Ark. Mat., **60** (2022), 351–364
DOI: 10.4310/ARKIV.2022.v60.n2.a7

On Hedenmalm-Shimorin type inequalities

Yong Han, Yanqi Qiu and Zipeng Wang

Abstract. We present a direct proof of an Hedenmalm-Shimorin inequality for short anti-diagonals proved recently in [HS20, Advances in Mathematics, 2020] and give the three tensor analogue of such inequality.

1. Introduction

1.1. Hedenmalm and Shimorin's inequality

Very recently, Hedenmalm and Shimorin proved the following:

Theorem A. (Hedenmalm and Shimorin [HS20]) *Let $M=\{m_{j,k}\}_{j,k=1}^{\infty}$ be an infinite complex-valued matrix which acts contractively on ℓ^2. Then*

$$\sum_{l=2}^{\infty} s^l \left| \sum_{j+k=l} \frac{m_{j,k}}{\sqrt{jk}} \right|^2 \le 2s \log\left(\frac{e}{1-s}\right), \quad 0 \le s < 1. \tag{1.1}$$

To prove Theorem A, Hedenmalm and Shimorin interpreted the bound (1.1) in terms of the correlation $\mathbb{E}\Phi(z)\Psi(z)$ of two coupled Gaussian analytic functions of Dirichlet type (simplified as $\mathcal{D}_0$-GAFs) with possibly intricate Gaussian correlation structure between them. More precisely, define a $\mathcal{D}_0$-GAF by

$$\Phi(z) = \sum_{j=1}^{\infty} \frac{\alpha_j}{\sqrt{j}} z^j, \quad z \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}, \tag{1.2}$$

where $(\alpha_j)_{j=1}^{\infty}$ are independent standard complex Gaussian variables, then Theorem A is equivalent to the following

Key words and phrases: Gaussian analytic functions, contractive operators, weighted Bergman spaces.
2010 Mathematics Subject Classification: primary 30B20; secondary 15A45, 15A60.

Theorem B. (Hedenmalm and Shimorin [HS20]) *For any two coupled $\mathcal{D}_0$-GAFs $\Phi(z)$ and $\Psi(z)$, with possibly intricate Gaussian correlation structure between them, we have*

$$\int_{\mathbb{T}} \left|\mathbb{E}\Phi(r\zeta)\Psi(r\zeta)\right|^2 dm(\zeta) \leq 2r^2 \log\left(\frac{e}{1-r^2}\right), \quad 0 \leq r < 1, \tag{1.3}$$

where dm is the normalized Lebesgue measure on the unit circle $\mathbb{T}$.

The inequality (1.3) follows immediately from the inequality (1.1). Indeed, if we write

$$\Phi(z) = \sum_{j=1}^{\infty} \frac{\alpha_j}{\sqrt{j}} z^j \quad \text{and} \quad \Psi(z) = \sum_{j=1}^{\infty} \frac{\beta_j}{\sqrt{j}} z^j,$$

with $(\alpha_j)_{j=1}^{\infty}$ and $(\beta_j)_{j=1}^{\infty}$ two sequences of independent standard complex Gaussian variables, with possibly intricate correlation structure between them, then the infinite matrix $M = \{m_{j,k}\}_{j,k=1}^{\infty}$ defined by

$$m_{j,k} := \mathbb{E}(\alpha_j \beta_k), \quad j,k \geq 1$$

acts contractively on ℓ^2 and the left hand side of the inequality (1.3) is given by

$$\int_{\mathbb{T}} \left|\mathbb{E}\Phi(r\zeta)\Psi(r\zeta)\right|^2 dm(\zeta) = \sum_{l=2}^{+\infty} r^{2l} \left| \sum_{j+k=l} \frac{m_{j,k}}{\sqrt{jk}} \right|^2. \tag{1.4}$$

Conversely, the inequality (1.3) also implies the inequality (1.1). The implication (1.3)$\Longrightarrow$(1.1) is rather simple by using a standard convexity argument and the fact that extreme points of the set of contractive operators on a Hilbert space are contained in the set of partial isometries.

1.2. Main results

Theorem 1.1. *For any infinite complex-valued matrix $M = \{m_{j,k}\}_{j,k=1}^{\infty}$, we have*

$$\sum_{l=2}^{\infty} s^l \left| \sum_{j+k=l} \frac{m_{j,k}}{\sqrt{jk}} \right|^2 \leq \left(\|M\|_{1\to 2}^2 + \|M\|_{2\to\infty}^2\right) s \log\left(\frac{1}{1-s}\right), \quad 0 \leq s < 1, \tag{1.5}$$

provided that the two quantities defined as follows

$$\|M\|_{1\to 2}^2 = \sup_{k\geq 1} \sum_{j=1}^{\infty} |m_{j,k}|^2 \quad \textit{and} \quad \|M\|_{2\to\infty}^2 = \sup_{j\geq 1} \sum_{k=1}^{\infty} |m_{j,k}|^2 \tag{1.6}$$

are both finite.

