Ark. Mat., **60** (2022), 195–212
DOI: 10.4310/ARKIV.2022.v60.n1.a9

A quantitative Gauss-Lucas theorem

Vilmos Totik

Abstract. A conjecture of T. Richards is proven which yields a quantitative version of the classical Gauss-Lucas theorem: if K is a convex set, then for every $\varepsilon>0$ there is an $\alpha_\varepsilon<1$ such that if a polynomial P_n of degree at most n has $k\geq\alpha_\varepsilon n$ zeros in K, then P_n' has at least $k-1$ zeros in the ε-neighborhood of K. Estimates are given for the dependence of α_ε on ε.

1. Introduction and results

The Gauss-Lucas theorem states that if K is a convex subset of the complex plane and all zeros of a polynomial P_n of degree n lie in K, then the same is true for P_n', i.e. all critical points belong to K. This is no longer true if a single zero of P_n is allowed to lie outside K, for then it may happen that all critical points lie outside K (see e.g. the simple example in the beginning of [11]). It was Boris Shapiro who conjectured that in this latter case even though the critical points may lie outside K, most of them lie close to K, and he formulated the following as a conjecture.

The asymptotic Gauss-Lucas theorem [11]. *If $\varepsilon>0$ and most of the zeros of P_n (i.e. with the exception of $o(n)$ of the zeros) lie in K, then most of the zeros of P_n' lie in the ε-neighborhood K_ε of K.*

This suggests that perhaps it is also true that if for some α at least αn of the zeros lie in K, then at least $(1+o(1))\alpha n$ (or at least βn with some β depending on α) of the critical points also lie in K_ε (as has been mentioned, none may lie in K). But for $\alpha<1/2$ this fails dramatically.

Example. If $P_n(z)=z^n-1$, and K is the square of side-length 2 and with center at the point $1+\sin((1/2-\alpha)\pi/2)$, then K contains for large n at least αn of the

---

Key words and phrases: zeros, critical points, Gauss-Lucas theorem, Cauchy transform, potential theory.

2010 Mathematics Subject Classification: primary 26C10, 31A15.

zeros of P_n (which are the n-th roots of unity), but all the critical points are at the origin, so K_ε with $\varepsilon = \frac{1}{2}\sin((1/2-\alpha)\pi/2)$ does not contain a single critical point.

Still, the asymptotic Gauss-Lucas theorem suggests that this cannot happen when α is close to 1. In general, if $k \geq \alpha n$ of the zeros lie in K, how many critical points can be expected in K_ε? The following simple example shows that not more than $k-1$.

Example. Let K be the closed unit disk, $\varepsilon=1$ and $P_n(z)=z^k(2n-z)^{n-k}$. This $P_n(z)$ has k zeros in K and $k-1$ critical points in K_ε.

It is remarkable that for α sufficiently close to 1, the set K_ε contains this many critical points, as is shown by the following theorem that was conjectured by T. Richards [5], [6].

Theorem 1. *For any $\varepsilon>0$ there is an $\alpha_\varepsilon<1$ such that if a polynomial P_n of degree n has $k\geq\alpha_\varepsilon n$ zeros in K, then P_n' has at least $k-1$ zeros in K_ε.*

An immediate consequence of the theorem is the asymptotic Gauss-Lucas theorem stated above (although one should mention that the asymptotic Gauss-Lucas theorem is true not just for convex sets but also so-called polynomially convex sets, see [12, Corollary 1.9]).

The α_ε depends on ε and K, and in the next theorem we give quantitative bounds for it in terms of ε.

Theorem 2. *There is an absolute constant C_1 such that $\alpha_\varepsilon=1-C_1\varepsilon^2/\mathrm{diam}(K)^2$ suffices in Theorem 1 for all $\varepsilon\leq\mathrm{diam}(K)$. On the other hand, there is a C_2 (that depends on K) such that any α_ε necessarily satisfies $\alpha_\varepsilon\geq1-C_2\varepsilon$.*

Here $\mathrm{diam}(K)$ denotes the diameter of K. Note that the condition $\varepsilon\leq\mathrm{diam}(K)$ is a natural one in this question.

Remark. One could also consider numbers $\alpha_\varepsilon^*<1$ with the property that if a polynomial P_n of degree n has at least $k\geq\alpha_\varepsilon^* n$ zeros in K, then P_n' has at least $(1+o(1))k$ zeros in K_ε. Here $o(1)$ tends to 0 as $n\to\infty$. Clearly, one can choose $\alpha_\varepsilon^*=\alpha_\varepsilon$, so $\alpha_\varepsilon^*=1-C_1\varepsilon^2/\mathrm{diam}(K)^2$ suffices for this number by Theorem 2. On the other hand, the proof of Theorem 2 shows that any such α_ε^* necessarily satisfies $\alpha_\varepsilon^*\geq1-C_2\varepsilon$ provided K has non-empty interior.

Remark. A weaker version of Theorem 2 was proved in [6], where it was shown that the conclusion is true if P_n has at least $n(1-c_{\varepsilon,K}/\log n)$ zeros in K. The proof of Theorem 2 proceeds along similar ideas and verifies, in addition, a conjecture formulated in [6] that certain discrete Cauchy-transforms "cannot supercharge certain curves".

2. Proof of Theorem 1

For a positive Borel-measure μ of compact support on the complex plane let

$$C_\mu(z)=\int \frac{1}{t-z}d\mu(t)$$

be its Cauchy-transform. The proof of Theorem 2 is based on the following lemma.

Lemma 3. *If μ is a discrete measure of finite support, $\lambda>0$ and G is a connected component of the level set*

$$\Lambda_\lambda(\mu)=\{z||C_\mu(z)|>\lambda\},$$

then

$$\operatorname{diam}(G)\le 4\frac{\|\mu\|}{\lambda}, \tag{1}$$

where $\|\mu\|$ denotes the total mass of μ.

Note that the set $\Lambda_\lambda(\mu)$ is open, and so are its connected components.

The lemma proves in a quantitative form the conjecture from [6] mentioned above about "supercharging curves".

The formulation given in Lemma 3 is sufficient for our purposes, but there is a more general version, see Lemma 4 below.

