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On the double of the Jordan plane

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Abstract. We compute the simple finite-dimensional modules and the center of the Drinfeld double of the Jordan plane introduced in [AP1] assuming that the characteristic is zero.

1. Introduction

Let $\Bbbk$ be a field. The well-known Jordan plane is the quadratic algebra $J=\Bbbk\langle x,y|xy-yx-\frac{1}{2}x^2\rangle$; it bears a structure of braided Hopf algebra where x and y are primitive [G]. When $\Bbbk$ has characteristic zero, J is indeed a Nichols algebra (any primitive element belongs to $V=\Bbbk x\oplus\Bbbk y$) but if $\operatorname{char}\Bbbk=p>0$, then J covers the Nichols algebra $\mathscr{B}(V)$ which has now finite dimension [CLW]. In [AP1] $\mathscr{B}(V)$ was called the restricted Jordan plane and, assuming $p>2$, the Drinfeld double $D(\mathcal{H})$ of the bosonization $\mathcal{H}=\mathscr{B}(V)\#\Bbbk C_p$ was studied. It was shown that $D(\mathcal{H})$ fits into an exact sequence of Hopf algebras $\mathbf{R}\hookrightarrow D(\mathcal{H})\twoheadrightarrow\mathfrak{u}(\mathfrak{sl}_2(\Bbbk))$ where $\mathbf{R}$ is local commutative and $\mathfrak{u}(\mathfrak{sl}_2(\Bbbk))$ is the restricted enveloping algebra of $\mathfrak{sl}_2(\Bbbk)$. Hence the simple $D(\mathcal{H})$-modules coincide with those of $\mathfrak{u}(\mathfrak{sl}_2(\Bbbk))$ [AP1, 1.11].

In [AP1] a Hopf algebra $\mathcal{D}$ covering $D(\mathcal{H})$ was defined, see Section 2; $\mathcal{D}$ can be thought of as the Drinfeld double of the bosonization $J\#\Bbbk\mathbb{Z}$. Now $\mathcal{D}$ fits into an exact sequence of Hopf algebras $\mathcal{O}(\mathbf{G})\hookrightarrow\mathcal{D}\xrightarrow{\pi}U(\mathfrak{sl}_2(\Bbbk))$ where $\mathcal{O}(\mathbf{G})$ is the algebra of regular functions on $\mathbf{G}=(\mathbf{G}_a\times\mathbf{G}_a)\rtimes\mathbf{G}_m$ and $U(\mathfrak{sl}_2(\Bbbk))$ is the enveloping algebra.

Both the definition of $\mathcal{D}$ and the exact sequence are still valid in characteristic 0, which we assume from now on. In this paper, we offer two results on the structure of $\mathcal{D}$. First, we classify in Section 3 the finite-dimensional irreducible representations of $\mathcal{D}$; the outcome ressembles the case of $D(\mathcal{H})$, analogy supported by Lemma 3.10. Concretely, we prove:

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Theorem 3.11. *There is a bijection* $\operatorname{irrep}\mathcal{D} \simeq \operatorname{irrep}\mathfrak{sl}_2(\Bbbk)$ *induced by the morphism* $\pi : \mathcal{D} \to U(\mathfrak{sl}_2(\Bbbk))$.

Second, we consider in Section 4 the localization $\mathcal{D}'$ of $\mathcal{D}$ at the powers of two elements x and q which generate an Ore subset of $\mathcal{D}$. We show in Theorem 4.8 that $\mathcal{D}'$ is isomorphic to a localization of the Weyl algebra $A_2(S)$, where $S := \Bbbk[z^{\pm 1}, z']$ with z and z' algebraically independent. This result allows us to compute the center of $\mathcal{D}$, which turns out to be a Kleinian singularity of type A_1, see Theorem 4.10. Also, by Theorem 4.8, $\mathcal{D}$ satisfies the Gelfand-Kirillov property; see §4.4.

The classification of the Nichols algebras over abelian groups with finite Gelfand-Kirillov dimension (GK-dim for short) is essential for the classification of Hopf algebras with finite GK-dim; see [AAH] for details. Conjecturally, a Nichols algebra of diagonal type has finite GK-dim if and only if the corresponding Weyl groupoid is finite. Assuming the conjecture, the classification for Nichols algebras of 'blocks & points' was solved in [AAH]; the Jordan and super Jordan planes are crucial ingredients of this work.

A basic input for many questions is the Drinfeld double of a given Hopf algebra. Note that the definition of the Drinfeld double of an infinite-dimensional Hopf algebra relies on a suitable choice of a Hopf subalgebra of the Sweedler dual. For instance, the Drinfeld doubles of (bosonizations of) Nichols algebras of diagonal type are essentially multiparametric quantized (super) enveloping algebras, where the duals of the bosonizations are chosen starting from suitable groups of characters. The study of the Drinfeld doubles of Nichols algebras of 'blocks & points' began by the smallest and crucial examples of the Jordan and super Jordan planes [AP1], [AP2] which behave very differently from the diagonal case. As shown here, both the representation theory and the algebra structure of the Drinfeld double of the Jordan plane also deviate from the diagonal case. This confirms the intuition that the Jordan world is closer to the classical one than to the quantum one. We expect that the considerations of this article will guide us to set up a comprehensive general picture.

Conventions If $\ell < n \in \mathbb{N}_0$, we set $\mathbb{I}_{\ell,n} = \{\ell, \ell+1, \dots, n\}$, $\mathbb{I}_n = \mathbb{I}_{1,n}$. If Y is a subobject of an object X in a category $\mathcal{C}$, then we write $Y \leq X$.

In the rest of the paper $\Bbbk$ is an algebraically closed field of characteristic 0. The subspace of a vector space V generated by $S \subset V$ is denoted by $\Bbbk S$. Let A be an algebra and $a_1, \dots, a_n \in A$, $n \in \mathbb{N}$. Let $\mathcal{Z}(A)$ denote the center of A. We denote by $\Bbbk\langle a_1, \dots, a_n\rangle$ the subalgebra generated by $a_1, \dots, a_n$. An element $x \in A$ is normal if $Ax = xA$. If $A = \bigoplus_{n \in \mathbb{Z}} A^n$ is graded and $T \subseteq A$ is a subspace, then $T^n := T \cap A^n$. If M is an A-module and $m_1, \dots, m_n \in M$, $n \in \mathbb{N}$, then we denote by $\langle m_1, \dots, m_n\rangle$ the submodule generated by $m_1, \dots, m_n$.

Let L be a Hopf algebra. The kernel of the counit ε is denoted L^+, the antipode (always assumed bijective) by $\mathcal{S}$, the space of primitive elements by $\mathcal{P}(L)$ and the group of group-likes by $G(L)$. The space of (g,h)-primitives is $\mathcal{P}_{g,h}(L)=\{x\in L: \Delta(x)=x\otimes h+g\otimes x\}$ where $g,h\in G(L)$. The category of left-left, respectively right-right, Yetter-Drinfeld modules over L is denoted by ${}^L_L\mathcal{YD}$, respectively $\mathcal{YD}^L_L$. Our reference for Hopf algebras is [R].