Remark *Note that* $\|M\|_{1\to 2}$ *and* $\|M\|_{2\to\infty}$ *are in fact the operator norms*:

$$\|M\|_{1\to 2} = \|M : \ell^1 \longrightarrow \ell^2\| \quad \text{and} \quad \|M\|_{2\to\infty} = \|M : \ell^2 \longrightarrow \ell^\infty\|.$$

The inequality (1.5) *clearly implies the inequality* (1.1) *since*

$$\max(\|M\|_{1\to 2}, \|M\|_{2\to\infty}) \le \|M : \ell^2 \longrightarrow \ell^2\|.$$

A little-o version of the inequality (1.1) for compact operators on ℓ^2 is given in the following

Proposition 1.2. *Suppose that the complex matrix* $M=\{m_{j,k}\}_{j,k=1}^{\infty}$ *is a compact operator on* ℓ^2. *Then*

$$\sum_{l=2}^{\infty} s^l \left| \sum_{j+k=l} \frac{m_{j,k}}{\sqrt{jk}} \right|^2 \le o\Big(\log \frac{1}{1-s}\Big), \quad \text{as } s \longrightarrow 1^-. \tag{1.7}$$

Theorem 1.1 can be easily generalized to the case of higher tensors. Here we only state Hedenmalm and Shimorin-type inequalities for 3-tensors.

Theorem 1.3. *Let* $\{m_{i,j,k}\}_{i,j,k=1}^{\infty}$ *be a sequence of complex numbers such that*

$$\sup_{j,k\ge 1} \sum_{i=1}^{\infty} |m_{i,j,k}|^2 + \sup_{i,k\ge 1} \sum_{j=1}^{\infty} |m_{i,j,k}|^2 + \sup_{i,j\ge 1} \sum_{k=1}^{\infty} |m_{i,j,k}|^2 \le 1. \tag{1.8}$$

Then

$$\sum_{l=3}^{\infty} \frac{s^l}{l+1} \left| \sum_{i+j+k=l} \frac{m_{i,j,k}}{\sqrt{ijk}} \right|^2 \le \frac{s}{2}\Big(\log \frac{1}{1-s}\Big)^2, \quad 0 \le s < 1. \tag{1.9}$$

It is not known to us whether the inequality (1.9) is optimal. However, we have the following

Proposition 1.4. *There exists a sequence* $\{m_{i,j,k}\}_{i,j,k=1}^{\infty}$ *of complex numbers with*

$$\max\Big(\sup_{j,k\ge 1} \sum_{i=1}^{\infty} |m_{i,j,k}|^2, \ \sup_{i,k\ge 1} \sum_{j=1}^{\infty} |m_{i,j,k}|^2, \ \sup_{i,j\ge 1} \sum_{k=1}^{\infty} |m_{i,j,k}|^2\Big) \le 1 \tag{1.10}$$

such that for a constant $c>0$, *we have*

$$\sum_{l=3}^{\infty} \frac{s^l}{l+1} \left| \sum_{i+j+k=l} \frac{m_{i,j,k}}{\sqrt{ijk}} \right|^2 \ge c \log \frac{1}{1-s} \quad \text{for all } s \in [0,1). \tag{1.11}$$

Remark 1.5. In the case where the numbers $m_{i,j,k}$ arise as expectation of products of three random variables, the inequality can be improved significantly. The following result is rather simple, we include it here only for comparison.

Let α, β, γ be three centered real random variables with finite moments up to order 6. Let $(\alpha_i)_{i=1}^\infty$, $(\beta_j)_{j=1}^\infty$ and $(\gamma_k)_{k=1}^\infty$ be independent copies of α, β, γ respectively, possibly with intricate joint distribution. Then, for any $\delta>0$, we have

$$\sum_{l=3}^\infty \frac{1}{(\log l)^{3+\delta}} \left| \sum_{i+j+k=l} \frac{\mathbb{E}(\alpha_i\beta_j\gamma_k)}{\sqrt{ijk}} \right|^2 < \infty. \tag{1.12}$$

In particular, we have

$$\sum_{l=3}^\infty \frac{1}{l+1} \left| \sum_{i+j+k=l} \frac{\mathbb{E}(\alpha_i\beta_j\gamma_k)}{\sqrt{ijk}} \right|^2 < \infty.$$

2. Hedenmalm and Shimorin's inequality

Proof of Theorem 1.1. For any fixed integer $l\geq 2$, by Cauchy-Schwarz inequality,

$$\left| \sum_{j+k=l} \frac{m_{j,k}}{\sqrt{jk}} \right|^2 \leq \sum_{j+k=l} \frac{|m_{j,k}|^2}{jk} \cdot \sum_{j+k=l} 1 = \sum_{j+k=l} \frac{|m_{j,k}|^2}{jk} \cdot (l-1).$$