Consider the special case when $\mu=\mu_N$ is the sum of N unit point masses, so that $\|\mu_N\|=N$. The lemma says that if A is large, then any component of the level set

$$\Lambda_{AN}=\{z||C_{\mu_N}(z)|>AN\}$$

has diameter $\le 4/A$, i.e. even the largest diameter tends to 0 (uniformly in N and μ_N) if $A\to\infty$. This should be compared to the fact that the set Λ_{AN} does not have to be small in some other metric sense. Indeed, the example given in [1, Theorem 2.2'] shows that for every N there is a μ_N (which is the sum of N unit masses) supported in the unit disk such that the projection of $\Lambda_{(\log N)^{1/2}N}$ onto the real line has linear measure $\ge c$, where $c>0$ is an absolute constant. Still, in this case the largest diameter of the connected components of $\Lambda_{(\log N)^{1/2}N}$ is at most $\le 4/(\log N)^{1/2}$ by Lemma 3.

Proof. Let $A,B\in G$ be two points in G, and let E be a broken line connecting A and B inside G. The conformal map Φ from $\overline{\mathbf{C}}\setminus E$ onto the exterior of the unit disk that leaves the point infinity invariant is of the form (around ∞)

$$\Phi(z)=\frac{z}{\operatorname{cap}(E)}+c_0+\frac{c_{-1}}{z}+\dots, \tag{2}$$

where cap denotes logarithmic capacity. If Ω is the unbounded component of $\overline{\mathbf{C}}\setminus\overline{\Lambda_\lambda(\mu)}$, then the maximum modulus theorem applied to the function $(1/\Phi(z))/(C_\mu(z)/\lambda)$, which is analytic in Ω, gives that this function is at most 1 in absolute value in Ω, therefore

$$\mathrm{cap}(E)=\lim_{z\to\infty}\frac{|z|}{|\Phi(z)|}\le\lim_{z\to\infty}|zC_\mu(z)|/\lambda=\frac{\|\mu\|}{\lambda}.$$

For a continuum E we have (see Theorem 5.3.2,(a) in [4])

$$\frac{1}{4}\mathrm{diam}(E)\le\mathrm{cap}(E),$$

so we obtain

$$\mathrm{diam}(E)\le 4\frac{\|\mu\|}{\lambda}. \tag{3}$$

Since this is true for any two points A,B of G, the lemma follows. □

Let us point out what is behind the preceding lemma. For a positive Borel-measure μ of compact support on the complex plane let

$$C_\mu^*(z)=\sup_{\varepsilon>0}\left|\int_{|t-z|\ge\varepsilon}\frac{1}{t-z}d\mu(t)\right|$$

be the maximal Cauchy-transform. The following extension of Lemma 3 follows from some classical results of X. Tolsa on analytic capacity.

Lemma 4. *Let μ be a positive measure of compact support. If $\lambda>0$ and G is a connected component of the level set*

$$\Lambda_\lambda^*(\mu)=\left\{z\,\middle|\,C_\mu^*(z)>\lambda\right\},$$

then

$$\mathrm{diam}(G)\le C\frac{\|\mu\|}{\lambda}, \tag{4}$$

where $\|\mu\|$ denotes the total mass of μ, and C is an absolute constant.

Note that the set $\Lambda_\lambda^*(\mu)$ is open, hence so are its connected components.

To prove (4) we need the concept of analytic capacity of a set E. Actually, there are two notions of analytic capacity in the literature denoted by $\gamma(E)$ and $\gamma_+(E)$, but by the fundamental theorem of X. Tolsa [9, (1.1) and Theorem 1.1] they are of the same size: $\gamma(E)\approx\gamma_+(E)$, so in what follows we shall only work with $\gamma(E)$. If E is a compact set, then $\gamma(E)$ is defined as the supremum

$$\gamma(E)=\sup_f |f'(\infty)|,$$

where the supremum is taken for all functions f that are analytic in the unbounded component of $\overline{\mathbf{C}}\setminus E$ and $|f(z)|\le 1$ there. Note also that

$$f'(\infty):=\lim_{z\to\infty} z(f(z)-f(\infty)).$$

The analytic capacity of a Borel-set E is then defined as the supremum of the analytic capacities of all compact sets lying in E.

Consider, for example, a continuum (connected compact set) E that has at least two points. The conformal map from the unbounded component Ω of $\overline{\mathbf{C}}\setminus E$ onto the exterior of the unit disk is of the form (2). Therefore, setting $f(z)=1/\Phi(z)$ as a test function in the definition of $\gamma(E)$ we obtain

$$\gamma(E)\ge \operatorname{cap}(E).$$

There is also a converse inequality, namely if f is as in the definition of $\gamma(E)$, then $((f(z)-f(\infty))/2)\Phi(z)$ is of modulus ≤ 1 in Ω by the maximal principle, and hence

$$|f'(\infty)|\le 2\lim_{z\to\infty} z/\Phi(z)=2\operatorname{cap}(E),$$

giving $\gamma(E)\le 2\operatorname{cap}(E)$. Since for a continuum E we have (see Theorems 5.3.2,(a) and 5.3.4 in [4])

$$\frac{1}{4}\operatorname{diam}(E)\le \operatorname{cap}(E)\le \frac{1}{2}\operatorname{diam}(E),$$

we obtain as before

$$\operatorname{diam}(E)\le 4\gamma(E). \tag{5}$$

The reverse inequality $\gamma(E)\le\operatorname{diam}(E)$ also follows from the just given discussion, and in view of $\gamma(E)\approx\gamma_+(E)$ this yields $\gamma_+(E)\approx\operatorname{diam}(E)$, which is attributed in [9] to P. Jones.

The relevance of all these to Lemma 4 is that by [9, Theorem 1] and [10, Proposition 2.1]

$$\gamma(\Lambda_\lambda^*(\mu))\le D\frac{\|\mu\|}{\lambda} \tag{6}$$

with some absolute constant D. Hence, if G is a component of $\Lambda_\lambda^*(\mu)$, $A,B\in G$ are any two points and E is a broken line connecting A and B in G as in the proof of Lemma 3, then applying (5) and (6) we obtain Lemma 4.

Proof of Theorem 1. The proof easily follows from Lemma 3 and from Rouché's theorem (cf. [5], [6]). Since we need a quantitative estimate in Theorem 2, we give some details.

We may assume $\varepsilon < \mathrm{diam}(K)/100$.

