Ark. Mat., **60** (2022), 67–106
DOI: 10.4310/ARKIV.2022.v60.n1.a4

On the arithmetic of monoids of ideals

Alfred Geroldinger and M. Azeem Khadam

Abstract. We study the algebraic and arithmetic structure of monoids of invertible ideals (more precisely, of r-invertible r-ideals for certain ideal systems r) of Krull and weakly Krull Mori domains. We also investigate monoids of all nonzero ideals of polynomial rings with at least two indeterminates over noetherian domains. Among others, we show that they are not transfer Krull but they share several arithmetic phenomena with Krull monoids having infinite class group and prime divisors in all classes.

1. Introduction

Let R be a (commutative integral) domain, r be an ideal system on R, $\mathcal{I}_r(R)$ be the semigroup of nonzero r-ideals with r-multiplication, and $\mathcal{I}_r^*(R)\subset\mathcal{I}_r(R)$ be the subsemigroup of r-invertible r-ideals. As usual, we denote by v the system of divisorial ideals and for the d-system of usual ring ideals we omit all suffices (i.e., $\mathcal{I}(R)=\mathcal{I}_d(R)$, and so on). Factoring ideals into finite products of special ideals (such as prime ideals, radical ideals, and more) is a central topic of multiplicative ideal theory. The monograph [32] of Fontana, Houston, and Lucas shows the rich variations of this theme. Algebraic and arithmetic properties of ideal semigroups help to understand the multiplicative structure of the underlying domain. To mention some classical results, R is a Dedekind domain if and only if $\mathcal{I}(R)=\mathcal{I}^*(R)$ if and only if $\mathcal{I}(R)$ is a factorial monoid, and R is a Krull domain if and only if $\mathcal{I}_v^*(R)$ is a factorial monoid. Recent progress in such directions can be found in the work by Anderson, Chang, Juett, Kim, Klingler, Olberding, Reinhart, and others (e.g., [21],[54], [63], [53], [3], [55], [61], [62] and [57]).

In the present paper, we first study factorizations of r-invertible r-ideals into multiplicatively irreducible r-ideals of weakly Krull Mori domains and, in particular, of Krull domains. Clearly, monoids $\mathcal{I}_r^*(R)$ of r-invertible r-ideals are commutative

This work was supported by the Austrian Science Fund FWF, Project P33499-N.

cancellative monoids. After some preparations in the setting of abstract monoids in Section 3, we show in Section 4 that the monoid of invertible ideals of a weakly Krull Mori domain is a weakly Krull Mori monoid again and that a domain R is Krull if and only if the monoid $\mathcal{I}_r^*(R)$ (with r-multiplication) is a Krull monoid (Theorems 4.3 and 4.5). Much is known about the arithmetic of weakly Krull monoids and, in particular, of Krull monoids. The arithmetic of the latter is uniquely determined by its class group and the distribution of prime divisors in the classes. We focus on two arithmetical properties, namely on the structure of unions of sets of lengths and on being fully elastic (definitions are recalled at the beginning of Section 3). Our arithmetic results on the monoids of invertible ideals (as given in Corollary 4.4 and Proposition 4.9) are based on our understanding of their algebraic structure.

In Section 5, we study the semigroup of all nonzero ideals. In general, these semigroups are not cancellative, and for this reason only first steps have been made towards the understanding of their arithmetic. However, under natural ideal-theoretic assumptions they are unit-cancellative and even BF-monoids (Propositions 2.1 and 2.2). In the last years, parts of the existing machinery of factorization theory, developed in the setting of cancellative monoids, was generalized to the setting of unit-cancellative monoids (for a first paper, see [30]). In Section 2, we introduce all the required arithmetical concepts in the setting of unit-cancellative monoids. The monoid of all nonzero ideals was studied for orders in Dedekind domains with finite class group [18], [42], [15]. In this setting, monoids of all nonzero ideals share arithmetical finiteness properties with monoids of invertible ideals and, more generally, with Krull monoids having finite class group. Our main result in Section 5 deals with the monoid of nonzero ideals of polynomial rings R with at least two variables over noetherian domains, and they show a completely different behaviour. Theorem 5.1 shows that $\mathcal{I}(R)$ is not transfer Krull and that factorizations in $\mathcal{I}(R)$ are as wild as possible. Indeed, they share arithmetical phenomena with Krull monoids having infinite class group and prime divisors in all classes (see Theorem 5.1, Conjecture 5.12, Example 5.13 and the preceding discussion). The methods, used in the proof of Theorem 5.1, stem from the theory of Gröbner bases in polynomial ideal theory.

2. Background on the ideal theory and the arithmetic of monoids

We denote by $\mathbb{N}$ the set of positive integers and we set $\mathbb{N}_0=\mathbb{N}\cup\{0\}$. For real numbers $a,b\in\mathbb{R}$, we let $[a,b]=\{x\in\mathbb{Z}\colon a\le x\le b\}$ denote the discrete interval between a and b. Let A and B be sets. We use the symbol $A\subset B$ to mean that A is contained in B but may be equal to B. Suppose that A and B are subsets of $\mathbb{Z}$. Then $A+B=\{a+b\colon a\in A, b\in B\}$ denotes their sumset and the set of distances $\Delta(A)\subset\mathbb{N}$ is the set of all $d\in\mathbb{N}$ for which there is $a\in A$ such that $A\cap[a,a+d]=\{a,a+d\}$. If

$A\subset\mathbb{N}$, then $\rho(A)=\sup A/\min A\in\mathbb{Q}_{\ge 1}\cup\{\infty\}$ denotes the elasticity of A, and we set $\rho(\{0\})=1$.

Let H be a multiplicatively written commutative semigroup with identity element. We denote by $H^\times$ the group of invertible elements of H, and we say that H is reduced if $H^\times=\{1\}$. An element $a\in H$ is said to be

– *cancellative* if $b,c\in H$ and $ab=ac$ implies that $b=c$, and

– *unit-cancellative* if $a\in H$ and $a=au$ implies that $u\in H^\times$.

By definition, every cancellative element is unit-cancellative. The semigroup H is said to be *cancellative* (resp. *unit-cancellative*) if every element $a\in H$ is cancellative (resp. unit-cancellative).

Throughout this paper, a monoid means a commutative unit-cancellative semigroup with identity element.

For a set P, we denote by $\mathcal{F}(P)$ the free abelian monoid with basis P. Elements $a\in\mathcal{F}(P)$ are written in the form

$$a=\prod_{p\in P}p^{\mathsf{v}_p(a)}\,,\quad \text{where } \mathsf{v}_p\colon\mathcal{F}(P)\longrightarrow\mathbb{N}_0$$

is the p-adic valuation. We denote by $|a|=\sum_{p\in P}\mathsf{v}_p(a)\in\mathbb{N}_0$ the length of a and by $\operatorname{supp}(a)=\{p\in P\colon \mathsf{v}_p(a)>0\}\subset P$ the support of a. Let H be a monoid. A monoid H is cancellative if and only if it has a quotient group, which will be denoted by $\mathsf{q}(H)$. Let H be a cancellative monoid. We denote by

– $H'=\{x\in\mathsf{q}(H)\colon$ there is $N\in\mathbb{N}$ such that $x^n\in H$ for all $n\ge N\}\subset\mathsf{q}(H)$ the *seminormalization* of H, and by

– $\widehat{H}=\{x\in\mathsf{q}(H)\colon$ there is $c\in H$ such that $cx^n\in H$ for all $n\in\mathbb{N}\}\subset\mathsf{q}(H)$ the *complete integral closure* of H.

Then $H\subset H'\subset\widehat{H}\subset\mathsf{q}(H)$, and H is called

– *seminormal* if $H=H'$ (equivalently, if $x\in\mathsf{q}(H)$ and $x^2,x^3\in H$, then $x\in H$), and

– *completely integrally closed* if $H=\widehat{H}$.

A submonoid $S\subset H$ is called *divisor-closed* if $a\in S$ and $b\in H$ with $b|a$ implies that $b\in S$. For a subset $S\subset H$, we denote by $[\![S]\!]$ the smallest divisor-closed submonoid generated by S. Let $\varphi\colon H\to D$ be a monoid homomorphism to a cancellative monoid D. We set $H_\varphi=\{a^{-1}b\colon a,b\in H,\varphi(a)|_D\varphi(b)\}$ and we say that φ is a *divisor homomorphism* if $a,b\in H$ and $\varphi(a)|\varphi(b)$ in D implies that $a|b$ in H (equivalently, $H_\varphi=H$).

Ideal Theory of Monoids. Our notation of ideal theory follows [51], but note that the monoids in this paper do not contain a zero element. An *ideal system* on a cancellative monoid H is a map $r\colon\mathcal{P}(H)\to\mathcal{P}(H)$, where $\mathcal{P}(H)$ is the power set of H, such that the following conditions are satisfied for all subsets $X, Y \subset H$ and all elements $a \in H$:

– $X \subset X_r$,
– $X \subset Y_r$ implies that $X_r \subset Y_r$,
– $aH \subset \{a\}_r$, and
– $aX_r = (aX)_r$.

We say that r is *finitary* if, for all $X \subset H$, X_r is the union of all E_r over all finite subsets $E \subset X$. As usual, the v-system denotes the system of divisorial ideals. The monoid H is said to be

– a *Mori monoid* if it is cancellative and satisfies the ACC on divisorial ideals, and

– a *Krull monoid* if it is a completely integrally closed Mori monoid.

Let r be any ideal system on H. A subset $I \subset H$ is called an r-ideal if $I_r = I$, and $\mathcal{I}_r(H)$ is the set of nonempty r-ideals. Then $\mathcal{I}_r(H)$ together with r-multiplication (defined by $I \cdot_r J = (IJ)_r$ for all $I, J \in \mathcal{I}_r(H)$) is a reduced semigroup with identity element H. Let $\mathcal{F}_r(H)$ denote the semigroup of fractional r-ideals and $\mathcal{F}_r(H)^\times$ the group of r-invertible fractional r-ideals. Then $\mathcal{I}_r^*(H) = \mathcal{F}_r(H)^\times \cap \mathcal{I}_r(H)$ is the cancellative monoid of r-invertible r-ideals with r-multiplication and $\mathcal{I}_r^*(H) \subset \mathcal{I}_r(H)$ is a divisor-closed submonoid. If q is a further ideal system on H with $\mathcal{I}_q(H) \subset \mathcal{I}_r(H)$, then, by [51, Theorem 12.1],

$$\begin{aligned} &\mathcal{F}_r(H)^\times \subset \mathcal{F}_q(H)^\times \subset \mathcal{F}_v(H)^\times \quad \text{are subgroups and} \\ &\mathcal{I}_r^*(H) \subset \mathcal{I}_q^*(H) \subset \mathcal{I}_v^*(H) \quad \text{are submonoids}\,. \end{aligned} \tag{2.1}$$

The cokernel of the group homomorphism $\mathsf{q}(H) \to \mathcal{F}_r(H)^\times$, $a \mapsto aH$, is called the r-class group of H. It will be denoted by $\mathcal{C}_r(H)$ and written additively. Thus, if $I, J \in \mathcal{F}_r(H)^\times$, then $[I \cdot_r J] = [I] + [J] \in \mathcal{C}_r(H)$. We denote by $\mathfrak{X}(H)$ the set of all minimal nonempty prime s-ideals of H and note that $\mathfrak{X}(H) \subset t\text{-spec}(H)$. We say that H satisfies the *r-Krull Intersection Theorem* if

$$\bigcap_{n \ge 0} (I^n)_r = \emptyset \quad \text{for all } I \in \mathcal{I}_r(H) \setminus \{H\}\,.$$

Arithmetic of Monoids. Let H be a monoid. An element $p \in H$ is said to be

– *irreducible* (an *atom*) if $p \notin H^\times$ and $p = ab$ with $a, b \in H$ implies that $a \in H^\times$ or $b \in H^\times$,

– *primary* if $p \notin H^\times$ and $p \mid ab$ with $a, b \in H$ implies that $p \mid a$ or $p \mid b^n$ for some $n \in \mathbb{N}$, and

– *prime* if $p \notin H^\times$ and $p \mid ab$ with $a, b \in H$ implies that $p \mid a$ or $p \mid b$.

We denote by $\mathcal{A}(H)$ the set of atoms of H. The free abelian monoid $\mathsf{Z}(H)=\mathcal{F}(\mathcal{A}(H_{\mathrm{red}}))$ is the *factorization monoid* of H and $\pi\colon \mathsf{Z}(H)\to H_{\mathrm{red}}$, defined by $\pi(u)=u$ for all $u\in\mathcal{A}(H_{\mathrm{red}})$, denotes the *factorization homomorphism* of H. For $a\in H$,

– $\mathsf{Z}_H(a)=\mathsf{Z}(a)=\pi^{-1}(aH^\times)\subset\mathsf{Z}(H)$ is the *set of factorizations* of a,

– $\mathsf{L}_H(a)=\mathsf{L}(a)=\{|z|\colon z\in\mathsf{Z}(a)\}\subset\mathbb{N}_0$ is the *set of lengths* of a, and

– $\mathcal{L}(H)=\{\mathsf{L}(a)\colon a\in H\}$ is the *system of sets of lengths* of H.

If $S\subset H$ is a divisor-closed submonoid and $a\in S$, then $\mathsf{Z}(S)\subset\mathsf{Z}(H)$, $\mathsf{Z}_S(a)=\mathsf{Z}_H(a)$, and $\mathsf{L}_S(a)=\mathsf{L}_H(a)$. We say that H is

– *atomic* if $\mathsf{L}(a)$ is nonempty for all $a\in H$,

– *half-factorial* if $|\mathsf{L}(a)|=1$ for all $a\in H$,

– a BF-*monoid* if $\mathsf{L}(a)$ is finite and nonempty for all $a\in H$,

– an FF-*monoid* if $\mathsf{Z}(a)$ is finite and nonempty for all $a\in H$, and

– *locally finitely generated* if $[\![a]\!]_{\mathrm{red}}\subset H_{\mathrm{red}}$ is finitely generated for all $a\in H$.

Every Mori monoid H is a BF-monoid and if, in addition, r-$\max(H)=\mathfrak{X}(H)$, then H is of finite r-character ([38, Theorems 2.2.5.1 and 2.2.9]). Krull monoids are locally finitely generated and locally finitely generated monoids are FF-monoids ([38, Proposition 2.7.8]).

The next lemma gathers the properties of ideal semigroups needed in the sequel.

Proposition 2.1. *Let H be a cancellative monoid and r be an ideal system on H.*

1. If H is a Mori monoid, then

$$\bigcap_{n\ge 0}(I^n)_r=\varnothing \quad \text{for all } I\in\mathcal{I}_r^*(H)\setminus\{H\}\,.$$

2. If H is r-noetherian, then $(\mathcal{I}_r^*(H),\cdot_r)$ is a Mori monoid.

3. If $(\mathcal{I}_r^*(H),\cdot_r)$ is a Mori monoid, then H is a Mori monoid.

4. If r is finitary and H satisfies the r-Krull Intersection Theorem, then $\mathcal{I}_r(H)$ is unit-cancellative and if, in addition, H has finite r-character, then $\mathcal{I}_r(H)$ is a BF-*monoid.*

Proof. 1. See [51, Theorem 12.5].

2. and 3. follow from [39, Example 2.1].

4. See [15, Lemma 4.1] and [42, Section 4]. □

Rings. By a ring, we mean a commutative ring with identity element and by a domain, we mean a commutative integral domain with identity element. Let R be a ring. Then its multiplicative semigroup $R^\bullet$ of regular elements is a cancellative monoid. All arithmetic concepts introduced for monoids will be used for the monoids of regular elements of rings. Thus, we say that R is atomic (factorial, and so on) if $R^\bullet$ has the respective property and we set $\rho(R):=\rho(R^\bullet)$ and similarly for all arithmetical invariants. If R is a v-Marot ring, then R is a Mori ring resp. a Krull ring if and only if $R^\bullet$ is a Mori monoid resp. a Krull monoid ([41, Theorem 3.5]).

Let R be a domain. We denote by $\mathcal{H}(R)$ the monoid of nonzero principal ideals of R, and note that $\mathcal{H}(R)\cong R^\bullet_{\text{red}}$. Let r be an ideal system on R. Then r restricts to an ideal system r' on $R^\bullet$, whence for every subset $I\subset R$, we have $I_r=(I^\bullet)_{r'}\cup\{0\}$. We use all ideal theoretic concepts introduced for monoids for domains. In particular, $\mathcal{I}_r(R)$ is the semigroup of nonzero r-ideals of R, $\mathcal{I}_r^*(R)$ is the subsemigroup of r-invertible r-ideals of R, and $\mathcal{C}_r(R)=\mathcal{F}_r(R)^\times/\mathsf{q}(\mathcal{H}(R))$ is the r-class group of R.

The usual ring ideals form a finitary ideal system (the d-system), and for these ideals we omit all suffices, whence $\mathcal{I}(R)=\mathcal{I}_d(R)$, and the d-class group $\mathcal{C}_d(R)=\mathcal{F}(R)^\times/\mathsf{q}(\mathcal{H}(R))$ is the Picard group $\operatorname{Pic}(R)$ of R. Throughout this paper, we suppose that $\mathcal{I}_r(R)\subset\mathcal{I}(R)$, whence

$$\mathcal{I}^*(R)\subset\mathcal{I}_r^*(R)\subset\mathcal{I}_v^*(R)\,. \tag{2.2}$$

If R satisfies the r-Krull Intersection Theorem, then $\mathcal{I}_r(R)$ is unit-cancellative by Proposition 2.1.4. If R is a Mori domain, then $(\mathcal{I}_v^*(R),\cdot_v)$ is a Mori monoid by Proposition 2.1.2. If R is a one-dimensional Mori domain, then $\mathcal{I}_v^*(R)=\mathcal{I}^*(R)$, R has finite character by [35, Lemma 3.11], $\mathcal{I}(R)$ is unit-cancellative by [42, Corollary 4.4], whence $\mathcal{I}(R)$ is a BF-monoid by Proposition 2.1.4. For $I\in\mathcal{I}(R)$, we define $\omega'(I)\in\mathbb{N}_0\cup\{\infty\}$ to be the smallest N having the following property:

If $n\in\mathbb{N}$ and $J_1,\ldots,J_n\in\mathcal{I}(R)$ with $J_1\cdot\ldots\cdot J_n\subset I$, then there exists a subset $\Omega\subset[1,n]$ such that $|\Omega|\le N$ and $\prod_{\lambda\in\Omega}J_\lambda\subset I$.