Therefore, for any $s\in[0,1)$,

$$\begin{aligned} \sum_{l=2}^\infty s^l \left| \sum_{j+k=l} \frac{m_{j,k}}{\sqrt{jk}} \right|^2 &\leq \sum_{l=2}^\infty s^l \left(\sum_{j+k=l} \frac{|m_{j,k}|^2}{jk} \right) (l-1) \\ &\leq \sum_{l=2}^\infty s^l \left(\sum_{j+k=l} \frac{|m_{j,k}|^2}{jk} \right) l = \sum_{l=2}^\infty s^l \sum_{j+k=l} \frac{|m_{j,k}|^2}{jk} (k+j) \\ &= \underbrace{\sum_{l=2}^\infty s^l \sum_{j+k=l} \frac{|m_{j,k}|^2}{j}}_{\text{denoted by } I} + \underbrace{\sum_{l=2}^\infty s^l \sum_{j+k=l} \frac{|m_{j,k}|^2}{k}}_{\text{denoted by } II}. \end{aligned} \tag{2.13}$$

Now we estimate the summations I and II. Since $0\leq s<1$, for any $j\geq 1$, we have

$$\sum_{k=1}^\infty |m_{j,k}|^2 s^k = s \cdot \sum_{k=1}^\infty |m_{j,k}|^2 s^{k-1} \leq s \sup_{j\geq 1} \sum_{k=1}^\infty |m_{j,k}|^2 = s\|M\|_{2\to\infty}^2.$$

It follows that

$$I=\sum_{l=2}^{\infty}\sum_{j+k=l}\frac{|m_{j,k}|^2}{j}s^j s^k=\sum_{j,k=1}^{\infty}\frac{|m_{j,k}|^2}{j}s^j s^k=\sum_{j=1}^{\infty}\frac{s^j}{j}\sum_{k=1}^{\infty}|m_{j,k}|^2 s^k$$
$$\leq s\|M\|_{2\to\infty}^2\cdot\sum_{j=1}^{\infty}\frac{s^j}{j}=s\|M\|_{2\to\infty}^2\cdot\log\left(\frac{1}{1-s}\right).$$

Similarly, for all integers $k\geq 1$,

$$\sum_{j=1}^{\infty}|m_{j,k}|^2 s^j=s\cdot\sum_{j=1}^{\infty}|m_{j,k}|^2 s^{j-1}\leq s\cdot\sup_{k\geq 1}\sum_{j=1}^{\infty}|m_{j,k}|^2=s\|M\|_{1\to 2}^2,$$

then

$$II=\sum_{l=2}^{\infty}\sum_{j+k=l}\frac{|m_{j,k}|^2}{k}s^j s^k=\sum_{j,k=1}^{\infty}\frac{|m_{j,k}|^2}{k}s^j s^k=\sum_{k=1}^{\infty}\frac{s^k}{k}\sum_{j=1}^{\infty}|m_{j,k}|^2 s^j$$
$$\leq s\cdot\|M\|_{1\to 2}^2\cdot\sum_{k=1}^{\infty}\frac{s^k}{k}=s\|M\|_{1\to 2}^2\cdot\log\left(\frac{1}{1-s}\right).$$

This completes the whole proof. □

Proof of Proposition 1.2. Without loss of generality, we assume that $M:\ell^2\to\ell^2$ is a compact operator with operator norm $\|M\|_{2\to 2}\leq 1$. Recall the inequality (2.13):

$$\sum_{l=2}^{\infty}s^l\left|\sum_{j+k=l}\frac{m_{j,k}}{\sqrt{jk}}\right|^2\leq\underbrace{\sum_{l=2}^{\infty}s^l\sum_{j+k=l}\frac{|m_{j,k}|^2}{j}}_{\text{denoted by } I}+\underbrace{\sum_{l=2}^{\infty}s^l\sum_{j+k=l}\frac{|m_{j,k}|^2}{k}}_{\text{denoted by } II}.$$

Define

$$a_j(s)=\frac{s^j}{j},\quad b_j=\sum_{k=1}^{\infty}|m_{j,k}|^2\quad\text{and}\quad c_k=\sum_{j=1}^{\infty}|m_{j,k}|^2.$$

Since $\|M\|_{2\to 2}\leq 1$, we have $0\leq b_j\leq 1$ and $0\leq c_k\leq 1$. The compactness of M on ℓ^2 implies that

$$\lim_{j\to\infty}b_j=0\quad\text{and}\quad\lim_{k\to\infty}c_k=0.$$

For any $s\in[0,1)$, we have

$$I=\sum_{j=1}^{\infty}\frac{s^j}{j}\sum_{k=1}^{\infty}|m_{j,k}|^2 s^k\leq\sum_{k=1}^{\infty}a_j(s)b_j.$$

For any given $\varepsilon>0$, there exists an integer j_0 such that $b_j\leq\varepsilon$ for all $j\geq j_0$. Then we get

$$\begin{aligned}\sum_{j=1}^{\infty} a_j(s)b_j &\leq \varepsilon\sum_{j=j_0}^{\infty} a_j(s)+\sum_{j=1}^{j_0-1} a_j(s)b_j \leq \varepsilon\sum_{j=j_0}^{\infty} a_j(s)+\sum_{j=1}^{j_0-1} a_j(s)\\ &\leq \varepsilon\sum_{j=1}^{\infty} a_j(s)+\sum_{j=1}^{j_0-1} a_j(s)=\varepsilon\log\frac{1}{1-s}+\sum_{j=1}^{j_0-1} a_j(s).\end{aligned}$$