Let $P_n(z) = \prod_{j=1}^n (z - z_j)$, and assume that $k \geq n/2$ of the zeros, say $z_1, ..., z_k$, lie in K. For simpler pole and zero counting we assume that the z_j's are different — the general case follows from here by taking limits. We set

$$\mu_1 = \sum_{j=1}^k \delta_{z_j}, \qquad \mu_2 = \sum_{j=k+1}^n \delta_{z_j}, \qquad \mu = \mu_1 + \mu_2,$$

where δ_z denotes the Dirac mass at z.

The relevance of the Cauchy transform to our theorem is that

$$-C_\mu(z) = \sum_{j=1}^n \frac{1}{z - z_j} = \frac{P_n'(z)}{P_n(z)}.$$

In particular, the poles of C_μ are the zeros of P_n, and a zeros of C_μ are the zeros of P_n'.

Instead of ε we shall prove the result for 3ε. Let ∂K_ε be the boundary of the set K_ε. First we need that for $z \in K_{3\varepsilon} \setminus K_\varepsilon$, $\varepsilon \leq \mathrm{diam}(K)$, the inequality

$$|C_{\mu_1}(z)| \geq c_1 n \varepsilon, \tag{7}$$

holds, where c_1 depends only on the diameter of K. Indeed, let $z \in K_{3\varepsilon} \setminus K_\varepsilon$, and let w be the closest point to z from K. Let ℓ be the line that passes through w and is perpendicular to the segment zw. Since the open disk about z and of radius $|w - z|$ cannot contain a point of K, it follows that K must lie on different side of ℓ than z. Without loss of generality we may assume that ℓ is the imaginary axis, z belongs to the negative half of the real axis, and K lies in the half-plane $\Re z \geq 0$. Then for all $z_j \in K$ we have $\Re(z_j - z) \geq \varepsilon$, and hence

$$\Re \frac{1}{z_j - z} = \frac{\Re(z_j - z)}{|z_j - z|^2} \geq \frac{\varepsilon}{(3\varepsilon + \mathrm{diam}(K))^2} \geq \varepsilon \frac{1}{4\mathrm{diam}(K)^2}, \qquad 1 \leq j \leq k,$$

and (7) follows with $c_1 = 1/8\mathrm{diam}(K)^2$ since $k \geq n/2$.

Now assume that

$$n - k \leq \frac{\varepsilon^2 c_1}{4 \cdot 5} n, \tag{8}$$

which, for $\varepsilon \le \mathrm{diam}(K)$, also implies the $k \ge n/2$ assumption used above. By Lemma 3 any connected component G of the set

$$\Lambda = \Lambda_{c_1 n\varepsilon/2}(\mu_2) = \left\{ z \middle| |C_{\mu_2}(z)| > \frac{1}{2} c_1 n\varepsilon \right\}$$

satisfies

$$\mathrm{diam}(G) \le 4\frac{n-k}{c_1 n\varepsilon/2} < \varepsilon/2. \tag{9}$$

Thus, if such a component intersects $\partial K_{2\varepsilon}$, then it lies inside the set $K_{3\varepsilon} \setminus K_\varepsilon$.

Choose now an oriented Jordan curve (i.e. a homeomorphic image of the unit circle) Γ in $K_{3\varepsilon} \setminus K_\varepsilon$ that avoids the set Λ and that circles K once in the counterclockwise direction. The existence of Γ follows from the fact that each component of Λ has diameter $<\varepsilon/2$. We shall give a rigorous proof for the existence, but first let us finish the proof of Theorem 1. Thus, on Γ we have $|C_{\mu_2}(z)| \le c_1 n\varepsilon/2$, which is smaller than the absolute value $|C_{\mu_1}(z)| \ge c_1 n\varepsilon$ established above. Thus, by Rouché's theorem, the difference

$$\Delta = (\text{number of zeros inside } \Gamma - \text{number of poles inside } \Gamma)$$

is the same for $C_{\mu_1}(z)$ and for $C_{\mu_1}(z) + C_{\mu_2}(z) = C_\mu(z)$. By the Gauss-Lucas theorem this difference is -1 for $C_{\mu_1}(z)$ (all poles and zeros of $(\prod_1^k (z-z_j))' / (\prod_1^k (z-z_j))$ lie in K), hence this difference is again -1 for $C_\mu(z)$. By the assumption of the theorem the number of poles of C_μ inside Γ is at least k, therefore C_μ, and hence also $P_n'(z)$, has at least $k-1$ zeros inside Γ. Since Γ lies inside $K_{3\varepsilon}$, it follows that P_n' has at least $k-1$ zeros inside $K_{3\varepsilon}$, and that completes the proof of the theorem.

The existence of Γ is intuitively clear, but for completeness we give a rigorous proof. To do that, define the polynomial convex hull $\mathrm{Pc}(S)$ for a compact $S \subset \mathbf{C}$ as the complement $\mathbf{C} \setminus \Omega$ of the unbounded component Ω of the complement $\mathbf{C} \setminus S$ of S. This is nothing else than the union of S with the bounded components of $\mathbf{C} \setminus S$. The boundary of the polynomial convex hull is called the outer boundary of S and is denoted by $\partial_{\mathrm{out}} S$. Clearly, $\partial_{\mathrm{out}} S = \partial \Omega$.

We may assume without loss of generality that $n\varepsilon/2$ is not a critical value of C_{μ_2} i.e. $C'_{\mu_2}(z) \ne 0$ on the set

$$\left\{ z \middle| |C_{\mu_2}(z)| = \frac{c_1}{2} n\varepsilon \right\}$$

(if this is not the case, just decrease ε by a tiny amount — note that C_{μ_2} has only finitely many critical values). But then every component G of Λ is bounded by a finite number of disjoint analytic Jordan curves, and so the outer boundary $\partial_{\mathrm{out}} G$ of G is also an analytic Jordan curve.

