

Remarks on random walks on graphs and the Floyd boundary

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Abstract. We show that for a uniformly irreducible random walk on a graph, with bounded range, there is a Floyd function for which the random walk converges to its corresponding Floyd boundary. Moreover if we add the assumptions, $p^{(n)}(v, w) \leq C\rho^n$, where $\rho < 1$ is the spectral radius, then for any Floyd function f that satisfies $\sum_{n=1}^{\infty} nf(n) < \infty$, the Dirichlet problem with respect to the Floyd boundary is solvable.

1. Introduction

Different compactifications of classes of graphs, respectively groups, play important roles in the boundary theory of random walks, typically in connection with probabilistic and potential theoretic aspects of boundary convergence. The basic compactification of a locally finite, connected graph is its end compactification, see Freudenthal [4]. Any transient random walk with bounded range on such a graph converges to a random end, and under suitable irreducibility assumptions, its Martin compactification, see Martin [12] and Doob [2], covers the end compactification, see Picardello and Woess [13]. For a class of random walks on hyperbolic graph, respectively on hyperbolic groups (in the sense of Gromov [6]), Ancona [1] showed that the Martin boundary can be identified with the hyperbolic boundary. In case of invariance under a transitive group action, very general results are available for convergence to the boundary and the Dirichlet problem at infinity with respect to ends or the hyperbolic compactification. For many references, see Woess [15]. In absence of a group action, Kaimanovich and Woess [8] provided geometric criteria beyond bounded range to obtain such results.

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In 1980 Floyd [3] introduced the Floyd compactification of groups, respectively their Cayley graphs, in order to study limit sets of Kleinian groups, the Floyd compactification is defined for all graphs, respectively groups. Moreover there are groups whose Floyd boundary is bigger than their space of ends. More precisely the Floyd boundary of the fundamental group of a closed hyperbolic n -manifold is the $(n-1)$ -dimensional sphere.

In this context, it first was Karlsson [9], [10], [11] to relate the Floyd boundary with issues concerning random walks. In case of random walks on groups, or more generally, random walks on graphs which are invariant under a transitive group action, Karlsson has clarified the typical convergence questions. Only recently, there has been substantial new work in the group setting. Gekhtman, Gerasimov, Potyagailo and Yang [5] have shown that under natural assumptions, the Martin boundary of a random walk on a group covers the Floyd boundary.

On the other hand, for random walks on graphs that possess no group invariance, [11] only contains a few indications, just concerning simple random walk, under which conditions one may have convergence to the Floyd boundary.

The present short note aims at giving more general answers to those questions in the non-group invariant case, along the line of the methods of [8]. As a matter of fact, it needs a few modifications of those methods to get analogous results for the Floyd boundary.

In Section 2, convergence to the Floyd boundary is proved for uniformly irreducible random walks with bounded range, without the typical assumption that the spectral radius is smaller than 1. Instead, the Floyd function is assumed to depend on the Green function of the random walk.

In Section 3, the solution of the Dirichlet problem at infinity and convergence to the boundary are proved under uniform assumptions concerning irreducibility, first moment and exponential decay of transition probabilities, which relate the random walk with the underlying geometry.

2. Convergence to the boundary

We first describe some preliminaries. Let G be a locally finite graph and d the graph distance metric. Fix a vertex e and let f be a function $f:\mathbb{N}\rightarrow\mathbb{R}_{>0}$ such that $\lambda f(r)\leq f(r+1)\leq f(r)$ for some constant $\lambda>0$ and every $r>0$. It is required in addition for f to be summable $\sum_{j=0}^{\infty} f(r)<\infty$. The function f is called Floyd function.

We fix a vertex e in the Cayley graph and for an edge $[v, w]$ we denote $d(e, w)=|w|$ and $\min\{|v|, |w|\}=|[v, w]|$. To create the Floyd boundary, one needs to resize

each edge of the graph based to its distance from the fixed vertex e and the Floyd function. So the new length will be $f(|[v, w]|)$.