Therefore,

$$\limsup_{s\to 1^-}\frac{\sum_{j=1}^{\infty} a_j(s)b_j}{\log\frac{1}{1-s}}\leq\varepsilon+\limsup_{s\to 1^-}\frac{\sum_{j=1}^{j_0-1} a_j(s)}{\log\frac{1}{1-s}}=\varepsilon.$$

It follows that

$$\limsup_{s\to 1^-}\frac{I}{\log\frac{1}{1-s}}=0.$$

With similar arguments, we also have

$$\limsup_{s\to 1^-}\frac{II}{\log\frac{1}{1-s}}=0.$$

Consequently, we obtain

$$\lim_{s\to 1^-}\frac{1}{\log\frac{1}{1-s}}\sum_{l=2}^{\infty} s^l\left|\sum_{j+k=l}\frac{m_{j,k}}{\sqrt{jk}}\right|^2=0$$

and complete the proof. □

3. Hedenmalm and Shimorin-type inequalities for 3-tensors

Proof of Theorem 1.3. For any fixed integer $l\geq 3$, by Cauchy-Schwarz inequality, we have

$$\left|\sum_{i+j+k=l}\frac{m_{i,j,k}}{\sqrt{ijk}}\right|^2\leq\sum_{i+j+k=l}\frac{|m_{i,j,k}|^2}{ijk}\cdot\sum_{i+j+k=l}1=\sum_{i+j+k=l}\frac{|m_{i,j,k}|^2}{ijk}\frac{(l-1)(l-2)}{2}.$$

Therefore, for any $s\in[0,1)$, we have

$$\sum_{l=3}^{\infty}\frac{s^l}{l+1}\left|\sum_{i+j+k=l}\frac{m_{i,j,k}}{\sqrt{ijk}}\right|^2\leq\sum_{l=3}^{\infty}\frac{s^l}{l+1}\frac{(l-1)(l-2)}{2}\sum_{i+j+k=l}\frac{|m_{i,j,k}|^2}{ijk}$$

$$\leq \frac{1}{2}\sum_{l=3}^{\infty} s^l \sum_{i+j+k=l} \frac{|m_{i,j,k}|^2}{ijk}(i+j+k)$$

$$= \frac{1}{2}\Big(\underbrace{\sum_{l=3}^{\infty} s^l \sum_{i+j+k=l} \frac{|m_{i,j,k}|^2}{jk}}_{\text{denoted } T(1)} + \underbrace{\sum_{l=3}^{\infty} s^l \sum_{i+j+k=l} \frac{|m_{i,j,k}|^2}{ik}}_{\text{denoted } T(2)} + \underbrace{\sum_{l=3}^{\infty} s^l \sum_{i+j+k=l} \frac{|m_{i,j,k}|^2}{ij}}_{\text{denoted } T(3)} \Big).$$

We have

$$T(1) = \sum_{i,j,k=1}^{\infty} s^{i+j+k}\frac{|m_{i,j,k}|^2}{jk} = \sum_{j=1}^{\infty}\frac{s^j}{j}\sum_{k=1}^{\infty}\frac{s^k}{k}\sum_{i=1}^{\infty} s^i |m_{i,j,k}|^2$$

$$\leq \sum_{j=1}^{\infty}\frac{s^j}{j}\sum_{k=1}^{\infty}\frac{s^k}{k}\cdot s \sup_{j,k\geq 1}\sum_{i=1}^{\infty}|m_{i,j,k}|^2 = s\Big(\log\frac{1}{1-s}\Big)^2 \cdot \sup_{j,k\geq 1}\sum_{i=1}^{\infty}|m_{i,j,k}|^2.$$

Similarly, we have

$$T(2) \leq s\Big(\log\frac{1}{1-s}\Big)^2 \cdot \sup_{i,k\geq 1}\sum_{j=1}^{\infty}|m_{i,j,k}|^2$$

and

$$T(3) \leq s\Big(\log\frac{1}{1-s}\Big)^2 \cdot \sup_{i,j\geq 1}\sum_{k=1}^{\infty}|m_{i,j,k}|^2.$$

Under the assumption (1.8), we have

$$\sum_{l=3}^{\infty}\frac{s^l}{l+1}\left|\sum_{i+j+k=l}\frac{m_{i,j,k}}{\sqrt{ijk}}\right|^2 \leq \frac{s}{2}\Big(\log\frac{1}{1-s}\Big)^2.$$

This completes the proof of the theorem. □

The proof of Proposition 1.4 is based on a modified Zachary Chase's construction [HS20, p.35] described as follows. Let $\mathbb{N}$ denote the set of positive integers. For any even integer $d\geq 2$ and any integer $m\geq 2$, define

$$I_m(d) := \{(i,j,k)\in\mathbb{N}^3 : i,j,k \geq 2^{-1}d^{m-1} \text{ and } i+j+k = d^m\}.$$