The invariant $\omega'(I)$ is closely related to a well-studied invariant $\omega(\cdot)$ where, in the definition, containment is replaced by divisibility (see, for example, [30] and [39]). Thus, for invertible ideals $I\in\mathcal{I}^*(R)$, we have $\omega(I)=\omega'(I)$.

Suppose that R is noetherian. Then the integral closure $\overline{R}$ is a Krull domain by the Theorem of Mori-Nagata. Since noetherian domains are Mori, they are BF-domains but they need not be FF-domains. An algebraic characterization of when noetherian domains are locally finitely generated is given in [56, Theorem 1]. The next proposition shows that $\mathcal{I}(R)$ is a BF-monoid.

Proposition 2.2. *Let R be a noetherian domain and $I\in\mathcal{I}(R)$.*

1. If $J\in\mathcal{I}(R)$ with $I\subset J$ and $J/I\cong R/P$ (as R-modules) for some prime ideal $P\subset R$, then $\omega'(I)\le\omega'(J)+1$.

2. $\omega'(I)<\infty$.

3. $\sup\mathsf{L}_{\mathcal{I}(R)}(I)\le\omega'(I)$. In particular, $\mathcal{I}(R)$ is a BF-monoid.

Proof. Since R satisfies Krull's Intersection Theorem, $\mathcal{I}(R)$ is unit-cancellative by Proposition 2.1.4.

1. Since $J/I \cong R/P$, we obtain $PJ \subset I$ and $P = \{s \in R \colon sr \in I\}$ for every $r \in J \setminus I$. If $J_1, ..., J_n \in \mathcal{I}(R)$ with $J_1 \cdot ... \cdot J_n \subset I$, then $J_1 \cdot ... \cdot J_n \subset J$. Therefore, there exists a subset $\Omega \subset [1,n]$ such that $|\Omega| \le \omega'(J)$ and $\prod_{\lambda \in \Omega} J_\lambda \subset J$. If $\prod_{\lambda \in \Omega} J_\lambda \subset I$, then the statement follows. If $\prod_{\lambda \in \Omega} J_\lambda \not\subset I$, then there exists $r \in R$ with $r \in \prod_{\lambda \in \Omega} J_\lambda \setminus I \subset J \setminus I$. Thus, we have

$$r \cdot \prod_{\lambda \in [1,n] \setminus \Omega} J_\lambda \subset J_1 \cdot ... \cdot J_n \subset I\,,$$

and so $\prod_{\lambda \in [1,n] \setminus \Omega} J_\lambda \subset P$. Therefore, there exists $\ell \in [1,n] \setminus \Omega$ such that $J_\ell \subset P$, whence

$$J_\ell \cdot \prod_{\lambda \in \Omega} J_\lambda \subset PJ \subset I\,.$$

2. We need the following module theoretic result: if M is a nonzero finitely generated R-module and $N \subset M$ a submodule, then there exist a chain of submodules $N = N_0 \subset N_1 \subset ... \subset N_m = M$ and prime ideals $P_i \subset R$ such that $N_{i+1}/N_i \cong R/P_i$ for all $i \in [0, m-1]$ ([58, Theorem 6.4]). Thus since $\omega'(R) = 0$, the claim follows by 1.

3. Let $I, J_1, ..., J_n \in \mathcal{I}(R)$ with $I = J_1 \cdot ... \cdot J_n$. Then there exists a subset $\Omega \subset [1,n]$ such that $|\Omega| \le \omega'(I)$ and $J' = \prod_{\lambda \in \Omega} J_\lambda \subset I$. Setting $J'' = \prod_{\lambda \in [1,n] \setminus \Omega} J_\lambda$, we obtain that

$$J' \subset I = J'J'' \subset J'\,,$$

whence $J' = J'J''$. Therefore $J'' = R$, $\Omega = [1,n]$, and $n = |\Omega| \le \omega'(I)$. Finally, the in particular statement follows now by 2. □

3. On unions of sets of lengths and sets of elasticities

Let H be a BF-monoid. Then

$$\Delta(H) = \bigcup_{L \in \mathcal{L}(H)} \Delta(L) \ \subset \mathbb{N}$$

is the *set of distances* of H. Let $k \in \mathbb{N}_0$. If $H = H^\times$, then we set $\mathcal{U}_k(H) = \{k\}$, and otherwise we set

$$\mathcal{U}_k(H) = \bigcup_{k \in L,\, L \in \mathcal{L}(H)} L \ \subset \mathbb{N}_0$$

is the *union of sets of lengths* containing k. Then $\rho_k(H) = \sup \mathcal{U}_k(H)$ is the *k-th elasticity* of H and

$$\rho(H) = \sup\{\rho(L) \colon L \in \mathcal{L}(H)\} = \lim_{k \to \infty} \frac{\rho_k(H)}{k}$$

is the *elasticity* of H. We say that H has *accepted elasticity* if there is $L\in\mathcal{L}(H)$ with $\rho(L)=\rho(H)$. By definition, H is half-factorial if and only if $\Delta(H)=\emptyset$ if and only if $\rho(H)=1$. If H is not half-factorial, then $\min\Delta(H)=\gcd\Delta(H)$.

We start with a discussion on elasticities. In [23] and [14], Chapman et al. initiated the study of the set $\{\rho(L)\colon L\in\mathcal{L}(H)\}\subset\mathbb{Q}_{\ge 1}$ of elasticities of all sets of lengths. By definition, H is half-factorial if and only if $\{\rho(L)\colon L\in\mathcal{L}(H)\}=\{1\}$. The reverse extremal case, namely when the set of elasticities is as large as possible, found special attention. We say that H is *fully elastic* if for every rational number q with $1<q<\rho(H)$ there is an $L\in\mathcal{L}(H)$ such that $\rho(L)=q$. Thus, by definition, every half-factorial monoid is fully elastic. For a detailed study of sets of elasticities in the setting of locally finitely generated monoids we refer to [66].

Next we discuss the structure of sets of lengths and of their unions. To do so, we need the concept of almost arithmetic (multi) progressions. Let $d\in\mathbb{N}$, $M\in\mathbb{N}_0$, and $\{0,d\}\subset\mathcal{D}\subset[0,d]$. A subset $L\subset\mathbb{Z}$ is called an

– *almost arithmetic multiprogression* (AAMP) with difference d, period $\mathcal{D}$, and bound M if

$$L=y+(L'\cup L^*\cup L'')\subset y+\mathcal{D}+d\mathbb{Z}\,,$$

where $y\in\mathbb{Z}$ is a shift parameter, L^* is finite nonempty, with $\min L^*=0$ and $L^*=(\mathcal{D}+d\mathbb{Z})\cap[0,\max L^*]$, $L'\subset[-M,-1]$, and $L''\subset\max L^*+[1,M]$, and

– *almost arithmetic progression* (AAP) with difference d and bound M if L can be written in the form

$$L=y+(L'\cup L^*\cup L'')\subset y+d\mathbb{Z}\,,$$

where $y\in\mathbb{Z}$ is a shift parameter, $L'\subset[-M,-1]$, L^* is a nonempty arithmetic progression with difference d and $\min L^*=0$, $L''\subset\max L^*+[1,M]$ if L^* is finite, and $L''=\emptyset$ if L^* is infinite.

Note that AAPs need not be finite, whereas AAMPs are finite. Moreover, an AAMP with period $\mathcal{D}=\{0,d\}$ is an AAP with difference d. We say that H satisfies the

– *Structure Theorem for Sets of Lengths* if there are $M\in\mathbb{N}_0$ and a finite nonempty set $\Delta\subset\mathbb{N}$ such that every $L\in\mathcal{L}(H)$ is an AAMP with difference $d\in\Delta$ and bound M, and

– *Structure Theorem for Unions* if there are $d\in\mathbb{N}$ and $M\in\mathbb{N}_0$ such that $\mathcal{U}_k(H)$ is an AAP with difference d and bound M for all sufficiently large $k\in\mathbb{N}$.

We refer to [38, Chapter 4.7] for background on the Structure Theorem for Sets of Lengths and to [36], [30], [65], [67], [17, Theorem 6.6], [22, Proposition 4.9], [12, Theorem 1.1], [11, Theorem 3.9], [60, Theorem 5.4], [59, Theorem 5.5] for background and recent progress on the Structure Theorem for Unions.

The next lemma gathers simple properties of unions of sets of lengths which we need in the sequel.

Lemma 3.1. *Let H be a* BF*-monoid and $k, \ell \in \mathbb{N}$.*
1. $\mathcal{U}_k(H) + \mathcal{U}_\ell(H) \subset \mathcal{U}_{k+\ell}(H)$.
2. We have $k \in \mathcal{U}_\ell(H)$ if and only if $\ell \in \mathcal{U}_k(H)$.
3. If $\mathcal{U}_2(H) = \mathbb{N}_{\ge 2}$, then $\mathcal{U}_k(H) = \mathbb{N}_{\ge 2}$ for all $k \ge 2$.

Proof. 1. and 2. follow immediately by the definitions.

3. Suppose that $\mathcal{U}_2(H) = \mathbb{N}_{\ge 2}$. We proceed by induction on k. Let $k \ge 2$ and suppose that $\mathcal{U}_k(H) = \mathbb{N}_{\ge 2}$. Then 1. implies that

$$\mathbb{N}_{\ge 3} = 1 + \mathbb{N}_{\ge 2} = 1 + \mathcal{U}_k(H) \subset \mathcal{U}_{k+1}(H) .$$

Since $k+1 \in \mathcal{U}_2(H)$, 2. implies that $2 \in \mathcal{U}_{k+1}(H)$. Thus, we obtain that $\mathcal{U}_{k+1}(H) = \mathbb{N}_{\ge 2}$. $\square$

In [13] it was proved that every commutative cancellative monoid having a prime element and accepted elasticity is fully elastic. We generalize this result.

Proposition 3.2. *Let H be a* BF*-monoid. Consider the following conditions.*

(a) *There are submonoids H_1 and H_2 such that $H = H_1 \times H_2$, where H_1 is half-factorial but not a group. This holds true in particular if H has a cancellative prime element.*

(b) *For every $q \in \mathbb{Q}$ with $1 < q < \rho(H)$, there is an element $c \in H$ such that*

$$\rho(\mathsf{L}(c^k)) = \rho(\mathsf{L}(c)) > q \quad \text{for all } k \in \mathbb{N} .$$

This holds true in particular if H has accepted elasticity.

(b') *For every $m \in \mathbb{N}_{\ge 2}$ there is $L \in \mathcal{L}(H_2)$ with $\min L = 2$ and $\max L = m$.*

If Conditions (a) and (b) or Conditions (a) and (b') hold, then H is fully elastic.

Proof. If $\rho(H) = 1$, then Condition (b) holds trivially, H is half-factorial, and hence it is fully elastic. From now on we suppose that $\rho(H) > 1$.

1. We first check the two in particular statements stated in (a) and (b).

(i) If $p \in H$ is a cancellative prime element of H, then

$$H = \mathcal{F}(\{p\}) \times T \,, \text{where} \quad T = \{a \in H \colon p \nmid a\} .$$

If $a \in H$ and $k \in \mathbb{N}$, then

$$\max \mathsf{L}(a^k) \ge k \max \mathsf{L}(a) \quad \text{and} \quad \min \mathsf{L}(a^k) \le k \min \mathsf{L}(a) ,$$

whence

$$\rho(\mathsf{L}(a^k)) = \frac{\max \mathsf{L}(a^k)}{\min \mathsf{L}(a^k)} \ge \frac{k \max \mathsf{L}(a)}{k \min \mathsf{L}(a)} = \rho(\mathsf{L}(a)) . \tag{3.1}$$

(ii) Suppose that H has accepted elasticity, say $\rho(\mathsf L(a))=\rho(H)$. By (3.1), we infer that $\rho(\mathsf L(a^k))=\rho(\mathsf L(a))=\rho(H)$ for all $k\in\mathbb N$ and, if $\rho(\mathsf L(a^k))=\rho(\mathsf L(a))$ for some $k\in\mathbb N$, then

$$\max \mathsf L(a^k) = k \max \mathsf L(a) \quad \text{and} \quad \min \mathsf L(a^k) = k \min \mathsf L(a)\,. \tag{3.2}$$

(iii) Suppose that (a) and (b) hold. To show that H is fully elastic, we choose an atom $p\in H_1$. Then for every $a=a_1a_2\in H=H_1\times H_2$ and every $k\in\mathbb N$, we have

$$\mathsf L_H(p^k a) = \mathsf L_{H_1}(p^k a_1) + \mathsf L_{H_2}(a_2) = k + \mathsf L_{H_1}(a_1) + \mathsf L_{H_2}(a_2) = k + \mathsf L_H(a)\,. \tag{3.3}$$

Let $q\in\mathbb Q$ with $1<q<\rho(H)$, and let $c\in H$ with

$$\rho(\mathsf L(c^k)) = \rho(\mathsf L(c)) > q \quad \text{for all } k \in \mathbb N\,.$$

We set

$$q = \frac{r}{s} \quad \text{with} \quad r, s \in \mathbb N\,,$$

$$i = r - s, \quad j = s \max \mathsf L(c) - r \min \mathsf L(c), \quad \text{and} \quad b = c^i p^j\,.$$

Then, by (3.2) and (3.3),

$$\max \mathsf L(b) = j + i \max \mathsf L(c) \quad \text{and} \quad \min \mathsf L(b) = j + i \min \mathsf L(c)\,.$$

Putting all together we obtain that

$$\begin{aligned}
\rho(\mathsf L(b)) &= \frac{\max \mathsf L(b)}{\min \mathsf L(b)} = \frac{j + i \max \mathsf L(c)}{j + i \min \mathsf L(c)} \\
&= \frac{(r-s)\max \mathsf L(c) + s \max \mathsf L(c) - r \min \mathsf L(c)}{(r-s)\min \mathsf L(c) + s \max \mathsf L(c) - r \min \mathsf L(c)} \\
&= \frac{r(\max \mathsf L(c) - \min \mathsf L(c))}{s(\max \mathsf L(c) - \min \mathsf L(c))} \\
&= \frac{r}{s} = q\,.
\end{aligned}$$

2. Suppose that (a) and (b') hold and let $q\in\mathbb Q$ with $1<q<\rho(H)$. Then there are $r,s\in\mathbb N$ such that $q=r/s$. By assumption, there is $a_2\in H_2$ such that $\min \mathsf L(a_2)=2$ and $\max \mathsf L(a_2)=r-s+2$. We choose an atom $u\in H_1$ and define

$$b = u^{s-2} a_2\,.$$

Then

$$\rho(\mathsf L(b)) = \frac{\max \mathsf L(b)}{\min \mathsf L(b)} = \frac{\max \mathsf L(u^{s-2}) + \max \mathsf L(a_2)}{\min \mathsf L(u^{s-2}) + \min \mathsf L(a_2)} = \frac{(s-2)+(r-s+2)}{(s-2)+2} = q\,. \quad \square$$

Example 3.3.

1. Consider the additive monoid $H_2 = (\mathbb{N}^2 \cup \{(0,0)\}, +) \subset (\mathbb{N}_0^2, +)$. For every $m \ge 2$, $(m-1,1), (1,m-1) \in \mathcal{A}(H_2)$ and we have

$$(m,m) = (m-1,1) + (1,m-1) = \underbrace{(1,1) + \ldots + (1,1)}_{m-\text{times}},$$

whence $\min \mathsf{L}((m,m)) = 2$ and $\max \mathsf{L}((m,m)) = m$. Thus, Condition (b') of Proposition 3.2 is satisfied. Moreover, $\mathcal{U}_2(H) = \mathbb{N}_{\ge 2}$, whence $\mathcal{U}_k(H) = \mathbb{N}_{\ge 2}$ for all $k \ge 2$ by Lemma 3.1. The forthcoming Proposition 4.2 shows that H_2 is not fully elastic (because H_2 is strongly primary). However, if H_1 is any half-factorial monoid, then $H_1 \times H_2$ is fully elastic by Proposition 3.2.

2. In Section 5, we show that the monoid of nonzero ideals of a polynomial ring with at least two variables also satisfy Conditions (a) and (b') of Proposition 3.2 (Theorem 5.1).

4. On monoids of invertible ideals of weakly Krull domains

In this section we study the algebraic structure of monoids of invertible ideals and we derive some consequences for their arithmetic. Our focus will be on weakly Krull domains and Krull domains. We start with a result in the setting of r-invertible r-ideals. Then our discussion is divided into four subsections, namely on weakly Krull domains 4.1, Krull domains 4.2, transfer Krull monoids 4.3, and on the arithmetic of transfer Krull monoids 4.4.

Let H be a cancellative monoid and r be an ideal system on H. An ideal $I \in \mathcal{F}_r(H)$ is called an r-*cancellation ideal* if whenever $I \cdot_r J_1 = I \cdot_r J_2$ for $J_1, J_2 \in \mathcal{I}_r(H)$, we have $J_1 = J_2$. It is easily seen that I is an r-cancellative if and only if whenever $I \cdot_r J_1 \subset I \cdot_r J_2$ for all $J_1, J_2 \in \mathcal{I}_r(H)$, we have $J_1 \subset J_2$. All r-invertible ideals (whence all principal ideals) are r-cancellation ideals, whence $\mathcal{I}_r^*(H)$ is a cancellative monoid. A divisorial ideal is v-invertible if and only if it is v-cancellative ([51, Chapter 13.4]). Let R be a domain. A nonzero ideal I of R is a cancellation ideal if and only if I is locally principal ([5] and [34]), and $\mathcal{I}(R)$ is a cancellative monoid if and only if R is almost Dedekind ([51, Theorem 23.2]). If R is a Mori domain, then every nonzero locally principal ideal is invertible ([6, Corollary 1]). If $p \in R$ is prime, then pR is a cancellative prime element of $\mathcal{I}^*(R)$ and of $\mathcal{I}(R)$. Moreover, if R is noetherian, then p is also a prime element of $\overline{R}$ ([26, Lemma 4.7]; for more on prime elements in noetherian domains, we refer to [25]).

Theorem 4.1. *Let H be a cancellative monoid and let r be an r-noetherian ideal system of H.*

1. If $\mathcal{I}_r^*(H)$ has a prime element, and if either $\mathcal{I}_r^*(H)$ has accepted elasticity or is locally finitely generated, then $\mathcal{I}_r^*(H)$ is fully elastic.