Consequently a new length arises, let $\alpha = \{x_i\}_{i=1}^n$ be a path (where x_i and x_{i+1} are neighbours), now the Floyd length of α will be $L_f(\alpha) = \sum_{i=1}^{n-1} f(|[x_i, x_{i+1}]|)$. Subsequently a new metric arises $d_f(v, w) = \inf_{\alpha} L_f(\alpha)$, where α runs over all paths connecting v and w . With this metric G has finite diameter and the completion \bar{G} of G is a compact metric space and the Floyd boundary is defined as $\partial_f G = \bar{G} \setminus G$.

Let P be the irreducible transition matrix of a transient random walk Z_n on G . If there is a $\Gamma \leq \text{Aut}(G, P)$ that acts transitively on G , Karlsson [10] proved that \bar{G} satisfies Woess' axioms of projectivity and contractivity, meaning that \bar{G} is a contractive Γ -compactification of G (as defined in [14]). Therefore if the Floyd boundary $\partial_f G$ is infinite and does not have a fixed point by Γ , then the Dirichlet problem with respect to P and \bar{G} is solvable.

In order to prove this, Karlsson first proves [10, Lemma 1]. Since it will be needed in our proof, we briefly describe it. Let v, w be two vertices of G , set $v \wedge w = \frac{1}{2}(d(v, e) + d(w, e) - d(v, w))$ the Gromov product of v and w , let m be the nearest, to e , point of the d -geodesic $[v, w]$. Then

$$(1) \quad d_f(v, w) \leq d_f(v, m) + d_f(w, m) \leq 2 \left(2|m|f(|m|) + \sum_{i=|m|}^{|m|+d(v,w)} f(i) \right).$$

Define $\nu(|m|) := 4|m|f(|m|) + 2 \sum_{i=|m|}^{\infty} f(i)$ and observe that $\nu(|m|) \rightarrow 0$ as $|m| \rightarrow \infty$. Moreover if $(v \wedge w) \rightarrow \infty$, then $d_f(v, w) \rightarrow 0$, since $d_f(v, w) \leq \nu(|m|)$ and $|m| = d(e, m) \geq \frac{1}{2}(d(v, e) + d(w, e) - d(v, w))$. This inequality turns out equally useful in the general case.

With the hypothesis of uniform irreducibility and bounded range, we find a Floyd function for which the random walk converges to the corresponding Floyd boundary. This function depends on the distribution of the random walk. Uniform irreducibility means that there are $\varepsilon_0 > 0$ and $K > 0$ such that for every $v, w \in G$ with $v \sim w$ there is $\varkappa < K$ such that $p^{(\varkappa)}(v, w) \geq \varepsilon_0$ and bounded range means that there is an $M > 0$ such that $\sup\{d(v, w) : v, w \in G, p(v, w) > 0\} \leq M$. The Green function $g(v, w)$, for $v, w \in G$, of a random walk Z_n is the expected number of visits to w when $Z_0 = v$. If $g(v, w)$ is finite for every $v, w \in G$ then Z_n is called transient. For any set $W \subset G$ we set $g(v, W) = \sum_{w \in W} g(v, w)$. Let $S_n = \{w \in G : |w| = n\}$ and $B_n = \{w \in G : |w| \leq n\}$.

Lemma 2.1. *Let P be the transition matrix of a transient, uniformly irreducible random walk Z_n on G with bounded range. If $h(n)$ is a Floyd function, then $f(n) = \frac{h(n)}{g(e, B_{n+M})}$ is well defined and satisfies the requirements of a Floyd function.*

Proof. The graph is locally finite, thus S_n and B_n are finite sets for every $n \in \mathbb{N}$. The random walk Z_n is transient, therefore $g(e, B_{n+M})$ is finite for every $n \in \mathbb{N}$, hence f is well defined. Since the balls are an increasing sequence of sets, f is decreasing and summable. In order to show that f satisfies the requirements of a Floyd function, it has to be proven that there is a $\lambda > 0$ such that $\lambda f(n) \leq f(n+1)$ for every $n \in \mathbb{N}$.