Clearly, the subsets $I_m(d)\subset\mathbb{N}^3$ are mutually disjoint. Set

$$\mathcal{S}(d) := \bigsqcup_{m=2}^{\infty} I_m(d).$$

Lemma 3.1. *Let $d\geq 2$ be an integer. For any $(i,j)\in\mathbb{N}^2$, there exists at most one $k\in\mathbb{N}$ such that $(i,j,k)\in\mathcal{S}(d)$. That is,*

$$\sup_{i,j\in\mathbb{N}}\sum_{k\in\mathbb{N}}\mathbb{1}_{\mathcal{S}(d)}(i,j,k)\leq 1.$$

Similarly,

$$\sup_{j,k\in\mathbb{N}}\sum_{i\in\mathbb{N}}\mathbb{1}_{\mathcal{S}(d)}(i,j,k)\leq 1 \quad \textit{and} \quad \sup_{i,k\in\mathbb{N}}\sum_{j\in\mathbb{N}}\mathbb{1}_{\mathcal{S}(d)}(i,j,k)\leq 1.$$

Proof. We prove the lemma by contradiction. Suppose there exists $(i,j)\in\mathbb{N}^2$ and two distinct integers $k_1,k_2\in\mathbb{N}$ such that $(i,j,k_1),(i,j,k_2)$ are both inside the subset $\mathcal{S}(d)$. Then, by the definition of the set $\mathcal{S}(d)$, there exist two distinct integers $m_1,m_2\in\mathbb{N}$ with $m_1\geq 2, m_2\geq 2$ such that

$$\begin{cases} i,j,k_1\geq 2^{-1}d^{m_1-1} \\ i+j+k_1=d^{m_1}\end{cases} \quad \text{and} \quad \begin{cases} i,j,k_2\geq 2^{-1}d^{m_2-1} \\ i+j+k_2=d^{m_2}\end{cases}.$$

Without loss of generality, we assume that $m_2>m_1$. Then

$$d^{m_1}=i+j+k_1\geq 2^{-1}d^{m_2-1}+2^{-1}d^{m_2-1}+2^{-1}d^{m_1-1}=d^{m_2-1}+2^{-1}d^{m_1-1}.$$

That is,

$$1\geq d^{m_2-m_1-1}+\frac{1}{2d}.$$

Note that the assumption $m_2>m_1$ implies $m_2-m_1-1\geq 0$. Thus, we obtain

$$1\geq d^{m_2-m_1-1}+\frac{1}{2d}\geq 1+\frac{1}{2d},$$

which is absurd and we complete the proof of the lemma. □

Proof of Proposition 1.4. Let $d\geq 2$ be an even integer and take

$$m_{i,j,k}=\mathbb{1}_{\mathcal{S}(d)}(i,j,k), \quad i,j,k\in\mathbb{N}.$$

By Lemma 3.1, $\{m_{i,j,k}\}_{i,j,k=1}^{\infty}$ satisfies the assumption (1.10) of Proposition 1.4. We now show that this sequence $\{m_{i,j,k}\}_{i,j,k=1}^{\infty}$ satisfies the required lower estimation (1.11). For any integer $m\geq 2$ and any $(i,j,k)\in I_m(d)$, we have $i,j,k\leq d^m$ and hence

$$\sqrt{ijk}\leq d^{\frac{3m}{2}}. \tag{3.14}$$

Thus

$$\sum_{i+j+k=d^m}\frac{m_{i,j,k}}{\sqrt{ijk}}=\sum_{i+j+k=d^m}\frac{\mathbb{1}_{I_m(d)}(i,j,k)}{\sqrt{ijk}}\geq\frac{\sharp I_m(d)}{d^{\frac{3m}{2}}}, \tag{3.15}$$

where $\sharp I_m(d)$ denotes the cardinality of the finite set $I_m(d)$. Since d is an even integer and $m\in\mathbb{N}$ with $m\geq 2$, we have $2^{-1}d^{m-1}\in\mathbb{N}$. Now using the following equality

$$\sharp I_m(d)=\sharp\{(i,j)\in\mathbb{N}^2: i,j,d^m-i-j\geq 2^{-1}d^{m-1}\},$$

we obtain

$$\sharp I_m(d)=\sum_{\ell=1}^{d^m-\frac{3d^{m-1}}{2}+1}\ell=\frac{1}{2}\Big(d^m-\frac{3d^{m-1}}{2}+1\Big)\Big(d^m-\frac{3d^{m-1}}{2}+2\Big)$$
$$\geq\frac{1}{2}d^{2m}(1-\frac{3}{2d})^2\geq\frac{d^{2m}}{32}. \tag{3.16}$$