2. The double of the Jordan plane

Definition 2.1. ([AP1, 2.3]) The Hopf algebra $\mathcal{D}$ is presented by generators u, v, ζ, $g^{\pm 1}$, x, y and relations

$$g^{\pm 1}g^{\mp 1}=1, \quad \zeta g=g\zeta, \tag{2.1}$$

$$\begin{aligned} &gx=xg, \quad gy=yg+xg, \quad \zeta y=y\zeta+y, \quad \zeta x=x\zeta+x,\\ &ug=gu, \quad vg=gv+gu, \quad v\zeta=\zeta v+v, \quad u\zeta=\zeta u+u, \end{aligned} \tag{2.2}$$

$$\begin{aligned} &yx=xy-\frac{1}{2}x^2, \quad vu=uv-\frac{1}{2}u^2,\\ &ux=xu, \quad vx=xv+(1-g)+xu,\\ &uy=yu+(1-g), \quad vy=yv-g\zeta+yu. \end{aligned} \tag{2.3}$$

The comultiplication is defined by $g\in G(\mathcal{D})$, $u,\zeta\in\mathcal{P}(\mathcal{D})$, $x,y\in\mathcal{P}_{g,1}(\mathcal{D})$,

$$\Delta(v)=v\otimes 1+1\otimes v+\zeta\otimes u.$$

We list some basic properties of $\mathcal{D}$, cf. [AP1] for details.

- The following set is a PBW-basis of $\mathcal{D}$:

$$B=\{x^n\, y^r\, g^m\, \zeta^k\, u^i\, v^j : i,j,k,n,r\in\mathbb{N}_0,\ m\in\mathbb{Z}\}.$$

- $\mathcal{D}=\oplus_{n\in\mathbb{Z}}\mathcal{D}^{[n]}$ is $\mathbb{Z}$-graded by

$$\deg x=\deg y=-1, \quad \deg u=\deg v=1, \quad \deg g=\deg\zeta=0. \tag{2.4}$$

- Let Γ be the infinite cyclic group with generator g written multiplicatively and let $\mathfrak{h}$ be the one dimensional Lie algebra. The subalgebra $\mathcal{D}^0=\Bbbk\langle g^{\pm 1},\zeta\rangle$ is a Hopf subalgebra isomorphic to $\Bbbk\Gamma\otimes U(\mathfrak{h})$.
- The subalgebra $\mathcal{D}^{<0}=\Bbbk\langle x,y\rangle$ is isomorphic to the Jordan plane J. It is a Hopf algebra in ${}^{\Bbbk\Gamma}_{\Bbbk\Gamma}\mathcal{YD}$ and the bosonization $\mathcal{D}^{<0}\#\Bbbk\Gamma\simeq\Bbbk\langle g^{\pm 1},x,y\rangle$.
- The subalgebra $\mathcal{D}^{>0}=\Bbbk\langle u,v\rangle$ is isomorphic to J as algebra via $u\mapsto x$ and $v\mapsto y$, but with a different comultiplication; actually $\mathcal{D}^{>0}$ is the graded dual of $\mathcal{D}^{<0}$. Then $\mathcal{D}^{>0}$ is a Hopf algebra in $\mathcal{YD}^{U(\mathfrak{h})}_{U(\mathfrak{h})}$ and $U(\mathfrak{h})\#\mathcal{D}^{>0}\simeq\Bbbk\langle\zeta,u,v\rangle$.
- The subalgebras $\mathcal{D}^{>0}$, $\mathcal{D}^0$ and $\mathcal{D}^{<0}$ are graded and satisfy

(a) $\mathcal{D}^{>0}\subseteq\oplus_{n\in\mathbb{N}_0}\mathcal{D}^{[n]}$, $\mathcal{D}^{<0}\subseteq\oplus_{n\in\mathbb{N}_0}\mathcal{D}^{[-n]}$.
(b) $(\mathcal{D}^{>0})^{[0]}=\Bbbk=(\mathcal{D}^{<0})^{[0]}$.
(c) $\mathcal{D}^{\geq 0}:=\mathcal{D}^0\mathcal{D}^{>0}$ and $\mathcal{D}^{\leq 0}:=\mathcal{D}^{<0}\mathcal{D}^0$ are Hopf subalgebras of $\mathcal{D}$.

• The algebra $\mathcal{D}$ admits an exhaustive ascending filtration that satisfies $\operatorname{gr}\mathcal{D}\simeq\Bbbk[X_1,\dots,X_5,T^{\pm 1}]$. Hence $\mathcal{D}$ is a noetherian domain.
• The subalgebra $\mathcal{O}:=\Bbbk\langle x,u,g^{\pm 1}\rangle$ is a commutative Hopf subalgebra, hence $\mathcal{O}\simeq\mathcal{O}(\mathbf{G})$, where $\mathbf{G}$ is the algebraic group as in the Introduction.
• Let e,f,h be the Chevalley generators of $\mathfrak{sl}_2(\Bbbk)$, i.e. $[e,f]=h$, $[h,e]=2e$, $[h,f]=-2f$. The Hopf algebra map $\pi\colon\mathcal{D}\to U(\mathfrak{sl}_2(\Bbbk))$ determined by

$$\pi(v)=\frac{1}{4}e,\quad \pi(y)=2f,\quad \pi(\zeta)=-\frac{1}{2}h,\quad \pi(u)=\pi(y)=\pi(g-1)=0, \tag{2.5}$$

induces an isomorphism of Hopf algebras $\mathcal{D}/\mathcal{D}\mathcal{O}^+\simeq U(\mathfrak{sl}_2(\Bbbk))$.

Remark 2.2. The Hopf algebra $\mathcal{D}$ is pointed with coradical $\mathcal{D}_0=\Bbbk\langle g^{\pm 1}\rangle$. Indeed, by [M, 5.3.4] it suffices to show that $\mathcal{D}$ admits an exhaustive coalgebra filtration $\mathcal{D}=\cup_{n\in\mathbb{N}_0}\mathcal{D}_{[n]}$ with $\mathcal{D}_{[0]}=\Bbbk\langle g^{\pm 1}\rangle$. Let $\mathcal{D}_{[n]}$ defined recursively by

$$\begin{aligned}&\mathcal{D}_{[0]}=\Bbbk\langle g^{\pm 1}\rangle, && \mathcal{D}_{[1]}=\mathcal{D}_{[0]}+\Bbbk\{x,y,\zeta,u\},\\ &\mathcal{D}_{[2]}=\mathcal{D}_{[1]}+\Bbbk\{v\}, && \mathcal{D}_{[n]}=\mathcal{D}_{[2]}\mathcal{D}_{[n-1]},\quad n\geq 3.\end{aligned}$$

Using the PBW basis one can check this is an exhaustive coalgebra filtration.