2. Suppose that H has finite r-character and satisfies the r-Krull Intersection Theorem. If H has an r-invertible prime r-ideal, and if either $\mathcal{I}_r(H)$ or all divisor-closed submonoids generated by one element have accepted elasticity, then $\mathcal{I}_r(H)$ is fully elastic.

Proof. In both cases we verify the assumptions of Proposition 3.2.

1. Since r is r-noetherian, $\mathcal{I}_r^*(H)$ is a Mori monoid by Proposition 2.1.2 and hence it is a BF-monoid. Let $P \in \mathcal{I}_r^*(H)$ be a prime element. Since $\mathcal{I}_r^*(H)$ is a cancellative monoid, P is a cancellative prime element.

Let q be a rational number with $1 < q < \rho(\mathcal{I}^*(H))$. If $\mathcal{I}_r^*(H)$ has accepted elasticity, then there is $J \in \mathcal{I}_r^*(H)$ such that $\rho(\mathsf{L}(J)) = \rho(\mathcal{I}_r^*(H))$, whence

$$\rho(\mathsf{L}(J^k)) = \rho(\mathsf{L}(J)) > q \quad \text{for all } k \in \mathbb{N}\,.$$

Now suppose that $\mathcal{I}_r^*(H)$ is locally finitely generated. We choose an ideal $I \in \mathcal{I}_r^*(H)$ with $\rho(\mathsf{L}(I)) > q$. We consider the divisor-closed submonoid $S = [\![I]\!] \subset \mathcal{I}_r^*(H)$. Then S is a finitely generated monoid. Thus, by [38, Theorem 3.1.4], there is an ideal $J \in S$ with $\rho(S) = \rho(\mathsf{L}_S(J)) \ge \rho(\mathsf{L}_S(I)) = \rho(\mathsf{L}_{\mathcal{I}_r^*(H)}(I)) > q$. Therefore, we obtain that

$$\rho(\mathsf{L}(J^k)) = \rho(\mathsf{L}(J)) > q \quad \text{for all } k \in \mathbb{N}\,.$$

Thus, Conditions (a) and (b) of Proposition 3.2 are satisfied, whence the assertion follows.

2. Since H is r-noetherian, r is a finitary ideal system by [51, Theorem 3.5]. Thus $\mathcal{I}_r(H)$ is a BF-monoid by Proposition 2.1.4. Let P be an r-invertible prime r-ideal. Then P is a cancellative prime element of $\mathcal{I}_r(H)$. Arguing as in 1., we obtain an r-ideal J such that

$$\rho(\mathsf{L}(J^k)) = \rho(\mathsf{L}(J)) > q \quad \text{for all } k \in \mathbb{N}\,.$$

Thus, Conditions (a) and (b) of Proposition 3.2 are satisfied, whence the assertion follows. $\square$

4.1. Weakly Krull domains

In this subsection, we consider weakly Krull domains. We start with the local case and for this we need the concept of primary monoids. Let H be a cancellative monoid and $\mathfrak{m} = H \setminus H^\times$. Then H is called

– *primary* if $H \ne H^\times$ and for all $a, b \in \mathfrak{m}$ there is $n \in \mathbb{N}$ such that $b^n \in aH$, and

– *strongly primary* if $H \ne H^\times$ and for every $a \in \mathfrak m$ there is $n \in \mathbb N$ such that $\mathfrak m^n \subset aH$ (we denote by $\mathcal M(a)$ the smallest $n \in \mathbb N$ having this property).

Primary Mori monoids are strongly primary and strongly primary monoids are BF-monoids. The multiplicative monoid of nonzero elements of a domain is primary if and only if the domain is one-dimensional and local. If R is a one-dimensional local Mori domain, then $\mathcal I_v^*(R) = \mathcal I^*(R) \cong R^\bullet$ and $R^\bullet$ is locally tame strongly primary ([43, Corollary 3.10]).

Proposition 4.2. *Let H be a strongly primary monoid.*

1. If H is not half-factorial, then there is $\beta \in \mathbb Q_{>1}$ such that $\rho(L) \ge \beta$ for all $L \in \mathcal L(H)$ with $\rho(L) \ne 1$. In particular, H is fully elastic if and only if it is half-factorial.

2. If H is locally tame, then H satisfies the Structure Theorem for Sets of Lengths and the Structure Theorem for Unions.

Proof. 1. The first statement follows by [44, Theorem 5.5]. Since half-factorial monoids are fully elastic, the in particular statement holds.

2. This follows from [37, Theorem 4.1]. □

A family of monoid homomorphisms $\boldsymbol\varphi = (\varphi_p \colon H \to D_p)_{p \in P}$ is said to be

– of *finite character* if the set $\{p \in P \colon \varphi_p(a) \notin D_p^\times\}$ is finite for all $a \in H$, and

– a *defining family* (for H) if it is of finite character and

$$H = \bigcap_{p \in P} H_{\varphi_p} \,.$$

If $\boldsymbol\varphi$ is of finite character, then it induces a monoid homomorphism

$$\varphi \colon H \longrightarrow D = \coprod_{p \in P} (D_p)_{\rm red} \,, \quad \text{defined by} \quad \varphi(a) = \Big(\varphi_p(a) D_p^\times \Big)_{p \in P} \,,$$

and $\boldsymbol\varphi$ is a defining family if and only if φ is a divisor homomorphism.

We recall the concept of weak divisor theories and weakly Krull monoids ([50], [51, Chapter 22]). A monoid is said to be *weakly factorial* if it is cancellative and every nonunit is a finite product of primary elements. Every reduced weakly factorial monoid has a unique decomposition in the form

$$D = \coprod_{p \in P} D_p, \quad \text{where } D_p \subset D \text{ are reduced primary submonoids}\,.$$

Let D be a reduced weakly factorial monoid as above. If $(a^{(i)})_{i \in I}$ is a family of elements $a^{(i)} \in D$ with components $a_p^{(i)}$, then a is called a *strict greatest common divisor* of $(a^{(i)})_{i \in I}$, we write

$$a = \wedge (a^{(i)})_{i \in I} \,,$$

if the following two properties are satisfied for all $p \in P$:

– $a_p | a_p^{(i)}$ for all $i \in I$, and
– $a_p = a_p^{(i)}$ for at least one $i \in I$.

A monoid homomorphism $\partial\colon H \to D$ is called a *weak divisor theory* if the following two conditions are satisfied:

(a) ∂ is a divisor homomorphism and D is reduced weakly factorial.
(b) For every $a \in D$, there are $a_1, \ldots, a_m \in H$ such that $a = \partial(a_1) \wedge \ldots \wedge \partial(a_m)$.

A monoid is said to be a *weakly Krull monoid* if it is cancellative and one of the following equivalent conditions is satisfied:

– H has a weak divisor theory $\partial\colon H \to D$.
– The family of embeddings $(\varphi_{\mathfrak{p}}\colon H \hookrightarrow H_{\mathfrak{p}})_{\mathfrak{p} \in \mathfrak{X}(H)}$ is a defining family for H.
– $\varphi\colon H \to \coprod_{\mathfrak{p} \in \mathfrak{X}(H)} (H_{\mathfrak{p}})_{\mathrm{red}}$ is a weak divisor theory.

By the uniqueness of weak divisor theories, the *(weak divisor) class group* $\mathcal{C}(H) = \mathsf{q}(D)/\mathsf{q}(\partial(H))$ depends on H only and it is isomorphic to the t-class group $\mathcal{C}_t(H)$ of H ([51, Theorems 20.4 and 20.5]). If H is weakly Krull Mori, then $\mathcal{C}(H) \cong \mathcal{C}_t(H) = \mathcal{C}_v(H)$. A monoid is weakly factorial if and only if it is weakly Krull with trivial class group. The localizations $H_{\mathfrak{p}}$ are primary for all $\mathfrak{p} \in \mathfrak{X}(H)$.

A domain R is a weakly Krull domain if $R^\bullet$ is a weakly Krull monoid. If R is a one-dimensional Mori domain, then R is a weakly Krull Mori domain, $\mathcal{I}_v^*(R) = \mathcal{I}^*(R)$, and $\mathcal{C}_v(R) = \mathrm{Pic}(R)$ ([38, Proposition 2.10.5]). In particular, orders in holomorphy rings of global fields are weakly Krull Mori domains and every class of their Picard group contains infinitely many invertible prime ideals ([38, Corollary 2.11.16 and Proposition 8.9.7]). To mention higher-dimensional weakly Krull domains, recall that all Cohen-Macaulay domains are weakly Krull. We mention a recent characterization of when monoid algebras are weakly Krull. Let D be a domain with quotient field K and let S be a cancellative monoid with torsion-free quotient group $G = \mathsf{q}(S)$. Suppose that G satisfies the ACC on cyclic subgroups. Then the monoid algebra $D[S]$ is weakly Krull if and only if D is a weakly Krull domain satisfying the G-UMT property and S is a weakly Krull monoid satisfying the K-UMT property ([28, Theorem 3.7]). Monoid algebras, that are weakly Krull Mori and have height-one prime ideals in all classes, are studied in [27].

Theorem 4.3. *Let R be a weakly Krull Mori domain. Then $\mathcal{I}_v^*(R)$ is a reduced weakly factorial Mori monoid. The inclusion $\mathcal{I}^*(R) \hookrightarrow \mathcal{I}_v^*(R)$ is a weak divisor theory, $\mathcal{I}^*(R)$ is a weakly Krull Mori monoid and its class group is isomorphic to $\mathcal{C}_v(R)/\mathrm{Pic}(R)$. If every class of $\mathcal{C}_v(R)$ contains at least one (resp. infinitely many) $\mathfrak{p} \in \mathfrak{X}(R)$, then every class of $\mathcal{C}_v(\mathcal{I}^*(R))$ contains at least one (resp. infinitely many) $\mathfrak{q} \in \mathfrak{X}(\mathcal{I}^*(R))$.*

Proof. By [40, Proposition 5.3], we have a monoid isomorphism

$$\mathcal{I}_v^*(R) \longrightarrow \coprod_{\mathfrak{p}\in\mathfrak{X}(R)} (R_{\mathfrak{p}}^\bullet)_{\mathrm{red}}\,, \tag{4.1}$$

whence $\mathcal{I}_v^*(R)$ is a reduced weakly factorial Mori monoid (the Mori property follows from Proposition 2.1.2). Since R is a weakly Krull Mori domain, the inclusion $\mathcal{H}(R)\hookrightarrow\mathcal{I}_v^*(R)$ is a weak divisor theory. To verify that the inclusion $\mathcal{I}^*(R)\hookrightarrow\mathcal{I}_v^*(R)$ is a divisor homomorphism, let $I,J\subset R$ be invertible ideals such that $I\,|\,J$ in $\mathcal{I}_v^*(R)$. Then $I^{-1}J\in\mathcal{F}(R)^\times\cap\mathcal{I}_v^*(R)\subset\mathcal{F}(R)^\times\cap\mathcal{I}(R)=\mathcal{I}^*(R)$. Thus, the inclusion $\mathcal{I}^*(R)\hookrightarrow\mathcal{I}_v^*(R)$ is a divisor homomorphism. This implies that $\mathcal{I}^*(R)$ is a Mori monoid by [38, Proposition 2.4.4]. Since $\mathcal{H}(R)\hookrightarrow\mathcal{I}_v^*(R)$ is a weak divisor theory, every $I\in\mathcal{I}_v^*(R)$ is a strict greatest common divisor of principal ideals and hence a strict greatest common divisor of invertible ideals. Therefore, $\mathcal{I}^*(R)\hookrightarrow\mathcal{I}_v^*(R)$ is a weak divisor theory and for the class group we have

$$\begin{aligned}\mathsf{q}(\mathcal{I}_v^*(R))/\mathsf{q}(\mathcal{I}^*(R)) &= \mathcal{F}_v(R)^\times/\mathcal{F}(R)^\times \\ &\cong \Big(\mathcal{F}_v(R)^\times/\mathsf{q}(\mathcal{H}(R))\Big)\Big/\Big(\mathcal{F}(R)^\times/\mathsf{q}(\mathcal{H}(R))\Big) = \mathcal{C}_v(R)/\operatorname{Pic}(R)\,.\end{aligned}$$

The claim on the distribution of prime divisors $\mathfrak{q}\in\mathfrak{X}(\mathcal{I}^*(R))$ follows immediately from the above isomorphisms. □

Let D be a weakly Krull monoid. If $H\subset D$ is a submonoid such that $H\hookrightarrow D$ is a divisor homomorphism and the class group $\mathsf{q}(D)/D^\times\mathsf{q}(H)$ is torsion, then H is a weakly Krull monoid by [40, Lemma 5.1]. This abstract result applies to the setting $\mathcal{H}(R)\hookrightarrow\mathcal{I}^*(R)\hookrightarrow\mathcal{I}_v^*(R)$, provided that the respective class groups are torsion. But, we did not check the general case.

Let R be a weakly Krull Mori domain. Many aspects of the arithmetic of $\mathcal{I}_v^*(R)$ have been studied in a variety of settings, from orders in quadratic number fields to seminormal weakly Krull domains to stable weakly Krull domains (see [40, Theorem 5.8], [44, Theorem 5.8], [45, Corollary 4.6], [18, Theorem 1.1], [42, Theorem 5.13], [15, Theorem 5.10]). The following corollary characterizes when - under some additional assumptions - $\mathcal{I}_v^*(R)$ is fully elastic (compare with Proposition 4.9.1). For the sake of completeness and in order to compare it with Theorem 5.1, we also recall two results on the structure of sets of lengths and their unions.

Corollary 4.4. *Let R be a weakly Krull Mori domain with nonzero conductor $(R{:}\widehat{R})$.*

1. $\mathcal{I}_v^*(R)$ satisfies the Structure Theorem for Sets of Lengths.

2. Suppose that $\widehat{R_{\mathfrak p}}^{\times}/R_{\mathfrak p}^{\times}$ *is a torsion group for all* $\mathfrak p \in \mathfrak X(R)$ *and that* $\mathcal I_v^*(R)$ *has finite elasticity. Then* $\mathcal I_v^*(R)$ *satisfies the Structure Theorem for Unions, and it is fully elastic if and only if there is* $\mathfrak q \in \mathfrak X(R)$ *such that* $R_{\mathfrak q}$ *is half-factorial. If* $\mathcal I_v^*(R)$ *is not fully elastic, then* R *is a one-dimensional semilocal Mori domain with* $\mathcal C_v(R) = \mathbf 0$.

Proof. 1. See [47, Theorem 7.4.3].

2. The additional assumptions imply that $\mathcal I_v^*(R)$ has accepted elasticity ([45, Theorem 4.4.(ii)]), whence it satisfies the Structure Theorem of Unions by [65, Theorem 1.2]. We set $\mathfrak f = (R{:}\widehat R)$, $\mathcal P^* = \{\mathfrak p \in \mathfrak X(R) \colon \mathfrak p \supset \mathfrak f\}$, $\mathcal P = \mathfrak X(R) \setminus \mathcal P^*$, and

$$T = \coprod_{\mathfrak p \in \mathcal P^*} (R_{\mathfrak p}^{\bullet})_{\mathrm{red}} \,.$$

By (4.1), we obtain that

$$\mathcal I_v^*(R) \cong \coprod_{\mathfrak p \in \mathfrak X(R)} (R_{\mathfrak p}^{\bullet})_{\mathrm{red}} = \coprod_{\mathfrak p \in \mathcal P} (R_{\mathfrak p}^{\bullet})_{\mathrm{red}} \times T \cong \mathcal F(\mathcal P) \times T \,. \tag{4.2}$$

The localization $R_{\mathfrak p}$ is a discrete valuation domain if and only if $\mathfrak p \in \mathcal P$. For all $\mathfrak p \in \mathcal P^*$, $R_{\mathfrak p}^{\bullet}$ is a primary Mori domain, whence it is strongly primary.

(i) Let $\mathfrak q \in \mathfrak X(R)$ such that $R_{\mathfrak q}$ is half-factorial. Then

$$\mathcal I_v^*(R) \cong \underbrace{(R_{\mathfrak q}^{\bullet})_{\mathrm{red}}}_{H_1} \times \underbrace{\coprod_{\mathfrak p \in \mathcal P \setminus \{\mathfrak q\}} (R_{\mathfrak p}^{\bullet})_{\mathrm{red}}}_{H_{2,1}} \times \underbrace{\coprod_{\mathfrak p \in \mathcal P^* \setminus \{\mathfrak q\}} (R_{\mathfrak p}^{\bullet})_{\mathrm{red}}}_{H_{2,2}} \,.$$

If $\mathfrak X(R) = \{\mathfrak q\}$, then $\mathcal I_v^*(H)$ is half-factorial, whence it is fully elastic. Suppose that $\mathfrak X(R) \ne \{\mathfrak q\}$. Then $H_2 := H_{2,1} \times H_{2,2}$ is a nontrivial monoid. Since $H_{2,1}$ is free abelian, it has accepted elasticity. Since $\widehat{R_{\mathfrak p}}^{\times}/R_{\mathfrak p}^{\times}$ is a torsion group for all $\mathfrak p \in \mathfrak X(R)$ and since $\mathcal I_v^*(R)$ has finite elasticity, $(R_{\mathfrak p}^{\bullet})_{\mathrm{red}}$ has accepted elasticity by [45, Lemma 4.1 and Theorem 4.4] for all $\mathfrak p \in \mathcal P^* \setminus \{\mathfrak q\}$. Thus H_2 has accepted elasticity by [45, Lemma 2.6]. Therefore, $\mathcal I_v^*(R)$ is fully elastic by Proposition 3.2.

