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Weil–Poincaré series and topology of collections of valuations on rational double points

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Abstract. Earlier it was described to which extent the Alexander polynomial in several variables of an algebraic link in the Poincaré sphere determines the topology of the link. It was shown that, except some explicitly described cases, the Alexander polynomial of an algebraic link determines the combinatorial type of the minimal resolution of the curve and therefore the topology of the corresponding link. The Alexander polynomial of an algebraic link in the Poincaré sphere coincides with the Poincaré series of the corresponding set of curve valuations. The latter one can be defined as an integral over the space of divisors on the $\mathbb{E}_8$-singularity. Here, we consider a similar integral for rational double point surface singularities over the space of Weil divisors called the Weil–Poincaré series. We show that, except a few explicitly described cases, the Weil–Poincaré series of a collection of curve valuations on a rational double point surface singularity determines the topology of the corresponding link. We give analogous statements for collections of divisorial valuations.

1. Introduction

An algebraic link in the three-dimensional sphere is the intersection $K=C\cap\mathbb{S}^3_\varepsilon$ of a germ $(C,0)\subset(\mathbb{C}^2,0)$ of a complex analytic plane curve with the sphere $\mathbb{S}^3_\varepsilon$ of radius ε centred at the origin in $\mathbb{C}^2$ with ε small enough. The number r of the components of the link K is equal to the number of the irreducible components of the curve $(C,0)$. It is well-known that the Alexander polynomial in r variables determines the topological type of an algebraic link (or, equivalently, the (local)

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topological type of the triple $(\mathbb{C}^2, C, 0)$): [15]. This follows from the fact that the Alexander polynomial determines the combinatorial type of the minimal embedded resolution of the curve C. The Alexander polynomial is defined for links in three-dimensional manifolds which are homology spheres. One of them is the Poincaré sphere which is the intersection of the surface $S=\{(z_1, z_2, z_3)\in\mathbb{C}^3 : z_1^5+z_2^3+z_3^2=0\}$ (the $\mathbb{E}_8$ surface singularity) with the 5-dimensional sphere $\mathbb{S}^5_\varepsilon=\{(z_1, z_2, z_3)\in\mathbb{C}^3 : |z_1|^2+|z_2|^2+|z_3|^2=\varepsilon^2\}$. An algebraic link in the Poincaré sphere is the intersection of a germ $(C, 0)\subset(S, 0)$ of a complex analytic curve in $(S, 0)$ with the sphere $\mathbb{S}^5_\varepsilon$ of radius ε small enough.

A (reducible) curve singularity $(C, 0)\subset(S, 0)$ in a normal surface singularity determines a collection of (discrete rang one) valuations on the ring $\mathcal{O}_{S,0}$ of germs of functions on S (called curve valuations). To a collection $\{v_i\}$ of (discrete rank one) valuations on $\mathcal{O}_{S,0}$, $i=1, ..., r$, one may associate, as in [11], a Poincaré series $P_{\{v_i\}}(t_1, ..., t_r)\in\mathbb{Z}[[t_1, ..., t_r]]$ (see also Section 2). In [2] it was shown that, for $(S, 0)=(\mathbb{C}^2, 0)$, the Poincaré series $P_{\{v_{C_i}\}}(t_1, ..., t_r)$ of a collection of (different) curve valuations $\{v_{C_i}, i=1, ..., r\}$ coincides with the Alexander polynomial $\Delta^C(t_1, ..., t_r)$ in r variables of the algebraic link defined by the curve $C=\bigcup\limits_{i=1}^{r} C_i$ for $r>1$. (For $r=1$, one has $P_{v_C}(t)=\frac{\Delta^C(t)}{1-t}$.) In [5] it was shown that the same holds for an algebraic link in the Poincaré sphere. In [7] it was proved that the Poincaré series of a collection of divisorial valuations on $\mathcal{O}_{\mathbb{C}^2,0}$ (computed in [12]) determines the combinatorial type of the minimal resolution of the collection. (In general, this is not the case for a collection consisisting both of curve and divisorial valuations.)

In [9], it was discussed to which extent the Alexander polynomial in several variables of an algebraic link in the Poincaré sphere (that is the Poincaré series of the corresponding curve) determines the topology of the link or rather the combinatorial type of the minimal (embedded) resolution of the curve on the $\mathbb{E}_8$ surface singularity. It was shown that two curves (even irreducible ones) with combinatorially different minimal resolutions may have equal Alexander polynomials. However, under some restrictions on the curve (formulated in terms of the intersection of the strict transform of a curve with the exceptional divisor in the minimal resolution of the $\mathbb{E}_8$ surface singularity), its Poincaré series determines the combinatorial type of the minimal resolution of the curve and therefore the topology of the corresponding link. There were given analogues of these statements for collections of divisorial valuations on the $\mathbb{E}_8$ surface singularity.

For other surface singularities (say, for rational ones, for whom one has a formula for the Poincaré series) the (classical) Poincaré series for a collection of curve or divisorial valuations does not determine the combinatorial type of the minimal resolution even in the simplest case of the $\mathbb{A}_k$ singularities. The Poincaré series of a collection of valuations can be interpreted as an integral with respect to the Euler characteristic over the space of Cartier divisors (appropriately defined: see below). We consider an analogue of the Poincaré series which is the same integral over the space of all (that is Weil) divisors. One has an equation for this ("Weil–Poincaré") series similar to the one for the smooth case or for the $\mathbb{E}_8$ surface singularity. (The Weil–Poincaré series of a collection of curve or divisorial valuations is a fractional power series.) We show that, except a few cases (somewhat similar to the exceptions for the $\mathbb{E}_8$ singularity), the Weil–Poincaré series of a collection of curve valuations or of a collection of divisorial valuations on a rational double point surface singularity determines the combinatorial type of the minimal resolution (and thus the topology of the corresponding link in the curve case) up to the possible symmetry of the minimal resolution graph of the surface singularity. (It is somewhat curious that exceptions exist for the $\mathbb{E}_7$ and for the $\mathbb{E}_8$ surface singularities, i.e. precisely for those whose minimal resolution graphs have no non-trivial symmetries.)

2. The Weil–Poincaré series

A valuation (discrete of rank one) on the ring $\mathcal{O}_{V,0}$ of germs of functions on a complex analytic variety $(V,0)$ is a function $v:\mathcal{O}_{V,0}\to\mathbb{Z}_{\geq 0}\cup\{+\infty\}$ such that

1) $v(\lambda g)=v(g)$ for $\lambda\in\mathbb{C}$, $\lambda\neq 0$;
2) $v(g_1+g_2)\geq\min(v(g_1),v(g_2))$;
3) $v(g_1g_2)=v(g_1)+v(g_2)$.

We permit a valuation to have the value infinity for a non-zero element. (In this case, some authors speak about "semivaluations".)

An irreducible curve germ $(C,0)$ in a germ of a complex analytic variety $(V,0)$ defines a valuation v_C on the ring $\mathcal{O}_{V,0}$ of germs of functions on $(V,0)$ (called a curve valuation). Let $\varphi:(\mathbb{C},0)\to(V,0)$ be a parametrization (an uniformization) of the curve $(C,0)$, that is $\operatorname{Im}\varphi=(C,0)$ and φ is an isomorphism between punctured neighbourhoods of the origin in $\mathbb{C}$ and in C. For a function germ $f\in\mathcal{O}_{V,0}$, the value $v_C(f)$ is defined as the degree of the leading term in the Taylor series of the function $f\circ\varphi:(\mathbb{C},0)\to\mathbb{C}$:

$$f\circ\varphi(\tau)=a\tau^{v_C(f)}+\text{terms of higher degree},$$

where $a\neq 0$; if $f\circ\varphi\equiv 0$, one defines $v_C(f)$ to be equal to $+\infty$.

A collection $\{(C_i,0)\}$ of irreducible curves in $(V,0)$, $i=1,\dots,r$, defines the collection $\{v_{C_i}\}$ of valuations. To a collection $\{v_i\}$ of discrete rank one valuations on $\mathcal{O}_{V,0}$, $i=1,\dots,r$, one may associate, as in [11], a Poincaré series $P_{\{v_i\}}(t_1,\dots,t_r)\in\mathbb{Z}[[t_1,\dots,t_r]]$. The collection $\{v_i\}$ defines a multi-index filtration on $\mathcal{O}_{V,0}$ by

$$J(\underline{u})=\{g\in\mathcal{O}_{V,0}:\underline{v}(g)\geq\underline{u}\},\tag{1}$$

where $\underline{u}=(u_1,\dots,u_r)\in\mathbb{Z}^r_{\geq 0}$, $\underline{v}(g)=(v_1(g),\dots,v_r(g))$ and $\underline{u}'=(u'_1,\dots,u'_r)\geq\underline{u}=(u_1,\dots,u_r)$ if and only if $u'_i\geq u_i$ for all $i=1,\dots,r$. Equation (1) defines the subspaces $J(\underline{u})$

for all $\underline{u}\in\mathbb{Z}^r$. The Poincaré series of the filtration $\{J(\underline{u})\}$ (or of the collection $\{v_i\}$ of valuations) is defined by:

$$P_{\{v_i\}}(t_1,\dots,t_r)=\frac{\mathcal{L}(t_1,\dots,t_r)\cdot\prod_{i=1}^r(t_i-1)}{t_1\cdot\ldots\cdot t_r-1}, \tag{2}$$

where

$$\mathcal{L}(t_1,\dots,t_r):=\sum_{\underline{u}\in\mathbb{Z}^r}\dim(J(\underline{u})/J(\underline{u}+\underline{1}))\cdot\underline{t}^{\,\underline{u}},$$

$\underline{1}=(1,1,\dots,1)\in\mathbb{Z}^r$. This definition makes sense if and only if all the quotients $J(\underline{u})/J(\underline{u}+\underline{1})$ are finite-dimensional.