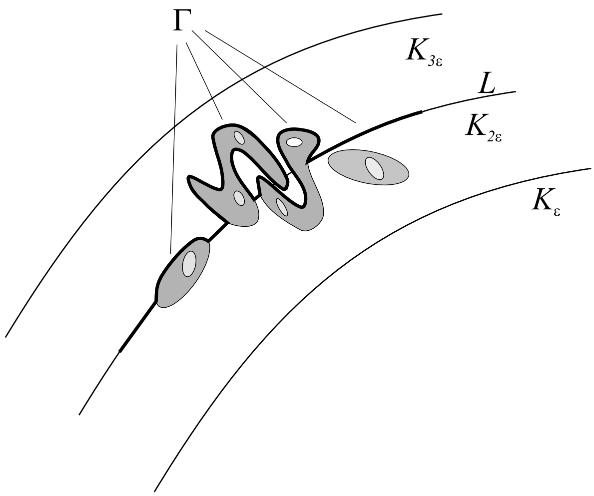

Figure 1. The sets $K_\varepsilon, K_{2\varepsilon}$ and $K_{3\varepsilon}$ and their boundaries (in particular, $L{=}\partial K_{2\varepsilon}$), some components of the level set $\Lambda_{c_1\varepsilon n/2}$ (shaded regions) and a possible path for Γ (the thick path).

For simpler notation we set $L{=}\partial K_{2\varepsilon}$ with its counterclockwise orientation.

We define Γ so that

• Γ consists of parts of L and parts of the outer boundaries of some components of Λ,

• Γ circles K once in the counterclockwise direction (i.e. the index of any point $z{\in}K$ with respect to Γ is 1),

• Γ does not have a point common with the interior of Λ, and

• Γ lies in $K_{3\varepsilon}\setminus K_\varepsilon$.

See Figure 1.

Let $G_1, ..., G_n$ be those connected components of Λ which intersect L (if there are no such components, then $L{=}\partial K_{2\varepsilon}$ oriented counterclockwise is suitable for Γ). Note that for two such G_j the polynomially convex hulls $\mathrm{Pc}(\overline{G_j})$ of their closures are either disjoint or one of them is part of the other one. Discard those G_j for which $\mathrm{Pc}(\overline{G_j})$ is part of some other $\mathrm{Pc}(\overline{G_k})$, and we may assume that G_j, $1{\le}j{\le}m$, are those components that remain. Then $\mathrm{Pc}(\overline{G_j})$, $1{\le}j{\le}m$, are disjoint, they have diameter $<\varepsilon/2$ (see (9)), and $L{\cap}\Lambda$ is part of $\cup_{j=1}^m \mathrm{Pc}(\overline{G_j})$. As has been said, $\partial_{\mathrm{out}}\overline{G_j}$ are analytic Jordan curves.

For each $j{=}0, 1, ..., m$ we shall construct an oriented Jordan curve Γ_j with the properties: either $\Gamma_j{=}L$, or Γ_j has the following structure. There are subarcs $\overline{A_tB_t}\big|_L$, $1{\le}t{\le}r{=}r_j$, of L in the counterclockwise orientation of L, so that their numbering reflects counterclockwise orientation, i.e. $A_1B_1A_2B_2...A_rB_rA_{r+1}B_{r+1}...$

follow each other in the counterclockwise orientation on L, where the indices are considered mod r (i.e. $A_{r+1}=A_1$). Each arc $\overline{A_tB_t}\Big|_L$ lies outside (the interior of) $\cup_{s=1}^{j}\mathrm{Pc}(\overline{G_s})$, and the curve Γ_j consists of these arcs as well as for each t of a subarc of some $\partial_{\mathrm{out}}\overline{G_{k_t}}$ that connects B_t and A_{t+1}, i.e. each arc $\overline{B_tA_{t+1}}\Big|_{\Gamma_j}$ of Γ_j is a subarc of the outer boundary of some $\overline{G_{k_t}}$, where the k_t's are different for different t's. The orientation of the arcs $\overline{A_tB_t}\Big|_L=\overline{A_tB_t}\Big|_{\Gamma_j}$ on Γ_j coincides with their (counterclockwise) orientation on L, while each $\overline{B_tA_{t+1}}\Big|_{\Gamma_j}$ is oriented from B_t to A_{t+1}. In other words, the structure of Γ_j is as follows: an arc of L is followed by an arc of some $\partial_{\mathrm{out}}\overline{G_k}$, followed by another arc of L followed by an arc of some other $\partial_{\mathrm{out}}\overline{G_{k'}}$ etc., and the arcs of L follow each other in the same order on Γ_j (in the orientation of the latter) as on L (in its counterclockwise orientation).

These Γ_j will have the properties:

1) Γ_j consists of parts of L and parts of the outer boundaries of $\overline{G_1},...,\overline{G_j}$,

2) Γ_j circles K once in the counterclockwise orientation,

3) Γ_j does not have a point common with the interior of $\mathrm{Pc}(\overline{G_1}),...,\mathrm{Pc}(\overline{G_j})$, and

4) Γ_j lies in $K_{3\varepsilon}\setminus K_\varepsilon$.

Then clearly, $\Gamma=\Gamma_m$ will satisfy all the requirements.

To do this, first choose points $X_1,...,X_M\in L\setminus\cup_{j=1}^{m}\mathrm{Pc}(\overline{G_j})$ in the counterclockwise direction on L such that the length $\ell(\overline{X_sX_{s+1}}\Big|_L)$ of the oriented arc $\overline{X_sX_{s+1}}\Big|_L$ of L from X_s to X_{s+1} satisfies $2\varepsilon\le\ell(\overline{X_sX_{s+1}}\Big|_L)<6\varepsilon$ for all $s=1,...M$, where we take the indices mod M (i.e. $X_{M+1}=X_1$). Indeed, let $X_1\in L\setminus\cup_j\mathrm{Pc}(\overline{G_j})$ be arbitrary, and then consider the points $P,Q\in L$ such that $\ell(\overline{X_1X_P}\Big|_L)=2\varepsilon$ and $\ell(\overline{X_PX_Q}\Big|_L)=2\varepsilon$, and X_1,P,Q follow each other in this order on L. Since the diameter of a convex arc is at least as large as $1/\pi$-times its length, ([1]) (see [8] or [2, Sec. 44, (5)]) it follows that $\overline{X_PX_Q}\Big|_L$ has diameter $>\varepsilon/2$. Therefore, this arc cannot lie entirely in a $\mathrm{Pc}(\overline{G_j})$ because these latter have diameter smaller than $\varepsilon/2$. But then $\overline{X_PX_Q}\Big|_L\subset\cup_j\mathrm{Pc}(\overline{G_j})$ is impossible, for then $\overline{X_PX_Q}\Big|_L$ would be the union of more than one of its non-empty disjoint closed subsets (namely of those $\overline{X_PX_Q}\Big|_L\cap\mathrm{Pc}(\overline{G_j})$ that are not empty) which is impossible since $\overline{X_PX_Q}\Big|_L$ is connected. As a consequence, there is an $X_2\in L\Big|_{PQ}\setminus\cup_j\mathrm{Pc}(\overline{G_j})$ giving the choice of X_2. Now do the same

([1]) This is usually stated for closed curves, but the arc-case then follows by simply connecting the two endpoints of the arc by a segment.

construction starting from X_2 to get X_3, then from X_3 to get X_4, etc. until we get to an X_M for which $\ell(\overline{X_MX_1}\big|_L)<6\varepsilon$ from X_1.