Combining (3.14), (3.15) and (3.16), we obtain, for any $m\geq 2$, that

$$\sum_{i+j+k=d^m}\frac{m_{i,j,k}}{\sqrt{ijk}}=\sum_{i+j+k=d^m}\frac{\mathbb{1}_{\mathcal{S}(d)}(i,j,k)}{\sqrt{ijk}}\geq\frac{d^{\frac{m}{2}}}{32}.$$

It follows that, for $d\geq 2$ and $m\geq 2$, we have

$$\frac{1}{d^m+1}\Big(\sum_{i+j+k=d^m}\frac{m_{i,j,k}}{\sqrt{ijk}}\Big)^2\geq\frac{1}{32}\frac{d^m}{d^m+1}\geq\frac{1}{40}.$$

Therefore, for any $s\in[0,1)$, we have

$$\sum_{l=3}^{\infty}\frac{s^l}{l+1}\Big(\sum_{i+j+k=l}\frac{m_{i,j,k}}{\sqrt{ijk}}\Big)^2=\sum_{m=2}^{\infty}\frac{s^{d^m}}{d^m+1}\Big(\sum_{i+j+k=d^m}\frac{\mathbb{1}_{\mathcal{S}(d)}(i,j,k)}{\sqrt{ijk}}\Big)^2$$
$$\geq\frac{1}{40}\sum_{m=2}^{\infty}s^{d^m}.$$

Finally, by applying the well-known equality (cf. [HS20, p.36])

$$\lim_{s\to 1^-}\frac{1}{\log\frac{1}{1-s}}\sum_{m=2}^{\infty}s^{d^m}=\frac{1}{\log d},$$

we see that there exists a constant $c_d>0$ depending on d such that

$$\sum_{l=3}^{\infty}\frac{s^l}{l+1}\Big(\sum_{i+j+k=l}\frac{m_{i,j,k}}{\sqrt{ijk}}\Big)^2\geq c_d\log\frac{1}{1-s}\quad\text{for } s\in[0,1).$$

This completes the proof of the proposition. □

We now proceed to the proof of inequality (1.12). The following elementary lemma will be useful for us.

Lemma 3.2. *For any $\delta>0$, there exist two constants $c_1, c_2>0$ depending on δ such that for any integer $n\geq 1$, we have*

$$\frac{c_1}{\log(n+1)^{3+\delta}} \leq \int_0^1 \frac{t^n}{(1-t)[\log \frac{2}{1-t}]^{4+\delta}} dt \leq \frac{c_2}{\log(n+1)^{3+\delta}}. \tag{3.17}$$

Proof. By change of variables, we have

$$\int_0^1 \frac{t^n}{(1-t)[\log \frac{2}{1-t}]^{4+\delta}} dt = \int_{\log 2}^{\infty} H_n(x)dx,$$

where

$$H_n(x) = \frac{(1-2e^{-x})^n}{x^{4+\delta}}.$$

Note that

$$H_n'(x) = \frac{(1-2e^{-x})^{n-1}}{x^{5+\delta}} xe^{-x}(4+\delta)\Big(\frac{2n}{4+\delta} - \frac{e^x-2}{x}\Big).$$

It is easy to see that the function $(e^x-2)/x$ is increasing for $x\in(0,\infty)$. Therefore, for any integer n such that $\log(n)>4+\delta$ and any $x\in[\log 2, \log n]$, we have

$$\frac{2n}{4+\delta} - \frac{e^x-2}{x} \geq \frac{2n}{4+\delta} - \frac{n-2}{\log n} > n\Big(\frac{2}{4+\delta} - \frac{1}{\log n}\Big) > 0.$$

It follows that for all integer $n\geq e^{4+\delta}$, the function $H_n(x)$ is increasing on $[\log 2, \log n]$. Consequently, we have

$$0 \leq \int_{\log 2}^{\log n} H_n(x)dx \leq H_n(\log n)\log n = \Big(1-\frac{2}{n}\Big)^n (\log n)^{-3-\delta} \leq c(\log n)^{-3-\delta},$$

where $c>0$ is a numerical constant. We thus obtain, for all integer $n\geq e^{4+\delta}$, that

$$\int_{\log 2}^{\infty} H_n(x)dx \leq c(\log n)^{-3-\delta} + \int_{\log n}^{\infty} \frac{1}{x^{4+\delta}} dx = \Big(c + \frac{1}{3+\delta}\Big)(\log n)^{-3-\delta}$$

and

$$\int_{\log 2}^{\infty} H_n(x)dx \geq \int_{\log n}^{\infty} \frac{(1-2e^{-x})^n}{x^{4+\delta}} dx \geq \int_{\log n}^{\infty} \frac{(1-2/n)^n}{x^{4+\delta}} dx \geq \frac{c'}{3+\delta}(\log n)^{-3-\delta},$$

where $c'>0$ is a numerical constant (for instance, take $c'=\inf_{n\geq e^{4+\delta}}(1-2/n)^n>0$). For the finitely many integers $1\leq n<e^{4+\delta}$, the inequalities (3.17) clearly hold for suitable $c_1, c_2>0$, hence by modifying the two constants c_1, c_2 if necessary, the inequalities (3.17) hold for all integers $n\geq 1$. □