Let $(S,0)$ be a normal surface singularity and let v_i, $i=1,\dots,r$, be either a curve valuation on $\mathcal{O}_{S,0}$ defined by an irreducible curve $(C_i,0)\subset(S,0)$, or a divisorial valuation on $\mathcal{O}_{S,0}$ defined by a component of the exceptional divisor $\mathcal{D}$ of a resolution $\pi:(\mathcal{X},\mathcal{D})\to(S,0)$ of the surface S. Assume that $\pi:(\mathcal{X},\mathcal{D})\to(S,0)$ is a resolution of the surface S which is, at the same time, a resolution of the collection $\{v_i\}$ of valuations, that is the total transform of the union of the curves C_i (such that v_i is the curve valuation v_{C_i}) is a normal crossing divisor on $\mathcal{X}$ and each divisor defining the divisorial valuation from the collection $\{v_i\}$ is present in $\mathcal{D}$.

Let $\mathcal{D}=\bigcup_{\sigma\in\Gamma}E_\sigma$ be the representation of the exceptional divisor $\mathcal{D}$ as the union of its irreducible components. For $\sigma\in\Gamma$, let $\overset{\circ}{E}_\sigma$ be the "smooth part" of the component E_σ in the total transform of the curve $\bigcup_i C_i$, i.e., the component E_σ itself minus the intersection points with all other components of the exceptional divisor $\mathcal{D}$ and with the strict transforms of the curves C_i. A *curvette* corresponding to a component E_σ of the resolution is the blow-down of a germ of a smooth curve transversal to E_σ at a point of $\overset{\circ}{E}_\sigma$. For $i\in\{1,2,\dots,r\}$, let $\tau(i)$ be either the index of the component $E_{\tau(i)}$ which intersects the strict transform of the curve C_i (if v_i is a curve valuation), or the index of the component which defines the divisorial valuation v_i. Let $(E_\sigma\circ E_\delta)$ be the intersection matrix of the components E_σ. The diagonal entries of this matrix are negative integers and a non-diagonal entry is equal to 1 if the components E_σ and E_δ intersect and is equal to 0 otherwise. Let $(m_{\sigma\delta}):=-(E_\sigma\circ E_\delta)^{-1}$. The entries $m_{\sigma\delta}$ are positive rational numbers whose denominators divide the determinant d of the matrix $-(E_\sigma\circ E_\delta)$. For $\sigma\in\Gamma$, let $\underline{m}_\sigma:=(m_{\sigma\tau(1)},m_{\sigma\tau(2)},\dots,m_{\sigma\tau(r)})\in\mathbb{Q}^r_{>0}$.

Definition 1. The *Weil–Poincaré series* (*W–Poincaré series* for short) of the collection of valuations $\{v_i\}$ is

$$P^W_{\{v_i\}}(\underline{t}):=\prod_{\sigma\in\Gamma}(1-\underline{t}^{\,\underline{m}_\sigma})^{-\chi(\overset{\circ}{E}_\sigma)}\in\mathbb{Z}[[t_1^{1/d},t_2^{1/d},\dots,t_r^{1/d}]], \tag{3}$$

where $\underline{t}=(t_1,t_2,\dots,t_r)$, for $\underline{m}=(m_1,\dots,m_r)\in\mathbb{Q}^r_{\geq0}$, $\underline{t}^{\,\underline{m}}=t_1^{m_1}\cdot\ldots\cdot t_r^{m_r}$.

Remark 2. One can see that the W–Poincaré series $P^W_{\{v_i\}}(\underline{t})$ is well-defined, i.e., does not depend on the choice of the resolution $\pi\colon(\mathcal{X},\mathcal{D})\to(S,0)$. This follows from the fact that a resolution of the collection $\{v_i\}$ can be obtained from the minimal one by additional blow-ups either at smooth points of the total transform of the curve $\bigcup_i C_i$ or at intersection points of it.

If $(S,0)$ is smooth or if it is the $\mathbb{E}_8$-surface singularity, the W–Poincaré series of the collection $\{v_i\}$ coincides with the usual Poincaré series of $\{v_i\}$ described above: [2], [5], [9].

Remark 3. For the case when $(S,0)$ was a rational surface singularity and v_i were the divisorial valuations corresponding to all the components of the exceptional divisor of a resolution of $(S,0)$, the series $P^W_{\{v_i\}}(\underline{t})$ was defined in [4] and [5] and used in [6]; see also [14].

3. Weil–Poincaré series and integrals with respect to the Euler characteristic

In [3] it was (essentially) shown that the Poincaré series $P_{\{v_i\}}(\underline{t})$ of a collection $\{v_i\}$ of valuations (curve or divisorial ones) on the ring $\mathcal{O}_{X,0}$ of germs of functions on a variety X can be given by the equation

$$P_{\{v_i\}}(\underline{t}) = \int_{\mathbb{P}\mathcal{O}_{X,0}} \underline{t}^{\underline{v}}\, d\chi\,,$$

where the right hand side is the integral with respect to the Euler characteristic over the projectivization of $\mathcal{O}_{X,0}$ (properly defined), $t_i^{\infty}:=0$ (see also [10, Proposition 1.1]). In [3] it was shown that the Poincaré series of a collection of curve valuations on $\mathcal{O}_{\mathbb{C}^2,0}$ can be written as an integral with respect to the Euler characteristic over the configuration space of effective divisors on the smooth part of the exceptional divisors of the embedded resolution of the union of curves.

Let $(S,0)$ be a normal surface singularity, let $\{v_i\}$ be a collection of curve or divisorial valuations on $\mathcal{O}_{S,0}$, and let $\pi\colon(\mathcal{X},\mathcal{D})\to(S,0)$ be a resolution of the collection (not the minimal one, in general). Let E_σ, $\sigma\in\Gamma$, be the irreducible components of the exceptional divisor $\mathcal{D}$ and let $\overset{\circ}{E}_\sigma$ be the "smooth part" of E_σ in the total transform of the union $(C,0)=\bigcup_i(C_i,0)$ of the irreducible curves $(C_i,0)$ defining curve valuations from the collection (i.e. E_σ minus the intersection points with other components of $\mathcal{D}$ and with the total transforms of the curves C_i). Let

$$Y^\pi := \prod_{\sigma\in\Gamma}\left(\coprod_{k=0}^{\infty} S^k\,\overset{\circ}{E}_\sigma\right) = \coprod_{\{k_\sigma\}\in\mathbb{Z}^\Gamma_{\geq 0}}\ \prod_{\sigma} S^{k_\sigma}\,\overset{\circ}{E}_\sigma$$

be the configuration space of effective divisors on $\overset{\circ}{\mathcal{D}}=\bigcup_\sigma \overset{\circ}{E}_\sigma$.

Let $\underline{v}:Y^\pi\to\mathbb{Q}^r_{\geq 0}$ be the function which sends the component $\prod_\sigma S^{k_\sigma}\overset{\circ}{E}_\sigma$ of Y^π to $\sum\limits_{\sigma\in\Gamma} k_\sigma \underline{m}_\sigma$. Let $\mathcal{O}^\pi_{S,0}$ be the set of non-zero function germs on $(S,0)$ such that the strict transform of the zero-level curve $\{f=0\}$ intersects $\mathcal{D}$ only at points of $\overset{\circ}{\mathcal{D}}$. One has a map I^π from $\mathcal{O}^\pi_{S,0}$ to Y^π which sends a function f to the intersection of the strict transform of the curve $\{f=0\}$ with $\overset{\circ}{\mathcal{D}}$. Let $Y^\pi_{\mathcal{C}}$ be the image of I^π. The set $\{\pi\}$ of resolutions of the collection $\{v_i\}$ is a partially ordered set: a resolution is bigger than another one if it can be obtained from the latter by a sequence of blow-ups.

Let $\mathfrak{W}_{S,0}$ be the set of effective divisors (that is Weil divisors) on $(S,0)$ and let $\mathfrak{C}_{S,0}\subset\mathfrak{W}_{S,0}$ be the set of effective Cartier divisors on $(S,0)$. There is a natural map J from $\mathbb{P}\mathcal{O}_{S,0}$ to $\mathfrak{C}_{S,0}$. For a resolution π of the collection $\{v_i\}$, let $\mathfrak{W}^\pi_{S,0}$ be the set of divisors in $\mathfrak{W}_{S,0}$ whose strict transforms intersect the exceptional divisor $\mathcal{D}$ only at points of $\overset{\circ}{\mathcal{D}}$. One has the natural map $\check{I}^\pi$ from $\mathfrak{W}^\pi_{S,0}$ onto Y^π and the map I^π factorizes through it: $I^\pi=\check{I}^\pi\circ J$.

Let $\underline{v}:\mathfrak{W}_{S,0}\to\mathbb{Q}^r_{\geq 0}$ be the composition $\underline{v}\circ I^\pi$, where $\underline{v}:Y^\pi\to\mathbb{Q}^r_{\geq 0}$ is described above. The map $\underline{v}$ sends $\mathfrak{C}_{S,0}$ to $\mathbb{Z}^r_{\geq 0}$. For a rational surface singularity S the map $\underline{v}:\mathfrak{W}_{S,0}\to\mathbb{Q}^r_{\geq 0}$ can be defined in the following way. For any divisor $C\in\mathfrak{W}_{S,0}$, a multiple kC of it, $k>0$, is a Cartier divisor, i.e. the divisor of a holomorphic function $f:S\to\mathbb{C}$. (It is possible to take $k=\det(E_\sigma,E_\delta)$ for a resolution of the singularity S.) Then $\underline{v}(C)=\underline{v}(f)/k$. For two divisors C and C' on S ($C=\bigcup C_i$, $C'=\bigcup C'_j$, where C_i and C'_j are irreducible curves) the (rational) number $\sum_i v_{C_i}(C')=\sum_j v_{C'_j}(C)$ can be regarded as the intersection number (C,C') of the curves C and C' and will be called (and denoted) in this way. (In these terms, the number $m_{\sigma\delta}$ is the intersection number of curvettes at the components E_σ and E_δ.)