The random walk is uniformly irreducible therefore there is an $\varepsilon_0 > 0$ and a $K < \infty$ such that $v \sim w \Rightarrow p^{(\varkappa)}(v, w) \geq \varepsilon_0$ for some $\varkappa \leq K$. This also implies that $\deg(v) \leq \frac{K+1}{\varepsilon_0}$. Let $v, w \in G$ with $v \sim w$, then:

$$\varepsilon_0 g(e, w) = \sum_{k=1}^{\infty} \varepsilon_0 p^{(k)}(e, w) \leq \sum_{k=1}^{\infty} p^{(\varkappa)}(w, v) p^{(k)}(e, w) \leq \sum_{k=1}^{\infty} p^{(k)}(e, v) = g(e, v)$$

Combining the bounded degree of each vertex with the inequality for the Green function of neighbouring vertices we gain:

$$g(e, S_{n+1}) = \sum_{w \in S_{n+1}} g(e, w) \leq \sum_{v_w} \frac{1}{\varepsilon_0} g(e, v_w) \leq \frac{K+1}{\varepsilon_0} \frac{1}{\varepsilon_0} g(e, S_n)$$

We note that v_w is an arbitrary neighbour of w such that $v_w \in S_n$, moreover the same vertex can appear multiple times since many $w \in S_{n+1}$ could be spanned from the same vertex v_w , but at most $\frac{K+1}{\varepsilon_0}$, since we have bounded degree.

Since h is a Floyd function, there is a $\lambda' > 0$ such that $\lambda' h(n) \leq h(n+1)$. We conclude:

$$\begin{aligned} \frac{h(n+1)}{h(n)} \frac{g(e, B_{n+M})}{g(e, B_{n+M+1})} &= \frac{h(n+1)}{h(n)} \frac{\sum_{k=0}^{n+M} g(e, S_k)}{\sum_{k=0}^{n+M+1} g(e, S_k)} \\ &= \frac{h(n+1)}{h(n)} \frac{\sum_{k=0}^{n+M} g(e, S_k)}{\sum_{k=0}^{n+M} g(e, S_k) + g(e, S_{n+M+1})} \\ &\geq \frac{h(n+1)}{h(n)} \frac{\sum_{k=0}^{n+M} g(e, S_k)}{\sum_{k=0}^{n+M} g(e, S_k) + \frac{K+1}{\varepsilon_0} g(e, S_{n+M})} \\ &\geq \lambda' \frac{\varepsilon_0^2}{(K+1 + \varepsilon_0^2)} > 0 \quad \square \end{aligned}$$

Theorem 2.2. *Let P be the transition matrix of a transient, uniformly irreducible random walk Z_n on G with bounded range. Then there exist a Floyd function f such that, for any starting point, Z_n converges to a point of $\partial_f G$ almost surely.*

Proof. The random walk Z_n has bounded range, so there is an $M > 0$ such that $d(Z_k, Z_{k+1}) < M$ for every $k \in \mathbb{N}$. We define a new random variable based on the random walk:

$$X_k := \max\{|Z_k| - M, 0\}$$

The Gromov product is bounded below:

$$\begin{aligned} Z_k \wedge Z_{k+1} &= \frac{1}{2}(|Z_k| + |Z_{k+1}| - d(Z_k, Z_{k+1})) \\ &\geq \frac{1}{2}(|Z_k| + |Z_{k+1}| - M) \\ &\geq \frac{1}{2}(|Z_k| + |Z_k| - M - M) \\ &= X_k \end{aligned}$$

Let h be a Floyd function such that $nh(n)$ is decreasing and summable, we define $f(n) = \frac{h(n)}{g(e, B_{n+M})}$, f is a well defined from Lemma 2.1. Observe that $nf(n)$ is a decreasing function, since $nh(n)$ is decreasing, hence so is $\nu(n)$.

Let $m < n$ for $m, n \in \mathbb{N}$.

$$\begin{aligned} (2) \quad d_f(Z_m, Z_n) &\leq \sum_{k=m}^n d_f(Z_k, Z_{k+1}) \leq \sum_{k=m}^n \nu(\lfloor Z_k \wedge Z_{k+1} \rfloor) \leq \sum_{k=m}^n \nu(X_k) \\ &\leq \sum_{k=m}^n 4(X_k)f(X_k) + 2 \sum_{k=m}^n \sum_{i=X_k}^{X_k+M} f(i) \\ &\leq \sum_{k=m}^n 4X_k f(X_k) + 2 \sum_{k=m}^n M f(X_k) \end{aligned}$$

The second sum in this inequality becomes small as m grows, since f satisfies the requirements of a Floyd function.