Proof of inequality (1.12). Fix a number $\delta>0$. Let $(\alpha_j)_{j=1}^{\infty}$, $\beta=(\beta_j)_{j=1}^{\infty}, \gamma=(\gamma_j)_{j=1}^{\infty}$ be three sequence of random variables as stated in Remark 1.5. For any $r\in[0,1)$, define

$$S(r):=\sum_{l=3}^{\infty}\left|\sum_{i+j+k=l}\frac{\mathbb{E}(\alpha_i\beta_j\gamma_k)}{\sqrt{ijk}}\right|^2\frac{r^{2l}}{(\log l)^{3+\delta}}.$$

Then, to prove the inequality (1.12), it suffices to prove

$$\sup_{0\le r<1} S(r)<\infty. \tag{3.18}$$

By Lemma 3.2, there exists a constant $C>0$ such that

$$\frac{1}{(\log l)^{3+\delta}}\le C\int_{\mathbb{D}}|z|^{2l}\frac{dA(z)}{(1-|z|^2)[\log\frac{2}{1-|z|^2}]^{4+\delta}}\quad\text{for all integers } l\ge 3,$$

where $dA(z)$ is the normalized Lebesgue measure on $\mathbb{D}$. Therefore, for any $r\in[0,1)$,

$$S(r)\le \underbrace{C\sum_{l=3}^{\infty}\left|\sum_{i+j+k=l}\frac{\mathbb{E}(\alpha_i\beta_j\gamma_k)}{\sqrt{ijk}}\right|^2 r^{2l}\int_{\mathbb{D}}|z|^{2l}\frac{dA(z)}{(1-|z|^2)[\log\frac{2}{1-|z|^2}]^{4+\delta}}}_{\text{denoted } I(r)}.$$

Consequently, the inequality (3.18) would be a consequence of the following inequality

$$\sup_{0\le r<1} I(r)<\infty. \tag{3.19}$$

Now define three random analytic functions on $\mathbb{D}$ by

$$F_\alpha(z)=\sum_{j=1}^{\infty}\frac{\alpha_j}{\sqrt{j}}z^j,\quad F_\beta(z)=\sum_{j=1}^{\infty}\frac{\beta_j}{\sqrt{j}}z^j\quad\text{and}\quad F_\gamma(z)=\sum_{j=1}^{\infty}\frac{\gamma_j}{\sqrt{j}}z^j.$$

Set

$$f_r(z):=\mathbb{E}[F_\alpha(rz)F_\beta(rz)F_\gamma(rz)]=\sum_{l=3}^{\infty}\left(\sum_{i+j+k=l}\frac{\mathbb{E}(\alpha_i\beta_j\gamma_k)}{\sqrt{ijk}}\right)r^l z^l. \tag{3.20}$$

Then clearly, we have

$$I(r)=\int_{\mathbb{D}}|f_r(z)|^2\frac{dA(z)}{(1-|z|^2)[\log\frac{2}{1-|z|^2}]^{4+\delta}}. \tag{3.21}$$

Write the integral in (3.21) in the polar coordinate system $z = \rho e^{i\theta}$ with $0 \le \rho < 1$ and $\theta \in [0, 2\pi)$, we obtain

$$\begin{aligned} I(r) &= 2 \int_0^1 \Big[\int_0^{2\pi} |f_r(\rho e^{i\theta})|^2 d\theta \Big] \frac{\rho d\rho}{(1-\rho^2)[\log \frac{2}{1-\rho^2}]^{4+\delta}} \\ &= 2 \int_0^1 \Big[\int_0^{2\pi} |f_{\rho r}(e^{i\theta})|^2 d\theta \Big] \frac{\rho d\rho}{(1-\rho^2)[\log \frac{2}{1-\rho^2}]^{4+\delta}} \\ &= 2 \int_0^1 \|f_{\rho r}\|^2_{L^2(\mathbb{T})} \frac{\rho d\rho}{(1-\rho^2)[\log \frac{2}{1-\rho^2}]^{4+\delta}}. \end{aligned}$$

Let us proceed to the estimate of $\|f_{\rho r}\|^2_{L^2(\mathbb{T})}$. From the definition (3.20), for any $\rho \in [0, 1)$ and $r \in [0, 1)$, by Jensen's inequality and then by Hölder's inequality, we have

$$\begin{aligned} \|f_{\rho r}\|^2_{L^2(\mathbb{T})} &= \|\mathbb{E}[F_\alpha(\rho r \cdot) F_\beta(\rho r \cdot) F_\gamma(\rho r \cdot)]\|^2_{L^2(\mathbb{T})} \le \mathbb{E}\Big[\|F_\alpha(\rho r \cdot) F_\beta(\rho r \cdot) F_\gamma(\rho r \cdot)\|^2_{L^2(\mathbb{T})} \Big] \\ &\le \mathbb{E}\Big[\|F_\alpha(\rho r \cdot)\|^2_{L^6(\mathbb{T})} \|F_\beta(\rho r \cdot)\|^2_{L^6(\mathbb{T})} \|F_\beta(\rho r \cdot)\|^2_{L^6(\mathbb{T})} \Big]. \end{aligned}$$

