5e_6e_{10}+e_2e_5e_7e_{10}) \pmod{2^6}.$$

Now we have $e_1^{16}\equiv 2e_7e_9-2e_6e_{10} \pmod{2^2}$ by (2.4). As $e_{10}^2=0$ and $e_7^2\equiv 0 \pmod 2$, the second summand vanishes in the last formula for $A_{10}[55]$ and we come to the congruence (modulo 2^6)

$$\begin{aligned}\mathrm{res}(A_{10}[55]) &\equiv 2^4e_3e_5e_6e_{10}\cdot e_1^{1+2+4+8}\cdot e_1^{16} \equiv 2^4e_3e_5e_6e_{10}\cdot e_1e_2e_4e_8\cdot(2e_7e_9-2e_6e_{10})\\ &\equiv 2^4e_3e_5e_6e_{10}\cdot e_1e_2e_4e_8\cdot 2e_7e_9 = 2^5\cdot p.\end{aligned}$$

Hence, by (2.1) the statement follows. □

3b. Divisibility in $GK(X_n)$

Now we show that the images under the map

$$\varphi\colon\ \mathrm{CH}(X_n)\longrightarrow GK(X_n)$$

of the elements given in Proposition 3.1 are divisible by 2. This will complete the proof of Theorem 1.3.

Proposition 3.4. *The classes of the elements*

$$c_2c_3c_6e^{31}\in K(X_9)^{(42)} \quad \text{and} \quad c_2c_3c_6c_{10}e^{31}\in K(X_{10})^{(52)}$$

in the quotients $K(X_9)^{(42/43)}$ and $K(X_{10})^{(52/53)}$, respectively, are divisible by 2.

Proof. We shall use the (extended) Rees ring $\widetilde{K}(X_n)$ of $K(X_n)$ and the ideal $I(X_n)\subset\widetilde{K}(X_n)$, introduced in §2b. Let

$$B_9' := c_2c_3c_6e^{31}\in K(X_9)^{(42)} \quad \text{and} \quad B_{10}' := c_2c_3c_6c_{10}e^{31}\in K(X_{10})^{(52)}.$$

We have $B_9'u^{42}\in\widetilde{K}^{42}(X_9)$ and $B_{10}'u^{52}\in\widetilde{K}^{52}(X_{10})$, where $u=t^{-1}$. We will show that

$$B_9'u^{42}\in I(X_9) \quad \text{and} \quad B_{10}'u^{52}\in I(X_{10}).$$

Consider the elements

$$B_9 := (2e_2-e_3)(2e_3-e_4)(2e_6-e_7)e_1^{31}\in K(\overline{X}_9)^{(42)} \quad \text{and} \quad B_{10} := 2e_{10}B_9\in K(\overline{X}_{10})^{(52)},$$

where the equality defining B_{10} is an equality of formal expressions. We have $B_9\in K(X_9)^{(42)}$ and $B_{10}\in K(X_{10})^{(52)}$ by Lemma 2.12. (For the sake of clarification, let us recall that $K(X_n)=K(\overline{X}_n)$ and $e_i\in K(\overline{X}_n)^{(i)}$; however the inclusion $K(X_n)^{(i)}\subset$

$K(\overline{X}_n)^{(i)}$ can be strict.) Moreover, the classes of $B_9'u^{42}\in\widetilde{K}^{42}(X_9)$ and $B_{10}'u^{52}\in\widetilde{K}^{52}(X_{10})$ modulo $t\widetilde{K}^{43}(X_9)$, $t\widetilde{K}^{53}(X_{10})$ are represented by

$$B_9u^{42}\in\widetilde{K}^{42}(X_9) \quad\text{and}\quad B_{10}u^{52}\in\widetilde{K}^{52}(X_{10}),$$

respectively. Since

$$t\widetilde{K}^{43}(X_9)\subset I(X_9) \quad\text{and}\quad t\widetilde{K}^{53}(X_{10})\subset I(X_{10}),$$

it suffices to show that

$$B_9u^{42}\in I(X_9) \quad\text{and}\quad B_{10}u^{52}\in I(X_{10}).$$

We are going to prove first that

$$B_9u^{42}\in I(\overline{X}_9)^5 \quad\text{and}\quad B_{10}u^{52}\in I(\overline{X}_{10})^6. \tag{3.5}$$

For this, we expand the element B_9 as follows:

$$\big(2^3e_2e_3e_6-2^2(e_2e_3e_7+e_2e_4e_6+e_3^2e_6)+2(e_2e_4e_7+e_3^2e_7+e_3e_4e_6)-e_3e_4e_7\big)\cdot e_1^{31}. \tag{3.6}$$