(ii) Suppose that $R_{\mathfrak q}$ is not half-factorial for all $\mathfrak q \in \mathfrak X(R)$. Then $\mathfrak X(R) = \mathcal P^*$ is finite, whence $\mathcal I_v^*(R) \cong T$ is a finite product of non-half-factorial strongly primary monoids, say $T = T_1 \times \ldots \times T_n$. Let $i \in [1,n]$. By Proposition 4.2, there is $\beta \in \mathbb Q_{>1}$ such that $\rho(L) \ge \beta$ for all $L \in \mathcal L(T_i)$ with $\rho(L) \ne 1$. We set $\mathfrak m_i = T_i \setminus T_i^{\times}$ and we choose $a_i^* \in \mathfrak m_i$ with $|\mathsf L(a_i^*)| > 1$. If $M = \max\{\mathcal M(a_1^*), \ldots, \mathcal M(a_n^*)\}$, then

$$\mathfrak m_i^M \subset \mathfrak m_i^{\mathcal M(a_i^*)} \subset a_i^* T_i \,.$$

Now let $a \in T$ with $\rho(\mathsf L(a)) > 1$. Then $a = a_1 \cdot \ldots \cdot a_n$ and, after renumbering if necessary, we may assume that $\rho(\mathsf L(a_i)) > 1$ for all $i \in [1,m]$ and $\rho(\mathsf L(a_i)) = 1$ for all $i \in [m+1,n]$

with $m\in[1,n]$. If $i\in[m+1,n]$, then $\rho(\mathsf L(a_i))=1$ implies that $a_i\notin\mathfrak m_i^M$, whence $\mathsf L(a_i)=\{k_i\}$ for some $k_i\in[0,M-1]$. Then

$$\rho(\mathsf L(a_1\cdot\ldots\cdot a_m))=\frac{\max\mathsf L(a_1\cdot\ldots\cdot a_m)}{\min\mathsf L(a_1\cdot\ldots\cdot a_m)}=\frac{\max\mathsf L(a_1)+\ldots+\max\mathsf L(a_m)}{\min\mathsf L(a_1)+\ldots+\min\mathsf L(a_m)}\geq\beta\,,$$

and

$$\rho(\mathsf L(a))=\frac{\max\mathsf L(a_1\cdot\ldots\cdot a_m)+k_{m+1}+\ldots+k_n}{\min\mathsf L(a_1\cdot\ldots\cdot a_m)+k_{m+1}+\ldots+k_n}\,.$$

Thus, there is $\beta^*\in\mathbb Q_{>1}$ such that $\rho(\mathsf L(a))\geq\beta^*$ for all $a\in T$ with $\rho(\mathsf L(a))>1$, whence $T\cong\mathcal I_v^*(R)$ is not fully elastic.

(iii) Suppose that $\mathcal I_v^*(R)$ is not fully elastic. Since v-spec$(R)=\mathfrak X(R)$ ([51, Theorem 24.5]), the equivalence of (i) and (ii) shows that v-spec(R) is finite. Thus, v-max$(R)=\max(R)$ and R is one-dimensional with $\mathcal C_v(R)=\mathbf 0$ by [38, Propositions 2.10.4 and 2.10.5]. □

4.2. Krull domains

Let H be a cancellative monoid. A *divisor theory* for H is a weak divisor theory $\varphi\colon H\to D$, where D is a free abelian monoid. The following statements are equivalent ([38, Chapter 2.4].

(a) H is a Krull monoid (i.e., H is a completely integrally closed Mori monoid).

(b) The map $\partial\colon H\to\mathcal I_v^*(H)$, defined by $a\mapsto aH$ for all $a\in H$, is a divisor theory.

(c) H has a divisor theory $\varphi\colon H\to\mathcal F(P)$.

(d) There is a divisor homomorphism $\varphi\colon H\to D$, where D is a factorial monoid.

Let H be a Krull monoid. Then there is a free abelian monoid $\mathcal F(P)$ such that the inclusion $H_{\mathrm{red}}\hookrightarrow\mathcal F(P)$ is a divisor theory. Then

$$\mathcal C(H)=\mathsf q(\mathcal F(P))/\mathsf q(H_{\mathrm{red}})$$

is called the *(divisor) class group* of H and $G_P=\{[p]=p\mathsf q(H_{\mathrm{red}})\colon p\in P\}\subset\mathcal C(H)$ is the set of classes containing prime divisors. By the uniqueness of divisor theories, $\mathcal C(H)$ and the set $G_P\subset\mathcal C(H)$ depend on H only. In particular, $\mathcal C(H)$ is isomorphic to $\mathcal C_v(H)$.

Theorem 4.5. *Let R be a domain and r be an ideal system on R with $\mathcal I_r(R)\subset\mathcal I(R)$. Then the following statements are equivalent.*

(a) R is a Krull domain.

(b) The monoid of principal ideals $\mathcal H(R)$ is a Krull monoid.

(c) The monoid of invertible ideals $\mathcal I^*(R)$ is a Krull monoid.

(d) The monoid of r-invertible r-ideals $\mathcal I_r^*(R)$ is a Krull monoid.

(e) The monoid of v-invertible v-ideals $\mathcal I_v^*(R)$ is a Krull monoid.

If these conditions hold, then the inclusion $\mathcal{I}_r^*(R) \hookrightarrow \mathcal{I}_v^*(R)$ *is a divisor theory with class group* $\mathcal{C}(\mathcal{I}_r^*(R))$ *being isomorphic to* $\mathcal{C}_v(R)/\mathcal{C}_r(R)$*. Moreover, if every class of* $\mathcal{C}_v(R)$ *contains at least one prime divisor resp. infinitely many prime divisors, then the same is true for* $\mathcal{C}(\mathcal{I}_r^*(R))$*.*

Remark. Clearly, there is a redundancy in the above formulation. If $r=d$ is the system of usual ideals, then $\mathcal{I}_r^*(R)=\mathcal{I}^*(R)$ and if $r=v$ is the system of divisorial ideals, then $\mathcal{I}_r^*(R)=\mathcal{I}_v^*(R)$. But, we want to emphasize these two important special cases.

Proof. A monoid H is Krull if and only if the associated reduced monoid H_{red} is Krull and, clearly, $\mathcal{H}(R) \cong (R^\bullet)_{\mathrm{red}}$. Thus (a) and (b) are equivalent by [38, Chapter 2]. Therefore, it remains to verify that (a) and (b) are equivalent to (d). We have

$$\mathsf{q}(\mathcal{I}_r^*(R)) = \mathcal{F}_r(R)^\times \quad \text{and} \quad \mathsf{q}(\mathcal{I}_v^*(R)) = \mathcal{F}_v(R)^\times .$$

By Equation (2.1), $\mathcal{F}_r(R)^\times$ is a subgroup of $\mathcal{F}_v(R)^\times$ and, in particular, r-ideal multiplication in $\mathcal{F}_r(R)^\times$ coincides with the v-multiplication. We continue with three assertions.

A1. $\mathcal{H}(R) \hookrightarrow \mathcal{I}_r^*(R)$ is a divisor homomorphism.

A2. $\mathcal{I}_r^*(R) \hookrightarrow \mathcal{I}_v^*(R)$ is a divisor homomorphism.

A3. If D is a Krull monoid and $H \hookrightarrow D$ is a divisor homomorphism, then H is a Krull monoid.

Proof of **A1**. If $a, b \in R^\bullet$ such that $aR | bR$ in $\mathcal{I}_r^*(R)$, then $bR = aR \cdot_r I$ for some $I \in \mathcal{I}_r^*(R)$, and $a^{-1}bR = I \subset R$ implies that $a|b$ in R.

Proof of **A2**. Let $I, J \in \mathcal{I}_r^*(R)$ such that I divides J in $\mathcal{I}_v^*(R)$. Then $I^{-1} \cdot_r J \in \mathcal{F}_r(R)^\times \cap \mathcal{I}_v^*(R) \subset \mathcal{F}_r(R)^\times \cap \mathcal{I}(R) = \mathcal{I}_r^*(R)$. Thus, the inclusion $\mathcal{I}_r^*(R) \hookrightarrow \mathcal{I}_v^*(R)$ is a divisor homomorphism.

Proof of **A3**. If D is Krull, then there is a divisor homomorphism $\varphi \colon D \to F$, where F is a factorial monoid. Since the composition of divisor homomorphisms is a divisor homomorphism again, we obtain a divisor homomorphism $H \hookrightarrow D \to F$ from H to a factorial monoid, whence H is a Krull monoid.

Suppose that (a) and (b) hold. Then $\mathcal{I}_v^*(R)$ is free abelian by [38, Theorem 2.3.11] and hence Krull. Thus, $\mathcal{I}_r^*(R)$ is Krull by **A2** and **A3**, which means that (d) holds. Conversely, if (d) holds, then $\mathcal{H}(R)$ is Krull by **A1** and **A3**.

Now suppose that (a) – (e) hold. Since the inclusion $\mathcal{H}(R) \hookrightarrow \mathcal{I}_v^*(R)$ is a divisor theory, every $I \in \mathcal{I}_v^*(R)$ is a greatest common divisor of principal ideals and hence a greatest common divisor of r-invertible r-ideals. This, together with **A2**, shows that the inclusion $\mathcal{I}_r^*(R) \hookrightarrow \mathcal{I}_v^*(R)$ is a divisor theory with class group

$$\begin{aligned}\mathsf{q}(\mathcal{I}_v^*(R))/\mathsf{q}(\mathcal{I}_r^*(R)) &= \mathcal{F}_v(R)^\times/\mathcal{F}_r(R)^\times \\ &\cong \Big(\mathcal{F}_v(R)^\times/\mathsf{q}(\mathcal{H}(R))\Big)\Big/\Big(\mathcal{F}_r(R)^\times/\mathsf{q}(\mathcal{H}(R))\Big) = \mathcal{C}_v(R)/\mathcal{C}_r(R)\,.\end{aligned}$$

The monoid $\mathcal{I}_v^*(R)$ is free abelian with basis v-spec(R). If every class of $\mathcal{C}_v(R)$ contains at least one resp. infinitely many prime v-ideals, then the same is true for the factor group $\mathcal{C}_v(R)/\mathcal{C}_r(R)$. □

Thus, if R is a Krull domain, then the monoid $\mathcal{I}^*(R)$ of invertible ideals is a Krull monoid with class group isomorphic to $\mathcal{C}_v(R)/\operatorname{Pic}(R)$. This factor group shows a wide range of behavior. Daniel D. Anderson gave a characterization when the factor group is a torsion group and he showed that the factor group is trivial if and only if $R_\mathfrak{m}$ is factorial for all $\mathfrak{m}\in\max(R)$ ([2, Theorems 3.1 and 3.3]; see also [7]). We consider monoid algebras that are Krull. For a domain D and a cancellative monoid S, the monoid algebra $D[S]$ has the following properties:

– $D[S]$ is Krull if and only if D is Krull, S is Krull, $\mathsf{q}(S)$ is torsion-free, and $S^\times$ satisfies the ACC on cyclic subgroups (this was first proved by Chouinard in [24]; see also [48, Theorem 15.6]).

– $D[S]$ is seminormal if and only if D and S are seminormal ([19, Theorem 4.76]).

Corollary 4.6. *Let D be a domain and S be a cancellative monoid such that the monoid algebra $R=D[S]$ is Krull. Then $\mathcal{I}^*(R)$ is a Krull monoid with class group $\mathcal{C}(\mathcal{I}^*(R))\cong\mathcal{C}_v(R)/\operatorname{Pic}(R)$, $\mathcal{C}_v(R)\cong\mathcal{C}_v(D)\oplus\mathcal{C}_v(S)$, $\operatorname{Pic}(R)\cong\operatorname{Pic}(D)$, and every class of $\mathcal{C}_v(\mathcal{I}^*(R))$ contains infinitely many prime divisors.*

Proof. Since $D[S]$ is Krull, the previous remark implies that D is a Krull domain and S is a Krull monoid, whence D and S are seminormal. Thus, the natural map

$$\operatorname{Pic}(D[S]) \longrightarrow \operatorname{Pic}(D)$$

is an isomorphism by [8, Corollary 1] (note, since D is completely integrally closed, D is strongly quasinormal in the sense of [8]). We have $\mathcal{C}_v(R)\cong\mathcal{C}_v(D)\oplus\mathcal{C}_v(S)$ by [48, Corollary 16.8]. Since every class of $\mathcal{C}_v(R)$ contains infinitely many prime divisors by [29, Theorem], the same is true for the factor group $\mathcal{C}_v(\mathcal{I}^*(R))$ by Theorem 4.5. □

Most arithmetical results, valid for weakly Krull Mori monoids H, are established under the additional assumption that H has nonempty conductor $(H{:}\widehat{H})$ to its complete integral closure. Now let R be a weakly Krull Mori domain. We already know that $\mathcal{I}^*(R)$ is a weakly Krull Mori monoid. The next corollary shows that, if R has nonzero conductor $(R{:}\widehat{R})$, then also $\mathcal{I}^*(R)$ has nonempty conductor, whence all arithmetical results, valid for weakly Krull Mori monoids with nonempty conductor, also apply to $\mathcal{I}^*(R)$.

Corollary 4.7. *Let R be a weakly Krull Mori domain with $(R{:}\widehat{R})\neq\{0\}$. Then $\mathcal{I}_v^*(R)$ and $\mathcal{I}^*(R)$ are weakly Krull Mori monoids with $(\mathcal{I}_v^*(R){:}\widehat{\mathcal{I}_v^*(R)})\neq\emptyset$ and with $(\mathcal{I}^*(R){:}\widehat{\mathcal{I}^*(R)})\neq\emptyset$.*

Proof. $\mathcal{I}_v^*(R)$ and $\mathcal{I}^*(R)$ are weakly Krull Mori monoids by Theorem 4.3, whence it remains to prove the statements on the conductor.

(i) To prove the claim on $\mathcal{I}_v^*(R)$, we use the same notation as in the proof of Corollary 4.4. Thus, we set $\mathfrak{f}=(R{:}\widehat{R})$, $\mathcal{P}^*=\{\mathfrak{p}\in\mathfrak{X}(R)\colon \mathfrak{p}\supset\mathfrak{f}\}$, $\mathcal{P}=\mathfrak{X}(R)\setminus\mathcal{P}^*$, and by Equation (4.2) we have

$$\mathcal{I}_v^*(R)\cong\coprod_{\mathfrak{p}\in\mathfrak{X}(R)}(R_\mathfrak{p}^\bullet)_{\mathrm{red}}\cong\mathcal{F}(\mathcal{P})\times\coprod_{\mathfrak{p}\in\mathcal{P}^*}(R_\mathfrak{p}^\bullet)_{\mathrm{red}}\,. \tag{4.3}$$

Note that $\mathcal{P}^*$ is finite. We use the following simple facts on the complete integral closure of cancellative monoids S_1, S_2, and S.

(a) If $S=S_1\times S_2$, then $\widehat{S}=\widehat{S_1}\times\widehat{S_2}$. Thus, if $(S_i{:}\widehat{S_i})\neq\emptyset$ for all $i\in[1,2]$, then $(S{:}\widehat{S})\neq\emptyset$.

(b) If $(S{:}\widehat{S})\neq\emptyset$ and $\mathfrak{p}\in\mathfrak{X}(S)\neq\emptyset$, then $(S_\mathfrak{p}{:}\widehat{S_\mathfrak{p}})\neq\emptyset$.

Since $(R{:}\widehat{R})\neq\{0\}$, it follows that $(R_\mathfrak{p}^\bullet{:}\widehat{R_\mathfrak{p}^\bullet})\neq\emptyset$ for all $\mathfrak{p}\in\mathcal{P}^*$. Thus, by the isomorphism in (4.3) and by Property (a), it follows that $(\mathcal{I}_v^*(R){:}\widehat{\mathcal{I}_v^*(R)})\neq\emptyset$.

(ii) Next we show that $(\mathcal{I}^*(R){:}\widehat{\mathcal{I}^*(R)})\neq\emptyset$. By Theorem 4.5, the inclusion $\mathcal{I}^*(R)\hookrightarrow\mathcal{I}_v^*(R)$ is a divisor theory, whence $\mathcal{I}_v^*(R)\cap\mathsf{q}\big(\mathcal{I}^*(R)\big)=\mathcal{I}^*(R)$. By (i), there is $I\in(\mathcal{I}_v^*(R){:}\widehat{\mathcal{I}_v^*(R)})\neq\emptyset$. Then there is $J\in\mathcal{I}_v^*(R)$ and $a\in R$ such that $I\cdot_v J=(IJ)_v=aR\in(\mathcal{I}_v^*(R){:}\widehat{\mathcal{I}_v^*(R)})$. Then

$$(aR)\widehat{\mathcal{I}^*(R)}\subset(aR)\widehat{\mathcal{I}_v^*(R)}\cap\mathsf{q}\big(\mathcal{I}^*(R)\big)\subset\mathcal{I}_v^*(R)\cap\mathsf{q}\big(\mathcal{I}^*(R)\big)=\mathcal{I}^*(R)\,,$$

whence $aR\in(\mathcal{I}^*(R){:}\widehat{\mathcal{I}^*(R)})\neq\emptyset$. $\square$

4.3. Transfer Krull monoids

A monoid homomorphism $\theta\colon H\to B$ is called a *transfer homomorphism* if it has the following properties:

(T 1) $B=\theta(H)B^\times$ and $\theta^{-1}(B^\times)=H^\times$.

(T 2) If $u \in H$, b, $c \in B$ and $\theta(u)=bc$, then there exist v, $w \in H$ such that $u=vw$, $\theta(v) \in bB^\times$, and $\theta(w) \in cB^\times$.

Transfer homomorphisms allow to pull back arithmetical properties from B to H. In particular, we have $\mathsf{L}_H(a)=\mathsf{L}_B(\theta(a))$ for all $a \in H$, whence $\mathcal{L}(H)=\mathcal{L}(B)$. This implies that an element $a \in H$ is an atom of H if and only if $\theta(a)$ is an atom of B. A monoid H is called *transfer Krull* if there are a Krull monoid B and a transfer homomorphism $\theta\colon H \to B$. A commutative ring R is said to be transfer Krull if its monoid $R^\bullet$ of regular elements is a transfer Krull monoid.

If H is half-factorial, then $\theta\colon H \to (\mathbb{N}_0,+)$, defined by $\theta(u)=1$ for all $u \in \mathcal{A}(H)$ and $\theta(\varepsilon)=0$ for every $\varepsilon \in H^\times$, is a transfer homomorphism, whence all half-factorial monoids are transfer Krull. Furthermore, all Krull monoids are transfer Krull (with θ being the identity). Since transfer homomorphisms preserve lengths of factorizations, all transfer Krull monoids are BF-monoids but they need neither be v-noetherian nor completely integrally closed. A list of transfer Krull monoids and domains that are not Krull can be found in [47, Example 5.4], and we refer to [16] for a systematic study of the transfer Krull property. On the other hand, here are some monoids that are not transfer Krull.

– $\mathrm{Int}(\mathbb{Z})$ is not transfer Krull, by [33, Remark 12].