3. Simple finite-dimensional modules

3.1. Overview

Let A be an algebra and B a subalgebra. Let ${}_A\mathcal{M}$ (respectively $\operatorname{Irrep} A$, $\operatorname{irrep} A$) denote the category of left A-modules (respectively, the set of isomorphism classes of simple objects in ${}_A\mathcal{M}$, the finite-dimensional ones). Often we do not distinguish a class in $\operatorname{Irrep} A$ and one of its representatives. Let $\operatorname{Ind}_B^A\colon{}_B\mathcal{M}\to{}_A\mathcal{M}$ and $\operatorname{Res}_A^B\colon{}_A\mathcal{M}\to{}_B\mathcal{M}$ denote the induction and restriction functors, e.g. $\operatorname{Ind}_B^A(M)=A\otimes_B M$. Given $S\in\operatorname{irrep} A$, there exists $T\in\operatorname{irrep} B$ such that $T\leq\operatorname{Res}_A^B S$. By the standard adjunction

$$\operatorname{Hom}_A(\operatorname{Ind}_B^A M,N)\simeq\operatorname{Hom}_B(M,\operatorname{Res}_A^B N),\quad N\in{}_A\mathcal{M},M\in{}_B\mathcal{M}, \tag{3.1}$$

S is a quotient of $\operatorname{Ind}_B^A T$. We apply this (classical) remark twice to compute $\operatorname{irrep}\mathcal{D}$. First we compute $\operatorname{irrep}\mathcal{D}^{\geq 0}$ by determining all simple quotients of $\operatorname{Ind}_H^{\mathcal{D}^{\geq 0}} W$ for each $W\in\operatorname{irrep} H$, where $H:=\Bbbk\langle g^{\pm 1},u,v\rangle$, cf. [ABFF]. Then we compute $\operatorname{irrep}\mathcal{D}$ from $\operatorname{irrep}\mathcal{D}^{\geq 0}$ in the same way.

3.2. Determination of irrep $\mathcal{D}^{\geq 0}$

For each $(a,b)\in\Bbbk^\times\times\Bbbk$ there is a one-dimensional H-module, denoted by $\Bbbk_{a,b}$, with basis $\mathtt{x}_{a,b}$ and action

$$g\cdot\mathtt{x}_{a,b}=a\mathtt{x}_{a,b},\quad v\cdot\mathtt{x}_{a,b}=b\mathtt{x}_{a,b},\quad u\cdot\mathtt{x}_{a,b}=0. \tag{3.2}$$

Then irrep $H=\{\Bbbk_{a,b}\colon (a,b)\in\Bbbk^\times\times\Bbbk\}$ [ABFF, 3.3]. Let $W_{a,b}:=\operatorname{Ind}_H^{\mathcal{D}^{\geq 0}}\Bbbk_{a,b}$.

Lemma 3.1. *The elements $\mathtt{x}_{a,b}^{(n)}:=\zeta^n\cdot\mathtt{x}_{a,b}$, $n\geq 0$, form a basis of $W_{a,b}$.*

Proof. Indeed, $\mathcal{D}^{\geq 0}\otimes_H\Bbbk_{a,b}\simeq(U(\mathfrak{h})\otimes_\Bbbk H)\otimes_H\Bbbk_{a,b}\simeq U(\mathfrak{h})\otimes_\Bbbk\Bbbk_{a,b}$. □

Lemma 3.2. *If $b\neq 0$, then $W_{a,b}$ is simple for any $a\in\Bbbk^\times$.*

Proof. We use that $(v-b)^n\zeta=\zeta(v-b)^n+nv(v-b)^{n-1}$ for all $n\geq 0$, which is straightforward. Notice that $(v-b)^n\cdot\mathtt{x}_{a,b}^{(n)}=b^nn!\mathtt{x}_{a,b}^{(0)}$ for all $n\geq 0$. Indeed,

$$\begin{aligned}(v-b)^{n+1}\cdot\mathtt{x}_{a,b}^{(n+1)}&=(v-b)(v-b)^n\zeta\cdot\mathtt{x}_{a,b}^{(n)}\\&=(v-b)\left(\zeta(v-b)^n+nv(v-b)^{n-1}\right)\cdot\mathtt{x}_{a,b}^{(n)}\\&=b^nn!(v-b)\zeta\cdot\mathtt{x}_{a,b}^{(0)}+nb^nn!v\cdot\mathtt{x}_{a,b}^{(0)}=b^{n+1}(n+1)!\mathtt{x}_{a,b}^{(0)}.\end{aligned}$$

Let $0\neq z\in W_{a,b}$ and write $z=\sum_{k=0}^n c_k\mathtt{x}_{a,b}^{(k)}$ with $n\geq 0$ and $c_n\neq 0$. Then $(v-b)^n\cdot z=b^nn!c_n\mathtt{x}_{a,b}^{(0)}$. Thus $\langle z\rangle=W_{a,b}$ and the claim follows. □

We next study the simple quotients of $W_{a,0}$, $a\in\Bbbk^\times$. Let

$$V_{a,c}=\langle\mathtt{x}_{a,0}^{(1)}+\frac{1}{2}c\mathtt{x}_{a,0}^{(0)}\rangle\leq W_{a,0},\quad T_{a,c}:=W_{a,0}/V_{a,c},\quad c\in\Bbbk.$$

The choice of the coefficient $\frac{1}{2}$ is convenient for calculations with K_c; see Remark 3.12. Then $T_{a,c}$ is one-dimensional with basis $z_{a,c}$ and action

$$g\cdot z_{a,c}=az_{a,c},\quad \zeta\cdot z_{a,c}=-\frac{1}{2}cz_{a,c},\quad u\cdot z_{a,c}=0,\quad v\cdot z_{a,c}=0. \tag{3.3}$$

Lemma 3.3. *The set of maximal submodules of $W_{a,0}$ is $\{V_{a,c}\colon c\in\Bbbk\}$.*

Proof. The subcategory of $\mathcal{D}^{\geq 0}$-modules where u, v and $g-a$ act by 0 is equivalent to the category of modules over $\mathcal{D}^{\geq 0}/(u,v,g-a)\simeq\Bbbk[\zeta]$. Now $W_{a,0}$ belongs to this subcategory, because the action in the basis $\left\{\mathtt{x}_{a,b}^{(n)}\right\}$ is

$$g\cdot\mathtt{x}_{a,b}^{(n)}=g\zeta^n\cdot\mathtt{x}_{a,b}=\zeta^ng\cdot\mathtt{x}_{a,b}=a\mathtt{x}_{a,b}^{(n)},$$

$$u\cdot \mathtt{x}_{a,b}^{(n)} = u\zeta^n\cdot \mathtt{x}_{a,b} = \sum_{k=0}^{n}\binom{n}{k}\zeta^k u\cdot \mathtt{x}_{a,b} = 0,$$
$$v\cdot \mathtt{x}_{a,b}^{(n)} = v\zeta^n\cdot \mathtt{x}_{a,b} = \sum_{k=0}^{n}\binom{n}{k}\zeta^k v\cdot \mathtt{x}_{a,b} = 0.$$

Under this correspondence, $W_{a,0}$ goes to the regular $\Bbbk[\zeta]$-module; the claim follows. □

Proposition 3.4. $\operatorname{irrep}\mathcal{D}^{\geq 0} \simeq \{T_{a,c} : (a,c)\in\Bbbk^\times\times\Bbbk\}$.