In order to prove convergence to the boundary, we will prove that the sequence is Cauchy almost surely. We compute

$$\begin{aligned} \sum_{k=1}^{\infty} \mathbb{E}[X_k f(X_k)] &= \sum_{k=1}^{\infty} \sum_{n=1}^{\infty} n f(n) \mathbb{P}(X_k = n) \\ &= \sum_{n=1}^{\infty} n f(n) \sum_{k=1}^{\infty} \mathbb{P}(X_k = n) \end{aligned}$$

$$\begin{aligned}
 &= \sum_{n=1}^{\infty} n \frac{h(n)}{g(e, B_{n+M})} \sum_{k=1}^{\infty} \mathbb{P}(X_k = n) \\
 &= \sum_{n=1}^{\infty} n \frac{h(n)}{g(e, B_{n+M})} g(Z_0, S_{n+M}) \\
 &\leq \sum_{n=1}^{\infty} nh(n) \frac{1}{g(e, B_{n+M})} \varepsilon_0^{-|Z_0|} g(e, S_{n+M}) \\
 &< \varepsilon_0^{-|Z_0|} \sum_{n=1}^{\infty} nh(n) < \infty
 \end{aligned}$$

This leads to $\sum_{k=1}^{\infty} \mathbb{E}[X_k f(X_k)] < \infty$, so $\sum_{k=1}^{\infty} X_k f(X_k) < \infty$ almost surely. Similarly, $\sum_{k=1}^{\infty} f(X_k) < \infty$ almost surely. We conclude that

$$\sum_{k=1}^{\infty} 4X_k f(X_k) + 2 \sum_{k=1}^{\infty} M f(X_k) < \infty \quad \text{almost surely}$$

From the inequality (2), this implies that (Z_n) is a d_f -Cauchy sequence. \square

In the Lemma 2.1 we require for the random walk to have bounded range, in order to fix M for the proof of the Theorem 2.2. This assumption could be removed, and M could be taken as an arbitrary natural number.

Since $f(n) < Ch(n)$ for some constant $C > 0$, there is a surjection from the Floyd boundary with respect to h to the Floyd boundary with respect to f . Therefore the Floyd boundary for the function f is potentially smaller than the one for the function h .

3. Dirichlet problem

In Theorem 2.2, we have achieved convergence to the boundary for a weaker Floyd function; essentially this function depends on the rate at which the Gromov product of consecutive points of the random walk tends to infinity.

In the case of the graph that satisfies the strong isoperimetric inequality, Karlsson [11] mentions that maybe, following the methods of [8], one could achieve convergence to the boundary and solvability for the Dirichlet problem if the Floyd function $f = o(n^{-3})$ and the Floyd boundary is infinite.

The conditions imposed in [8] imply asymptotically linear speed for the Gromov product. This will allow us to prove convergence to the boundary and solvability of the Dirichlet problem for every Floyd function f that satisfies $\sum_{n=1}^{\infty} nf(n) < \infty$

(independently the boundary’s cardinality). Therefore for the rest of the statements we require that f satisfies this condition.

For both cases of space of ends and hyperbolic boundary, [8] proves lemmas about the asymptotic behavior of the random walk. Those lemmas are proved for the graph and they do not depend on a specific compactification, therefore they can be used in the case of the Floyd compactification. A slightly more general version can be found in [15] and we shall use those in order to gain a more general result.

A graph G satisfies the strong isoperimetric inequality if there is a $\eta > 0$ such that $|\partial W| \geq \eta|W|$, for every finite subset of its vertices $W \subseteq G$, where $\partial W = \{w \in W : \text{there exist a } v \sim w \text{ with } v \in G \setminus W\}$. Let P be the transition matrix of a random walk Z_n on G . If P is irreducible, the spectral radius of P is defined as $\rho(P) = \limsup_{n \rightarrow \infty} p^n(v, w)^{\frac{1}{n}}$ for $v, w \in G$ and it is independent of the choice of the two vertices. We say that (G, P) has uniform first moment if $\bar{m} = \sum_{n=1}^{\infty} \phi(n) < \infty$, where $\phi(n) = \sup_{v \in G} \sigma_v([n, \infty))$ are the tails of the step length distribution $\sigma_v(n) = \sum_{w: d(v,w)=n} p(v, w)$.