Using the relations

$$(e_iu^i)^2\equiv e_{2i}u^{2i},\ e_{12}u^{12}\equiv 0,\quad\text{and}\quad e_{16}u^{16}\equiv 0 \mod I(\overline{X}_n) \tag{3.7}$$

for $i\geq 1$ and $n=9,10$, we easily see that each term in (3.6) multiplied by u^{42} is contained in $I(\overline{X}_9)^5$ except for the term

$$2(e_3^2e_7e_1^{31})u^{42}=2t^2\cdot(e_3^2e_1^7)u^{13}\cdot(e_7e_1^{24})u^{31}. \tag{3.8}$$

Similarly, each term in (3.6), considered as an element of $K(\overline{X}_{10})$ and multiplied by $2e_{10}u^{52}$, is contained in $I(\overline{X}_{10})^6$ except for the term

$$2^2e_{10}(e_3^2e_7e_1^{31})u^{52}=2^2t^2\cdot(e_3^2e_1^7)u^{13}\cdot(e_7e_{10}e_1^{24})u^{41}. \tag{3.9}$$

By the following Lemmas 3.10 and 3.14, the terms in (3.8) and (3.9) are contained in $I(\overline{X}_9)^5$ and $I(\overline{X}_{10})^6$, respectively. This proves inclusions (3.5).

It follows by (2.1) and (2.11) that B_9u^{42} is congruent modulo $I(X_9)$ to an element of

$$(2^3t^2)\widetilde{K}^{44}(\overline{X}_9)+(2^2t^3)\widetilde{K}^{45}(\overline{X}_9),$$

whereas $B_{10}u^{52}$ is congruent modulo $I(X_{10})$ to an element of

$$(2^4t^2)\widetilde{K}^{54}(\overline{X}_{10})+(2^3t^3)\widetilde{K}^{55}(\overline{X}_{10}).$$

According to (2.8) with $n=9,10$, these elements are of the shape $\big((2^3l)a+(2^2p)b\big)u^{42}$ and $\big((2^4l)c+(2^3p)d\big)u^{52}$, respectively (for some integers a, b, c, d). Therefore, by Lemma 3.16, we conclude that $B_9u^{42}\in I(X_9)$ and $B_{10}u^{52}\in I(X_{10})$. $\square$

Lemma 3.10. *For X_9, one has $(e_7\cdot e_1^{24})u^{31}\in I(\overline{X}_9)^2$.*

Proof. Let $x=(e_7\cdot e_1^{24})u^{31}\in\widetilde{K}^{31}(\overline{X}_9)$ and define $f_i:=e_iu^i\in\widetilde{K}^i(\overline{X}_9)$ for $i\geq 1$. Then, by (3.7), we get

$$x=f_7f_1^8f_1^8f_1^8\equiv f_7\cdot f_8^2\cdot f_8 \mod I(\overline{X}_9)^2. \tag{3.11}$$

We need to calculate e_8^2 modulo $K(\overline{X}_9)^{(18)}+2K(\overline{X}_9)^{(17)}$. This can be done using the K-theoretical Pieri formula [4, Theorem 1.2] involving some combinatorial calculations. To avoid them, we provide an alternative method using a partially split generic quadratic form.

Let us consider a quadratic form q' of dimension 19 which is the orthogonal sum of three hyperbolic planes with a 13-dimensional generic quadratic form q'_6. The generic quadratic form here (without any condition like the triviality of its discriminant or Clifford invariant, considered so far) is given by a generic torsor under the orthogonal group $\mathrm{O}(13)$. It can also be defined in an elementary way using free variables for its coefficients (see [10, §9] for details).

Let X'_9 be the highest orthogonal grassmannian of q'. The Grothendieck rings $K(\overline{X}'_9)$ and $K(\overline{X}_9)$ are identified canonically. For $i=7,8,9$, the element e_i is in $K(X'_9)^{(i)}$. By (2.3), the difference $e_8^2-2e_7e_9$ is in $K(X'_9)\cap K(\overline{X}_9)^{(17)}$.

We claim that $K(X'_9)\cap K(\overline{X}_9)^{(i)}=K(X'_9)^{(i)}$ for any $i\in\mathbb{Z}$. The claim is a consequence of the fact that the Chow group $\mathrm{CH}(X'_9)$ is free of torsion. To prove the fact, one uses the decomposition [3, Theorem 7.5] of the Chow motive of X'_9 in a direct sum of shifted motives of X'_6 – the highest orthogonal grassmannian of q'_6, implying that the graded group $\mathrm{CH}(X'_9)$ is a direct sum of shifted copies of $\mathrm{CH}(X'_6)$. The latter group is torsion free by [10, Corollary 6.2].