After this we turn to the construction of the Jordan curves Γ_j for all j. Let $\Gamma_0=L$ oriented counterclockwise, and suppose that for some $0\le j<m$ the Γ_j has already been constructed. If $\Gamma_j\cap G_{j+1}=\emptyset$ (which is equivalent to the fact that $\Gamma_j\cap\mathrm{Pc}(\overline{G_{j+1}})$ can contain only boundary points of $\mathrm{Pc}(\overline{G_{j+1}})$), then set $\Gamma_{j+1}=\Gamma_j$. If this is not the case, then let A_0 be a point in $\Gamma_j\cap G_{j+1}$. Note that since different $\mathrm{Pc}(\overline{G_k})$ are disjoint, every point of $\Gamma_j\cap\mathrm{Pc}(\overline{G_{j+1}})$ lies in one of the arcs $\overline{A_tB_t}\big|_L$. So this A_0 lies in one of the arcs $\overline{X_sX_{s+1}}\big|_L$ of L. Then, by the construction of the points X_k and by $\mathrm{diam}(G_{j+1})<\varepsilon/2$, the intersection $\Gamma_j\cap\mathrm{Pc}(\overline{G_{j+1}})$ is part of the arc $\overline{X_{s-1}X_{s+2}}\big|_L$ of L. Now let A and B be the first and last points in the orientation of Γ_j that lie in $\mathrm{Pc}(\overline{G_{j+1}})$ (i.e. $A,B\in\Gamma_j\cap\mathrm{Pc}(\overline{G_{j+1}})$, AA_0B follow each other on Γ_j in this order, and the arc $\overline{BA}\big|_{\Gamma_j}$ of Γ_j from B to A does not intersect $\mathrm{Pc}(\overline{G_{j+1}})$ except for its endpoints B,A). Then $A\ne B$ (since $A_0\in G_{j+1}$ and G_{j+1} is open), A and B lie on the outer boundary $\partial_{\mathrm{out}}\overline{G_{j+1}}$ of $\overline{G_{j+1}}$, and since this outer boundary is a Jordan curve, there is a Jordan arc J on that boundary that connects A and B (actually, there are two such arcs, it does not matter which one we choose). Orient J so that J is an arc from A to B, see Figure 2. The points A and B also lie on Γ_j, and they divide Γ_j into two Jordan arcs J_1 and J_2, say J_1 is the arc from A to B (in the orientation inherited from Γ_j). Replace now the arc $J_1=\overline{AB}\big|_{\Gamma_j}$ on Γ_j from A to B by J to get the Jordan-curve Γ_{j+1} (note that J does not intersect the other arc $J_2=\overline{BA}\big|_{\Gamma_j}$ of Γ_j because of the definition of the points A and B, so $J\cup J_2$ is, indeed, a Jordan-curve). It is clear that this Γ_{j+1} has the structure described above. Properties 1), 3) and 4) are obvious for Γ_{j+1} from the induction hypothesis and from the fact that J lies in the $\varepsilon/2$-neighborhood of the arc $\overline{X_{s-1}X_{s+2}}\big|_L$ of L (recall that the points A and B belong to L).

As for property 2), note first of all that J_1 consists of subarcs of L and of some subarcs $\mathcal{J}_k$ of some $\partial_{\mathrm{out}}\overline{G_k}$'s. Each of the latter ones connect some two points C_k,D_k of L that lie in between A and B in the counterclockwise orientation on L. If $\Delta_\varepsilon(C_k)$ is the disk of radius ε about C_k, then $\mathcal{J}_k\subset\partial_{\mathrm{out}}\overline{G_k}\subset\Delta_\varepsilon(C_k)$ (recall that G_k has diameter $<\varepsilon/2$ and $C_k\in\overline{G_k}$), so the arcs $\mathcal{J}_k$ and $\overline{C_kD_k}\big|_L$ can be continuously deformed into each other within $\Delta_\varepsilon(C_k)$. Since $\Delta_\varepsilon(C_k)$ is also part of the ε-neighborhood of $\overline{AB}\big|_L$, we obtain that J_1 can be continuously deformed into $\overline{AB}\big|_L$ within the ε-neighborhood of $\overline{AB}\big|_L$. Clearly the same is true for J and $\overline{AB}\big|_L$ (for the same reason), hence Γ_j and Γ_{j+1} can be continuously deformed into