We call (G, P) reversible if there is a measure $m: G \rightarrow (0, \infty)$ such that $m(v)p(v, w) = m(w)p(w, v)$ for every $v, w \in G$ and strongly reversible if there is a constant $M > 0$ such that $M^{-1} \leq m(v) \leq M$ for every $v \in G$. For a strongly reversible, uniformly irreducible random walk on a graph, Kaimanovich [7] proved that the strong isoperimetric inequality implies spectral radius strictly smaller than 1 and the existence of a constant $C > 0$ such that $p^n(v, w) \leq C\rho^n$ for every $v, w \in G$ and $n \in \mathbb{N}$.

Lemma 3.1. *Let $\tau(n) := 10 \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^{\infty} f(i)$, then τ is a decreasing function, $\nu(n) \leq \tau(n)$ for every $n \in \mathbb{N}$ and $\sum_{i=1}^{\infty} \tau(i) < \infty$.*

Proof. Obviously τ is decreasing, since f is also decreasing we gain,

$$nf(n) \leq 2 \left(f \left(\left\lfloor \frac{n}{2} \right\rfloor + 1 \right) + f \left(\left\lfloor \frac{n}{2} \right\rfloor + 2 \right) \dots f \left(2 \left\lfloor \frac{n}{2} \right\rfloor + 1 \right) \right) < 2 \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^{\infty} f(i).$$

This yields

$$\nu(n) = 4nf(n) + 2 \sum_{i=n}^{\infty} f(i) < 8 \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^{\infty} f(i) + 2 \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^{\infty} f(i) = \tau(n).$$

To complete the proof we make the following calculations

$$\sum_{n=1}^{\infty} \tau(n) = \sum_{n=1}^{\infty} 10 \sum_{i=\lfloor \frac{n}{2} \rfloor + 1}^{\infty} f(i) \leq 10 \sum_{n=1}^{\infty} 2nf(n) < \infty. \quad \square$$

Lemma 3.2. *Let P be the transition matrix of a uniformly irreducible random walk Z_n on G with uniform first moment and $\rho(P) < 1$. Then Z_n converges to a random point ξ of $\partial_f G$ almost surely.*

Proof. Since Z_n satisfies those conditions, we gain the following asymptotic properties $\lim_{n \rightarrow \infty} \frac{1}{n} d(Z_n, Z_{n+1}) = 0$ and $0 < \underline{m} \leq \liminf_{n \rightarrow \infty} \frac{1}{n} |Z_n|$ almost surely (from [15, Lemma 8.8 and Corollary 8.9]). Combining those two properties, we obtain that $\liminf_{n \rightarrow \infty} \frac{1}{n} (Z_n \wedge Z_{n+1}) \geq \underline{m}$ almost surely.

So there are $c_0 > 0$ and $n_0 \in \mathbb{N}$ such that $Z_n \wedge Z_{n+1} > c_0 n$ for every $n \geq n_0$ almost surely. We set $c_1 = \min\{c_0, 1\}$. From inequality (1) and Lemma 3.1 we conclude that:

$$\begin{aligned} \sum_{k=1}^{\infty} d_f(Z_k, Z_{k+1}) &\leq \sum_{k=1}^{\infty} \tau(\lfloor Z_k \wedge Z_{k+1} \rfloor) \\ &\leq \sum_{k=1}^{n_0} \tau(\lfloor Z_k \wedge Z_{k+1} \rfloor) + \sum_{k=n_0}^{\infty} \tau(\lfloor c_0 n \rfloor) \\ &\leq \sum_{k=1}^{n_0} \tau(\lfloor Z_k \wedge Z_{k+1} \rfloor) + \sum_{k=1}^{\infty} c_1^{-1} \tau(n) < \infty \end{aligned}$$

This yields that $Z_n \rightarrow \xi$ almost surely for a random $\xi \in \partial_f G$. \square

The following Lemma has the same core as [8, Corollary 2], which was proved and used in order to show the solvability of the Dirichlet problem in both cases of ends of graphs and hyperbolic boundary. Once more we provide a slightly more general Lemma that will follow [15, Lemma 21.17]. We will make some adjustments in order to work for the case of Floyd compactification.