– The monoid of polynomials having nonnegative integer coefficients is not transfer Krull, by [20, Remark 54.].

– The monoid of finite nonempty subsets of the nonnegative integers (with set addition as operation) is not transfer Krull, by [31, Proposition 4.12] (see the discussion after Conjecture 5.12).

Moreover, let R be a weakly Krull Mori domain. Then $\mathcal{I}_v^*(R)$ is transfer Krull if and only if it is half-factorial (in the local case this follows from Proposition 4.2.1 and Proposition 4.9; the general case is a simple consequence, see [47, Proposition 7.3] and also [15, Theorem 5.9]). For more results of this flavor, see the references given in the discussion before Corollary 4.4).

We continue with a simple lemma which we will use to show that the monoid of nonzero ideals over a polynomial ring with at least two variables is not transfer Krull (Theorem 5.1).

Lemma 4.8. *Let H be a cancellative monoid and r be an ideal system on H. Suppose there is a non-r-cancellative ideal $I \in \mathcal{I}_r(H)$, an ideal $J_1 \in \mathcal{A}(\mathcal{I}_r(H))$, and a $J_2 \in \mathcal{I}_r(H) \setminus \mathcal{A}(\mathcal{I}_r(H))$ such that $I \cdot_r J_1 = I \cdot_r J_2$. Then $\mathcal{I}_r(H)$ is not a transfer Krull monoid.*

Proof. Assume to the contrary that there are a Krull monoid B and a transfer homomorphism $\theta\colon \mathcal{I}_r(H) \to B$. Then $\theta(I)\theta(J_1)=\theta(I)\theta(J_2)$, whence $\theta(J_1)=\theta(J_2)$

because B is cancellative. Since J_1 is an atom of $\mathcal{I}_r(H)$, $\theta(J_1)=\theta(J_2)$ is an atom of B and hence J_2 is an atom of $\mathcal{I}_r(H)$, a contradiction. $\square$

4.4. Arithmetic of transfer Krull monoids

The arithmetic of Krull monoids is determined by their class groups and the distribution of prime divisors in the classes. There is an abundance of literature on the arithmetic of Krull monoids (see [38] and the survey [64]). We briefly summarize some results valid not only for Krull monoids but more generally for transfer Krull monoids, but we restrict for results on sets of lengths. This will allow us to compare them with the arithmetic of the monoids of ideals discussed in Section 5. In order to do so we recall the monoid of zero-sum sequences over an abelian group.

Let G be an additive abelian group and $G_0 \subset G$ be a subset. An element $S = g_1 \cdot \ldots \cdot g_\ell \in \mathcal{F}(G_0)$, with $\ell \in \mathbb{N}_0$ and $g_1, \ldots, g_\ell \in G_0$, is called a sequence over G_0. Then $|S| = \ell \in \mathbb{N}_0$ is the length of S, $\sigma(S) = g_1 + \ldots + g_\ell \in G$ is the sum of S, and

$$\mathcal{B}(G_0) = \{T \in \mathcal{F}(G_0) \colon \sigma(T) = 0\} \subset \mathcal{F}(G_0)$$

is the *monoid of zero-sum sequences* over G_0. Since the inclusion $\mathcal{B}(G_0) \hookrightarrow \mathcal{F}(G_0)$ is a divisor homomorphism, $\mathcal{B}(G_0)$ is a Krull monoid. As usual, we set $\mathcal{L}(G_0) := \mathcal{L}(\mathcal{B}(G_0))$, $\rho(G_0) := \rho(\mathcal{B}(G_0))$, and $\mathcal{U}_k(G_0) := \mathcal{U}_k(\mathcal{B}(G_0))$ for all $k \in \mathbb{N}$.

Let B be a Krull monoid, $\varphi \colon B \to D = \mathcal{F}(P)$ be a divisor theory, and let $G_P = \{[p] \colon p \in P\} \subset G$ denote the set of classes containing prime divisors. The map

$$\boldsymbol{\beta} \colon B \longrightarrow \mathcal{B}(G_P), \quad \text{defined by} \quad \boldsymbol{\beta}(a) = [p_1] \cdot \ldots \cdot [p_\ell]\,,$$

where $\varphi(a) = p_1 \cdot \ldots \cdot p_\ell$ with $p_1, \ldots, p_\ell \in P$, is a transfer homomorphism.

Let H be a transfer Krull monoid and $\theta_1 \colon H \to B$ be a transfer homomorphism to a Krull monoid B. If G is an abelian group, $G_0 \subset G$ a subset, and $\theta_2 \colon B \to \mathcal{B}(G_0)$, then $\theta = \theta_2 \circ \theta_1 \colon H \to \mathcal{B}(G_0)$ is a transfer homomorphism from H to the monoid of zero-sum sequences over G_0. In this case, we say that H is a transfer Krull monoid over G_0. Since every Krull monoid has a transfer homomorphism onto a monoid of zero-sum sequences, every transfer Krull monoid has a transfer homomorphism to a monoid of zero-sum sequences. If H is a Krull monoid with class group G and every class contains at least one prime divisor, then H is a transfer Krull monoid over the class group G.

Proposition 4.9. *Let H be a transfer Krull monoid and let $\theta \colon H \to \mathcal{B}(G_0)$ be a transfer homomorphism, where $G_0 \subset G$ is a subset of an abelian group.*

1. *H is fully elastic.*

2. *If* G_0 *is finite, then the elasticity* $\rho(H)<\infty$, H *satisfies the Structure Theorem for Sets of Lengths as well as the Structure Theorem for Unions.*

3. *If* G_0 *contains an infinite abelian group, then for every finite subset* $L\subset\mathbb{N}_{\ge 2}$, *there is* $a\in H$ *such that* $\mathsf{L}(a)=L$, *whence* $\mathcal{U}_k(H)=\mathbb{N}_{\ge 2}$ *for all* $k\ge 2$.

Proof. 1. This follows from [46, Theorem 3.1].

2. Suppose that G_0 is finite. We have $\mathcal{L}(H)=\mathcal{L}(G_0)$, whence $\rho(H)=\rho(G_0)$, and $\mathcal{U}_k(H)=\mathcal{U}_k(G_0)$ for all $k\in\mathbb{N}$. Since G_0 is finite, $\mathcal{B}(G_0)$ is finitely generated, whence $\rho(G_0)<\infty$ by [38, Theorem 3.1.4]. Furthermore, H satisfies the Structure Theorem for Sets of Lengths by [38, Chapter 4.7] and the Structure Theorem for Unions by [36, Corollary 3.6 and Theorem 4.2].

3. Suppose that G_0 contains an infinite abelian group G_1. By [38, Theorem 7.4.1], every finite subset $L\subset\mathbb{N}_{\ge 2}$ lies in $\mathcal{L}(G_1)$, and hence $L\in\mathcal{L}(G_1)\subset\mathcal{L}(G_0)=\mathcal{L}(H)$. Since $\{2,k\}\in\mathcal{L}(H)$ for every $k\in\mathbb{N}_{\ge 2}$, it follows that $\mathcal{U}_2(H)=\mathbb{N}_{\ge 2}$, whence $\mathcal{U}_k(H)=\mathbb{N}_{\ge 2}$ for every $k\ge 2$ by Lemma 3.1. □

Let $\theta\colon H\to\mathcal{B}(G_0)$ be as above. If G_0 is a finite abelian group, then there is a rich literature on invariants controlling the structure of sets of lengths ([64]). The elasticity $\rho(H)$ can be finite even if G_0 is infinite (if G is finitely generated, then [49] offers a characterization of when $\rho(H)$ is finite, and if $\rho(H)<\infty$, then also the Structure Theorem for Unions holds). The unions $\mathcal{U}_k(H)$ are intervals if G_0 is a group, but they need not be intervals in general. There are Krull monoids that do not satisfy the Structure Theorem for Unions ([30, Theorem 4.2]), and there are Krull monoids that neither satisfy the Structure Theorem for Sets of Lengths nor does every finite subset $L\subset\mathbb{N}_{\ge 2}$ occur as a set of lengths.

5. On the monoid of nonzero ideals of polynomial rings

The main goal of this section is to prove the result given in Theorem 5.1. We start with a couple of remarks. Let D be a noetherian domain, $n\ge 2$, $S=(\mathbb{N}_0^n,+)$, and $R=D[S]=D[X_1,\dots,X_n]$. Then D is Krull if and only if D is integrally closed if and only if R is Krull if and only if $\mathcal{I}^*(D)$ resp. $\mathcal{I}^*(R)$ are Krull (see Theorem 4.5). Furthermore, $\mathcal{C}(D)$ and $\mathcal{C}(R)$ are isomorphic, D is factorial if and only if R is factorial if and only if $\mathcal{C}(D)$ is trivial. If D is factorial (for example, if D is a field), then

(5.1)
$$\{aR\colon a\in R^\bullet\}=\mathcal{I}^*(R)=\mathcal{I}_v^*(R)=\mathcal{I}_v(R)\quad\text{and all these monoids are factorial.}$$

In orders of Dedekind domains with finite class group, monoids of all nonzero ideals and monoids of invertible ideals have similar arithmetical properties ([18], [42] and

[15]). In contrast to (5.1) and in contrast to orders in Dedekind domains, our conjecture (Conjecture 5.12) is that the arithmetic of the monoid $\mathcal{I}(R)$ is completely different from the arithmetic of $\mathcal{I}^*(R)$ and that it is as wild as it is for Krull monoids with infinite class group and prime divisors in all classes (see Proposition 4.9.3). The main result of this section (Theorem 5.1) is a first step towards this conjecture.

Theorem 5.1. *Let $R=D[X_1,...,X_n]$ be the polynomial ring in $n\geq 2$ indeterminates over a domain D, and suppose that $\mathcal{I}(R)$ is a* BF*-monoid.*

1. $\mathcal{I}(R)$ is neither transfer Krull nor locally finitely generated. Moreover, if $D^\times$ is infinite, then $\mathcal{I}(R)$ is not an FF*-monoid.*

2. $\mathcal{U}_k(\mathcal{I}(R))=\mathbb{N}_{\geq 2}$ for all $k\geq 2$.

3. $\mathsf{L}_{\mathcal{I}(R)}(\langle X_1,X_2\rangle^k)=[2,k]$ for all $k\geq 2$.

4. $\mathcal{I}(R)$ is fully elastic.

We briefly discuss the assumption that $\mathcal{I}(R)$ is a BF-monoid (made in Theorem 5.1, Lemma 5.3, Proposition 5.10, and Conjecture 5.12). If R is noetherian or a one-dimensional Mori domain, then $\mathcal{I}(R)$ is a BF-monoid (see Proposition 2.2 and the discussion after Proposition 2.1). But the property, that $\mathcal{I}(R)$ is a BF-monoid, seems to be much weaker than the above two assumptions. A crucial property in this context is Krull's Intersection Theorem, which guarantees that the semigroup $\mathcal{I}(R)$ is unit-cancellative. We mention two further classes of polynomial rings which satisfy Krull's Intersection Theorem (for more on the validity of Krull's Intersection Theorem, we refer to [4] and [51]).

(i) Let D be a domain, $\overline{D}$ its integral closure (in the quotient field of D), and let D^* be any domain with $D\subset D^*\subset\overline{D}$. The integral closure of $D[X_1,...,X_n]$ equals $\overline{D}[X_1,...,X_n]$, and we have

$$D[X_1,...,X_n]\subset D^*[X_1,...,X_n]\subset\overline{D}[X_1,...,X_n]\,.$$

If D is noetherian, then $D[X_1,...,X_n]$ is noetherian, whence $D^*[X_1,...,X_n]$ satisfies Krull's Intersection Theorem by [1, Theorem 5]. Moreover, if D is noetherian, then $\overline{D}$ is Krull and if D is Krull, then $D[X_1,...,X_n]$ is a Krull domain with class group isomorphic to $\mathcal{C}(D)$ and infinitely many prime divisors in all classes (compare with Corollary 4.6).

(ii) If the integral closure $\overline{R}$ of $R=D[X_1,...,X_n]$ in some field extension of the quotient field of R is noetherian, then R satisfies Krull's Intersection Theorem by [52, Proposition 2.6].

We proceed in a series of lemmas. Let R be a domain and $X_1,X_2\in R^\bullet$. For all $i\in\mathbb{N}$, we consider the following four families of nonzero ideals of R:

(i) $\mathfrak{a}_i(X_1,X_2):=\langle X_1,X_2\rangle^i$,

(ii) $\mathfrak{b}_i(X_1,X_2):=\langle X_1^i,X_2^i\rangle$,

(iii) $\mathfrak{c}_{2i+1}(X_1,X_2){:=}\langle\{X_1^{2i+1},X_1^{2i}X_2\}\cup\{X_1^{2i-j}X_2^{j+1} \ : \ j{\in}[1,2i{+}1] \text{ is even}\}\rangle$, and

(iv) $\mathfrak{c}_{2i}(X_1,X_2){:=}\langle\{X_1^{2i},X_1^{2i-1}X_2\}\cup\{X_1^{2i-j}X_2^{j} : j{\in}[1,2i] \text{ is even}\}\rangle$.

Lemma 5.2. *Let R be a domain and $X_1,X_2{\in}R^\bullet$.*
1. $\mathfrak{a}_1(X_1,X_2)^k{=}\mathfrak{a}_k(X_1,X_2)$ for all $k{\in}\mathbb{N}$.
2. $\mathfrak{a}_k(X_1,X_2){\cdot}\mathfrak{b}_\ell(X_1,X_2){=}\mathfrak{a}_{k+\ell}(X_1,X_2)$ for all $k,\ell{\in}\mathbb{N}$ with $k{\geq}\ell{-}1$.
3. $\mathfrak{a}_1(X_1,X_2){\cdot}\mathfrak{c}_{2k+1}(X_1,X_2){=}\mathfrak{a}_{2k+2}(X_1,X_2)$ for all $k{\in}\mathbb{N}$.
4. $\mathfrak{a}_1(X_1,X_2){\cdot}\mathfrak{c}_{2k}(X_1,X_2){=}\mathfrak{a}_{2k+1}(X_1,X_2)$ for all $k{\in}\mathbb{N}$.

Proof. This follows by direct calculations. □

Lemma 5.3. *Let R be a domain such that $\mathcal{I}(R)$ is a* BF*-monoid. Suppose there exist distinct $X_1,X_2{\in}R^\bullet$ such that $\mathfrak{a}_1(X_1,X_2)$, $\mathfrak{b}_2(X_1,X_2)$, and $\mathfrak{c}_{2i+1}(X_1,X_2)$ are atoms of $\mathcal{I}(R)$ for all $i{\in}\mathbb{N}$. Then $\mathcal{I}(R)$ has the following properties.*
1. $\mathcal{I}(R)$ is not a transfer Krull monoid.
2. $\mathcal{I}(R)$ is not locally finitely generated.
3. $\mathcal{U}_k(\mathcal{I}(R)){=}\mathbb{N}_{\geq 2}$ for all $k{\geq}2$.

Proof. 1. By Lemma 5.2 (items 1 and 2), we obtain that

$$\mathfrak{a}_1(X_1,X_2){\cdot}\mathfrak{b}_2(X_1,X_2)=\mathfrak{a}_3(X_1,X_2)=\mathfrak{a}_1(X_1,X_2){\cdot}\mathfrak{a}_2(X_1,X_2)\,.$$

Therefore, Lemma 4.8 implies that $\mathcal{I}(R)$ is not transfer Krull.

2. By Lemma 5.2.3, the divisor-closed submonoid $[\![\mathfrak{a}_1(X_1,X_2)]\!]{\subset}\mathcal{I}(R)$ contains infinitely many atoms, whence $\mathcal{I}(R)$ is not locally finitely generated.

3. By Lemma 3.1.3, it suffices to prove that $\mathcal{U}_2(\mathcal{I}(R)){=}\mathbb{N}_{\geq 2}$. Since $\mathfrak{c}_{2i+1}(X_1,X_2)$ is an atom of $\mathcal{I}(R)$ for all $i{\in}\mathbb{N}$, Lemma 5.2.3 implies that

$$\{\nu\in\mathbb{N} \ : \ \nu\equiv 0 \mod 2\}\subset\mathcal{U}_2(\mathcal{I}(R)).$$

Furthermore, by Lemma 5.2 (items 2 and 3), we observe that

$$\begin{aligned}\mathfrak{a}_{2i}(X_1,X_2){\cdot}\mathfrak{b}_2(X_1,X_2)&=\mathfrak{a}_{2i-1}(X_1,X_2){\cdot}\mathfrak{a}_3(X_1,X_2)=\mathfrak{a}_{2i+2}(X_1,X_2)\\&=\mathfrak{a}_1(X_1,X_2){\cdot}\mathfrak{c}_{2i+1}(X_1,X_2)\end{aligned}$$

for all $i{\in}\mathbb{N}$, whence

$$\{\nu\in\mathbb{N}_{\geq 3} \ : \ \nu\equiv 1 \mod 2\}\subset\mathcal{U}_2(\mathcal{I}(R))\,. \quad \square$$

Lemma 5.4. *Let R be a domain and $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal of R for some $\mathfrak{p}{\in}\mathrm{spec}(R)$. Let $\mathfrak{q}{=}IJ$, where I and J are ideals of R such that $\mathfrak{q}{\subsetneq}I,J{\subsetneq}R$. Then* $\mathrm{Rad}\,I{=}\mathrm{Rad}\,J{=}\mathfrak{p}$.