Proof. Let $T\in\operatorname{irrep}\mathcal{D}^{\geq 0}$. Then there exists $(a,b)\in\Bbbk^\times\times\Bbbk$ such that $\Bbbk_{a,b}$ is isomorphic to a submodule of $\operatorname{Res}^H_{\mathcal{D}^{\geq 0}} T$. By (3.1) T is a quotient of $W_{a,b}$. If $b\neq 0$, then $T\simeq W_{a,b}$ by Lemma 3.2, contradicting $\dim T<\infty$. Hence $b=0$ and $T\simeq T_{a,c}$ for some $c\in\Bbbk$. □

3.3. Calculation of irrep $\mathcal{D}$

The *Verma module* $M_{a,c}$, $(a,c)\in\Bbbk^\times\times\Bbbk$, is

$$M_{a,c} := \operatorname{Ind}^{\mathcal{D}}_{\mathcal{D}^{\geq 0}} T_{a,c} = \mathcal{D}\otimes_{\mathcal{D}^{\geq 0}} T_{a,c}.$$

Lemma 3.5. *The elements* $z_{a,c}^{(i,j)} := y^i x^j \cdot z_{a,c}$, $i,j\geq 0$, *form a basis of* $M_{a,c}$.

Proof. Indeed, $\mathcal{D}\otimes_{\mathcal{D}^{\geq 0}} T_{a,c} \simeq (\mathcal{D}^{<0}\otimes_\Bbbk \mathcal{D}^{\geq 0})\otimes_{\mathcal{D}^{\geq 0}} T_{a,c} \simeq \mathcal{D}^{<0}\otimes_\Bbbk T_{a,c}$. □

For the next proof we need the following formulas from [AP1, Lemma 2.5]:

$$uy^n = y^n u + ny^{n-1} - \sum_{k=0}^{n-1}\binom{n}{k+1}\frac{(k+1)!}{2^k} y^{n-1-k}x^k g, \quad n\geq 1; \tag{3.4}$$

$$vx^m = x^m v + mx^{m-1}(1-g) + mx^m u, \quad m\geq 1. \tag{3.5}$$

Lemma 3.6. *If* $a\neq 1$ *then* $M_{a,c}$ *is simple.*

Proof. From (3.4) we get the following formula for $i\geq 1$ and $j\geq 0$:

$$u\cdot z_{a,c}^{(i,j)} = (1-a)iz_{a,c}^{(i-1,j)} - a\sum_{k=1}^{i-1}\binom{i}{k+1}\frac{(k+1)!}{2^k} z_{a,c}^{(i-1-k,j+k)};$$

clearly $u\cdot z_{a,c}^{(0,j)}=0$ for all $j\geq 0$. Next, we prove by induction that

$$u^i\cdot z_{a,c}^{(i,j)} = (1-a)^i i! z_{a,c}^{(0,j)}, \quad i,j\geq 0.$$

Indeed, this is clear for $i=0,1$; for $i\geq 1$, we argue by induction on i:

$$\begin{aligned} u^{i+1}\cdot z_{a,c}^{(i+1,j)} &= (1-a)(i+1)u^i z_{a,c}^{(i,j)} - a\sum_{k=1}^{i}\binom{i+1}{k+1}\frac{(k+1)!}{2^k}u^i z_{a,c}^{(i-k,j+k)} \\ &= (1-a)^{i+1}(i+1)!z_{a,c}^{(0,j)} \\ &\quad - a\sum_{k=1}^{i}\binom{i+1}{k+1}\frac{(k+1)!}{2^k}(i-k)!(1-a)^{i-k}u^k z_{a,c}^{(0,j+k)} \\ &= (1-a)^{i+1}(i+1)!z_{a,c}^{(0,j)}. \end{aligned}$$

Thus $u^n\cdot z_{a,c}^{(i,j)}=0$ if $n>i$. From (3.5) we get $v\cdot z_{a,c}^{(0,j)}=j(1-a)z_{a,c}^{(0,j-1)}$, and it becomes evident that

$$v^j u^i\cdot z_{a,c}^{(i,j)} = (1-a)^{i+j}i!j!z_{a,c}^{(0,0)}.$$

Then $v^m u^i\cdot z_{a,c}^{(i,j)}=0$ for $m>j$ and $i\geq 0$. Given $z\in M_{a,c}$, $z\neq 0$, write

$$z=\sum_{i=0}^{N}\sum_{j=0}^{M_i} d_{i,j}z_{a,c}^{(i,j)}\neq 0 \quad \text{with } d_{N,M_N}\neq 0,$$

$$\text{hence}\quad v^{M_N}u^N\cdot z = d_{N,M_N}(1-a)^{M_N+N}N!M_N!z_{a,c}^{(0,0)}.$$

Since $a\neq 1$ and $M_{a,c}=\langle z_{a,c}^{(0,0)}\rangle$, the Lemma follows. □

From (3.1) with Proposition 3.4 and Lemma 3.6 we deduce:

Corollary 3.7. *Every $S\in\operatorname{irrep}\mathcal{D}$ is a quotient of $M_{1,c}$ for some $c\in\Bbbk$.* □

Thus we need to study the Verma modules $M_{1,c}$. For the next lemma we use the following commutation relation from [AP1, Lemma 2.5]:

$$g^n y^\ell = \sum_{k=0}^{\ell}\binom{\ell}{k}\frac{[2n]^{[k]}}{2^k}y^{\ell-k}x^k g^n, \quad n,\ell\in\mathbb{N}_0. \tag{3.6}$$

Here $[t]^{[k]}$ denotes the raising factorial $[t]^{[k]}:=\prod_{i=1}^{k}(t+i-1)$ for $t\in\Bbbk$ and $k\in\mathbb{N}_0$.

Lemma 3.8. *The action of $g-1$ on $M_{1,c}$ is locally nilpotent.*

Proof. We prove recursively on i that for each $i,j\geq 0$ there exists $n_{i,j}\in\mathbb{N}$ such that $(g-1)^{n_{i,j}}\cdot z_{1,c}^{(i,j)}=0$. If $i=0$, then $g\cdot z_{1,c}^{(0,j)}=z_{1,c}^{(0,j)}$ for $j\in\mathbb{N}$, hence we take $n_{0,j}=1$. Given such $n_{k,j}$ for every $j\in\mathbb{N}$ and $k<i$, we have

$$(g-1)\cdot z_{1,c}^{(i,j)} = (g-1)y^i z_{1,c}^{(0,j)} = \sum_{k=1}^{i}\binom{i}{k}\frac{[2]^{[k]}}{2^k}z_{1,c}^{(i-k,j+k)}$$

by (3.6). Taking $n_{i,j}=\max_{0\leq k<i}\{n_{k,i+j-k}\}+1$, the Lemma follows. □

Proposition 3.9. *Let* $M \in {}_{\mathcal{D}}\mathcal{M}$, $\dim M < \infty$, *with associated representation* $\rho\colon \mathcal{D} \to \operatorname{End} M$. *Then* $g-1$, x *and* u *act nilpotently on* M.

Proof. Arguing by induction on $\dim M$, we may assume that $M \in \operatorname{irrep} \mathcal{D}$. Then $g-1$ acts nilpotently on M by Corollary 3.7 and Lemma 3.8. Recall that a linear operator T on a finite-dimensional space is nilpotent if and only if $\operatorname{Tr}(T^n)=0$ for every $n \in \mathbb{N}$. Since $x^{n+1} = \frac{2}{n}(x^n y - y x^n)$ for $n \in \mathbb{N}$ we get $\operatorname{Tr}(\rho(x)^{n+1}) = \frac{2}{n} \operatorname{Tr}(\rho(x)^n \rho(y) - \rho(y)\rho(x)^n) = 0$. Since $x = \zeta x - x\zeta$ we also have that $\operatorname{Tr}(\rho(x)) = 0$. So $\rho(x)$ is nilpotent. The argument for u is similar using the relations $u^{n+1} = \frac{2}{n}(u^n v - v u^n)$ and $u = u\zeta - \zeta u$. $\square$

We now determine $\operatorname{irrep} \mathcal{D}$ via an argument connecting with [AP1, 1.11].