We shall show that the Poincaré series $P_{\{v_i\}}(\underline{t})$ can be interpreted as an integral

$$\int_{\mathfrak{C}_{S,0}} \underline{t}^{\underline{v}}\,d\chi$$

with respect to the Euler characteristic. Moreover, we shall show that in the same way the W-Poincaré series $P^W_{\{v_i\}}(\underline{t})$ is equal to

$$\int_{\mathfrak{W}_{S,0}} \underline{t}^{\underline{v}}\,d\chi\,.$$

For that we have to define such integrals.

To give the definition we shall consider arcs and divisors on $(S,0)$ as arcs and divisors on a resolution $\pi:(\mathcal{X},\mathcal{D})\to(S,0)$. In this case, an arc is represented locally (in some local coordinates) by a pair of power series in a parameter τ. Let $\mathcal{L}_{S,0}$ be the space of arcs (that is parametrized curves) on $(S,0)$ and let $B=\mathcal{L}_{S,0}/\mathrm{A}ut(\mathbb{C},0)$ be the space of branches, i.e. non-parametrized arcs. Let $\mathcal{B}=\bigsqcup_{k=0}^{\infty} S^k B$. Each element of $\mathcal{B}$ represents a (effective) divisor on $(S,0)$. However, some divisors are represented by different elements of $\mathcal{B}$. This can be explained by the following situation. Let $\breve{\gamma}\in B$ be a branch. It is represented by an arc $\gamma=\gamma(\tau)$. Then the branch $\breve{\gamma}^k$ represented by the arc $\gamma(\tau^k)$ defines the same divisor as the collection of k copies of $\breve{\gamma}$. Let us call a branch $\breve{\gamma}$ (represented by an arc γ) *primitive* if γ is an uniformization of its image and let $B_0\subset B$ be the set of primitive branches. One has $\mathfrak{W}_{S,0}=\bigsqcup_{k=0}^{\infty} S^k B_0$.

Let $J^m B$ be the space of m-jets of branches on $(\mathcal{X},\mathcal{D})$. The restriction of the truncation map to B_0 is surjective. Therefore, the image of $\bigsqcup_{k=0}^{\infty} S^k B_0$ in $\bigsqcup_{k=0}^{\infty} J^m B$ is the same as the one of $\bigsqcup_{k=0}^{\infty} S^k B$. This produces a problem to use the image to define the integral over $\mathfrak{W}_{S,0}$ through this truncation. To avoid this problem, let us consider the subspace $J^m_{prim} B\subset J^m B$ consisting of the jets each representative of whom is primitive (a jet is a class of branches). Let $\mathbb{J}^m=\bigsqcup_{k=0}^{\infty} S^k J^m_{prim} B$.

Let $w_i : \mathfrak{W}_{S,0}\to\mathbb{Q}_{\geq 0}\cup\{\infty\}$, $i=1,2,\dots,r$ be functions on the set of Weil divisors. Let $w_i^m : \mathbb{J}^m\to\mathbb{Q}_{\geq 0}\cup\{\infty\}$ be the function defined by $w_i^m([a])=\sup_{a\in[a]} w_i(a)$ where $[a]\subset\bigsqcup_{k=0}^{\infty} S^k B_0\subset\mathfrak{W}_{S,0}$ is the equivalence class of the m-jet a. Let $\underline{w}:=(w_1, \dots, w_r) : \mathfrak{W}_{S,0}\to(\mathbb{Q}_{\geq 0}\cup\{\infty\})^r$ and $\underline{w}^m :=(w_1^m, \dots, w_r^m) : \mathbb{J}^m\to(\mathbb{Q}_{\geq 0}\cup\{\infty\})^r$. We shall say that the function $\underline{w}$ is constructible if $\underline{w}^m$ is constructible for all m (and therefore integrable with respect to the Euler characteristic).

Definition 4. The integral with respect to the Euler characteristic of the function $\underline{t}^{\underline{w}(-)}$ over the space $\mathfrak{W}_{S,0}$ is defined by

$$\int_{\mathfrak{W}_{S,0}} \underline{t}^{\underline{w}(-)} d\chi = \lim_{m\to\infty} \int_{\mathbb{J}^m_{S,0}} \underline{t}^{\underline{w}^m(-)} d\chi \in \mathbb{Z}[[t_1^{1/d}, \dots, t_r^{1/d}]]\,; \tag{4}$$

where the limit in the right hand side is in the sense of the $\langle t_1, \dots, t_r\rangle$-adic topology on $\mathbb{Z}[[t_1^{1/d}, \dots, t_r^{1/d}]]$, $\langle t_1, \dots, t_r\rangle$ is the ideal generated by $t_1, \dots, t_r$.

If the right hand side of (4) makes no sense, i.e. the limit does not exist, we regard the function $\underline{t}^{\underline{w}(-)}$ as a non-integrable one. For a subset $\mathcal{A}\subset\mathfrak{W}_{S,0}$ and a function $\underline{w} : \mathcal{A}\to\mathbb{Z}[[t_1^{1/d}, \dots, t_r^{1/d}]]$, the integral $\int_{\mathcal{A}} \underline{t}^{\underline{w}(-)} d\chi$ is understood as $\int_{\mathfrak{W}_{S,0}} \underline{t}^{\widehat{\underline{w}}(-)} d\chi$, where $\widehat{\underline{w}}(-)$ is the extension of the function $\underline{w}$ by $+\infty$ outside of $\mathcal{A}$ (recall that $t^{+\infty}=0$).

Now let v_i, $i=1,\dots,r$ be curve and/or divisorial valuations on $(S,0)$. They define natural maps (also denoted by v_i) from $\mathfrak{W}_{S,0}$ to $\mathbb{Z}[[t_1^{1/d}, \dots, t_r^{1/d}]]$. (In this case, one has $v_i(a+b)=v_i(a)+v_i(b)$.) Moreover, in this case, one assumes that $d=\det(-(E_\sigma\cdot E_\delta))$. One has the following statement.

Proposition 5.

$$\int_{\mathfrak{W}_{S,0}} \underline{t}^{\underline{v}(-)} d\chi = \lim_{\{\pi\}} \int_{Y_C^\pi} \underline{t}^{\underline{v}(-)} d\chi = P_{\{v_i\}}(\underline{t})\,. \tag{5}$$

Proof. Let $\mathbb{J}_\pi^m$ be the set of jets of divisors which intersect the exceptional divisor $\mathcal{D}$ of the resolution π only at points of $\overset{\circ}{\mathcal{D}}$. There is a natural map from $\mathbb{J}_\pi^m$ to Y^π. One can see that the preimage of a point from Y^π has the Euler characteristic equal to 1. This follows from the following arguments. An arc at a point of $\overset{\circ}{\mathcal{D}}$ in some local coordinates (u, v) such that $\overset{\circ}{\mathcal{D}}$ is given by the equation $u{=}0$ can be written as $u{=}\tau^s$, and $v{=}v(\tau)$ is a (truncated) series in τ. For a fixed s the set of jets of these arcs has the Euler characteristic equal to 1 (being isomorphic to a complex affine space). If $s{=}1$, the jet belongs to $J^\bullet_{prim}B$. If $s{>}1$, the set of jets not belonging to $J^\bullet_{prim}B$ has the Euler characteristic equal to 1 (being also isomorphic to a complex affine space). Therefore, the set of jets belonging to $J^\bullet_{prim}B$ has the Euler characteristic equal to 0. The preimage of a point from Y^π is the union of products of symmetric powers of these spaces. All of them have the Euler characteristics equal to zero except the product of the symmetric powers of the spaces of the spaces of arcs with $s{=}1$ whose Euler characteristic is equal to 1.

The fact that the preimage of a point from Y^π has the Euler characteristic equal to 1 (alongside with the Fubini formula) implies the statements. □

The direct computation of the mid term in Equation (5) (see, e.g., [12, Equation 4]) gives the following equation.

Proposition 6.

$$\int_{\mathfrak{W}_{S,0}} \underline{t}^{\underline{v}(-)} d\chi = \lim_{\{\pi\}} \int_{Y^\pi} \underline{t}^{\underline{v}(-)} d\chi = \prod_{\sigma\in\Gamma} (1 - \underline{t}^{\underline{m}_\sigma})^{-\chi(\overset{\circ}{E}_\sigma)}.$$

Corollary 7.

$$P^W_{\{v_i\}}(\underline{t}) = \int_{\mathfrak{W}_{S,0}} \underline{t}^{\underline{v}(-)} d\chi\,.$$

4. Curves and divisors on the $\mathbb{E}_7$-singularity whose Weil–Poincaré series do not determine the minimal resolution

Example 8. The dual graph of the minimal resolution of the $\mathbb{E}_7$-singularity $(S, 0)$ is shown in Figure 5. Let C' be a curvette at the component E_2 and let C'' be the blow down of a smooth curve $\widetilde{C}''$ on the surface of the resolution tangent to

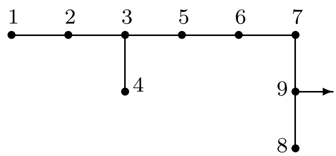

Figure 1. The minimal resolution graph of the curve C''.

the component E_7 at a smooth point (i.e. not at the intersection point with E_6) with the intersection multiplicity equal to 2 (i.e. the tangency of $\widetilde{C}''$ and E_7 is simple). The minimal resolution of the curve C' coincides with the minimal resolution of the surface $(S,0)$. The dual graph of the minimal resolution of the curve C'' is shown in Figure 1. One can show that

$$P^W_{C'}(t) = P^W_{C''}(t) = \frac{(1-t^6)(1-t^8)}{(1-t^2)(1-t^3)(1-t^4)}\,.$$

(The data for these computations and for those in the next example can be taken from the matrix 7 on page 311.) Therefore, the W–Poincaré series of a curve on the $\mathbb{E}_7$ surface singularity does not determine, in general, the combinatorial type of its minimal resolution.