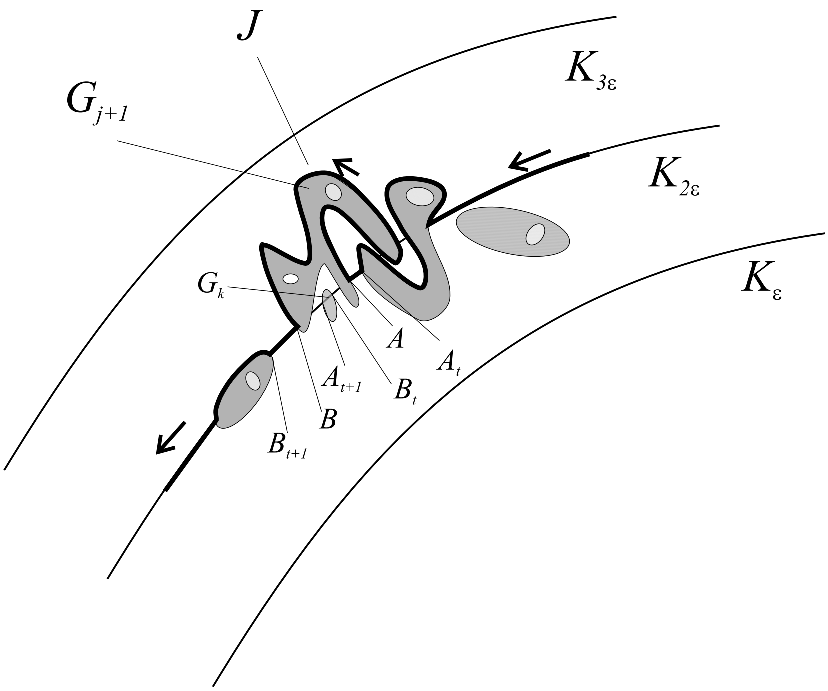

Figure 2. The points A and B and the arcs J and J_1 in the definition of Γ_{j+1}. In the figure we assume that $1 \le k \le j$, and then J_1 consists of the arc $\overline{AB_t}\big|_L$ of L, one of the subarcs of the boundary of G_k that connects B_t with A_{t+1}, and from the arc $\overline{A_{t+1}B}\big|_L$. When defining Γ_{j+1} from Γ_j, these three arcs are replaced by the single arc J connecting A and B on the boundary of G_{j+1} (as has been said, there are two choices for J, in the figure we chose the longer one). Note also if we had $k > j$ in the figure, then J_1 would be simply the arc of L from A to B.

each other in $K_{3\varepsilon} \setminus K_\varepsilon$. Since Γ_j circles K once in the counterclockwise direction by the induction hypothesis, the same is true of Γ_{j+1}, proving 2). □

3. Proof of Theorem 2

The first part follows from the just given proof for Theorem 1. Indeed, we have seen that if $\varepsilon \le \mathrm{diam}(K)/100$ and (see (8))

$$n - k \le \frac{1}{4 \cdot 5 \cdot 8} \frac{\varepsilon^2}{\mathrm{diam}(K)^2},$$

then $k-1$ critical points are guaranteed in $K_{3\varepsilon}$, so

$$C_1 = \frac{1}{9 \cdot 4 \cdot 5 \cdot 8}$$

suffices in Theorem 2 for such 3ε. Now to cover the range $3\mathrm{diam}(K)/100 \le 3\varepsilon \le \mathrm{diam}(K)$, just divide this C_1 by $100^2/3^2$.

We shall prove the second part first for a square of side-length 2.

We shall use some basic notions and results from logarithmic potential theory (see for example the books [3], [4] and [7]), among others the notion of equilibrium measure and of balayage (for the latter see [7, Sec. II.4] or [3, Ch. IV]). In particular, we shall use that if $R_N(z)$ is a polynomial of degree N with leading coefficient 1, μ is the normalized counting measure on its zeros, then the equilibrium measure of a level set $L=\{z \| R_N(z)|=\tau\}$ is the balayage $\hat{\nu}$ of ν out of the bounded components of $\mathbf{C}\setminus L$ (i.e. "onto" L). Indeed, on L the logarithmic potential

$$U^{\hat{\mu}}(z) = \int \log \frac{1}{|z-t|} d\hat{\mu}(t)$$

coincides with

$$U^{\mu}(z) = \int \log \frac{1}{|z-t|} d\mu(t) = \frac{1}{N} \log \frac{1}{|R_N(z)|} = \frac{1}{N} \log \frac{1}{\tau}$$

(the logarithmic potential does now change on L when forming balayage out of the components of $\mathbf{C}\setminus L$), i.e. it is constant on L, and that characterizes equilibrium measures among unit measures on L.

For an integer $s \ge 2$ set

$$R(z) = z^s(1-z).$$

This has $s-1$ critical points at 0 and one critical point at $s/(s+1)$. Let

$$\rho_0 = \left(\frac{s}{s+1}\right)^s \frac{1}{s+1}$$

be the value of R at the critical point $s/(s+1)$. Then the level set

$$L_{\rho_0} := \{z \, | \, |R(z)| = \rho_0\}$$

passes through the point $s/(s+1)$ and consists of two loops, say ℓ_0 around 0 and ℓ_1 around 1, that meet at the point $s/(s+1)$ (see Figure 3). If we set

$$\mu_0 = \frac{s}{s+1}\delta_0 + \frac{1}{s+1}\delta_1$$

(the normalized zero counting measure of the zeros of R), then, as has been mentioned, the equilibrium measure $\omega_{L_{\rho_0}}$ of L_{ρ_0} is the balayage of μ_0 out of the two bounded domains encircled by L_{ρ_0}. During this balayage process $(s/(s+1))\delta_0$ is moved entirely to ℓ_0, and $(1/(s+1))\delta_1$ is moved entirely to ℓ_1, hence

$$\omega_{L_{\rho_0}}(\ell_0) = \frac{s}{s+1}, \qquad \omega_{L_{\rho_0}}(\ell_1) = \frac{1}{s+1}.$$

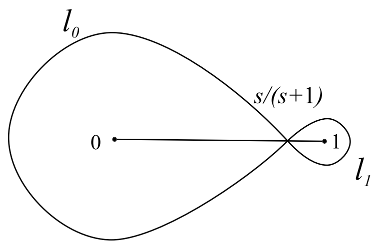

Figure 3. The level set L_{ρ_0} and its two loops.

Consider now the square with center at the origin and of side-length 2, and shift it horizontally so that its right-hand side passes through the point $1-2/3(s+1)$. This will be our set K. If we also set $\varepsilon=1/3(s+1)$, then the "right-hand side" K_ε of K_ε passes through the point $1-1/3(s+1)$ (see Figure 4). The point is that the loop ℓ_0 lies inside K, but K also contains some part of ℓ_1, hence

$$\omega_{L_{\rho_0}}(K)=(1+3\tau)\frac{s}{s+1}$$

with some $\tau>0$. But then for some $\rho^*>\rho_0$ we shall have

$$\omega_{L_{\rho^*}}(K)\geq(1+2\tau)\frac{s}{s+1} \tag{10}$$

as well (note that, as $\rho^*\searrow\rho_0$, $\omega_{L_{\rho^*}}$ converges in the weak* topology to $\omega_{L_{\rho_0}}$). We fix this ρ^*. For it the level set L_{ρ^*} is an analytic Jordan curve (this is the case for all the level sets L_ρ with $\rho>\rho_0$).