Lemma 3.10. *Let* A *be an algebra and* $\mathcal{F} \subset A$ *a family of elements satisfying*
(a) *the elements of* $\mathcal{F}$ *commute with each other*;
(b) *for any* $M \in {}_A\mathcal{M}$, $\dim M < \infty$, *any* $x \in \mathcal{F}$ *acts nilpotently on* M;
(c) $\mathcal{F}$ *is normal, i.e. the vector subspace* I *of* A *generated by* $\mathcal{F}$ *satisfies*

$$AI = IA. \tag{3.7}$$

Let $S \in \operatorname{irrep} A$. *Then the representation* $\rho\colon A \to \operatorname{End} S$ *factorizes through* A/IA. *Thus the projection* $A \to A/IA$ *induces a bijection*

$$\operatorname{irrep} A \simeq \operatorname{irrep} A/IA.$$

Proof. Let $\tilde{I} = \rho(I)$ and $\mathcal{I} = \rho(AI) = \rho(IA)$. By (a) and (b) there exists $r \in \mathbb{N}$ such that $\tilde{I}^r = 0$. Then $\mathcal{I}^r = 0$ by (3.7). Hence the ideal $\mathcal{I}$ is contained in the Jacobson radical of $\rho(A)$. But by Burnside's Theorem [CR, (3.3.2)] $\rho(A) = \operatorname{End} S$ since S is simple. So IA acts by 0 on S. $\square$

Recall the map $\pi\colon \mathcal{D} \to U(\mathfrak{sl}_2(\Bbbk))$ from (2.5).

Theorem 3.11. *The map* π *induces a bijection* $\operatorname{irrep} \mathcal{D} \simeq \operatorname{irrep} U(\mathfrak{sl}_2(\Bbbk))$.

Proof. Let $\mathcal{F} = \{x, u, g-1\}$. By the defining relations of $\mathcal{D}$, $\mathcal{F}$ satisfies (a) and (c). By Proposition 3.9 $\mathcal{F}$ satisfies (b). Thus Lemma 3.10 applies. $\square$

Remark 3.12. For $n \in \mathbb{N}_0$, let $L_n \in \operatorname{irrep} \mathcal{D}$ correspond to the simple $\mathfrak{sl}_2(\Bbbk)$-module of highest weight n. Then L_n has a basis $t_0, \dots, t_n$ where the action is given by

$$\begin{aligned} &y \cdot t_i = t_{i+1}, && v \cdot t_i = \frac{i}{2}(n-i+1)t_{i-1}, && \zeta \cdot t_i = -\frac{1}{2}(n-2i)t_i, \\ &x \cdot t_i = 0, && u \cdot t_i = 0, && g \cdot t_i = t_i. \end{aligned} \tag{3.8}$$

It can be shown that L_n can be presented as quotient of the Verma module $M_{1,n}$. Indeed, let $\mathtt{M}_n$ be the Verma module over $\mathfrak{sl}_2(\Bbbk)$ of highest weight n. Then $\mathtt{M}_n \simeq K_n := M_{1,n}/\widetilde{M}_n$ where $\widetilde{M}_n := xM_{1,n} = \langle z_{1,n}^{(0,1)} \rangle \leq M_{1,n}$.

Recall from [BaW] that a spherical Hopf algebra is a pair (H, ω), where H is a Hopf algebra and $\omega \in G(H)$ (called the *pivot*) such that

$$\mathcal{S}^2(x) = \omega x \omega^{-1}, \qquad x \in H, \tag{3.9}$$

$$\mathrm{Tr}_V(\theta\omega) = \mathrm{Tr}_V(\theta\omega^{-1}), \quad \theta \in \mathrm{End}_H(V), \quad V \in {}_H\mathcal{M}. \tag{3.10}$$

Corollary 3.13. *The Hopf algebra $\mathcal{D}$ is spherical with pivot g^{-1}.*

Proof. By direct calculation in the generators we see that $\mathcal{S}^2(h) = g^{-1}hg$ for every $h \in \mathcal{D}$. It remains to show that for every $V \in {}_{\mathcal{D}}\mathcal{M}$ with $\dim V < \infty$, $\mathrm{Tr}_V(fg) = \mathrm{Tr}_V(fg^{-1})$ for every $f \in \mathrm{End}_{\mathcal{D}}(V)$. By [AAGTV, Prop. 2.1] we only need to consider $V \in \operatorname{irrep} \mathcal{D}$. Since $\mathrm{End}_{\mathcal{D}}(V) \simeq \Bbbk$, and $\mathrm{Tr}_V(g) = \mathrm{Tr}_V(g^{-1}) = \dim V$ by 3.8, the claim follows. $\square$

4. A localization of the double of the Jordan plane

4.1. Weyl algebras and iterated Ore extensions

We refer to [MCR] for the notations and basic notions used here. Let R be a commutative ring. Recall that the Weyl algebra $A_1(R)$ is the R-algebra generated by p and q satisfying $pq - qp = 1$. Alternatively, it can be described as the Ore extension $A_1(R) \simeq R[q][p\,;\, \partial_q]$, where, here and below, ∂_q denotes $\frac{\partial}{\partial q}$.

We shall also consider the algebra $A_1'(R) = R[q^{\pm 1}][p\,;\, \partial_q]$, which is the localization of $A_1(R)$ with respect of the multiplicative set generated by q. Observing that $(qp)q - q(qp) = q$, we have an alternative description of $A_1'(R)$ as a Laurent extension:

$$A_1'(R) = R[q^{\pm 1}][p\,;\, \partial_q] = R[qp][q^{\pm 1}\,;\, \sigma^{\pm 1}] \tag{4.1}$$

with σ the R-automorphism of $R[qp]$ defined by $\sigma(qp) = qp - 1$.

The Weyl algebras $A_n(R)$ and their localizations are defined similarly for $n \in \mathbb{N}$; then $A_n(R) = R[q_1, \dots, q_n][p_1\,;\, \partial_{q_1}] \dots [p_1\,;\, \partial_{q_1}]$ and

$$\begin{aligned} A_n'(R) &= R[q_1^{\pm 1}, \dots, q_n^{\pm 1}][p_1\,;\, \partial_{q_1}] \dots [p_n\,;\, \partial_{q_n}] \\ &= R[q_1 p_1, \dots, q_n p_n][q_1^{\pm 1}\,;\, \sigma_1^{\pm 1}] \dots [q_n^{\pm 1}\,;\, \sigma_n^{\pm 1}]. \end{aligned} \tag{4.2}$$

The proof of the following Lemma is straightforward.