Remark 9. We do not know how to prove (or to refute) that the triples $(S,C',0)$ and $(S,C'',0)$ are not homeomorphic. However, the knots $K' = C' \cap \mathbb{S}^5_\varepsilon$ and $K'' = C'' \cap \mathbb{S}^5_\varepsilon$ in (the rational homology sphere) $L = S \cap \mathbb{S}^5_\varepsilon$ are not isotopic. This follows from the fact that the linking numbers of the knots K' and K'' with the classes $[E_i] \in H_2(\mathcal{X};\mathbb{Z})$ ($\mathcal{X}$ is the space of the minimal resolution of the surface singularity) are different. The same applies to the curves C' and C'' from [9, Example 2].

Example 10. Let D' be the divisor created by the blow-up of a smooth point of the component E_9 of the resolution shown in Figure 1 and let D'' be the divisor created after 3 blow-ups starting at a smooth point of the component E_2 and produced at each step at a smooth point of the previously created divisor. One can show that for the divisorial valuations v' and v'' defined by the divisors D' and D'' respectively one has

$$P^W_{v'}(t) = P^W_{v''}(t) = \frac{(1-t^6)(1-t^8)}{(1-t^2)(1-t^3)(1-t^4)(1-t^9)}\,.$$

Therefore, the W–Poincaré series of a divisorial valuation on the $\mathbb{E}_7$-singularity does not determine, in general, the combinatorial type of the minimal resolution.

5. Main statements

Let $(S,0)$ be a rational double point and let $\{v_i\}$, $i{=}1,\ldots,r$, be a collection of different curve valuations on $\mathcal{O}_{S,0}$ defined by irreducible curves $(C_i,0)\subset(S,0)$.

Let us make the following additional assumptions. If the singularity $(S,0)$ is of type $\mathbb{E}_7$ (respectively of type $\mathbb{E}_8$), we either assume that the (minimal) resolution process does not contain a blow-up at a smooth point of the exceptional divisor of the minimal resolution of $(S,0)$ lying on the component E_7 (on the component E_8 respectively) or assume that does not contain a blow-up at the similar point lying on the component E_2 (on the component E_6 respectively).

Theorem 11. *Under the previous assumptions the W–Poincaré series* $P^W_{\{C_i\}}(\underline{t}){:=}P^W_{\{v_i\}}(\underline{t})$, $\underline{t}{=}(t_1,\ldots,t_r)$, *of the curve* $C{=}\bigcup_{i=1}^r C_i$ *determines, up to the symmetry of the dual graph of the minimal resolution of* $(S,0)$, *the combinatorial type of the minimal (embedded) resolution of the curve* $(C,0){=}\bigcup_{i=1}^r(C_i,0)$ *and therefore the topological type of the link* $C\cap L$ *in* $L{=}S\cap\mathbb{S}^5_\varepsilon$, *i.e. the topological type of the pair* $(L,C\cap L)$.

Now let $\{v_i\}$, $i{=}1,\ldots,r$, be a collection of different divisorial valuations on $\mathcal{O}_{S,0}$. In the cases of $\mathbb{E}_7$ and of $\mathbb{E}_8$ singularities we assume the same restrictions on the resolution process of the collection of divisorial valuations as for curves above.

Theorem 12. *Under the previous assumptions the W–Poincaré series* $P^W_{\{v_i\}}(\underline{t})$ *determines, up to the symmetry of the dual resolution graph of* $(S,0)$, *the combinatorial type of the minimal resolution of the collection* $\{v_i\}$.

Remark 13. To a divisorial valuation v_i one can associate a curve C_i: a curvette at the component $E_{\tau(i)}$ defining the valuation v_i. Theorem 12 implies that the series $P^W_{\{v_i\}}(\underline{t})$ determines the topological type of the link $(\bigcup_{i=1}^r C_i)\cap L$ in L.

Remark 14. In a statement like Theorems 11 and 12 one cannot mix curve and divisorial valuations in one collection; see an example in [7].

6. The case of one valuation

Let $(S,0)$ be a rational double point (of type $\mathbb{A}_k$, $\mathbb{D}_k$, $\mathbb{E}_6$, $\mathbb{E}_7$, or $\mathbb{E}_8$) and let v be either a curve valuation (defined by a curve germ $(C,0)\subset(S,0)$) or a divisorial valuation on $\mathcal{O}_{S,0}$. In the latter case, let $(C,0)\subset(S,0)$ be a curvette at the divisor defining the valuation. (A resolution of a divisorial valuation is at the same time a resolution of the corresponding curvette, but not vice versa.) The minimal resolution of the valuation v is obtained from the minimal resolution of the surface $(S,0)$

by a sequence of blow-ups made (at each step) at intersection points of the strict transform of the curve C and the exceptional divisor. Let $\pi' : (\mathcal{X}', \mathcal{D}') \to (S, 0)$ be the minimal resolution of the surface $(S, 0)$ such that the strict transform of the curve C intersects the exceptional divisor $\mathcal{D}'$ at a smooth point of it. This resolution is obtained from the minimal resolution of $(S, 0)$ by blow-ups made (at each step) at intersection points of the components of the exceptional divisor.

Definition 15. The resolution $\pi' : (\mathcal{X}', \mathcal{D}') \to (S, 0)$ of $(S, 0)$ will be called the *pre-resolution* of the valuation v.

Let E_{σ_0} be the component of the exceptional divisor $\mathcal{D}'$ of the pre-resolution π' intersecting the strict transform of curve C and let ℓ be the intersection number of them in the space $\mathcal{X}'$ of the pre-resolution. We shall use the numbering of the components of the exceptional divisor of the minimal resolution of $(S, 0)$ shown in Figures 2, 3, 4, 5, and 7. (They are at the same time components of the exceptional divisor of the pre-resolution π'.) In the case of the $\mathbb{E}_7$ ($\mathbb{E}_8$) singularity we either assume that σ_0 is not 7 (8 respectively) or assume that it is not 2 (6 respectively). Pay attention that in all the cases excluded from consideration the pre-resolution π' is the minimal resolution of the surface $(S, 0)$.

Lemma 16. *Under the conditions above, the W–Poincaré series $P_C^W(t)$ determines, up to the symmetry of the resolution graph of the minimal resolution of $(S, 0)$, the pre-resolution π', the component E_{σ_0} of the exceptional divisor $\mathcal{D}'$, and the intersection multiplicity ℓ.*

Proof. The proof is based on the analysis of the matrix $(m_{\sigma\delta})$ which has to be made separately for different cases. In all the cases, let us write the Poincaré series $P_C^W(t)$ in the form

$$\prod_{i=1}^{q} (1 - t^{m_i})^{-1} \prod_{m>0} (1 - t^m)^{s_m}, \tag{6}$$

where $m_1 \le m_2 \le \ldots \le m_q$ (thus the first product may have repeated factors) and, in the second product, the (integer) exponents s_m are non-negative and are equal to zero for $m = m_i$, $i = 1, \ldots, q$. Let us recall that the representation of the Poincaré series in this form is unique.

Case of $\mathbb{A}_k$ singularity. Let $(S, 0)$ be the singularity of type $\mathbb{A}_k$. The minimal resolution graph is shown in Figure 2.

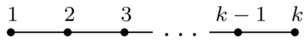

Figure 2. The dual resolution graph of the $\mathbb{A}_k$-singularity.

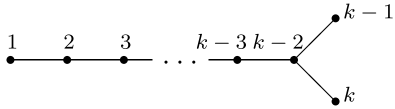

Figure 3. The dual resolution graph of the $\mathbb{D}_k$-singularity.

The matrix (m_{ij}) is

$$\frac{1}{k+1}\cdot\begin{pmatrix} k & k-1 & k-2 & \dots & i & \dots & 2 & 1\\ k-1 & 2(k-1) & 2(k-2) & \dots & 2i & \dots & 4 & 2\\ k-2 & 2(k-2) & 3(k-2) & \dots & 3i & \dots & 6 & 3\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\ i & 2i & 3i & \dots & i(k-i+1) & \dots & 2(k-i+1) & k-i+1\\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 1 & 2 & 3 & \dots & k-i+1 & \dots & k-1 & k \end{pmatrix}.$$

To identify the components of the exceptional divisor $\mathcal{D}'$, let us mark them by the indices σ being rational numbers in between 1 and k, naming the component created by the blow-up of the intersection point of the components E_{σ_1} and E_{σ_2} by $E_{\frac{\sigma_1+\sigma_2}{2}}$. (This methods can be applied to other rational double points under some restrictions.) Since we have to find π' and E_{σ_0} up to the symmetry of the graph in Figure 2, we can assume that $\sigma_0 \geq \frac{k+1}{2}$. We shall consider the following two cases:

1) $\sigma_0 = k$;

2) $\frac{k+1}{2} \leq \sigma_0 < k$.

Let us consider Case 1. For the curve case either $q=1$ or $\frac{m_2}{m_1} > k$ (the first option takes place if C is a curvette at E_k). For the divisorial case one has $\frac{m_2}{m_1} > k$.

In Case 2 one has $q \geq 2$, $m_1 = \ell m_{\sigma_0 1}$, $m_2 = \ell m_{\sigma_0 k}$ and therefore $\frac{m_2}{m_1} < k$ (in contrast with Case 1). The fact that the series $P^W_\bullet(t)$ determines the component E_{σ_0} follows from the fact that the ratio $\frac{m_2}{m_1}$ is strictly increasing with σ_0.

In both cases the intersection multiplicity ℓ is determined by the equation $m_1 = \ell m_{\sigma_0 1}$.

Case of $\mathbb{D}_k$ singularity. Let $(S,0)$ be the singularity of type $\mathbb{D}_k$. The minimal resolution graph is shown in Figure 3.