In what follows we need the following lemma for the integrals of R^n with large n.

Lemma 5. *If $z\in L_\rho$ with some $\rho>\rho_0$, then*

$$\int_{s/(s+1)}^{z} R^n(u)du=(1+o(1))\frac{R^{n+1}(z)}{(n+1)R'(z)}, \tag{11}$$

where $o(1)$ tends uniformly to 0 in $z\in L_\rho$ (with any fixed $\rho>\rho_0$) as $n\to\infty$.

Taking this lemma for granted for the time being, we continue the proof, and set

$$S_n(z)=(n(s+1)+1)\int_{s/(s+1)}^{z} R^n(u)du-(\rho^*)^n,$$

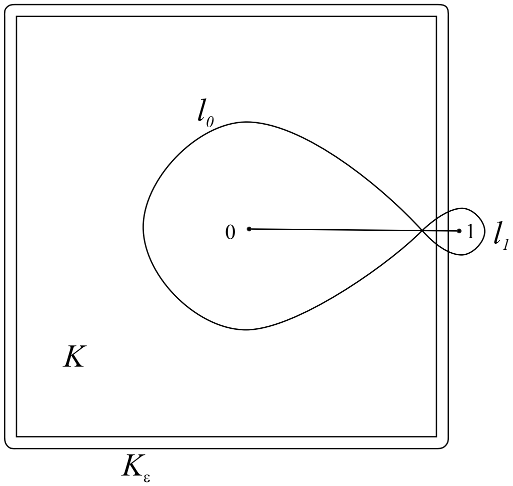

Figure 4. The position of the sets K and K_ε.

which is a polynomial of degree $n(s+1)+1$ with leading coefficient 1. We claim that if $\rho_0<\rho_1<\rho^*<\rho_2$, then for large n all the zeros of S_n lie in the strip in between the level sets L_{ρ_1} and L_{ρ_2}. Indeed, in view of (11) for $z\in L_{\rho_1}$ we have

$$S_n(z)=O\left(\rho_1^{n+1}\right)-(\rho^*)^n,$$

so S_n has no zero inside L_{ρ_1} by Rouché's theorem (if n is sufficiently large). On the other hand, if $z\in L_{\rho_2}$, then again (11) gives that

$$(n(s+1)+1)\left|\int_{s/(s+1)}^{z} R^n(u)du\right|=(1+o(1))(s+1)\left|\frac{R^{n+1}(z)}{R'(z)}\right|>c\rho_2^{n+1}$$

with some $c>0$ (that is uniform in $z\in L_{\rho_2}$), hence, by Rouché's theorem, for both $S_n(z)$ and $S_n(z)+(\rho^*)^n$ the number of zeros inside L_{ρ_2} is the same as the number

$$D=(\text{number of zeros inside } L_{\rho_2}-\text{number of poles inside } L_{\rho_2})$$

for the function

$$(n(s+1)+1)\frac{R^{n+1}(z)}{(n+1)R'(z)},$$

which is clearly $(n+1)(s+1)-s=n(s+1)+1$. Thus, all zeros of S_n lie inside L_{ρ_2}, and the claim follows.

Let

$$S_n(z)=\prod_{j=1}^{n(s+1)+1}(z-w_{j,n}),$$

and consider the zero counting measure

$$\nu_n = \frac{1}{n(s+1)+1} \sum_{j=1}^{n(s+1)+1} \delta_{w_{j,n}}.$$

We claim that these converge in the weak* topology to the equilibrium measure $\omega_{L_{\rho^*}}$, and to do that it is enough to show that if ν is a weak* limit of $\{\nu_n\}$, say $\nu_n \to \nu$ as $n\to\infty$, $n\in\mathcal{N}$, then $\nu=\omega_{L_\rho^*}$. We have just shown that ν is supported on L_{ρ^*}. Furthermore, if z lies outside L_{ρ^*}, then from (11) and from what we have just shown about the location of the zeros of S_n, it follows that

$$\begin{aligned}\int \log \frac{1}{|z-t|} d\nu(t) &= \lim_{n\to\infty,\ n\in\mathcal{N}} \int \log \frac{1}{|z-t|} d\nu_n(t) \\ &= \lim_{n\to\infty,\ n\in\mathcal{N}} \frac{1}{n(s+1)+1} \log \frac{1}{|S_n(z)|} = \frac{1}{(s+1)} \log \frac{1}{|R(z)|}.\end{aligned}$$

However, the right-hand side is the same as

$$\int \log \frac{1}{|z-t|} d\mu_0(t) = \int \log \frac{1}{|z-t|} d\omega_{L_{\rho^*}}(t),$$

where, in the last step, we used that the equilibrium measure of the level set L_{ρ^*} is the balayage of the measure μ_0 from the inner domain of L_{ρ^*}, hence its logarithmic potential outside L_{ρ^*} coincides with the logarithmic potential of μ_0. Thus, the logarithmic potentials of the measures ν and $\omega_{L_{\rho^*}}$, both of which are supported on L_{ρ^*}, coincide outside L_{ρ^*}, and the equality $\nu=\omega_{L_{\rho^*}}$ follows from Carleson's unicity theorem [7, Theorem II.4.13].

Now in view of (10) and of the convergence $\nu_n \to \omega_{L_\rho^*}$ in the weak* topology, for all large n the polynomial $S_n(z)$ of degree $n(s+1)+1$ has at least

$$(1+\tau)\frac{s}{s+1}(n(s+1)+1) \geq (1+\tau)sn$$

zeros in K, but $S_n'(z)=(n(s+1)+1)R^n(z)$ has only sn zeros (the ones at the origin) in K_ε. This shows that the number $\alpha_\varepsilon=\alpha_{1/3(s+1)}$ for K must be greater than

$$\frac{(1+\tau)sn}{n(s+1)+1} > \frac{s}{s+1} = 1 - \frac{3}{3(s+1)} = 1-3\varepsilon.$$

Finally, for ε not of the form $1/3(s+1)$ select the largest s for which $\varepsilon<1/3(s+1)$.

This proves the second part of Theorem 2 for a square of side-length 2.