Proof. Since $\mathfrak{p}=\operatorname{Rad}\mathfrak{q}=\operatorname{Rad}I\cap\operatorname{Rad}J$, so either $\operatorname{Rad}I=\mathfrak{p}$ or $\operatorname{Rad}J=\mathfrak{p}$, say $\operatorname{Rad}I=\mathfrak{p}$. It remains to prove that $\operatorname{Rad}J=\mathfrak{p}$. Clearly $\mathfrak{p}\subset\operatorname{Rad}J$. Conversely, if $g\in\operatorname{Rad}J$, then there exists some positive integer s such that $g^s\in J$. Since $\mathfrak{q}\subsetneq I$, there exists an element $f\in I\setminus\mathfrak{q}$. As $fg^s\in IJ=\mathfrak{q}$ and $f\notin\mathfrak{q}$, there exists some positive integer t such that $g^{st}\in\mathfrak{q}$ and hence $g\in\operatorname{Rad}\mathfrak{q}=\mathfrak{p}$. □

From now on till the end of the proof of Theorem 5.1, we fix the following notation. Let D be a domain with quotient field K, and let $R=D[X_1,\dots,X_n]$ and $S=K[X_1,\dots,X_n]$ be polynomial rings in $n\ge 2$ variables $X_1,\dots,X_n$. They are equipped with the natural $\mathbb{N}$-grading such that $\deg(X_1)=\dots=\deg(X_n)=1$. We set

$$R=\bigoplus_{t\ge 0}R_t, \quad \text{and} \quad S=\bigoplus_{t\ge 0}S_t\,,$$

where $R_t\subset S_t$ are the corresponding t-components. Every $f\in S$ can be written uniquely in the form $f=\sum_{i\ge 0}f_i$, where $f_i\in S_i$ for all $i\in\mathbb{N}_0$ and $f_i=0$ for all but finitely many $i\in\mathbb{N}_0$. We denote by $<$ the lexicographic order on monomials of $K[X_1,\dots,X_n]$ with $X_1>X_2>\dots>X_n$. For $f\in K[X_1,\dots,X_n]$, we denote by $\mathsf{in}(f)$ the *initial monomial* of f with respect to the order $<$.

The *min-degree* $\mathsf{m\text{-}deg}(f)$ of a nonzero polynomial $f\in S$ is the smallest nonnegative integer d such that f_d is nonzero. We set $\mathsf{m\text{-}deg}(0)=+\infty$. The $\mathsf{m\text{-}deg}$ function satisfies the following two properties for all $f,g\in S$:

(i) $\mathsf{m\text{-}deg}(fg)=\mathsf{m\text{-}deg}(f)+\mathsf{m\text{-}deg}(g)$, and

(ii) $\mathsf{m\text{-}deg}(f+g)\ge\min\{\mathsf{m\text{-}deg}(f),\,\mathsf{m\text{-}deg}(g)\}$, with equality if $\mathsf{m\text{-}deg}(f)\neq\mathsf{m\text{-}deg}(g)$.

Next we introduce the minimal degree of an ideal of R. Let $I\subset R$ be a nonzero ideal. We define the *min-degree* $\mathsf{m\text{-}deg}(I)$ of I to be the smallest nonnegative integer d such that I contains a polynomial whose min-degree is equal to d. We set the min-degree of the zero ideal equal to $+\infty$.

The next lemma says that the map

$$\mathsf{m\text{-}deg}\ :\ \mathcal{I}(R)\longrightarrow\mathbb{N}_0 \quad \text{given by} \quad I\longmapsto\mathsf{m\text{-}deg}(I)$$

is a semigroup homomorphism.

Lemma 5.5. *For every $I,J\in\mathcal{I}(R)$, we have* $\mathsf{m\text{-}deg}(IJ)=\mathsf{m\text{-}deg}(I)+\mathsf{m\text{-}deg}(J)$.

Proof. We set $d=\mathsf{m\text{-}deg}(I)$, $e=\mathsf{m\text{-}deg}(J)$, $m=\mathsf{m\text{-}deg}(IJ)$, and we choose $f\in I$ with $\mathsf{m\text{-}deg}(f)=d$, $g\in J$ with $\mathsf{m\text{-}deg}(g)=e$, and $h\in IJ$ such that $\mathsf{m\text{-}deg}(h)=m$. By the above property (i), we have $m\le d+e$. On the other hand, we have $h=f_1g_1+\dots+f_sg_s$, where $f_i\in I$ and $g_i\in J$ for all $i\in[1,s]$. Then the above property (ii) implies that

$$m=\mathsf{m\text{-}deg}(h)\ \ge\ \min\{\mathsf{m\text{-}deg}(f_i)+\mathsf{m\text{-}deg}(g_i)\colon i\in[1,s]\}\ge d+e\,. \quad \square$$

Let $I, J \in \mathcal{I}(R)$. We set $I[i]=\{f_i : f \in I\}$ and note that $I[i] \subset R_i$ is a D-module. For $i, j \in \mathbb{N}_0$,

$$I[i] \cdot J[j] = \Big\{ \sum_{s=1}^{k} a_s b_s \, : \, k \geq 1, \, a_s \in I[i], \, b_s \in J[j] \Big\}$$

is also a D-module. Let $I_K[i]=\mathrm{Span}_K\{ I[i] \}$, $J_K[j]=\mathrm{Span}_K\{ J[j] \}$ and, for $m \in \mathbb{N}_0$, $(IJ)_K[m]=\mathrm{Span}_K\{ (IJ)[m] \}$. Clearly, $I_K[i] \subset S_i$, $J_K[j] \subset S_j$, and $(IJ)_K[m] \subset S_m$, whence $I_K[i], J_K[j], (IJ)_K[m]$ are finite dimensional K-vector spaces.

Lemma 5.6. *Let* $I, J \in \mathcal{I}(R)$ *with* $\mathsf{m\text{-}deg}(I)=d$, $\mathsf{m\text{-}deg}(J)=e$, *and* $\mathsf{m\text{-}deg}(IJ)=m$.

1. $(IJ)[m]=I[d] \cdot J[e]$.

2. $(IJ)_K[m]=I_K[d] \cdot J_K[e]$.

Proof. 1. Let $a \in (IJ)[m]$. Then there exists $h \in IJ$ such that $h_m=a$, say $h=\sum_{i=1}^k f^{(i)} g^{(i)}$, where $f^{(i)} \in I$ and $g^{(i)} \in J$ for all $i \in [1,k]$. By Lemma 5.5, we obtain that $a=h_m=\sum_{i=1}^k f_d^{(i)} g_e^{(i)}$, whence $a \in I[d] \cdot J[e]$. Conversely, if $a \in I[d] \cdot J[e]$, then $a=\sum_{i=1}^k a_i b_i$ where $a_i \in I[d], b_i \in J[e]$ for all $i \in [1,k]$. Thus, there exist $f^{(i)} \in I, g^{(i)} \in J$ such that $f_d^{(i)}=a_i$ and $g_e^{(i)}=b_i$ for all $i \in [1,k]$. Hence $h=\sum_{i=1}^k f^{(i)} g^{(i)} \in IJ$ with $h_m=a \in (IJ)[m]$.

2. Let $a \in (IJ)_K[m]$. Then, by definition, $a=\sum_{\nu=1}^r (\alpha_\nu/\beta_\nu) h_\nu$, where $r \in \mathbb{N}$, $h_\nu \in (IJ)[m]$, $\alpha_\nu, \beta_\nu \in D$ and $\beta_\nu \neq 0$ for every $\nu \in [1,r]$. By 1., we obtain $h_\nu=\sum_{j_\nu=1}^{s_\nu} a_{j_\nu}^{(\nu)} b_{j_\nu}^{(\nu)}$, where $a_{j_\nu}^{(\nu)} \in I[d]$ and $b_{j_\nu}^{(\nu)} \in J[e]$ for every $j_\nu \in [1,s_\nu]$. Hence,

$$a = \sum_{\nu=1}^{r} \sum_{j_\nu=1}^{s_\nu} (\alpha_\nu/\beta_\nu) a_{j_\nu}^{(\nu)} b_{j_\nu}^{(\nu)} \in I_K[d] \cdot J_K[e] \, .$$

Conversely, let $\sum_{\nu=1}^r a_\nu b_\nu \in I_K[d] \cdot J_K[e]$. Then

$$a_\nu = \sum_{j_\nu=1}^{s_\nu} (\alpha_{\nu,j_\nu}/\beta_{\nu,j_\nu}) a_{j_\nu}^{(\nu)} \quad \text{and} \quad b_\nu = \sum_{k_\nu=1}^{t_\nu} (\gamma_{\nu,k_\nu}/\theta_{\nu,k_\nu}) b_{k_\nu}^{(\nu)} \, ,$$

where $a_{j_\nu}^{(\nu)} \in I[d]$, $b_{k_\nu}^{(\nu)} \in J[e]$ and $\alpha_{\nu,j_\nu}, \beta_{\nu,j_\nu}, \gamma_{\nu,k_\nu}, \theta_{\nu,k_\nu} \in D$ with $\beta_{\nu,j_\nu} \neq 0 \neq \theta_{\nu,k_\nu}$ for all ν, j_ν, k_ν. If we denote $\beta_\nu=\prod_{j_\nu=1}^{s_\nu} \beta_{\nu,j_\nu}$ and $\theta_\nu=\prod_{k_\nu=1}^{t_\nu} \theta_{\nu,k_\nu}$, then $a_\nu b_\nu$ can be written as

$$a_\nu b_\nu = (1/\beta_\nu \theta_\nu) \sum_{j_\nu=1}^{s_\nu} \sum_{k_\nu=1}^{t_\nu} \tau_{j_\nu,k_\nu} a_{j_\nu}^{(\nu)} b_{k_\nu}^{(\nu)} \, ,$$

where all $\tau_{j_\nu,k_\nu} \in D$. Setting $\beta=\prod_{\nu=1}^r \beta_\nu$ and $\theta=\prod_{\nu=1}^r \theta_\nu$ we obtain that

$$\sum_{\nu=1}^{r} a_\nu b_\nu = (1/\beta\theta) \sum_{\nu=1}^{r} \sum_{j_\nu=1}^{s_\nu} \sum_{k_\nu=1}^{t_\nu} \rho_{j_\nu,k_\nu} a_{j_\nu}^{(\nu)} b_{k_\nu}^{(\nu)},$$

where all $\rho_{j_\nu,k_\nu}\in D$. Therefore, $\beta\theta(\sum_{\nu=1}^r a_\nu b_\nu)\in I[d]\cdot J[e]$ and hence, again by using 1., we get

$$\beta\theta(\sum_{\nu=1}^r a_\nu b_\nu)=h_m,$$

where $h\in IJ$. Thus, we obtain that $\sum_{\nu=1}^r a_\nu b_\nu\in(IJ)_K[m]$. □

Before moving further we demonstrate in simple special cases how our techniques work for studying factorizations in the monoid of nonzero ideals of polynomial rings.

Example 5.7.

1. We claim that the ideal $\langle X^2,Y^2\rangle\subset K[X,Y]$ is an atom of $\mathcal{I}(K[X,Y])$. Assume to the contrary that there are two nonzero proper ideals I,J of $K[X,Y]$ with $\mathsf{m\text{-}deg}(I)=d$, $\mathsf{m\text{-}deg}(J)=e$ and

$$\langle X^2,Y^2\rangle=IJ.$$

By Lemma 5.6 we obtain

$$\mathrm{Span}_K\{X^2,Y^2\}=I_K[d]\cdot J_K[e], \tag{5.2}$$

and we have $d+e=2$ by Lemma 5.5. Furthermore, we have $\mathrm{Rad}\,I=\mathrm{Rad}\,J=\langle X,Y\rangle$, whence $d=e=1$. Clearly $\dim_K I[d]\ge1$ and $\dim_K J[e]\ge1$. Assume to the contrary that one of these dimensions equals one, say $I_K[d]=\mathrm{Span}_K\{f\}$ and $J_K[e]=\mathrm{Span}_K\{g_1,\dots,g_s\}$. Then $X^2=\sum_{j=1}^s\alpha_j f g_j$ where $\alpha_j\in K$ for all $j\in[1,s]$. This implies that $f=\alpha X$ for some $\alpha\in K^\times$, which is not possible since $Y^2\in I_K[d]\cdot J_K[e]$. Therefore, $\dim_K I_K[d]\ge2$ and $\dim_K J[e]\ge2$. Since $I_K[d],J_K[e]\subset\mathrm{Span}_K\{X,Y\}$, it follows that $I_K[d]=J_K[e]=\mathrm{Span}_K\{X,Y\}$. This implies that $I_K[d]\cdot J_K[e]=\mathrm{Span}_K\{X^2,XY,Y^2\}$, a contradiction to (5.2).

2. We claim that the ideal $\langle X^3+Y^3,X^2Y,XY^2\rangle\subset K[X,Y]$ is an atom of $\mathcal{I}(K[X,Y])$. Assume to the contrary that there are two nonzero proper ideals I,J of $K[X,Y]$ with $\mathsf{m\text{-}deg}(I)=d$, $\mathsf{m\text{-}deg}(J)=e$ and

$$\langle X^3+Y^3,X^2Y,XY^2\rangle=IJ.$$

By Lemma 5.6 we obtain

$$\mathrm{Span}_K\{X^3+Y^3,X^2Y,XY^2\}=I_K[d]\cdot J_K[e], \tag{5.3}$$

and we have $d+e=3$ by Lemma 5.5. Since $\langle X,Y\rangle^4\subset\langle X^3+Y^3,X^2Y,XY^2\rangle$, we infer that $\mathrm{Rad}\,I=\mathrm{Rad}\,J=\langle X,Y\rangle$. Thus, $d,e\ge1$, and after renumbering if necessary we suppose that $d=1$ and $e=2$. Clearly $\dim_K I[d]\ge1$ and $\dim_K J[e]\ge1$. Assume to the contrary that one of these dimensions equals one, say $I_K[d]=\mathrm{Span}_K\{f\}$ and $J_K[e]=\mathrm{Span}_K\{g_1,...,g_s\}$. Then $X^2Y=\sum_{j=1}^{s}\alpha_j f g_j$ where $\alpha_j\in K$ for all $j\in[1,s]$. This implies that $f=\alpha X$ or $f=\alpha' Y$ for some $\alpha,\alpha'\in K^\times$, which is not possible since $X^3+Y^3\in I_K[d]\cdot J_K[e]$. Therefore, $\dim_K I_K[d]\geq 2$ and $\dim_K J[e]\geq 2$. Since $I_K[d]\subset \mathrm{Span}_K\{X,Y\}$, it follows that $I_K[d]=\mathrm{Span}_K\{X,Y\}$. For every $i\in[1,s]$, we have $Xg_i\in I_K[d]\cdot J_K[e]$ and thus, by (5.3),

$$Xg_i=\beta_{i,1}(X^3+Y^3)+\beta_{i,2}X^2Y+\beta_{i,3}XY^2$$

where $\beta_{i,j}\in K$ for every $j\in[1,3]$. This implies that $\beta_{i,1}=0$ and $g_i=\beta_{i,2}XY+\beta_{i,3}Y^2$. Similarly, $Yg_i\in I_K[d]\cdot J_K[e]$ yields

$$Yg_i=\gamma_{i,1}(X^3+Y^3)+\gamma_{i,2}X^2Y+\gamma_{i,3}XY^2$$

where $\gamma_{i,j}\in K$ for every $j\in[1,3]$. This implies that $\gamma_{i,1}=0$ and $g_i=\gamma_{i,2}X^2+\gamma_{i,3}XY$. Hence $\beta_{i,3}=\gamma_{i,2}=0$ and $g_i=\beta_{i,2}XY$. Therefore, $J_K[e]=\mathrm{Span}_K\{XY\}$, a contradiction to $\dim_K J_K[e]\geq 2$.

3. The following equation

$$\langle X^2,Y^2\rangle\langle X^3+Y^3,X^2Y,XY^2\rangle=\langle X,Y\rangle^5$$

involves only atoms of $\mathcal{I}(K[X,Y])$, whence it shows that $2,5\in\mathsf{L}_{\mathcal{I}(K[X,Y])}(\langle X,Y\rangle^5)$.

Example 5.7.1 was already settled in [52, Proposition 4.6], but our techniques allow us to study polynomial ideals over domains whose semigroup of nonzero ideals is a BF-monoid (see Proposition 5.10). For $m\in\mathbb{N}_0$, let $\mathcal{M}_{m;1,2}$ denote the set of all monomials of the form $X_1^rX_2^s$ with $r+s=m$ and $r,s\in\mathbb{N}_0$.

Lemma 5.8. *The ideals*

$$\mathfrak{c}'(X_1,X_2)=\langle X_1^3+X_2^3,X_1^2X_2,X_1X_2^2\rangle \quad \text{and} \quad \mathfrak{a}(X_1,X_2)=\langle\{X_1^m,X_2^m\}\cup\mathcal{N}\rangle\,,$$

where $m\in\mathbb{N}$ and $\mathcal{N}\subset\mathcal{M}_{m;1,2}$ is any subset, are $\langle X_1,X_2\rangle$-primary in R. In particular, if $\mathfrak{c}'(X_1,X_2)=IJ$ or $\mathfrak{a}(X_1,X_2)=IJ$, where I and J are ideals of R such that $\mathfrak{c}'(X_1,X_2)\subsetneq I,J\subsetneq R$ or $\mathfrak{a}(X_1,X_2)\subsetneq I,J\subsetneq R$, then $\mathsf{m\text{-}deg}(I)\geq 1$ and $\mathsf{m\text{-}deg}(J)\geq 1$.

Proof. We set $\mathfrak{c}':=\mathfrak{c}'(X_1,X_2)$ and $\mathfrak{a}:=\mathfrak{a}(X_1,X_2)$. It is a well known fact that if an ideal $\mathfrak{q}\subset R$ is $\mathfrak{p}$-primary, then its extension $\mathfrak{q}[X]\subset R[X]$ is $\mathfrak{p}[X]$-primary, cf. [10, Exercise 4.7 (iii)]. We proceed by induction on the number of indeterminates $n\geq 2$.

Let $n=2$. Then the ideal extensions $\mathfrak{c}'K[X_1,X_2]$ and $\mathfrak{a}K[X_1,X_2]$ are $\langle X_1,X_2\rangle K[X_1,X_2]$-primary, because $\langle X_1,X_2\rangle K[X_1,X_2]^4\subset\mathfrak{c}'K[X_1,X_2]$, $\langle X_1,X_2\rangle K[X_1,X_2]^{2m-1}\subset\mathfrak{a}K[X_1,X_2]$ and $\langle X_1,X_2\rangle K[X_1,X_2]$ is a maximal ideal. Therefore the ideal contractions $\mathfrak{c}'K[X_1,X_2]\cap D[X_1,X_2]$ and $\mathfrak{a}K[X_1,X_2]\cap D[X_1,X_2]$ are $\langle X_1,X_2\rangle K[X_1,X_2]\cap D[X_1,X_2]$-primary. If we prove that

$$\langle X_1,X_2\rangle K[X_1,X_2]\cap D[X_1,X_2]\subset\langle X_1,X_2\rangle, \mathfrak{c}'K[X_1,X_2]\cap D[X_1,X_2]\subset\mathfrak{c}' \text{ and } \mathfrak{a}K[X_1,X_2]\cap D[X_1,X_2]\subset\mathfrak{a},$$

then we are done. Let $f/\alpha=g\in\langle X_1,X_2\rangle K[X_1,X_2]\cap D[X_1,X_2]$, where $f\in\langle X_1,X_2\rangle$, $g\in D[X_1,X_2]$ and $\alpha\in D^\bullet$. Then $\alpha g\in\langle X_1,X_2\rangle$ and hence $g\in\langle X_1,X_2\rangle$.