Having obtained the above claim, we conclude that the difference $e_8^2-2e_7e_9$ is in $K(X'_9)^{(17)}$. Taken modulo $K(X'_9)^{(18)}+2K(X'_9)^{(17)}$, it yields an element in the image of the restriction homomorphism $\mathrm{res}^{17}\colon \mathrm{Ch}^{17}(X'_9)\to\mathrm{Ch}^{17}(\overline{X}_9)$ of the modulo 2 Chow groups. By [17, Main Theorem 5.8] (see also [5, Theorem 87.7]), the image of the ring homomorphism $\mathrm{res}\colon \mathrm{Ch}(X'_9)\to\mathrm{Ch}(\overline{X}_9)$ of the total modulo 2 Chow groups is, as a ring, generated by e_7,e_8,e_9. In particular, any element of $\mathrm{Im}(\mathrm{res}^{17})$ is a multiple of e_8e_9. Thus we have the formula

$$e_8^2\equiv 2e_7e_9+me_8e_9 \mod K(\overline{X}_9)^{(18)}+2K(\overline{X}_9)^{(17)} \tag{3.12}$$

with some integer m.

Turning back to (3.11), since

$$e_7^2\in 2K(\overline{X}_9)^{(14)}+K(\overline{X}_9)^{(15)},$$

it follows from (3.12) that

$$f_7 \cdot (f_8)^2 \equiv t \cdot m e_7 e_8 e_9 u^{24} \mod I(\overline{X}_9)^2. \tag{3.13}$$

Furthermore, since

$$e_8^2 \in 2K(\overline{X}_9)^{(16)} + K(\overline{X}_9)^{(17)},$$

we get by (3.11) and (3.13) that

$$x \equiv t \cdot m e_7 e_8^2 e_9 u^{32} \equiv 0 \mod I(\overline{X}_9)^2. \quad \square$$

Lemma 3.14. *For X_{10}, one has $(e_7 \cdot e_{10} \cdot e_1^{24}) u^{41} \in I(\overline{X}_{10})^2$.*

Proof. Let $x = (e_7 \cdot e_{10} \cdot e_1^{24}) u^{41} \in \widetilde{K}^{41}(\overline{X}_{10})$ and $f_i := e_i u^i \in \widetilde{K}^i(\overline{X}_{10})$ for $i \geq 1$. Then, by (3.7) we get

$$x = f_7 f_{10} f_1^8 f_1^8 f_1^8 \equiv f_7 \cdot f_{10} \cdot f_8^2 \cdot f_8 \mod I(\overline{X}_{10})^2.$$

By the same arguments as in Lemma 3.10, using the orthogonal sum of (this time) four hyperbolic planes and q_6' (or the K-theoretical Pieri formula [4, Theorem 1.2]), we show that

$$e_8^2 \equiv 2(e_7 e_9 - e_6 e_{10}) + m e_8 e_9 + m' e_7 e_{10} \mod 2K(\overline{X}_{10})^{(17)} + K(\overline{X}_{10})^{(18)} \tag{3.15}$$

for some integers m and m'. Since, besides, $e_i^2 \in 2K(\overline{X}_{10})^{(2i)} + K(\overline{X}_{10})^{(2i+1)}$ for $i = 7, 8, 10$, it follows by (3.15) that $x \in I(\overline{X}_{10})^2$. $\square$

Lemma 3.16. *For X_9, one has $(2^3 l) u^{42}$, $(2^2 p) u^{42} \in I(X_9) \subset \widetilde{K}(X_9)$. For X_{10}, one has $(2^4 l) u^{52}$, $(2^3 p) u^{52} \in I(X_{10}) \subset \widetilde{K}(X_{10})$.*

Proof. Consider two elements $C_9 u^{42} \in \widetilde{K}(X_9)$ and $C_{10} u^{52} \in \widetilde{K}(X_{10})$, where

$$C_9 := (2e_2 - e_3)(2e_4 - e_5)(2e_6 - e_7) e_1^{30} \in K(X_9)^{(42)}$$

and $C_{10} \in K(X_{10})^{(52)}$, defined by the formal equality $C_{10} := 2 e_{10} C_9$.