If K is a convex set with non-empty interior, then the argument is the same. Clearly, it is sufficient to prove the claim for a homothetic copy of K. Now take a disk inside K and translate it so that we obtain a disk D which still lies in K, but contains a boundary point M. By scaling, rotating and translating we may achieve that M is the point $1-2/3(s+1)$, the tangent line to K at M is vertical and D is sufficiently large and lies to the left of that tangent line (which is necessarily the tangent line to D, as well). Now we are in the position that we can use the just given proof (which was for squares) using the same function R as before (in this situation ℓ_0 lies again in K).

Finally, if K has empty interior, then it is a segment, say $K=[-1,1]$. For an $s\geq 1$ set

$$S_n(z)=(z^2-1)^{sn}(z-i)^n,$$

which has $k=2sn$ zeros in $[-1,1]$, and

$$S_n'(z)=n(z^2-1)^{sn-1}(z-i)^{n-1}\left(s(z-i)2z+(z^2-1)\right)$$

has $2sn-2$ zeros in $[-1,1]$, $n-1$ zeros at i and two other zeros lying outside $[-1,1]$, the closest of which to $[-1,1]$ is

$$\frac{i}{s+\sqrt{s^2-2s-1}},$$

which is of distance $1/(s+\sqrt{s^2-2s-1})>1/2s$ from $[-1,1]$. Thus, if $\varepsilon=1/2s$, then S_n has at most $k-2$ critical points in K_ε, therefore $\alpha_{1/2s}$ must be bigger than $2sn/(2sn+n)=2s/(2s+1)$, which proves the second part of Theorem 2 for the segment $K=[-1,1]$.

We still need to prove Lemma 5.

Proof of Lemma 5. Let $\rho>\rho_0$ be fixed, $z\in L_\rho$, and select $\rho_0<\rho_1<\rho$ close to ρ. The mapping $\xi\to R(\xi)$ maps L_ρ into the circle $C_\rho=\{w \| w|=\rho\}$, and L_{ρ_1} into the circle C_{ρ_1} in a $(s+1)$–to–1 fashion. We may assume that $R(z)=\rho$ (if this is not the case then just multiply R by a suitable number θ of modulus 1 to achieve that and then divide the integral by θ^n). The inverse image of the segment $[\rho_1,\rho]$ under this mapping consists of $(s+1)$ Jordan arcs, one of which, say J, has z as one of its endpoints. Let z_1 be the other endpoint of J. Then $z_1\in L_{\rho_1}$. If the path of the integration lies in the inner domain of L_{ρ_1}, then it is immediate that

$$\int_{s/(s+1)}^{z_1} R^n(u)du=O(\rho_1^n).$$

We also have

$$\frac{1}{R'(z)}\int_{z_1}^{z} R^n(u)R'(u)du=\frac{R^{n+1}(z)}{(n+1)R'(z)}-\frac{R^{n+1}(z_1)}{(n+1)R'(z_1)}=\frac{R^{n+1}(z)}{(n+1)R'(z)}-O(\rho_1^n).$$

Thus, it is left to show that

$$\int_J R^n(u)\left(1-\frac{R'(u)}{R'(z)}\right)du = O\left(\frac{\rho^n}{n^2}\right), \tag{12}$$

because the right-hand side is

$$o\left(\left|\frac{R^{n+1}(z)}{(n+1)R'(z)}\right|\right).$$

If we make the substitution $t=R(u)$ in the integral on the left of (12), the integral becomes

$$\int_{\rho_1}^{\rho} t^n\left(1-\frac{R'(R^{-1}(t))}{R'(R^{-1}(\rho))}\right)\frac{1}{R'(R^{-1}(t))}dt$$

with some local branch of R^{-1}, which, in view of

$$\left|\frac{1}{R'(R^{-1}(t))}-\frac{1}{R'(R^{-1}(\rho))}\right|\leq C|t-\rho|,$$

is in absolute value at most

$$C\int_{\rho_1}^{\rho} t^n(\rho-t)dt \leq C\int_0^{\rho} t^n(\rho-t)dt = C\frac{\rho^{n+2}}{(n+1)(n+2)}$$

(apply integration by parts). □

Acknowledgments. The author thanks János Kincses for valuable discussions regarding convex curves.

References

1. Anderson, J. M. and Eiderman, V. Ya., Cauchy transforms of point masses: The logarithmic derivative of polynomials, *Annals of Math.* **163** (2006), 1057–1076. MR2215139
2. Bonnesen, T. and Fenchel, W., *Theory of Convex Bodies*, BCS Associates, Moscow USA, 1987. MR0920366
3. Landkof, N. S., *Foundations of Modern Potential Theory*, Grundlehren der mathematischen Wissenschaften **180**, Springer, Berlin, 1972. MR0350027
4. Ransford, T., *Potential Theory in the Complex Plane*, Cambridge University Press, Cambridge, 1995. MR1334766
5. Richards, T., On approximate Gauss-Lucas theorems. arXiv:1706.05410. MR4261771
6. Richards, T. and Steinerberger, S., Leaky roots and stable Gauss-Lucas theorems, *Complex Var. Elliptic Equ.* **64** (2019), 1898–1904. MR3992156

7. Saff, E. B. and Totik, V., *Logarithmic Potentials with External Fields*, Grundlehren der mathematischen Wissenschaften **316**, Springer, Berlin, 1997. MR1485778
8. Rosenthal, A. and Szász, O., Eine Extremaleigenschaft der Kurven konstanten Breite, *Jber. Deutsch. Math. Vereinig.* **25** (1917), 278–282. MR0123455
9. Tolsa, X., On the analytic capacity γ_+, *Indiana Univ. Math. J.*. **51** (2002), 317–343. MR1909292
10. Tolsa, X., Painlevé's problem and the semiadditivity of analytic capacity, *Acta Math.* **190** (2003), 105–149. MR1982794
11. Totik, V., The Gauss-Lucas theorem in an asymptotic sense, *Bull. London Math. Soc.* **48** (2016), 848–854. MR3556367
12. Totik, V., Distribution of critical points of polynomials, *Trans. Amer. Math. Soc.* **372** (2019), 2407–2428. MR3988581

Vilmos Totik
MTA-SZTE
Analysis and Stochastics Research Group
Bolyai Institute
University of Szeged
Aradi v. tere 1, HU-6720
Szeged
Hungary
totik@math.u-szeged.hu

Received February 6, 2021