Let now $f/\alpha=g\in\mathfrak{c}'K[X_1,X_2]\cap D[X_1,X_2]$, where $f\in\mathfrak{c}', g\in D[X_1,X_2]$ and $\alpha\in D^\bullet$. Then $\alpha g\in\mathfrak{c}'$ and it only requires to show that $g\in\mathfrak{c}'$. But $\alpha g\in\mathfrak{c}'$ implies that $g=\beta(X_1^3+X_2^3)+g_{21}X_1^2X_2+g_{12}X_1X_2^2+g'$, where $\beta,g_{21},g_{12}\in D$ and $g'\in D[X_1,X_2]$ with $\mathsf{m}\text{-}\mathsf{deg}(g')=4$. Since $g'\in\langle X_1,X_2\rangle^4\subset\mathfrak{c}'$, so $g\in\mathfrak{c}'$.

Let now $f/\alpha=g\in\mathfrak{a}K[X_1,X_2]\cap D[X_1,X_2]$, where $f\in\mathfrak{a}, g\in D[X_1,X_2]$ and $\alpha\in D^\bullet$. Then $\alpha g\in\mathfrak{a}$ and hence by using similar calculations as above we get $g\in\mathfrak{a}$.

Assume now the result is true for $n\geq 2$ and consider the ring $R[X_{n+1}]$. So $\mathfrak{c}'[X_{n+1}]$ and $\mathfrak{a}[X_{n+1}]$ are $\langle X_1,X_2\rangle[X_{n+1}]$-primary ideals of $R[X_{n+1}]$, but in $R[X_{n+1}]$ we have

$$\mathfrak{c}'[X_{n+1}]=\mathfrak{c}', \quad \mathfrak{a}[X_{n+1}]=\mathfrak{a} \quad \text{and} \quad \langle X_1,X_2\rangle[X_{n+1}]=\langle X_1,X_2\rangle.$$

Thus $\mathfrak{a}$ and $\mathfrak{c}'$ are $\langle X_1,X_2\rangle$-primary. The in particular statement follows by Lemma 5.4. $\square$

Lemma 5.9. *Let $V\subset S_d$, with $d\in\mathbb{N}$, be a K-vector subspace of dimension $\dim_K(V)=s$. Then there are linearly independent elements $f_1,\ldots,f_s\in V$ such that $\mathsf{in}(f_1)>\ldots>\mathsf{in}(f_s)$.*

Proof. Let $g_1,\ldots,g_s$ be linearly independent elements of V. If we consider the canonical isomorphism $S_d\cong K^{\binom{d+n-1}{d}}$, then $g_1,\ldots,g_s$ will represent the corresponding linearly independent vectors of $V\subset K^{\binom{d+n-1}{d}}$. Let A be an $s\times\binom{d+n-1}{d}$ matrix whose rows are $g_1,\ldots,g_s$. Now we apply the Gaussian elimination on A and reduce it to row echelon form. The resulting matrix is represented by rows, say $f_1,\ldots,f_s\in V$, which are again linearly independent elements and satisfy our requirement $\mathsf{in}(f_1)>\ldots>\mathsf{in}(f_s)$. $\square$

Proposition 5.10. *Suppose that $\mathcal{I}(R)$ is a* BF*-monoid. Then the following ideals are atoms of $\mathcal{I}(R)$.*

1. *$\mathfrak{b}_i(X_1,X_2)$ for every $i\in\mathbb{N}$,*
2. *$\mathfrak{c}_{2i+1}(X_1,X_2)$ for every $i\in\mathbb{N}$,*
3. *$\mathfrak{c}_{2i}(X_1,X_2)$ for every $i\in\mathbb{N}_{\geq 3}$, and*
4. *$\mathfrak{c}'(X_1,X_2)=\langle X_1^3+X_2^3, X_1^2X_2, X_1X_2^2\rangle$.*

Proof. For all $i\in\mathbb{N}$, we use the abbreviations $\mathfrak{b}_i:=\mathfrak{b}_i(X_1,X_2)$, $\mathfrak{c}_{2i+1}:=\mathfrak{c}_{2i+1}(X_1,X_2)$, $\mathfrak{c}_{2i}:=\mathfrak{c}_{2i}(X_1,X_2)$, and $\mathfrak{c}':=\mathfrak{c}'(X_1,X_2)$. In order to show that an ideal $\mathfrak{a}$ is an atom of $\mathcal{I}(R)$, it suffices to show that there are no ideals $I,J\in\mathcal{I}(R)$ with $\mathfrak{a}\subsetneq I,J\subsetneq R$ such that $\mathfrak{a}=IJ$, because $\mathcal{I}(R)$ is a reduced unit-cancellative semigroup.

1. Let $i\in\mathbb{N}$ and assume to the contrary that $\mathfrak{b}_i=IJ$ with $\mathfrak{b}_i\subsetneq I,J\subsetneq R$ such that $\mathsf{m\text{-}deg}(I)=d$ and $\mathsf{m\text{-}deg}(J)=e$. Then, by Lemma 5.8, we have $d\geq 1$ and $e\geq 1$, and by Lemma 5.6.2 we obtain that

$$\text{Span}_K\{X_1^i,X_2^i\}=I_K[d]\cdot J_K[e]. \tag{5.4}$$

Note that $I_K[d]$ and $J_K[e]$ are finite vector spaces and by Equation 5.4 it is not possible that $\dim_K I_K[d]=\dim_K J_K[e]=1$. Thus, without loss of generality, we may assume that $\dim_K I_K[d]\geq 2$ and $\dim_K J_K[e]\geq 1$. Let $f_1,f_2\in I_K[d]$ be linearly independent and $g\in J_K[e]$ any nonzero element.

By Lemma 5.9, we may assume that $\mathsf{in}(f_1)>\mathsf{in}(f_2)$. If $\mathsf{in}(f_i)$ does not equal X_1^d or X_2^d, then $\mathsf{in}(f_ig)$ $(=\mathsf{in}(f_i)\,\mathsf{in}(g))$ does not equal X_1^i or X_2^i, which is not possible by (5.4). This means we must have $\mathsf{in}(f_1)=X_1^d$ and $\mathsf{in}(f_2)=X_2^d$. If $\mathsf{in}(g)=X_1^e$, then $\mathsf{in}(f_2g)=X_1^eX_2^d$, a contradiction to (5.4). If $\mathsf{in}(g)\neq X_1^e$, say $\mathsf{in}(g)=X_1^{a_1}\cdot\ldots\cdot X_n^{a_n}$ such that $\sum_{j=1}^n a_i=e$ with $a_1<e$ and $a_j\geq 1$ for some $j\in[2,n]$, then $\mathsf{in}(f_1g)$ is divisible by X_1X_j, again a contradiction to (5.4).

2. Let $i\in\mathbb{N}$ and assume to the contrary that $\mathfrak{c}_{2i+1}=IJ$ with $\mathfrak{c}_{2i+1}\subsetneq I,J\subsetneq R$ such that $\mathsf{m\text{-}deg}(I)=d$ and $\mathsf{m\text{-}deg}(J)=e$. Then, by Lemma 5.8, we have $d\geq 1$ and $e\geq 1$, by Lemma 5.5, we have $2i+1=d+e$, and Lemma 5.6.2 implies that

$$\begin{aligned}\text{Span}_K\{\{X_1^{2i+1},X_1^{2i}X_2\}\cup\{X_1^{2i-j}X_2^{j+1}\ :\ j\in[1,2i+1]\text{ and }j\equiv 0\mod 2\}\}\\ =I_K[d]\cdot J_K[e].\end{aligned} \tag{5.5}$$

We claim that $\dim_K I[d]\geq 2$ and $\dim_K J[e]\geq 2$. Indeed, if $I_K[d]=\text{Span}_K\{f\}$ and $J_K[e]=\text{Span}_K\{g_1,\ldots,g_s\}$, then $I_K[d]\cdot J_K[e]=\text{Span}_K\{fg_1,\ldots,fg_s\}$. From (5.5), we get

$$X_1^{2i+1}=\sum_{j=1}^s\alpha_jfg_j,\ \text{where }\alpha_j\in K\text{ for all }j\in[1,s]\,,$$

and we deduce $f=\alpha X_1^d$ for some $\alpha\in K^\times$. Hence X_2^{2i+1} cannot belong to $I_K[d]\cdot J_K[e]$, a contradiction to (5.5).

Let $\dim_K I[d]=r\geq 2$ and $f_1,f_2,\ldots,f_r\in I_K[d]$ be linearly independent such that $\mathsf{in}(f_1)>\mathsf{in}(f_2)>\ldots>\mathsf{in}(f_r)$ (we use Lemma 5.9). Similarly, let $\dim_K J_K[e]=s\geq 2$ and let $g_1,\ldots,g_s\in J_K[e]$ be linearly independent such that $\mathsf{in}(g_1)>\ldots>\mathsf{in}(g_s)$. If $\mathsf{in}(f_1)\neq X_1^d$, then $\mathsf{in}(f_1g)\neq X_1^{2i+1}$ for any $g\in J_K[e]$, which is not possible by 5.5. Thus, we have $\mathsf{in}(f_1)=X_1^d$, and hence $\mathsf{in}(g_1)=X_1^e$. Without loss of generality assume that d is odd, while e is even. If $\mathsf{in}(f_2)=X_1^{d-u}X_2^u$ for an even $u\in\mathbb{N}$, then

$\mathsf{in}(f_2g_1)=X_1^{d+e-u}X_2^u$, a contradiction to (5.5). If $\mathsf{in}(f_2)=X_1^{d-u}X_2^u$ for an odd $u\in\mathbb{N}$, then we consider $\mathsf{in}(g_2)$. If $\mathsf{in}(g_2)=X_1^{e-v}X_2^v$ for an odd $v\in\mathbb{N}$, we get a contradiction as $\mathsf{in}(f_2g_2)=X_1^{d+e-u-v}X_2^{u+v}$. If $\mathsf{in}(g_2)=X_1^{e-v}X_2^v$ for an even $v\in\mathbb{N}$, we get a contradiction as $\mathsf{in}(f_1g_2)=X_1^{d+e-v}X_2^v$. Note that a case of $\mathsf{in}(f_{i_0})=X_1^{j_1}\cdot\ldots\cdot X_n^{j_n}$ (similarly for g_{i_0}) with $\sum_{\nu=1}^n j_\nu=d$ and $j_\nu>0$ for some $\nu\in[3,n]$ is not possible by the same argument.

3. Let $i\in\mathbb{N}_{\ge 3}$ and assume to the contrary that $\mathfrak{c}_{2i}=IJ$ with $\mathfrak{c}_{2i}\subsetneq I, J\subsetneq R$ such that $\mathsf{m\text{-}deg}(I)=d$ and $\mathsf{m\text{-}deg}(J)=e$. Then, by Lemma 5.8, we have $d\ge 1$ and $e\ge 1$, by Lemma 5.5, we have $2i=d+e$, and Lemma 5.6.2 implies that

(5.6)

$$\mathrm{Span}_K\{\{X_1^{2i}, X_1^{2i-1}X_2\}\cup\{X_1^{2i-j}X_2^j \;:\; j\in[1,2i] \text{ and } j\equiv 0 \mod 2\}\}=I_K[d]\cdot J_K[e].$$

As in 2., $I_K[d]$ and $J_K[e]$ are finite dimensional vector spaces of dimension at least two. Moreover, since $i\ge 3$, $\dim_K I_K[d]$ or $\dim_K J_K[e]$ must be at least three, say $\dim_K I[d]\ge 3$. Let $\dim_K I[d]=r\ge 3$ and let $f_1,\ldots,f_r\in I_K[d]$ be linearly independent such that $\mathsf{in}(f_1)>\mathsf{in}(f_2)>\ldots>\mathsf{in}(f_r)$ (again we use Lemma 5.9). Similarly, let $\dim_K J_K[e]=s\ge 2$ and let $g_1,\ldots,g_s\in J_K[e]$ be linearly independent such that $\mathsf{in}(g_1)>\ldots>\mathsf{in}(g_s)$. As in 2., we have $\mathsf{in}(f_1)=X_1^d$, and hence $\mathsf{in}(g_1)=X_1^e$.

Consider now f_2 and g_2 such that $\mathsf{in}(f_2)=X_1^{d-a}X_2^a$ and $\mathsf{in}(g_2)=X_1^{e-b}X_2^b$, where $a,b\in\mathbb{N}$. We distinguish two cases.

Case 1. ($a=1$) If b is even, then $\mathsf{in}(f_2g_2)=X_1^{d+e-b-1}X_2^{b+1}$ which is not possible, see (5.6). If $b\ge 3$ is odd, then $\mathsf{in}(f_1g_2)=X_1^{d+e-b}X_2^b$ which again is not possible. Thus, $b=1$ and $\dim_K J_K[e]=2$. Consider now f_3 such that $\mathsf{in}(f_3)=X_1^{d-c}X_2^c$ with $c\in\mathbb{N}_{\ge 2}$. If c is even, then $\mathsf{in}(f_3g_2)=X_1^{d+e-c-1}X_2^{c+1}$, a contradiction to (5.6), and if c is odd, then $\mathsf{in}(f_3g_1)=X_1^{d+e-c}X_2^c$, again a contradiction to (5.6). Hence $\dim_K I_K[d]=2$, which is not possible.

Case 2. ($a\ge 2$) If a is odd, then $\mathsf{in}(f_2g_1)=X_1^{d+e-a}X_2^a$, which is a contradiction to (5.6). Therefore, a is even and, hence b is also even. In general, there are only the following possibilities for $\mathsf{in}(f_u)$ and $\mathsf{in}(g_v)$:

$$\mathsf{in}(f_u)=X_1^{d-a_u}X_2^{a_u} \text{ with all } a_u \text{ even and } 0=a_1<\ldots<a_r\le d$$

and

$$\mathsf{in}(g_v)=X_1^{e-b_v}X_2^{b_v} \text{ with all } b_v \text{ even and } 0=b_1<\ldots<b_s\le e.$$

Now as $X_1^{2i-1}X_2\in I_K[d]\cdot J_K[e]$, so $X_1^{2i-1}X_2=\sum_{u=1}^r\sum_{v=1}^s \alpha_{uv}f_ug_v$ with $\alpha_{uv}\in K$ for every u,v. If $\alpha_{11}\ne 0$, then initial monomial of the right hand side would be X_1^{2i}, a contradiction. If $\alpha_{11}=0$, then

$$X_1^{2i-1}X_2=\mathsf{in}\left(\sum_{u=1}^r\sum_{v=1}^s \alpha_{uv}f_ug_v\right)\le\max\left\{\mathsf{in}(f_ug_v)\colon u\in[1,r], v\in[1,s]\right\}$$

which is not possible, since the right hand side of the above inequality is $X_1^{d+e-a_{u_0}-b_{v_0}}X_2^{a_{u_0}+b_{v_0}}$ which is always less than $X_1^{2i-1}X_2$.

4. Assume to the contrary that $\mathfrak{c}'=IJ$ with $\mathfrak{c}'\subsetneq I, J\subsetneq R$ such that $\mathsf{m\text{-}deg}(I)=d$ and $\mathsf{m\text{-}deg}(J)=e$. Then, by Lemma 5.8, we have $d\geq 1$ and $e\geq 1$, by Lemma 5.5, we have $d+e=3$, and Lemma 5.6.2 implies that

$$\text{Span}_K\{X_1^3+X_2^3, X_1^2X_2, X_1X_2^2\}=I_K[d]\cdot J_K[e]. \tag{5.7}$$

We may assume that $d=1$ and $e=2$. We claim that $\dim_K I[d]\geq 2$ and $\dim_K J[e]\geq 2$. Indeed, if $I_K[d]=\text{Span}_K\{f\}$ and $J_K[e]=\text{Span}_K\{g_1,\ldots,g_s\}$, then $I_K[d]\cdot J_K[e]=\text{Span}_K\{fg_1,\ldots,fg_s\}$. From (5.7), we get

$$X_1^2X_2=\sum_{j=1}^{s}\alpha_j fg_j,\ \text{ where } \alpha_j\in K \text{ for all } j\in[1,s]\,,$$

and we deduce either $f=\alpha X_1$ or $f=\alpha' X_2$ for some $\alpha,\alpha'\in K^\times$. Hence $X_1^3+X_2^3$ cannot belong to $I_K[d]\cdot J_K[e]$, a contradiction to (5.7). Note that similar argument works if we consider $d=2$.

Let $\dim_K I[d]=r\geq 2$ and let $f_1,\ldots,f_r\in I_K[d]$ be linearly independent such that $\mathsf{in}(f_1)>\ldots>\mathsf{in}(f_r)$ (we use Lemma 5.9). Similarly, let $\dim_K J_K[e]=s\geq 2$ and let $g_1,\ldots,g_s\in J_K[e]$ be linearly independent such that $\mathsf{in}(g_1)>\ldots>\mathsf{in}(g_s)$. On the other hand, for $j\in[3,n]$ $\mathsf{in}(f_{i_0})=X_j$ is not possible, whence $\dim_K I[d]=2$ and $\mathsf{in}(f_1)=X_1, \mathsf{in}(f_2)=X_2$. Similarly, we obtain that $\dim_K J[e]=2$ and $\mathsf{in}(g_1)=X_1^2, \mathsf{in}(g_2)=X_1X_2$. Assume $g_1=X_1^2+aX_2^2+\ldots$. Then by (5.7)

$$f_2g_1=\alpha_1(X_1^3+X_2^3)+\alpha_2X_1^2X_2+\alpha_3X_1X_2^2, \qquad \text{where } \alpha_1,\alpha_2,\alpha_3\in K$$

which implies that $a=0$. Similarly, the coefficient of the term X_2^2 in g_2 is zero as well. This shows that $X_1^3+X_2^3$ does not belong to $I_K[d]\cdot J_K[e]$, a contradiction. □

Remark 5.11. If 2 is a unit of D, then

$$\mathfrak{c}_4(X_1,X_2)=\langle X_1^4, X_1^3X_2, X_1^2X_2^2, X_2^4\rangle=\langle X_1^2, X_1X_2+X_2^2\rangle\langle X_1^2, X_1X_2-X_2^2\rangle\,,$$

whence $\mathfrak{c}_4(X_1,X_2)$ is not an atom of $\mathcal{I}(R)$.