Hence, by Hölder's inequality again, we have

(3.22)

$$\begin{aligned} \|f_{\rho r}\|_{L^2(\mathbb{T})} &\le \Big(\mathbb{E}\Big[\|F_\alpha(\rho r \cdot)\|^2_{L^6(\mathbb{T})} \|F_\beta(\rho r \cdot)\|^2_{L^6(\mathbb{T})} \|F_\gamma(\rho r \cdot)\|^2_{L^6(\mathbb{T})} \Big] \Big)^{1/2} \\ &\le \Big[\mathbb{E} \|F_\alpha(\rho r \cdot)\|^6_{L^6(\mathbb{T})} \Big]^{1/6} \Big[\mathbb{E} \|F_\beta(\rho r \cdot)\|^6_{L^6(\mathbb{T})} \Big]^{1/6} \Big[\mathbb{E} \|F_\gamma(\rho r \cdot)\|^6_{L^6(\mathbb{T})} \Big]^{1/6}. \end{aligned}$$

By Khintchine's inequality for centered i.i.d. random variables, there exists a constant $C_\alpha > 0$ such that for any $r \in [0, 1)$ and any $\zeta \in \mathbb{T}$, we have

$$\Big(\mathbb{E} |F_\alpha(\rho r \zeta)|^6 \Big)^{1/6} = \Big(\mathbb{E} \Big| \sum_{j=1}^\infty \alpha_j \frac{\rho^j r^j \zeta^j}{\sqrt{j}} \Big|^6 \Big)^{1/6} \le C_\alpha \Big(\mathbb{E} \Big| \sum_{j=1}^\infty \alpha_j \frac{\rho^j r^j \zeta^j}{\sqrt{j}} \Big|^2 \Big)^{1/2}.$$

Since $(\alpha_j)_{j=1}^\infty$ are centered i.i.d. random variables, they are orthogonal and with a common L^2-norm $\|\alpha\|_2$. Then

$$\mathbb{E} \Big| \sum_{j=1}^\infty \alpha_j \frac{\rho^j r^j \zeta^j}{\sqrt{j}} \Big|^2 = \|\alpha\|_2^2 \sum_{j=1}^\infty \frac{\rho^{2j} r^{2j}}{j}.$$

Therefore,

$$\sup_{0 \le r < 1} \Big[\mathbb{E} \|F_\alpha(\rho r \cdot)\|^6_{L^6(\mathbb{T})} \Big]^{1/6} \le C_\alpha \|\alpha\|_2 \sup_{0 \le r < 1} \Big(\sum_{j=1}^\infty \frac{\rho^{2j} r^{2j}}{j} \Big)^{1/2}$$

$$= C_\alpha \|\alpha\|_2 \Big(\log \frac{1}{1-\rho^2} \Big)^{1/2}.$$

Similar inequalities hold for the counterparts of β, γ and hence there exists a constant $C=C(\alpha,\beta,\gamma)>0$ such that for any $\rho\in[0,1)$,

$$\sup_{0\le r<1} \|f_{\rho r}\|_{L^2(\mathbb{T})}^2 \le C(\alpha,\beta,\gamma) \Big(\log \frac{1}{1-\rho^2} \Big)^3.$$

It follows that

$$\begin{aligned} \sup_{0\le r<1} I(r) &\le 2 \int_0^1 \sup_{0\le r<1} \|f_{\rho r}\|_{L^2(\mathbb{T})}^2 \frac{\rho d\rho}{(1-\rho^2)[\log \frac{2}{1-\rho^2}]^{4+\delta}} \\ &\le C(\alpha,\beta,\gamma) \int_0^1 \frac{dt}{(1-t)[\log \frac{2}{1-t}]^{1+\delta}} < \infty. \end{aligned}$$

This completes the proof of the desired inequality (3.19). □

Acknowledgements

Y. Qiu is supported by grants NSFC Y7116335K1, NSFC 11801547 and NSFC 11688101 of National Natural Science Foundation of China. Z. Wang is supported by NSFC 11601296.

References

[HS20] Hedenmalm, H. and Shimorin, S., Gaussian analytic functions and operator symbols of dirichlet type, *Adv. Math.* **372**, 107301 (2020). MR4126719

Yong Han
College of Mathematics and Statistics
Shenzhen University
Shenzhen 518060
Guangdong
China
hanyongprobability@gmail.com

Yanqi Qiu
School of Mathematics and Statistics
Wuhan University
Wuhan 430072
Hubei
China
and
Institute of Mathematics, AMSS
Chinese Academy of Sciences
Beijing 100190
China
yanqi.qiu@hotmail.com

Zipeng Wang
College of Mathematics and Statistics
Chongqing University
Chongqing 401331
China
zipengwang2012@gmail.com,
zipengwang@cqu.edu.cn

Received September 14, 2021
in revised form November 28, 2021