Lemma 3.3. *Let P be the transition matrix of a uniformly irreducible random walk Z_n on G with uniform first moment, $\rho(P) < 1$ and $p^{(n)}(v, w) \leq C\rho^n$ for a constant $C > 0$ and every $v, w \in G, n \in \mathbb{N}$. For $\alpha, \varepsilon > 0$ and $v \in G$ we define $\mathcal{A}_v = \mathcal{A}_v(\alpha, \varepsilon)$ as the event that:*

1. $Z_0 = v$
2. $d(Z_0, Z_n) \leq (\overline{m} + \varepsilon)n, n \geq \alpha|v|$
3. $d(Z_n, Z_{n+1}) \leq \varepsilon n, n > \alpha|v|$
4. $|Z_n| \geq (\underline{m} - \varepsilon)n, n \geq \alpha|v|$
5. $|Z_n| > \varepsilon|v|, n \geq 0$

Then there is $\varepsilon_0 > 0$ such that for all $\varepsilon \leq \varepsilon_0$ and $\alpha > 0$ $\lim_{|v| \rightarrow \infty} \mathbb{P}_v(\mathcal{A}_v) = 1$.

Proof. Let $\mathcal{A}_{v,i}$ be the event that the i -th property is satisfied, for $i=2, 3, 4, 5$. Then from [15, Lemma 21.17], it is known that $\lim_{|v| \rightarrow \infty} \mathbb{P}_v(\mathcal{A}_{v,i})=1$ for $i=3, 5$. As mentioned before uniform first moment yields that

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \sup_{k \leq n} d(Z_0, Z_k) \leq \overline{m} \text{ and } \liminf_{n \rightarrow \infty} \frac{1}{n} |Z_n| \geq \underline{m} \text{ almost surely}$$

(see [15, Lemma 8.8 and Corollary 8.9]) Therefore $\limsup_{n \rightarrow \infty} \frac{1}{n} d(Z_0, Z_n) \leq \overline{m}$ almost surely so $\lim_{n \rightarrow \infty} \mathbb{P}(\mathcal{A}_{v,i})=1$ for $i=2, 4$. \square

The main idea in [8] is to bound the probability that the random walk converges to a neighbourhood of ξ by the probability that an event \mathcal{A}_v occurs. We follow this plan.

Theorem 3.4. *Let P be the uniformly irreducible transition matrix of a random walk Z_n on G with uniform first moment, $\rho(P) < 1$ and $p^{(n)}(v, w) \leq C\rho^n$ for a constant $C > 0$ and every $v, w \in G, n \in \mathbb{N}$. Then the Dirichlet problem with respect to the Floyd compactification is solvable.*

Proof. Let μ_v be the Borel measure in $\partial_f G$ for the \mathbb{P}_v -distribution of $\lim_{n \rightarrow \infty} Z_n$, meaning $\mu_v(B) = \mathbb{P}[\lim_{n \rightarrow \infty} Z_n \in B | Z_0 = v]$. The Dirichlet problem is solvable if and only if the random walk converges to the boundary and the measures μ_v converges weakly to the Dirac measure δ_ξ as $v \rightarrow \xi$, for every $\xi \in \partial_f G$ (see Theorem (20.3) [15]). From Lemma 3.2 we already know that the random walk converges to a point on the boundary almost surely.

Let $\xi \in \partial_f G$ and $B(\xi, r)$ be the open ball in \overline{G} , let $v \in B(\xi, \frac{r}{2}) \cap G$ and we take $\mathcal{A}_v = \mathcal{A}_v(\varepsilon, \alpha)$ as in the previous Lemma, where $\alpha = \frac{1}{2\overline{m}}, \varepsilon < \frac{1}{3}\underline{m}$. If $|v| > \frac{\varepsilon + \overline{m}}{\varepsilon}$ and the event \mathcal{A}_v occurs then:

Set $k = \lceil \alpha|v| \rceil = \lceil \frac{1}{2\overline{m}}|v| \rceil$, then

$$\begin{aligned} 2(Z_0 \wedge Z_k) &= |Z_0| + |Z_k| - d(Z_0, Z_k) = |v| + |Z_k| - d(v, Z_k) \\ &\geq |v| + \varepsilon|v| - (k+1)(\overline{m} + \varepsilon) \geq |v|(1 + \varepsilon) - (\frac{1}{2\overline{m}}|v| - 1)(\overline{m} + \varepsilon) \\ &\geq |v|(1 + \varepsilon - \frac{1}{2} - \frac{\varepsilon}{2\overline{m}} - \frac{\overline{m} + \varepsilon}{|v|}) \geq |v|(1 - \frac{1}{2} - \frac{1}{6} + \frac{\varepsilon|v| - \varepsilon - \overline{m}}{|v|}) \\ &\geq \frac{|v|}{3} \end{aligned}$$

If $n \geq \alpha|v|$, then

$$\begin{aligned} 2(Z_n \wedge Z_{n+1}) &= |Z_n| + |Z_{n+1}| - d(Z_n, Z_{n+1}) \\ &\geq (\underline{m} - \varepsilon)n + (\underline{m} - \varepsilon)(n+1) - \varepsilon n \end{aligned}$$

$$\begin{aligned} &\geq (2\underline{m} - 3\varepsilon)n + \underline{m} - \varepsilon \\ &\geq (2\underline{m} - 3\varepsilon)n \end{aligned}$$

Consequently for $n \geq \alpha|v|$

$$\begin{aligned} d_f(Z_0, Z_n) &\leq d_f(Z_0, Z_k) + d_f(Z_k, Z_n) \\ &\leq \tau(\lfloor v \wedge Z_k \rfloor) + \sum_{i=k}^n \tau(\lfloor Z_i \wedge Z_{i+1} \rfloor) \\ &\leq \tau\left(\left\lfloor \frac{|v|}{6} \right\rfloor\right) + \sum_{i=k}^n \tau\left(\left\lfloor \left(\underline{m} - \frac{3\varepsilon}{2}\right)i \right\rfloor\right) \end{aligned}$$

Now we set $c = \min\{\underline{m} - \frac{3\varepsilon}{2}, 1\}$ and the inequality becomes:

$$\begin{aligned} d_f(Z_0, Z_n) &\leq \tau\left(\left\lfloor \frac{|v|}{6} \right\rfloor\right) + \sum_{i=k}^n \tau(\lfloor ci \rfloor) \\ &\leq \tau\left(\left\lfloor \frac{|v|}{6} \right\rfloor\right) + \sum_{i=\lfloor ck \rfloor}^{\infty} c^{-1} \tau(i) \end{aligned}$$

Since we have from Lemma 3.1 for τ that $\sum_{i=1}^{\infty} \tau(i) < \infty$, there is a $|v| > \frac{\underline{m} + \varepsilon}{\varepsilon}$ large enough such that $d_f(Z_0, Z_n) < \frac{r}{2}$ for every $n > \alpha|v|$. This means that $Z_n \in B(\xi, r)$ for all large n . This can be expressed as $\mu_v(B(\xi, r) \cap \partial_f G) \geq \mathbb{P}_v(\mathcal{A}_v)$, but from Lemma 3.3 $\lim_{|v| \rightarrow \infty} \mathbb{P}_v(\mathcal{A}_v) = 1$. This is true for every $r > 0$, therefore $\lim_{v \rightarrow \xi} \mu_v = \delta_\xi$. \square

Corollary 3.5. *Let G be a graph that satisfies the strong isoperimetric inequality and P be the uniformly irreducible transition matrix of a strongly reversible random walk Z_n that has uniform first moment. Then the random walk Z_n converges to the boundary $\partial_f G$ and the Dirichlet problem with respect to the Floyd compactification is solvable.*

Therefore there exist an analogue of probabilistic propositions for the Floyd boundaries as there is for hyperbolic boundaries and the space of ends. As already mentioned in the introduction, for a random walk on a locally finite hyperbolic graph there is a relation between the hyperbolic boundary and the Martin boundary. Also, for a random walk on an arbitrary graph, the Martin boundary covers the space of ends. The recent work of Gekhtman, Gerasimov, Potyagailo and Yang sparks the question of whether the Martin boundary of a random walk covers the Floyd boundary (for suitable Floyd functions) on a general graph under geometric adaptedness conditions in the place of group invariance.

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