The matrix (m_{ij}) is

$$\frac{1}{4}\cdot\begin{pmatrix} 4 & 4 & 4 & \dots & 4 & 2 & 2 \\ 4 & 8 & 8 & \dots & 8 & 4 & 4 \\ 4 & 8 & 12 & \dots & 12 & 6 & 6 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 4 & 8 & 12 & \dots & 4(k-2) & 2(k-2) & 2(k-2) \\ 2 & 4 & 6 & \dots & 2(k-2) & k & k-2 \\ 2 & 4 & 6 & \dots & 2(k-2) & k-2 & k \end{pmatrix}.$$

Assume first that $k>4$. Because of the symmetry of the graph $\mathbb{D}_k$ we can assume that σ_0 does not belong to the lower right tail of the graph, that is $1\le\sigma_0\le k-1$ (we use the same numbering of newly created components as above for $\mathbb{A}_k$). We shall consider the following four cases:

1) $1<\sigma_0\le k-2$;

2) $k-2<\sigma_0<k-1$;

3) $\sigma_0=1$;

4) $\sigma_0=k-1$.

In Cases 1 and 2 (both in the curve and the divisorial cases) one has $q\ge 3$ and m_1, m_2, and m_3 are $\ell m_{\sigma_0 1}$, $\ell m_{\sigma_0 k-1}$, and $\ell m_{\sigma_0 k}$ in a certain order.

In Case 1 at least two of the exponents m_1, m_2, and m_3 coincide (and are equal to m'). Let us denote the third component by m''. We always have $\frac{m'}{m''}>\frac{1}{2}$. All three exponents coincide, that is $m''=m'$, if and only if $\sigma_0=2$. If $m''>m'$, then $\sigma_0<2$ and $m_1=m_2=\ell m_{\sigma_0 k-1}$, $m_3=\ell m_{\sigma_0 1}$. If $m''<m'$, then $2<\sigma_0\le k-2$, $m_1=\ell m_{\sigma_0 1}$, $m_2=m_3=\ell m_{\sigma_0 k-1}$. In this case, the ratio $\frac{m_{\sigma_0(k-1)}}{m_{\sigma_0 1}}$ is strictly increasing with σ_0 and therefore determines σ_0.

In Case 2 the exponents m_1, m_2, and m_3 are different and moreover $m_1=\ell m_{\sigma_0 1}$, $m_2=\ell m_{\sigma_0 k}$, $m_3=\ell m_{\sigma_0(k-1)}$, $\frac{m_3}{m_2}<\frac{5}{4}$. Again the ratio $\frac{m_{\sigma_0(k-1)}}{m_{\sigma_0 1}}$ is strictly increasing and therefore determines σ_0.

The equations above determine ℓ.

In Case 3 one has: in the curve case either $q=2$ with $m_1=m_2=2\ell$ or $m'=m_1=m_2=2\ell$, $m''=m_3>4\ell$ with $\frac{m'}{m''}<\frac{1}{2}$; in the divisorial case $\frac{m'}{m''}\le\frac{1}{2}$, $m_1=m_2=2\ell$.

In Case 4 one has: in the curve case either $q=2$ with $m_1=2\ell$ or m_1, m_2, and m_3 are different with $\frac{m_3}{m_2}>\frac{5}{4}$, $m_1=2\ell$; in the divisorial case m_1, m_2, and m_3 are different with $\frac{m_3}{m_2}\ge\frac{5}{4}$, $m_1=2\ell$.

Now let $k=4$, i.e. $(S,0)$ is the singularity of type $\mathbb{D}_4$. Because of the symmetry of the graph, we can assume that $1\le\sigma_0\le 2$. We shall consider the following two cases:

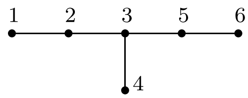

Figure 4. The dual resolution graph of the $\mathbb{E}_6$-singularity.

1) $1<\sigma_0\leq 2$.
2) $\sigma_0=1$;

In Case 1 one has (both for the curve and for the divisorial cases) $m_1=m_2=\ell m_{\sigma_0,3}$, $m_3=\ell m_{\sigma_0,1}$ and $m_3/m_1<2$. The ratio $\frac{m_{\sigma_0 3}}{m_{\sigma_0 1}}$ is strictly increasing with σ_0 and therefore determines the latter one. In Case 2 one has: in the curve case either $q=2$ with $m_1=m_2=2\ell$ or $m_1=m_2=2\ell$ and $m_3/m_1>2$; in the divisorial case $m_1=m_2=2\ell$ and $m_3/m_1\geq 2$.

Case of $\mathbb{E}_6$ singularity. Let $(S,0)$ be the singularity of type $\mathbb{E}_6$. The minimal resolution graph is shown in Figure 4.

The matrix (m_{ij}) is

$$\begin{pmatrix} 4/3 & 5/3 & 2 & 1 & 4/3 & 2/3 \\ 5/3 & 10/3 & 4 & 2 & 8/3 & 4/3 \\ 2 & 4 & 6 & 3 & 4 & 2 \\ 1 & 2 & 3 & 2 & 2 & 1 \\ 4/3 & 8/3 & 4 & 2 & 10/3 & 5/3 \\ 2/3 & 4/3 & 2 & 1 & 5/3 & 4/3 \end{pmatrix}.$$

Because of the symmetry of the graph, we can assume that σ_0 does not belong to the right tail of the graph, i.e. (using the same rule of numbering of the components of the exceptional divisor $\mathcal{D}'$ as above) $1\leq\sigma_0\leq 4$. We shall consider the following four cases:

1) $1<\sigma_0<3$;
2) $3\leq\sigma_0<4$;
3) $\sigma_0=1$;
4) $\sigma_0=4$.

In Cases 1 and 3 one has $m_1<m_2$; in Cases 2 and 4 $m_1=m_2$.

In Case 1 $\frac{m_3}{m_1}<2$. In Case 3 one has: in the curve case either $q=2$ or $\frac{m_3}{m_2}>2$; in the divisorial case $\frac{m_3}{m_2}\geq 2$ with $m_1=\ell$ in all the cases.

In Case 2 $\frac{m_3}{m_1}<2$. In Case 4 one has: in the curve case either $q=2$ or $\frac{m_3}{m_1}>2$; in the divisorial case $\frac{m_3}{m_1}\geq 2$ with $m_1=\ell$ in all the cases.

In Case 1 (both for the curve and for the divisorial valuation) one has either $\frac{m_2}{m_1}=\frac{3}{2}$ or $\frac{m_3}{m_1}=\frac{3}{2}$. If $\frac{m_2}{m_1}=\frac{3}{2}$, then $\sigma_0\leq 1.5$, the ratio $\frac{m_3}{m_1}$ is strictly increasing with σ_0

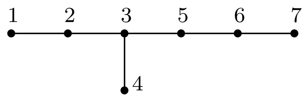

Figure 5. The dual resolution graph of the $\mathbb{E}_7$-singularity.

and therefore determines σ_0; $m_1 = \ell m_{\sigma_0 4}$. If $\frac{m_3}{m_1} = \frac{3}{2}$, then $\sigma_0 \geq 1.5$, the ratio $\frac{m_2}{m_1}$ is strictly decreasing with σ_0 and therefore determines σ_0; $m_1 = \ell m_{\sigma_0 6}$. In Case 2 (both for the curve and for the divisorial valuation) the ratio $\frac{m_3}{m_1}$ is strictly increasing with σ_0 and therefore determines σ_0; $m_1 = \ell m_{\sigma_0 1}$.

Case of $\mathbb{E}_7$ singularity. Let $(S, 0)$ be the singularity of type $\mathbb{E}_7$. The minimal resolution graph is shown in Figure 5.

The matrix (m_{ij}) is

$$\begin{pmatrix} 2 & 3 & 4 & 2 & 3 & 2 & 1 \\ 3 & 6 & 8 & 4 & 6 & 4 & 2 \\ 4 & 8 & 12 & 6 & 9 & 6 & 3 \\ 2 & 4 & 6 & 7/2 & 9/2 & 3 & 3/2 \\ 3 & 6 & 9 & 9/2 & 15/2 & 5 & 5/2 \\ 2 & 4 & 6 & 3 & 5 & 4 & 2 \\ 1 & 2 & 3 & 3/2 & 5/2 & 2 & 3/2 \end{pmatrix}. \tag{7}$$

Let us analyze first the situation when we assume that $\sigma_0 \neq 7$. We shall consider the following cases:

1. $\sigma_0 \neq 1, 4$;
2. $\sigma_0 = 1$;
3. $\sigma_0 = 4$.

In Case 1, both for the curve and for the divisorial valuations one has $m_3 = \ell m_{\sigma_0,4}$ and m_1 and m_2 are $\ell m_{\sigma_0 1}$ and $\ell m_{\sigma_0,7}$ in a certain order. Moreover $m_2/m_1 < 2$ and $m_3/m_1 < 7/3$.

In Figure 6, the ratio $(m_{\sigma_0,1} : m_{\sigma_0,4} : m_{\sigma_0,7}) \in \mathbb{RP}^2$ is shown (by the bold lines) in the affine chart $(m_{\sigma_0,1}/m_{\sigma_0,4}, m_{\sigma_0,7}/m_{\sigma_0,4})$. (The fact that the edges meeting at a vertex of valency 2 lie on a straight line is a general feature of pictures of this sort.) The figure shows that the ratio $(m_{\sigma_0,1} : m_{\sigma_0,4} : m_{\sigma_0,7})$ determines σ_0. However, the explanation above says that the W-Poincaré series determines this ratio only up to the exchange of the first and the third components. The graph obtained by exchanging $m_{\sigma_0,1}$ and $m_{\sigma_0,7}$ is drawn by thin lines. One can see that they intersect only at a point on the diagonal $\frac{m_{\sigma_0,1}}{m_{\sigma_0,4}} = \frac{m_{\sigma_0,7}}{m_{\sigma_0,4}}$ and therefore the ratios $(m_{\sigma_0,1} : m_{\sigma_0,4} : m_{\sigma_0,7})$ determines σ_0.