We expand C_9 as follows:

$$[2^3(e_2 e_4 e_6) - 2^2(e_2 e_4 e_7 + e_2 e_5 e_6 + e_3 e_4 e_6) + 2(e_2 e_5 e_7 + e_3 e_4 e_7 + e_3 e_5 e_6) - (e_3 e_5 e_7)] \cdot e_1^{30}. \tag{3.17}$$

By (3.7), we see that each term in (3.17), multiplied by u^{42}, is contained in $I(\overline{X}_9)^5$ except for the following two terms

$$2(e_3 e_5 e_6 e_1^{30}) u^{42} \quad \text{and} \quad (e_3 e_5 e_7 e_1^{30}) u^{42}. \tag{3.18}$$

Similarly, each term in (3.17), considered in $K(\overline{X}_{10})$ and multiplied by $2e_{10}u^{52}$, is contained in $I(\overline{X}_{10})^6$ except for the following two terms

$$2^2(e_3e_5e_6e_{10}e_1^{30})u^{52} \quad \text{and} \quad 2(e_3e_5e_7e_{10}e_1^{30})u^{52}. \tag{3.19}$$

By Lemmas 3.10 and 3.14, the second terms in (3.18) and in (3.19) are contained in $I(\overline{X}_9)^5$ and $I(\overline{X}_{10})^6$, respectively. By Lemmas 3.24 and 3.25, we obtain

$$C_9u^{42} \equiv 2^2t^2(e_2 \ldots e_9)u^{44} \mod I(\overline{X}_9)^5$$

and

$$C_{10}u^{52} \equiv 2^3t^2(e_2 \ldots e_{10})u^{54} \mod I(\overline{X}_{10})^6. \tag{3.20}$$

Thus for X_9 we have

$$(C_9 - 2^2l)u^{42} \in \sum_{i=0}^{3}(2^{5-i}t^i)\widetilde{K}^{42+i}(\overline{X}_9). \tag{3.21}$$

By (2.1) and (2.11), both groups $2^5\widetilde{K}^{42}(\overline{X}_9)$ and $(2^4t)\widetilde{K}^{43}(\overline{X}_9)$ are contained in $I(X_9)$. As $e_1C_9u^{42} = t(e_1C_9u^{43}) \in I(X_9)$, multiplying the element in (3.21) by e_1 and 2, respectively, we get from (3.21) and (2.8) with $n=9$ that

$$(2^2p + 2^3ap)u^{42} \in I(X_9) \quad \text{and} \quad (2^3l + 2^3bp)u^{42} \in I(X_9) \tag{3.22}$$

for some integers a and b. As $(2^3p)u^{42} = e_1 \cdot (2^3l + 2^3bp)u^{42} \in I(X_9)$, the statement for X_9 follows by (3.22).

For X_{10}, by (3.20), we similarly have

$$(C_{10} - 2^3l)u^{52} \in \sum_{i=0}^{3}(2^{6-i}t^i)\widetilde{K}^{52+i}(\overline{X}_{10}).$$

Thus, multiplying the element $(C_{10} - 2^3l)u^{52}$ by e_1 and 2, respectively, we get

$$(2^3p + 2^4cp)u^{52} \in I(X_{10}) \quad \text{and} \quad (2^4l + 2^4dp)u^{52} \in I(X_{10}) \tag{3.23}$$

for some integers c and d. Again, since $(2^4p)u^{52} = e_1 \cdot (2^4l + 2^4dp)u^{52} \in I(X_{10})$, by (3.23) the proof of Lemma 3.16 is complete. □

Lemma 3.24. $e_1^{30}u^{30} \equiv 2(e_2e_4e_7e_8e_9)u^{30} \mod I(\overline{X}_9)^2$.

Proof. By (3.7) we have

$$e_1^{30}u^{30} = f_1^2 f_1^4 f_1^8 f_1^8 f_1^8 \equiv f_2 f_4 f_8 f_8^2 \mod I(\overline{X}_9)^2.$$

Hence, it follows by (3.12) that

$$e_1^{30}u^{30} \equiv f_2 f_4 f_8 (2e_7 e_9 + m e_8 e_9)u^{16} \equiv 2 f_2 f_4 f_8 (e_7 e_9 u^{16}) = 2 f_2 f_4 f_8 f_7 f_9 \mod I(\overline{X}_9)^2.$$

Here we use that $f_i^2 \in I(\overline{X}_n)$ for $i > n/2$ which follows from (2.3). $\square$

Lemma 3.25. $(e_{10} \cdot e_1^{30})u^{40} \equiv 2(e_2 e_4 e_7 e_8 e_9 e_{10})u^{40} \mod I(\overline{X}_{10})^2$.

Proof. Let $x = (e_{10} \cdot e_1^{30})u^{40} = f_{10} f_1^2 f_1^4 f_1^8 f_1^8 f_1^8$. Then, by (3.7) we have

$$x \equiv f_{10} f_2 f_4 f_8 f_8^2 \mod I(\overline{X}_{10})^2.$$

Therefore, it follows by (3.15) and (2.3) that

$$x \equiv 2 f_{10} f_2 f_4 f_8 (e_7 e_9 u^{16}) = 2 f_{10} f_2 f_4 f_8 f_7 f_9 \mod I(\overline{X}_{10})^2. \quad \square$$

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