Lemma 4.1. *The algebra $\mathcal{D}$ can be described as an iterated Ore extension:*

$$\mathcal{D} \simeq \underbrace{\Bbbk[g^{\pm 1}, x, u]}_{\mathcal{O}\ \textit{commutative}} [y\,;\, d][\zeta\,;\, \delta][v\,;\, \sigma, \mathfrak{d}] \tag{4.3}$$

where d is the derivation of $\mathcal{O}{:=}\Bbbk[g^{\pm 1}, x, u]$, δ is the derivation of $\mathcal{O}[y\,;\, d]$, σ is the automorphism and $\mathfrak{d}$ is the σ-derivation of $\mathcal{O}[y\,;\, d][\zeta\,;\, \delta]$ defined by:

$$\begin{array}{llll} d(x) = -\frac{1}{2}x^2, & \delta(x) = x, & \sigma(x) = x, & \mathfrak{d}(x) = 1-g+xu, \\ d(u) = g-1, & \delta(u) = -u, & \sigma(u) = u, & \mathfrak{d}(u) = -\frac{1}{2}u^2, \\ d(g) = -xg, & \delta(g) = 0, & \sigma(g) = g, & \mathfrak{d}(g) = gu, \\ & \delta(y) = y, & \sigma(y) = y, & \mathfrak{d}(y) = -g\zeta + yu, \\ & & \sigma(\zeta) = \zeta + 1, & \mathfrak{d}(\zeta) = 0. \end{array} \tag{4.4}$$

Corollary 4.2. *The algebra $\mathcal{D}$ is strongly noetherian, AS-regular and Cohen-Macaulay.*

Proof. $\mathcal{D}$ is strongly noetherian by [ASZ, Proposition 4.10]; AS-regular by [AST, Proposition 2] and Cohen-Macaulay by [ZZ, Lemma 5.3]. $\square$

4.2. Localizing

We consider the following elements of the subalgebra $\mathcal{O}$:

$$q := ux + 2(1+g) \quad \text{and} \quad z := q^2 g^{-1}. \tag{4.5}$$

Lemma 4.3.
(i) *x is ad-locally nilpotent in $\mathcal{D}$;*
(ii) *q is normal in $\mathcal{D}$,*
(iii) *z is central in $\mathcal{D}$.*

Proof. (i): We have that $\operatorname{ad}_x(x) = \operatorname{ad}_x(u) = \operatorname{ad}_x(g^{\pm 1}) = \operatorname{ad}_x^2(\zeta) = 0$; then $\operatorname{ad}_x^2(y) = \operatorname{ad}_x(-\frac{1}{2}x^2) = 0$ and $\operatorname{ad}_x^2(v) = \operatorname{ad}_x(-xu+g-1) = 0$. By the Leibniz rule, the claim follows.

(ii): Clearly q commmutes with $u, x, g^{\pm 1}$ and we have

$$qy = (y + \frac{1}{2}x)q, \quad qv = (v - \frac{1}{2}u)q, \quad q\zeta = \zeta q \tag{4.6}$$

by straightforward calculations. Thus $\mathcal{D}q = q\mathcal{D}$, i.e. q is normal.

(iii): The generator g commutes with x, u and ζ, and satisfies: $gy = (y+x)g$, $gv = (v-u)g$. Hence $z = q^2 g^{-1}$ is central in $\mathcal{D}$. $\square$

By Lemma 4.3 (ii) the multiplicative set generated by q is an Ore subset of $\mathcal{D}$. Now by Lemma 4.3 (i) this is also the case for x by [KL, Lemma 4.7] which is a particular case of [BR]. This allows us to consider the localization $\mathcal{D}'$ of $\mathcal{D}$ with respect to the multiplicative set generated by x and q. Let us introduce the element

$$t := qx^{-1} = u + 2(1+g)x^{-1} \in \mathcal{O}' := \mathcal{D}' \cap \mathcal{O}. \tag{4.7}$$

Observe that the preceding proof implies at once that

$$tq = qt, \quad tz = zt. \tag{4.8}$$

Then $\Bbbk[g^{\pm 1}, x^{\pm 1}, u] = \Bbbk[g^{\pm 1}, x^{\pm 1}, t]$. In $\mathcal{O}'$, the element t is invertible with $t^{-1} = q^{-1}x$ and the element z is invertible with $z^{-1} = gq^{-2}$. Then:

$$\mathcal{O}' = \Bbbk[g^{\pm 1}, x^{\pm 1}, t^{\pm 1}] = \Bbbk[g^{\pm 1}, q^{\pm 1}, t^{\pm 1}] = \Bbbk[z^{\pm 1}, q^{\pm 1}, t^{\pm 1}]. \tag{4.9}$$

We deduce the following description of $\mathcal{D}'$ as an iterated Ore extension:

$$\mathcal{D}' = \underbrace{\Bbbk[z^{\pm 1}, q^{\pm 1}, t^{\pm 1}]}_{\mathcal{O}' \text{ commutative}}[y\,;\, d][\zeta\,;\, \delta][v\,;\, \sigma, \mathfrak{d}]; \tag{4.10}$$

here d, δ, σ, $\mathfrak{d}$ denote the canonical extensions to $\mathcal{D}'$ of d, δ, σ, $\mathfrak{d}$ as in (4.3).

Let $R := \Bbbk[z^{\pm 1}, t^{\pm 1}]$. We introduce

$$p := -2q^{-2}ty. \tag{4.11}$$

Lemma 4.4. *The subalgebra* $\Bbbk[z^{\pm 1}, q^{\pm 1}, t^{\pm 1}][y\,;\, d] = \Bbbk[z^{\pm 1}, q^{\pm 1}, t^{\pm 1}][p\,;\, \partial_q]$ *of* $\mathcal{D}'$ *is isomorphic to* $A'_1(R)$.

Proof. We have $d(z)=0$ by Lemma 4.3(iii). We compute using (4.4):

$$\begin{aligned} d(q) &= d(u)x + ud(x) + 2d(g) = -gx - x - \frac{1}{2}ux^2 = -\frac{1}{2}xq = -\frac{1}{2}q^2t^{-1}, \\ d(t) &= d(q)x^{-1} - qx^{-2}d(x) = 0. \end{aligned}$$

Then the change of variable $p := -2q^{-2}ty$ leads to the commutation relations

$$pz = zp, \quad pt = tp \quad \text{and} \quad pq - qp = 1. \tag{4.12}$$

This proves that $\Bbbk[z^{\pm 1}, q^{\pm 1}, t^{\pm 1}][y\,;\, d] = \Bbbk[z^{\pm 1}, q^{\pm 1}, t^{\pm 1}][p\,;\, \partial_q]$ since the powers of p form a basis of the left-hand side as $R[q^{\pm 1}]$-module. Now the right-hand side is isomorphic to $A'_1(R)$ by definition; see (4.1). $\square$

Lemma 4.5. *The following subalgebras of* $\mathcal{D}'$ *are equal*:

$$\Bbbk[z^{\pm 1}, q^{\pm 1}, t^{\pm 1}][y\,;\, d][\zeta\,;\, \delta] = \Bbbk[z^{\pm 1}, q^{\pm 1}, t^{\pm 1}][p\,;\, \partial_q][\zeta\,;\, -t\partial_t].$$