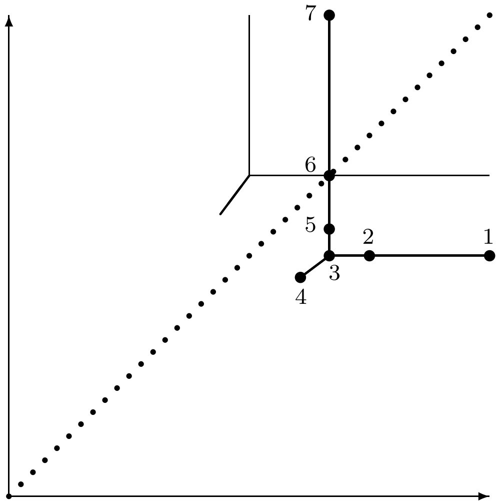

Figure 6. The points $(m_{\sigma_0,1}\!:\!m_{\sigma_0,4}\!:\!m_{\sigma_0,7})\in\mathbb{RP}^2$.

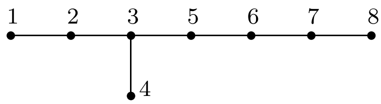

Figure 7. The dual resolution graph of the $\mathbb{E}_8$-singularity.

In Case 2, one has $m_2/m_1{=}2$ (in contrast to Case 1 above and Case 3 below); $m_1{=}\ell$. In Case 3, one has $m_2/m_1{=}4/3$ and for the curve case, either $q{=}2$ or $m_3/m_1{>}7/3$; for the divisorial case $m_3/m_1{\geq}7/3$; $m_2{=}2\ell$.

In the situation when we assume that $\sigma_0{\neq}2$, we shall consider the following four cases:

1. $\sigma_0{\neq}1, 4, 7$;
2. $\sigma_0{=}1$;
3. $\sigma_0{=}4$;
4. $\sigma_0{=}7$.

The analysis of the Cases 1 to 3 is the same as above. In Case 4, one has $m_2/m_1{=}3/2$. This differs Case 4 from Cases 2 and 3 and in Case 1 the value $m_2/m_1{=}3/2$ holds only if $\sigma_0{=}2$. In this case, $m_1{=}\ell$.

Case of $\mathbb{E}_8$ singularity. Let $(S,0)$ be the singularity of type $\mathbb{E}_8$. The minimal resolution graph is shown in Figure 7.

The matrix (m_{ij}) is:

$$\begin{pmatrix} 4 & 7 & 10 & 5 & 8 & 6 & 4 & 2 \\ 7 & 14 & 20 & 10 & 16 & 12 & 8 & 4 \\ 10 & 20 & 30 & 15 & 24 & 18 & 12 & 6 \\ 5 & 10 & 15 & 8 & 12 & 9 & 6 & 3 \\ 8 & 16 & 24 & 12 & 20 & 15 & 10 & 5 \\ 6 & 12 & 18 & 9 & 15 & 12 & 8 & 4 \\ 4 & 8 & 12 & 6 & 10 & 8 & 6 & 3 \\ 2 & 4 & 6 & 3 & 5 & 4 & 3 & 2 \end{pmatrix}$$

The situation when we assume that $\sigma_0 \neq 8$ was analyzed in [9]. One can see that, in this case, the possibility to restore ℓ easily follows from the discussion therein. In the situation when we assume that $\sigma_0 \neq 6$, we shall consider the following four cases:

1. $\sigma_0 \neq 1, 4, 8$;
2. $\sigma_0 = 1$;
3. $\sigma_0 = 4$;
4. $\sigma_0 = 8$.

In Case 1, one has $m_2/m_1 \leq 2$. The way to determine σ_0 in this case is described in [9]; $\mathfrak{m}_1 = \ell m_{\sigma_0 8}$. In Case 2, one has $m_2/m_1 = 5/2$. In Case 3, one has $m_2/m_1 = 5/3$ and either $q=2$ (in the curve case) or $m_3/m_1 \geq 8/3$. This can be met in Case 1 when $3 \leq \sigma_0 < 4$ (i.e. if σ_0 is on the lower tail of the diagram). In this case, one has $5/2 \leq m_3/m_1 < 8/3$. In case 3 $m_1 = 3\ell$.

In Case 4, one has $m_2/m_1 = 3/2$. This differs Case 4 from Cases 2 and 3. In Case 1 the value $m_2/m_1 = 3/2$ holds only if $\sigma_0 = 6$. In Case 4 $m_1 = 2\ell$. □

Let $(C, 0)$ be an irreducible curve on a rational double point $(S, 0)$.

Proposition 17. *Under the described assumptions, the W-Poincaré series $P^W_C(t)$ determines the combinatorial type of the minimal embedded resolution of the curve C and therefore the topological type ot the knot $C \cap L$ in $L = S \cap \mathbb{S}^5_\varepsilon$.*

Let v be a divisorial valuation on $(S, 0)$.

Proposition 18. *Under the described assumptions, the W-Poincaré series $P^W_{\{v\}}(t)$ determines the combinatorial type of the minimal resolution of the valuation v.*

Proofs of Propositions 17 and 18 are essentially the same as of Theorems 1 and 2 in [9].

Assume that C_1 and C_2 are two curves with the (known) W–Poincaré series $P^W_{C_i}(t)$ such that the components $E_{\sigma_0^i}$ of the exceptional divisors $\mathcal{D}'_i$ of the pre-resolutions emerge from the parts (one can say "tails" in all the cases but $\mathbb{A}_k$) of the resolution graph of the minimal resolution of $(S,0)$ exchangable by a symmetry of the graph acting non-trivially on them. (Pay attention that in this case S is one of the singularities $\mathbb{A}_k$, $\mathbb{D}_k$, and $\mathbb{E}_6$. It cannot be a singularity of type $\mathbb{E}_7$ or $\mathbb{E}_8$ when the series W–Poincaré series determines the component E_{σ_0} of the pre-resolution only under additional conditions.)

Lemma 19. *In the described situation (for $(S,0)$ of the type $\mathbb{A}_k$, $\mathbb{D}_k$, or $\mathbb{E}_6$), The W–Poincaré series $P^W_{C_i}(t)$, $i{=}1,2$, alongside with the intersection number $(C_1 \circ C_2)$ determines whether the strict transforms of the curves C_1 and C_2 intersect the same part of the graph of the minimal resolution of the surface singularity $(S,0)$ or different ones.*

Proof. If the strict transforms intersect the same part of the graph, then $(C_1 \circ C_2) \geq \ell_1 \cdot \ell_2 \cdot m_{\sigma_0^1 \sigma_0^2}$. (The sign $>$ may hold only if $\sigma_0^1 {=} \sigma_0^2$.) If the strict transforms intersect different parts of the graph, one has $(C_1 \circ C_2) {=} \ell_1 \cdot \ell_2 \cdot m_{\sigma_0^1 \sigma_0^2}$. Moreover, from the matrices $(m_{\sigma\delta})$ above one can see that, for fixed up to symmetry σ_0^1 and σ_0^2, the intersection number $m_{\sigma_0^1 \sigma_0^2}$ for the components σ_0^1 and σ_0^2 from the same part is strictly larger than the one for the components from different parts. $\square$

7. The case of several valuations

The idea of the proofs of Theorems 11 and 12 is the same as of Theorems 3 and 4 in [9]. Moreover, the proof of Theorem 12 is literally the same modulo one remark related with the symmetry of the minimal resolution graph of the surface singularity: see at the end of the section. This is explained by the fact that the "projection formula" (the equation connecting the Weil–Poincaré series of a collection of valuations with the one for the collection with one valuation excluded) is much simpler for a collection of divisorial valuations. In this case, the Weil–Poincaré series of the smaller collection is obtained from the other one simply by putting the value 1 for the corresponding variable t_i. Thus, the Weil–Poincaré series of the smaller collection is determined by the same series for the larger one. This is not the case, in general, for a collection of curve valuations. In this case, the projection formula includes a certain factor for the excluded valuation which, in general, cannot be directly obtained from the initial Weil–Poincaré series. (This also explains the fact that, in [9], the proof for divisorial valuations is much shorter than for curve ones.) For the curve case (Theorem 11) the proof contains some differences. Therefore, we include some parts of it.

Let C be an irreducible curve germ in the surface $(S,0)$ and let $\pi{:}(\mathcal{X},\mathcal{D})\to(S,0)$ be an embedded resolution of the curve $(C,0)\subset(S,0)$. Let Γ be the minimal resolution graph of C, i.e., the dual graph of π. Let $\overline{C}$ be the total transform of the curve C in $\mathcal{X}$. One has

$$\overline{C}=\widetilde{C}+\sum_{\sigma\in\Gamma} m_\sigma E_\sigma,$$

where $\widetilde{C}$ is the strict transform of the curve C. The rational numbers $\{m_\sigma{:}\sigma\in\Gamma\}$ are uniquely determined by the system of linear equations

$$\{\widetilde{C}\circ E_\alpha+\sum_{\sigma\in\Gamma} m_\sigma\, E_\sigma\circ E_\alpha=0\ :\ E_\alpha\in\mathcal{D}\}. \tag{8}$$

(Each equation is a consequence of the fact $\overline{C}\circ E_\alpha=0$; see [13, Equation (1)].)

Let $\rho\in\Gamma$ be an end and let $\Delta=\{\rho=\alpha_0,\alpha_1,\dots.\alpha_s=\sigma\}$ be the corresponding dead arc in Γ (i.e. the minimal connected subgraph of Γ such that σ is a star vertex). Then one has:

Lemma 20. *If $-E^2_{\alpha_i}\neq 1$ for $i=0,\dots,s-1$, then there exists an integer $N>1$, independent of the branch C, such that $m_\sigma=Nm_\rho$.*

Proof. Equation (8) for E_ρ gives $m_{\alpha_1}=(-E^2_\rho)m_\rho=N_1m_\rho$ with an integer $N_1>1$. Again, the same equation for E_{α_i} gives

$$\begin{aligned} m_{\alpha_{i+1}}&=(-E^2_{\alpha_i})m_{\alpha_i}-m_{\alpha_{i-1}}=-E^2_{\alpha_i}N_im_\rho-N_{i-1}m_\rho=\\ &=(-E^2_{\alpha_i}N_i-N_{i-1})m_\rho=N_{i+1}m_\rho. \end{aligned}$$

Since $E^2_{\alpha_i}\neq -1$, one has $N_{i+1}\geq 2N_i-N_{i-1}>N_i$ provided we assume $N_i>N_{i-1}$ by induction. □

Let $C=\bigcup_{i=1}^r C_i$ be a reducible (that is, $r>1$) curve germ in the surface $(S,0)$ and let $\pi{:}(\mathcal{X},\mathcal{D})\to(S,0)$ be the minimal embedded resolution of the curve $(C,0)\subset(S,0)$. Let Γ be the minimal resolution graph of C, i.e., the dual graph of π. Let $\tau(i)$ be the vertex of Γ such that the component $E_{\tau(i)}$ of the exceptional divisor $\mathcal{D}$ intersects the strict transform $\widetilde{C}_i$ of the curve C_i and let $m^i_\sigma:=m_{\sigma\tau(i)}$. One has $\underline{m}_\sigma=(m^1_\sigma,\dots,m^r_\sigma)$. The reason (somewhat psychological) for that is the fact that, for a multi-exponent of a term of the Poincaré series $P_C(t_1,\dots,t_r)$ or of a factor of its decomposition, one knows its components $m^1_\sigma,\dots,m^r_\sigma$, but does not know the vertex σ. One can say that our aim is to find vertices $\tau(i)$ corresponding to the curve.

Let $\overline{C}_k$ $(k=1, ..., r)$ be the total transform of the curve C_k in $\mathcal{X}$. One has

$$\overline{C}_k = \widetilde{C}_k + \sum_{\sigma\in\Gamma} m_\sigma^k E_\sigma,$$

where $\widetilde{C}_k$ is the strict transform of the curve C_k.

Let us fix a pair of branches C_i and C_j and let $q{:}\Gamma\to\mathbb{Q}$ be the function defined by $q(\alpha)=m_\alpha^j/m_\alpha^i$ for $\alpha\in\Gamma$. One has the following statement.

Lemma 21. *Let E_α be a component of the exceptional divisor $\mathcal{D}$ such that $\widetilde{C}_i\circ E_\alpha=0$ and let $\{\rho_1, ..., \rho_s\}\subset\Gamma$ be the set of all vertices connected by an edge with α. Let us assume that either $\widetilde{C}_j$ intersects E_α or there exists ρ_{i_0} such that $q(\rho_{i_0})>q(\alpha)$. Then there exists ρ_k such that $q(\alpha)>q(\rho_k)$.*

Proof. Assume that $q(\rho_k)\geq q(\alpha)$ for any $k=1, ..., s$. Applying (8) to C_j and C_i one gets:

$$\begin{aligned} 0 &= \widetilde{C}_j\circ E_\alpha + m_\alpha^j E_\alpha^2 + \sum_{k=1}^{s} m_{\rho_k}^j \geq \\ &\geq \widetilde{C}_j\circ E_\alpha + m_\alpha^j E_\alpha^2 + \sum_{k=1}^{s} q(\alpha) m_{\rho_k}^i = \\ &= \widetilde{C}_j\circ E_\alpha + q(\alpha)(m_\alpha^i E_\alpha^2 + \sum_{k=1}^{s} m_{\rho_k}^i) = \widetilde{C}_j\circ E_\alpha \geq 0 \end{aligned}$$

The inequality is strict if $\widetilde{C}_j\circ E_\alpha>0$ or if there exists i_0 such that $q(\rho_{i_0})>q(\alpha)$. This implies the statement. □

Let $[\tau(j), \tau(i)]\subset\Gamma$ be the (oriented) geodesic from $\tau(j)$ to $\tau(i)$ and let $\{\Delta_p\}$, $p\in\Pi$, be the connected components of $\Gamma\setminus[\tau(j), \tau(i)]$. For each $p\in\Pi$ there exists a unique $\rho_p\in[\tau(j), \tau(i)]$ connecting Δ_p with $[\tau(j), \tau(i)]$, i.e., such that $\Delta_p^*=\Delta_p\cup\{\rho_p\}$ is connected.

Proposition 22. *With the previous notations, one has:*
1. *The function q is strictly decreasing along the geodesic $[\tau(j), \tau(i)]$.*
2. *For each $p\in\Pi$, the function q is constant on Δ_p^*.*

Proof. Let α and β be two vertices of Γ connected by an edge and let $q(\alpha)>q(\beta)$. Lemma 21 permits to construct a maximal sequence $\alpha_0, \alpha_1, ..., \alpha_k$ of consecutive vertices starting with α and β (i.e., $\alpha_0=\alpha$, $\alpha_1=\beta$) such that $q(\alpha_i)>q(\alpha_{i+1})$. (We will call a sequence of this sort *a decreasing path.* If the inequality is in the other direction, the path will be called *increasing.*) The maximality means that either α_k is a deadend of Γ or $\widetilde{C}_i \cdot E_{\alpha_k} \neq 0$. If α_k is a deadend, α_{k-1} is the only vertex connected with α_k and Lemma 21 implies that $q(\alpha_k)=q(\alpha_{k-1})$. Therefore, the constructed path finishes by the vertex $\alpha_k=\tau(i)$. Note that, if $\alpha\in[\tau(j),\tau(i)]$ and $\beta\notin[\tau(j),\tau(i)]$, the end of a maximal decreasing (or increasing) path has to finish at a deadend and therefore $q(\alpha)=q(\beta)$. In particular, this implies that the function q is constant on each connected set Δ_p^*.

Assume that $\tau(i)\neq\tau(j)$. Lemma 21 implies that there exists a vertex α_1 connected with $\tau(j)$ such that $q(\tau(j))>q(\alpha_1)$. Therefore, the maximal decreasing path starting with $\tau(j)$ and α_1 coincides with the geodesic $[\tau(j),\tau(i)]$. □

Remark 23. Let $\rho\in\Gamma$ be an end and let $\Delta=\{\rho=\alpha_0,\alpha_1,\dots.\alpha_s=\sigma\}$ be the corresponding dead arc in Γ. In this case, Proposition 22 implies that the ratio m^i_α/m^j_α is constant for $\alpha\in\Delta$ and for any pair $i,j\in\{1,\dots,r\}$. In fact, from Lemma 20, one can easily deduce that $\underline{m}_\sigma=N\underline{m}_\rho$ for an integer $N>1$, in particular, $\underline{m}_\sigma>\underline{m}_\rho$.

Proof of Theorem 11. We have to show that the Weil-Poincaré series $P^W_{\{C_i\}}(\underline{t})$ determines the minimal resolution graph Γ of C. In the case under consideration, one has a projection formula different of the one for divisorial valuations.

Let $i_0\in\{1,\dots,r\}$. The A'Campo type formula (3) for $P^W_{\{C_i\}}(\underline{t})$ implies that

$$P^W_{\{C_i\}}(\underline{t})_{|t_{i_0}=1}=P^W_{C\setminus\{C_{i_0}\}}(t_1,\dots,t_{i_0-1},t_{i_0+1},\dots,t_r)\cdot(1-\underline{t}^{\underline{m}_{\tau(i_0)}})_{|t_{i_0}=1}. \tag{9}$$

Applying (9) several times one gets

$$P^W_C(\underline{t})_{|t_j=1\text{ for }j\neq i_0}=P^W_{C_{i_0}}(t_{i_0})\cdot\prod_{i\neq i_0}(1-t_{i_0}^{m^{i_0}_{\tau(i)}}). \tag{10}$$

Pay attention to the fact that $m^{i_0}_{\tau(i)}=m^i_{\tau(i_0)}$ and therefore the series $P^W_{C_{i_0}}(t_{i_0})$ can be determined from the Weil-Poincaré series $P_C(\underline{t})$ if one knows the multiplicity $\underline{m}_{\tau(i_0)}$. The strategy of the proof follows the steps from [7] (see also [8]):

1) To detect an index i_0 for which one can find the corresponding multiplicity $\underline{m}_{\tau(i_0)}$ from the A'Campo type formula for $P^W_C(\underline{t})$. Then Proposition 17 and equation (10) permit to recover the minimal resolution graph Γ_{i_0} of the curve C_{i_0}. Equation (9) gives the possibility to compute the Poincaré series $P^W_{C\setminus\{C_{i_0}\}}(t_1,\dots,t_{i_0-1},t_{i_0+1},\dots,t_r)$ of the curve $C\setminus\{C_{i_0}\}$. By induction one can assume that the resolution graph Γ^{i_0} of the curve $C\setminus\{C_{i_0}\}$ is known. Moreover, Lemma 19 implies that, for each j, the multiplicity $\underline{m}_{\tau(i_0)}$ determines whether the vertices of the minimal resolution graph corresponding to the curves C_{i_0} and C_j are on the same part from those exchanged by symmetries or on different ones.

2) To determine the separation vertex of the curves C_{i_0} and C_j for $j\neq i_0$ in order to join the graphs Γ_{i_0} and Γ^{i_0} to obtain the resolution graph Γ.

The second step literally repeats the same one in the proof of Theorem 3 in [9]. Therefore, we omit an analysis of it here.