Proof. We have $\delta(z)=0$ by Lemma 4.3(iii). We compute using (4.4):

$$\begin{aligned}
&\delta(t)=\delta(u)+2(1+g)\delta(x^{-1})=-u+2(1+g)(-x^{-1})=-t,\\
&\delta(q)=\delta(t)x+t\delta(x)=-tx+tx=0,\\
&\delta(p)=-2\delta(q^{-2})ty-2q^{-2}\delta(t)y-2q^{-2}t\delta(y)=2q^{-2}ty-2q^{-2}ty=0.
\end{aligned}$$

To sum up,

$$\delta(z)=\delta(q)=\delta(p)=0 \quad \text{and} \quad \delta(t)=-t, \tag{4.13}$$

which gives the desired result. □

Remark 4.6. The change of variable $s{:=}-t^{-1}\zeta$ leads to the commutation relations $sz-zs=sq-qs=sp-ps=0$ and $st-ts=1$. Then denoting $T{:=}\Bbbk[z^{\pm1}]$, we have in $\mathcal{D}'$ the equality of subalgebras:

$$\Bbbk[z^{\pm1},q^{\pm1},t^{\pm1}][y\,;\,d][\zeta\,;\,\delta]=\Bbbk[z^{\pm1}][q^{\pm1}][p\,;\,\partial_q][t^{\pm1}][s\,;\,\partial_t]\simeq A_2'(T).$$

Lemma 4.7. *With the change of variable $w{:=}t^{-1}v$, we have:*

$$\mathcal{D}'=\Bbbk[z^{\pm1},q^{\pm1},\zeta][p\,;\,\partial_q][w\,;\,\mathfrak{D}][t^{\pm1};\,\tau^{\pm1}],$$

where $\mathfrak{D}$ is the derivation of $\Bbbk[z^{\pm1},q^{\pm1},\zeta][p\,;\,\partial_q]$ such that:

$$\mathfrak{D}(z)=\mathfrak{D}(\zeta)=0, \tag{4.14}$$

$$\mathfrak{D}(q)=-1+\frac{1}{2}q-z^{-1}q^2, \tag{4.15}$$

$$\mathfrak{D}(p)=-\frac{1}{2}p+2qz^{-1}p+2z^{-1}\zeta+2q^{-2}-2z^{-1}, \tag{4.16}$$

and τ is the automorphism of $\Bbbk[z^{\pm1},q^{\pm1},\zeta][p\,;\,\partial_q][w\,;\,\mathfrak{D}]$ such that:

$$\tau(z)=z, \quad \tau(q)=q, \quad \tau(p)=p, \tag{4.17}$$

$$\tau(\zeta)=\zeta+1, \quad \tau(w)=w+\frac{1}{2}-2qz^{-1}. \tag{4.18}$$

Proof. We start with the description (4.10) of $\mathcal{D}'$ and recall Lemma 4.5. By direct calculations using (4.4), we show that:

$$\begin{aligned}
&vz=zv, \quad v\zeta=(\zeta+1)v, \quad vt=tv-\frac{1}{2}t^2+2qz^{-1}t^2,\\
&vq=qv-t+\frac{1}{2}tq-tz^{-1}q^2,\\
&vp=pv-\frac{1}{2}tp+2tz^{-1}qp+2tz^{-1}\zeta+2tq^{-2}-2tz^{-1}.
\end{aligned}$$

We replace in $\mathcal{D}'=\mathcal{O}'[p\,;\,\partial_q][\zeta\,;\,-t\partial_t][v\,;\,\sigma,\mathfrak{d}]$ the generator v by:

$$w:=t^{-1}v. \tag{4.19}$$

The last two of the above relations become:

$$\begin{aligned}wq&=qw-1+\frac{1}{2}q-z^{-1}q^2,\\ wp&=pw-\frac{1}{2}p+2qz^{-1}p+2z^{-1}\zeta+2q^{-2}-2z^{-1}.\end{aligned}$$

We still have $wz=zw$. By (4.13), we have $\zeta t=t\zeta-t$, thus $\zeta t^{-1}=t^{-1}\zeta+t^{-1}$. We deduce from this and the relation $\zeta v=v\zeta-v$ that $w\zeta=\zeta w$. Finally, the relation $wt=tw-\frac{1}{2}t+2qz^{-1}t$ can be rewritten as $tw=(w+\frac{1}{2}-2qz^{-1})t$, which gives rise to the desired description of $\mathcal{D}'$. $\square$

Next, we introduce the element

$$z':=q^{-1}\left[xv+uy+(-\frac{1}{2}ux+g-1)\zeta-2(1+g)\right] \tag{4.20}$$

$$=\left[xv+uy+(-\frac{1}{2}ux+g-1)\zeta-2(1+g)\right]q^{-1} \tag{4.21}$$

Theorem 4.8. *The algebra $\mathcal{D}'$ is isomorphic to the localized Weyl algebra $A_2'(S)$, with center $S:=\Bbbk[z^{\pm1},z']$. In particular, z' is central in $\mathcal{D}'$.*

Proof. The subalgebra $\Bbbk[z^{\pm1},q^{\pm1},\zeta][p\,;\,\partial_q]$ is isomorphic to the localized Weyl algebra $A_1'(S)$ for $S=\Bbbk[z^{\pm1},\zeta]$ by (4.12) and (4.13). Thus it is natural by [D, Lemma 4.6.8] to look for an element $f\in\Bbbk[z^{\pm1},q^{\pm1},\zeta][p\,;\,\partial_q]$ such that $\mathfrak{D}$ is the inner derivation ad_f. By (4.15) and (4.16), such an element satisfies:

$$\begin{aligned}fq-qf&=-1+\frac{1}{2}q-z^{-1}q^2,\\ fp-pf&=(-\frac{1}{2}+2z^{-1}q)p+2q^{-2}+2z^{-1}(\zeta-1).\end{aligned}$$

Using (4.13), a solution is clearly:

$$f:=-(1-\frac{1}{2}q+z^{-1}q^2)p+2q^{-1}-2z^{-1}(\zeta-1)q. \tag{4.22}$$

Then we have by construction for any $h\in\Bbbk[z^{\pm1},q^{\pm1},\zeta][p\,;\,\partial_q]$:

$$(w-f)h=wh-fh=hw+\mathfrak{D}(h)-fh=hw+fh-hf-fh=h(w-f).$$

Moreover, we deduce from $\tau(\zeta)=\zeta+1$ that $\tau(f)=f-2z^{-1}q$. Then the second identity of (4.18) implies that $\tau(w-f)=w-f+\frac{1}{2}$. A first consequence is that the element:

$$z' := w-f-\frac{1}{2}\zeta \tag{4.23}$$

is central in $\mathcal{D}'$. We can replace the generator w by z' to obtain:

$$\mathcal{D}' = \Bbbk[z^{\pm 1}, q^{\pm 1}, \zeta][p\,;\,\partial_q][z'][t^{\pm 1};\,\tau^{\pm 1}], \tag{4.24}$$

where all generators pairwise commute except:

$$pq-qp=1 \quad \text{and} \quad t\zeta-\zeta t=t. \tag{4.25}$$

We can replace the generator ζ by:

$$\xi := -t^{-1}\zeta \tag{4.26}$$

to obtain the following differential description:

$$\mathcal{D}' = \Bbbk[z^{\pm 1}, z'][q^{\pm 1}][p\,;\,\partial_q][t^{\pm 1}][\xi\,;\,\partial_t], \tag{4.27}$$

with

$$pq-qp=1 \quad \text{and} \quad \xi t-t\xi=1. \tag{4.28}$$

We can alternatively replace the generator p by:

$$r := -qp \tag{4.29}$$

to obtain the following automorphic description:

$$\mathcal{D}' = \Bbbk[z^{\pm 1}, z'][r][q^{\pm 1}\,;\,\sigma^{\pm 1}][\zeta][t^{\pm 1};\,\tau^{\pm 1}] \tag{4.30}$$

with

$$qr=(r+1)q \quad \text{and} \quad t\zeta=(\zeta+1)t. \tag{4.31}$$

We conclude from (4.27) or (4.30) that $\mathcal{D}'$ is isomorphic to the localized Weyl algebra $A'_2(S)$ as in (4.2) for $S=\Bbbk[z^{\pm 1}, z']$. Since $\operatorname{char}\Bbbk=0$, $S=\mathcal{Z}\,(A'_2(S))$.

The last step is to express the central element z' according to the initial generators of $\mathcal{D}$. Let us recall from (4.5) and (4.11) that

$$q=ux+2(1+g), \quad z=q^2g^{-1} \quad \text{and} \quad p=-2q^{-1}x^{-1}y.$$

It follows that the expression $(1-\frac{1}{2}q+z^{-1}q^2)p$ in formula (4.22) is equal to $q^{-1}uy$. Then: $f=q^{-1}[-uy+2(1+g)-2g\zeta]$. Moreover $w=q^{-1}xv$ by (4.7) and (4.19), and we obtain obviously the relation (4.20) from (4.23). The alternative expression (4.21) follows then from (4.6). $\square$

Remark 4.9. Observe in (4.20) that z' does not depend on negative powers of x; i.e. z' lies in the localization of $\mathcal{D}$ by inverting only the powers of q.

4.3. The center of $\mathcal{D}$

Because of Theorem 4.8 and (4.20), it is natural to introduce the following element of $\mathcal{D}$:

$$s := qz' = xv + uy + (-\frac{1}{2}ux + g - 1)\zeta - 2(1+g) \in \mathcal{O}v \oplus \mathcal{O}y \oplus \mathcal{O}\zeta \oplus \mathcal{O},$$

which is normal in $\mathcal{D}$ by Lemma 4.3 (ii) and Theorem 4.8.

Since $z = q^2 g^{-1}$ is central in $\mathcal{D}$, we are lead to introduce:

$$\theta := s^2 g^{-1} \in \mathcal{Z}(\mathcal{D}). \tag{4.32}$$

Now $z' = q^{-1}s = sq^{-1}$ is central in $\mathcal{D}'$ by Theorem 4.8, hence

$$\omega := zz' = qg^{-1}s \in \mathcal{Z}(\mathcal{D}). \tag{4.33}$$

Moreover $\omega \in \mathcal{O}v \oplus \mathcal{O}y \oplus \mathcal{O}\zeta \oplus \mathcal{O}$. The three elements z, θ, ω are not algebraically independent, since

$$z\theta = \omega^2. \tag{4.34}$$

Theorem 4.10. *The center of $\mathcal{D}$ is the commutative subalgebra generated by z, ω and θ, which is isomorphic to the quotient $\Bbbk[X,Y,Z]/(XZ - Y^2)$.*

Proof. Clearly, $\mathcal{Z}(\mathcal{D}) = \mathcal{Z}(\mathcal{D}') \cap \mathcal{D}$. Since $\mathcal{Z}(\mathcal{D}') = \Bbbk[z^{\pm 1}, z'] = \Bbbk[z^{\pm 1}, \omega]$, we need to determine $\Bbbk[z^{\pm 1}, \omega] \cap \mathcal{D}$. Since $\Bbbk[z, \omega] \subset \mathcal{Z}(\mathcal{D})$, we have to consider the $\Bbbk$-linear combinations of monomials $z^{-i}\omega^j$ for positive i. For any integer $j \geq 0$, it follows from the relations in the iterated Ore extension (4.3) that ω^j is of the form $\omega^j = (g^{-j}q^j x^j)v^j + \ldots$ where the rest if of degree $\leq j-1$ in $\mathcal{D}$. We deduce that a power z^i with $i \geq 1$ divides ω^j in $\mathcal{D}$ if and only if z^i divides $g^{-j}q^j x^j$ in $\mathcal{O}$, that is if and only if $j \geq 2i$. Then a monomial $m = z^{-i}\omega^j$ with $i \geq 1, j \geq 0$ is in $\mathcal{D}$ if and only if $j \geq 2i$ and we have in this case $m = \theta^i \omega^{j-2i}$. This is sufficient to complete the proof. □

The (spectrum of) $\Bbbk[X,Y,Z]/(XZ - Y^2)$ is the well-known Kleinian surface of type A_1, i.e. the algebra of invariants of the polynomial ring $\Bbbk[x_1, x_2]$ under the action of the involution $x_1 \mapsto -x_1, x_2 \mapsto -x_2$.

4.4. The skew field of fractions of $\mathcal{D}$

Let R be a commutative $\Bbbk$-algebra which is a domain. With the notations of §4.1, the algebra $A_n(R)$ admits a skew field of fractions $\mathrm{Frac}(A_n(R)) =: D_n(K)$ where K is the field of fractions of R. In particular, let $\mathscr{D}_{n,s}(\Bbbk) = D_n(K)$ when K is a purely transcendental extension $\Bbbk(z_1, \ldots, z_s)$ of degree s. Following the seminal paper [GK], we say that a noncommutative $\Bbbk$-algebra A which is a noetherian domain satisfies the Gelfand-Kirillov property when its skew field of fractions $\operatorname{Frac} A$ is $\Bbbk$-isomorphic to a Weyl skew field $\mathscr{D}_{n,s}(\Bbbk)$ for some integers $n\geq 1, s\geq 0$.

The Jordan plane J satisfies the Gelfand-Kirillov property since $\operatorname{Frac} J\simeq \mathscr{D}_{1,0}(\Bbbk)=D_1(\Bbbk)$. This is also the case for the bosonizations $\mathcal{D}^{<0}\#\Bbbk\Gamma$ and $U(\mathfrak{h})\#\mathcal{D}^{>0}$ because we can prove by easy technical calculations that $\operatorname{Frac}(\mathcal{D}^{<0}\#\Bbbk\Gamma)\simeq \operatorname{Frac}(U(\mathfrak{h})\#\mathcal{D}^{>0})\simeq\mathscr{D}_{1,1}(\Bbbk)$. Finally, we conclude that the algebra $\mathcal{D}$ itself satisfies the Gelfand-Kirillov property.

Corollary 4.11. *The skew field of fractions of $\mathcal{D}$ is $\Bbbk$-isomorphic to the Weyl skew field $\mathscr{D}_{2,2}(\Bbbk)$.*

Proof. By Theorem 4.8, we have $\operatorname{Frac}\mathcal{D}=\operatorname{Frac} A_2'(S)=\operatorname{Frac} A_2(S)=D_2(K)$ of center $K=\Bbbk(z,z')$. $\square$

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