Proposition 22 implies that, for any fixed i_0 and for any $j\neq i_0$ and $\sigma\in\Gamma$, one has $m^j_\sigma/m^{i_0}_\sigma\geq m^j_{\tau(i_0)}/m^{i_0}_{\tau(i_0)}$. Therefore, one has

$$\frac{1}{m^{i_0}_\sigma}\underline{m}_\sigma \geq \frac{1}{m^{i_0}_{\tau(i_0)}}\underline{m}_{\tau(i_0)}.$$

Let $P^W_C(\underline{t})=\prod_{k=1}^p(1-\underline{t}^{\underline{n}_k})^{s_k}$ be the Weil-Poincaré series of the curve C, where $s_k\neq 0$ for all k. Note that the only case in which $P^W_C(\underline{t})=1$ corresponds to the singularity $\mathbb{A}_k$ and two branches C_1, C_2 in such a way that they are curvettes at the end points named 1 and k of the dual graph of $\mathbb{A}_k$. In the sequel we omit this trivial situation. For $i\in I=\{1,\ldots,r\}$ let $\varkappa:I\to\{1,\ldots,p\}$ be the map defined by $k=\varkappa(i)$ be such that $s_k>0$ and

$$\frac{1}{n^i_j}\underline{n}_j \geq \frac{1}{n^i_k}\underline{n}_k \tag{11}$$

for all j. Note that if the inequalities (11) hold for $\underline{n}_k=\underline{m}_\rho$, ρ a deadend of Γ, then (see Proposition 22); one has the same condition for σ, the star vertex of Γ more close to ρ. Let $E=\varkappa(I)\subset\{1,\ldots,p\}$ be the set of indices k such that $k=\varkappa(i)$ for some $i\in\{1,\ldots,r\}$ and for $k\in E$ let $A(k)\subset\{1,\ldots,r\}$ denote the set of indices i such that $k=\varkappa(i)$. Note that $A(k)$ contains all the indices $i\in\{1,\ldots,r\}$ such that $\underline{n}_k=\underline{m}_{\tau(i)}$. Let $B(k)$ be the subset of such indices. Our aim is to show that one can find $k\in E$ such that $B(k)\neq\varnothing$.

Proposition 24. *Let us assume that $\#E\geq 2$. Then there exists $k\in E$ and $i\in A(k)$ such that*

1. *$n^i_k\geq n^j_k$ for any $j\in A(k)$*
2. *$n^j_k\leq n^i_{k'}$ for any $k'\in E$, $k'\neq k$, and $j\in A(k')$.*

Moreover, for any pair (k,i) satistying conditions 1) and 2) above, one has that $i\in B(k)$ and therefore $B(k)\neq\varnothing$.

Proof. Let $j\in A(k)$, $j\notin B(k)$. One has

$$\frac{1}{n^j_k}\underline{n}_k > \frac{1}{m^j_{\tau(j)}}\underline{m}_{\tau(j)}$$

and therefore $\chi(\overset{\circ}{E}_{\tau(j)})=0$. This implies that $\tau(j)$ is connected with only one vertex in Γ (plus the arrow corresponding to $\widetilde{C}_j$), i.e., $\tau(j)$ is a deadend of the resolution graph of the curve $C\setminus\{C_j\}$. Let $\sigma\in\Gamma$ be such that $\underline{n}_k=\underline{m}_\sigma$ and assume that $i\in$

$B(k)\neq\varnothing$. In this case, $\underline{n}_k=\underline{m}_{\tau(i)}$ and one has $n_k^i=m^i_{\tau(i)}=Nm^i_{\tau(j)}$, with an integer $N>1$, being $\tau(j)$ a deadend of the dual graph of $C\setminus\mathbb{C}_j\}$ (see 20). In particular, $n_k^i>m^i_{\tau(j)}=m^j_{\tau(i)}=n^j_k$. Notice that if $i,s\in B(k)$ one has also that $\underline{n}_k=\underline{m}_{\tau(s)}$ and therefore $n^i_k=n^s_k$. As a consequence, if $B(k)\neq\varnothing$ one can fix an index $i(k)\in B(k)$ taking a maximal one of the set $\{n^i_k:i\in A(k)\}$ as in the condition 1) of the statement.

Let $k'\in E$, $k'\neq k$, and $j\in A(k')$. If $j\in B(k')$, then $\underline{n}_{k'}=\underline{m}_{\tau(j)}$ and therefore $n^{i(k)}_{k'}=m^{i(k)}_{\tau(j)}=m^j_{\tau(i(k))}=n^j_k$. Otherwise, $j\in A(k')\setminus B(k')$ and one has that $n^j_k=m^j_{\tau(i(k))}=m^{i(k)}_{\tau(j)}$ and also $n^{i(k)}_{k'}=Nm^{i(k)}_{\tau(j)}$ for a positive integer N. As a consequence, $n^{i(k)}_{k'}>n^j_k$ and therefore $i(k)$ satisfies the second requirement of the proposition.

In order to finish the proof, one has to prove that $B(k)\neq\varnothing$ for some $k\in E$. Let us assume that $B(k)=\varnothing$ for some $k\in E$ and $\underline{n}_k=\underline{m}_\sigma$, for some $\sigma\in\Gamma$ with $v(\sigma)\geq 3$ (here $v(\sigma)$ is the valency of the vertex σ). Let $\pi':(\mathcal{X}',\mathcal{D}')\to(S,0)$ be the pre-resolution of C and let Γ' be the resolution graph of π'. For any $j\in A(k)$ one has that $\tau(j)$ is an end of $\Gamma\setminus\{\widetilde{C}_j\}$. We will distinguish two cases:

Let us assume that $\sigma\notin\Gamma'$. In this case, also $\tau(j)\notin\Gamma'$ is a deadend of the dual resolution graph of the curve $C\setminus\{C_j\}$ and $A(k)=\{j\}$. In particular, the vertex σ appears after $\tau(j)$ in the resolution process of a certain branch C_i, $i\neq j$, which is not a curvette at E_σ. It is clear that in this case $(1/n^i_k)\underline{n}_k>(1/m^i_{\tau(i)})\underline{m}_{\tau(i)}$ and $k'=\varkappa(i)$ provides a new element $k'\in E$. Using this new element k' we can repeat the argument up to the moment when we reach $e\in E$ such that $B(e)\neq\varnothing$ (note that $\underline{n}_k<\underline{n}_{k'}$).

Assume that $\sigma\in\Gamma'$. If there exists an irreducible component C_i such that its strict transform by π' intersects $\mathcal{D}'$ at E_σ one can repeat the same argument of the above case for $k'=\varkappa(i)$ and so there exists an element $e\in E$ such that $B(e)\neq\varnothing$. Otherwise, $\sigma\in\Gamma'$ must be a star vertex of Γ' and all the elements of $A(k)$ correspond to curvettes at some of the end points of Γ'. However, there is (at most) only one vertex in Γ' with such conditions: the vertex named $k-2$ if $S=\mathbb{D}_k$, the vertex named 3 for $\mathbb{E}_6,\mathbb{E}_7$ and $\mathbb{E}_8$ and no one for the case $\mathbb{A}_k$. Being $\#E\geq 2$ in order to finish it suffices to take a new $k'\in E$, $k'\neq k$. $\square$

In order to finish the proof of the Theorem it remains only to treat the case when $\#E=1$, i.e. $k\in E$ is such that $A(k)=\{1,\dots,r\}$. First of all, we will identify the cases when $B(k)=\varnothing$. Taking into account the discussion in the proof of the Proposition above about the indices $j\in A(k)\setminus B(k)$, the only possibility to have this situation is when $\underline{n}_k=\underline{m}_\sigma$ for $\sigma\in\Gamma'$ being a star vertex and all the branches are curvettes at some end points of Γ' and no two of them are in the same vertex. This situations can be described (and so detected) one by one for each of the singularities $\mathbb{D}_k$, $\mathbb{E}_6,\mathbb{E}_7,\mathbb{E}_8$ (note that $\mathbb{A}_k$ does not appear in this situation). The table below describe all the possible choices of the ends and the corresponding Weil-Poincaré series:

$$\begin{array}{lll}
\mathbb{D}_k & \{1,k\} & \mapsto (1-t_1t_2^{(k-2)/2})(1-t_1^{1/2}t_2^{(k-2)/4})^{-1} \\
 & \{k-1,k\} & \mapsto (1-t_1^{(k-2)/2}t_2^{(k-2)/2})(1-t_1^{1/2}t_2^{1/2})^{-1} \\
 & \{1,k-1,k\} & \mapsto (1-t_1t_2^{(k-2)/2}t_3^{(k-2)/2}) \\
\mathbb{E}_6 & \{1,4\} & \mapsto (1-t_1^2t_2^3)(1-t_1^{2/3}t_2^1)^{-1} \\
 & \{1,6\} & \mapsto (1-t_1^2t_2^2)(1-t_1t_2)^{-1} \\
 & \{1,4,6\} & \mapsto (1-t_1^2t_2^3t_3^2) \\
\mathbb{E}_7 & \{1,4\} & \mapsto (1-t_1^4t_2^6)(1-t_1t_2^{3/2})^{-1} \\
 & \{1,7\} & \mapsto (1-t_1^4t_2^3)(1-t_1^2t_2^{3/2})^{-1} \\
 & \{4,7\} & \mapsto (1-t_1^6t_2^3)(1-t_1^2t_2)^{-1} \\
 & \{1,4,7\} & \mapsto (1-t_1^4t_2^6t_3^3) \\
\mathbb{E}_8 & \{1,4\} & \mapsto (1-t_1^{10}t_2^{15})(1-t_1^2t_2^3)^{-1} \\
 & \{1,8\} & \mapsto (1-t_1^{10}t_2^6)(1-t_1^5t_2{}^3)^{-1} \\
 & \{4,8\} & \mapsto (1-t_1^{15}t_2^6)(1-t_1^5t_2^2)^{-1} \\
 & \{1,4,8\} & \mapsto (1-t_1^{10}t_2^{15}t_3^6)
\end{array}$$

Now, if the Weil-Poincaré series is not one of the above, then $B(k)\neq\varnothing$ and the we can choose an index $i\in B(k)$ as in the Proposition above. This finishes the proof of Theorem 11. $\square$

For the proof of Theorem 12 (an analogue of Theorem 11 for divisorial valuations) the projection formula permits to reduce (to split) the case of r valuations to the cases of one valuation v_1 and $(r-1)$ remaining valuations. For the $\mathbb{E}_8$-singularity in [9] this finishes the proof (due to the absence of symmetries of the $\mathbb{E}_8$ graph). In the case under consideration the multiplicity $\underline{m}_{1i}$ determines whether the vertices of the minimal resolution graph corresponding to the divisorial valuations v_1 and v_i are on the same part from those exchanged by symmetries or on different ones.

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