Proof of Theorem 5.1. 1. and 2. We need to show the claim concerning the finite factorization property. All other statements follow by Lemma 5.3 and Proposition 5.10 (clearly, $\mathfrak{a}_1(X_1,X_2)$ is an atom). Suppose that $D^\times$ is infinite. For every $\alpha\in D^\times$, we have the identity

$$\langle X_1, X_2\rangle\cdot\langle X_1^2+\alpha X_2^2, X_1X_2\rangle=\langle X_1,X_2\rangle^3\,.$$

If $\alpha, \alpha' \in D^\times$ are distinct, then $\langle X_1^2 + \alpha X_2^2, X_1X_2 \rangle \neq \langle X_1^2 + \alpha' X_2^2, X_1X_2 \rangle$. Thus, the element $\langle X_1, X_2 \rangle^3$ has infinitely many divisors, whence $\mathcal{I}(R)$ is not an FF-monoid by [38, Proposition 1.5.5].

3. In order to show that $\mathsf{L}_{\mathcal{I}(R)}\big(\mathfrak{a}_k(X_1, X_2)\big) = \mathsf{L}\big(\mathfrak{a}_k(X_1, X_2)\big) \subset [2, k]$ for all $k \ge 2$, we choose $\ell \in \mathbb{N}_{\ge 2}$ and consider a factorization $\mathfrak{a}_k(X_1, X_2) = I_1 \cdot \ldots \cdot I_\ell$, where $I_1, \ldots, I_\ell$ are atoms in $\mathcal{I}(R)$. By Lemma 5.8, we have $\mathsf{m\text{-}deg}(I_j) \ge 1$ for all $j \in [1, \ell]$. Using Lemma 5.5 we obtain that

$$k = \mathsf{m\text{-}deg}(\mathfrak{a}_k) = \sum_{j=1}^{\ell} \mathsf{m\text{-}deg}(I_j) \ge \ell \,,$$

whence $\ell \le k$ and $\mathsf{L}\big(\mathfrak{a}_k(X_1, X_2)\big) \subset [2, k]$. To verify the reverse inclusion we proceed by induction on k. This is clear for $k=2$. By Lemma 5.2.2 (with $\ell=2$ and $k=1$), we obtain that

$$\mathfrak{a}_3(X_1, X_2) = \mathfrak{a}_1(X_1, X_2) \cdot \mathfrak{b}_2(X_1, X_2) \,.$$

Since the involved ideals are atoms by Proposition 5.10.1, the assertion holds for $k = 3$. Suppose that claim holds for $k \ge 3$. Since $\mathfrak{a}_{k+1}(X_1, X_2) = \mathfrak{a}_1(X_1, X_2) \cdot \mathfrak{a}_k(X_1, X_2)$ and $\mathfrak{a}_1(X_1, X_2)$ is an atom, it follows that

$$[3, k+1] \subset 1 + \mathsf{L}\big(\mathfrak{a}_k(X_1, X_2)\big) \subset \mathsf{L}\big(\mathfrak{a}_{k+1}(X_1, X_2)\big) \,.$$

It remains to verify that $2 \in \mathsf{L}\big(\mathfrak{a}_{k+1}(X_1, X_2)\big)$. If $k \in \mathbb{N}_{\ge 3} \setminus \{4\}$, then the following identity (see Lemma 5.2)

$$\mathfrak{a}_{k+1}(X_1, X_2) = \mathfrak{a}_1(X_1, X_2) \cdot \mathfrak{c}_k(X_1, X_2)$$

together with Proposition 5.10 show that $2 \in \mathsf{L}\big(\mathfrak{a}_{k+1}(X_1, X_2)\big)$. If $k=4$, then $2 \in \mathsf{L}\big(\mathfrak{a}_{k+1}(X_1, X_2)\big)$ because $\mathfrak{a}_5(X_1, X_2) = \mathfrak{b}_2(X_1, X_2) \cdot \mathfrak{c}'(X_1, X_2)$ (use Proposition 5.10.4).

4. We set $\mathfrak{p}_0 = \langle X_1 \rangle$, $H_1 = \mathcal{F}(\{\mathfrak{p}_0\})$ and $H_2 = \{\mathfrak{a} \in \mathcal{I}(R) : \mathfrak{p}_0 \nmid \mathfrak{a}\}$. Since $\mathfrak{p}_0$ is an invertible prime ideal, it is a cancellative prime element of $\mathcal{I}(R)$, whence we get that

$$\mathcal{I}(R) = H_1 \times H_2 \,.$$

Note that $\mathfrak{a}_k(X_1, X_2) \in H_2$ for all $k \in \mathbb{N}$. Thus, 3. shows that Conditions (a) and (b') of Proposition 3.2 hold, whence $\mathcal{I}(R)$ is fully elastic. $\square$

Theorem 5.1 shows, among others, that unions of sets of lengths of the monoid of all nonzero ideals are equal to $\mathbb{N}_{\ge 2}$, as it is true for transfer Krull monoids (which include monoids of invertible ideals of Krull domains) with infinite class group and prime divisors in all classes (Proposition 4.9). We post the conjecture that also their sets of lengths coincide, namely that every finite subset $L \subset \mathbb{N}_{\ge 2}$ occurs as a set of lengths.

Conjecture 5.12. Let $R=D[X_1,\dots,X_n]$ be the polynomial ring in $n\geq 2$ indeterminates over a domain D, and suppose that $\mathcal{I}(R)$ is a BF-monoid. Then, for every finite subset $L\subset\mathbb{N}_{\geq 2}$, there is $\mathfrak{a}\in\mathcal{I}(R)$ such that $\mathsf{L}_{\mathcal{I}(R)}(\mathfrak{a})=L$.

To conclude this paper, we would like to compare the arithmetic of $\mathcal{I}(R)$, in particular Theorem 5.1 and Conjecture 5.12, with the arithmetic of the power monoid of $\mathbb{N}_0$. Following the terminology and notation of Fan and Tringali [31], we denote by

– $\mathcal{P}_{\text{fin}}(\mathbb{N}_0)$ the *power monoid* of $\mathbb{N}_0$, that is the semigroup of finite nonempty subsets of $\mathbb{N}_0$ with set addition as operation (i.e., for finite nonempty subsets $A,B\subset\mathbb{N}_0$, their sumset $A+B$ is defined as $A+B=\{a+b\colon a\in A, b\in B\}$), and by

– $\mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$ the *reduced power monoid* of $\mathbb{N}_0$, that is the subsemigroup of $\mathcal{P}_{\text{fin}}(\mathbb{N}_0)$ consisting of all finite nonempty subsets of $\mathbb{N}_0$ that contain 0.

Both, $\mathcal{P}_{\text{fin}}(\mathbb{N}_0)$ and $\mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$, are commutative reduced unit-cancellative semigroups (whence monoids in the present sense) and $\{0\}$ is their zero-element. Power monoids are objects of primary interest in additive combinatorics and their arithmetic is studied in detail by Antoniou, Fan, and Tringali in [31] and [9]. Among others, they show that $\mathcal{P}_{\text{fin}}(\mathbb{N}_0)$ is not transfer Krull, that unions of sets of lengths of $\mathcal{P}_{\text{fin}}(\mathbb{N}_0)$ are equal to $\mathbb{N}_{\geq 2}$, and that the set of distances equals $\mathbb{N}$. The standing conjecture is that every finite subset $L\subset\mathbb{N}_{\geq 2}$ occurs as a set of lengths of $\mathcal{P}_{\text{fin}}(\mathbb{N}_0)$ ([31, Section 5]). Thus, the arithmetic of $\mathcal{I}(R)$ and the arithmetic of $\mathcal{P}_{\text{fin}}(\mathbb{N}_0)$ seem to have pretty much in common. Our final result shows that the method, developed to show that $\mathcal{I}(R)$ is fully elastic, also allows to show that $\mathcal{P}_{\text{fin}}(\mathbb{N}_0)$ is fully elastic, a question that remained open in [31]. Moreover, $\mathcal{I}(R)$ has a submonoid that is isomorphic to $\mathcal{P}_{\text{fin}}(\mathbb{N}_0)$.

Proposition 5.13.

1. The element $\{1\}$ *is a cancellative prime element of* $\mathcal{P}_{\text{fin}}(\mathbb{N}_0)$, *whence* $\mathcal{P}_{\text{fin}}(\mathbb{N}_0)=F\times\mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$, *where* F *is the free abelian monoid generated by the prime element* $\{1\}$. *Moreover,* $\mathcal{P}_{\text{fin}}(\mathbb{N}_0)$ *is fully elastic.*

2. $\mathcal{P}_{\text{fin}}(\mathbb{N}_0)$ *is isomorphic to a submonoid of* $\mathcal{I}(R)$, *where* $\mathcal{I}(R)$ *is as in Conjecture 5.12.*

Proof. 1. It is straightforward to verify that $\{1\}$ is a cancellative prime element of $\mathcal{P}_{\text{fin}}(\mathbb{N}_0)$. Since $\mathcal{P}_{\text{fin},0}(\mathbb{N}_0)=\{A\in P_{\text{fin}}(\mathbb{N}_0)\colon \{1\}$ does not divide $A\}$, it follows that $\mathcal{P}_{\text{fin}}(\mathbb{N}_0)=F\times\mathcal{P}_{\text{fin},0}(\mathbb{N}_0)$. Proposition 4.8 in [31] shows that, for every $n\geq 2$,

$$[2,n]=\mathsf{L}_{\mathcal{P}_{\text{fin},0}(\mathbb{N}_0)}([0,n])=\mathsf{L}_{\mathcal{P}_{\text{fin}}(\mathbb{N}_0)}([0,n])\,.$$

Thus, Condition (a) and Condition (b') of Proposition 3.2 are satisfied, whence $\mathcal{P}_{\text{fin}}(\mathbb{N}_0)$ is fully elastic.

2. Let $R=D[X_1,\ldots,X_n]$ be the polynomial ring in $n\geq 2$ indeterminates over a domain D, and suppose that $\mathcal{I}(R)$ is a BF-monoid. We consider the monoid $\mathcal{M}(X_1,X_2)$ of all monomial ideals in the indeterminates X_1,X_2. This is a submonoid of $\mathcal{I}(R)$. Recall that, for every $m\in\mathbb{N}_0$, we denote by $\mathcal{M}_{m;1,2}$ the set of all monomials of the form $X_1^rX_2^s$ with $r+s=m$ and $r,s\in\mathbb{N}_0$. Let $\mathcal{M}_2(X_1,X_2)\subset\mathcal{M}(X_1,X_2)$ consist of all ideals

$$\langle\mathcal{N}\cup\{X_2^m\}\rangle\,,$$

where $m\in\mathbb{N}_0$ and $\mathcal{N}\subset\mathcal{M}_{m;1,2}$ is any subset (note that, for example, $\langle X_2\rangle$ and $\langle X_1^3,X_1^2X_2,X_2^3\rangle$ are in $\mathcal{M}_2(X_1,X_2)$, whereas $\langle X_1^3,X_1^2X_2\rangle$ and $\langle X_1^2\rangle$ do not belong to $\mathcal{M}_2(X_1,X_2)$). Then $\mathcal{M}_2(X_1,X_2)\subset\mathcal{I}(R)$ is a submonoid.

For $A\in\mathcal{P}_{\text{fin}}(\mathbb{N}_0)$, say $A=\{m_1,\ldots,m_\ell\}$ with $\ell\geq 1$ and $0\leq m_1<m_2<\ldots<m_\ell$. We denote by

$$I_A:=\langle X_1^{m_\ell-m_1}X_2^{m_1},X_1^{m_\ell-m_2}X_2^{m_2},\ldots,X_1^{m_\ell-m_{\ell-1}}X_2^{m_{\ell-1}},X_2^{m_\ell}\rangle$$

an ideal which clearly belongs to $\mathcal{M}_2(X_1,X_2)$. Thus, we obtain a map

$$\mathcal{P}_{\text{fin}}(\mathbb{N}_0)\longrightarrow\mathcal{M}_2(X_1,X_2)\,,\qquad\text{given by}\qquad A\longmapsto I_A\,,$$

which is easily seen to be a monoid isomorphism. □

Acknowledgement. We would like to thank Rob Eggermont, Azhar Farooq, Florian Kainrath, Andreas Reinhart, and Daniel Smertnig, for many helpful discussions. Furthermore, we would like to thank the reviewer for their careful work and all their suggestions.

References

1. Anderson, D. D., The Krull intersection theorem, *Pac. J. Math.* **57** (1975), 11–14. MR0376649
2. Anderson, D. D., Globalization of some local properties in Krull domains, *Proc. Am. Math. Soc.* **85** (1982), 141–145. MR0652428
3. Anderson, D. D., Juett, J. R. and Mooney, C. P., Factorization of ideals, *Commun. Algebra* **47** (2019), 1742–1772. MR3975966
4. Anderson, D. D., Matijevic, J. and Nichols, W., The Krull intersection theorem II, *Pac. J. Math.* **66** (1976), 15–22. MR0435062
5. Anderson, D. D. and Roitman, M., A characterization of cancellation ideals, *Proc. Am. Math. Soc.* **125** (1997), 2853–2854. MR1415571
6. Anderson, D. D. and Zafrullah, M., Integral domains in which nonzero locally principal ideals are invertible, *Commun. Algebra* **39** (2011), 933–941. MR2782574

7. Anderson, D. F., A general theory of class groups, *Commun. Algebra* **16** (1988), 805–847. MR0932636
8. Anderson, D. F., The Picard group of a monoid domain, *J. Algebra* **115** (1988), 342–351. MR0943260
9. Antoniou, A. and Tringali, S., On the arithmetic of power monoids and sumsets in cyclic groups, *Pacific J. Math.* **312** (2021), 279–308. MR4305774
10. Atiyah, M. F. and Macdonald, I. G., *Introduction to Commutative Algebra*, Addison-Wesley, Reading, 1969. MR0242802
11. Baeth, N. R. and Sampson, R., Upper triangular matrices over information algebras, *Linear Algebra Appl.* **587** (2020), 334–357. MR4036676
12. Baeth, N. R. and Smertnig, D., Arithmetical invariants of local quaternion orders, *Acta Arith.* **186** (2018), 143–177. MR3870710
13. Baginski, P., Chapman, S. T., Crutchfield, C., Kennedy, K. G. and Wright, M., Elastic properties and prime elements, *Result. Math.* **49** (2006), 187–200. MR2288240
14. Baginski, P., Chapman, S. T., Holden, M. and Moore, T., Asymptotic elasticity in atomic monoids, *Semigroup Forum* **72** (2006), 134–142. MR2215535
15. Bashir, A., Geroldinger, A. and Reinhart, A., On the arithmetic of stable domains, *Commun. Algebra* **49** (2021), 4763–4787. MR4304671
16. Bashir, A. and Reinhart, A., On transfer Krull monoids. `arXiv:2109.04764`.
17. Blanco, V., García-Sánchez, P. A. and Geroldinger, A., Semigroup-theoretical characterizations of arithmetical invariants with applications to numerical monoids and Krull monoids, *Illinois J. Math.* **55** (2011), 1385–1414. MR3082874
18. Brantner, J., Geroldinger, A. and Reinhart, A., On monoids of ideals of orders in quadratic number fields, in *Advances in Rings, Modules, and Factorizations* **321**, pp. 11–54, Springer, Berlin, 2020. MR4113956
19. Bruns, W. and Gubeladze, J., *Polytopes, Rings, and K-Theory*, Springer, Berlin, 2009. MR2508056
20. Campanini, F. and Facchini, A., Factorizations of polynomials with integral non-negative coefficients, *Semigroup Forum* **99** (2019), 317–332. MR4026385
21. Chang, G. W. and Kim, H., Integral domains with a free semigroup of $*$-invertible integral $*$-ideals, *Bull. Korean Math. Soc.* **48** (2011), 1207–1218. MR2894888
22. Chapman, S. T., Gotti, F. and Gotti, M., Factorization invariants of Puiseux monoids generated by geometric sequences, *Comm. Algebra* **48** (2020), 380–396. MR4060036
23. Chapman, S. T., Holden, M. and Moore, T., Full elasticity in atomic monoids and integral domains, *Rocky Mt. J. Math.* **36** (2006), 1437–1455. MR2285293
24. Chouinard, L. G. II., Krull semigroups and divisor class groups, *Can. J. Math.* **33** (1981), 1459–1468. MR0645239
25. Davis, E. D., Prime elements and prime sequences in polynomial rings, *Proc. Amer. Math. Soc.* **72** (1978), 33–38. MR0498533
26. Dutta, A. K. and Onoda, N., On finite generation of R-subalgebras of $R[X]$, *J. Algebra* **320** (2008), 57–80. MR2417977
27. Fadinger, V. and Windisch, D., On the distribution of prime divisors in class groups of affine monoid algebras. *Commun. Algebra* `https://doi.org/10.`

1080/00927872.2021.2023888 MR4308072

28. Fadinger, V. and Windisch, D., A characterization of weakly Krull monoid algebras, *J. Algebra* **590** (2022), 277–292. MR4325833
29. Fadinger, V. and Windisch, D., On the distribution of prime divisors in Krull monoid algebras, *J. Pure Appl. Algebra* **226** (2022). Article ID: 106887. MR4308072
30. Fan, Y., Geroldinger, A., Kainrath, F. and Tringali, S., Arithmetic of commutative semigroups with a focus on semigroups of ideals and modules, *J. Algebra Appl.* **11**, 1750234 (2017). (42 pages). MR3725094
31. Fan, Y. and Tringali, S., Power monoids: A bridge between factorization theory and arithmetic combinatorics, *J. Algebra* **512** (2018), 252–294. MR3841525
32. Fontana, M., Houston, E. and Lucas, T., *Factoring Ideals in Integral Domains*, Lecture Notes of the Unione Matematica Italiana **14**, Springer, Berlin, 2013. MR2976403
33. Frisch, S., A construction of integer-valued polynomials with prescribed sets of lengths of factorizations, *Monatsh. Math.* **171** (2013), 341–350. MR3090795
34. Gabelli, S., Locally principal ideals and finite character, *Bull. Math. Soc. Sci. Math. Roumanie (N. S.)* **56(104)** (2013), 99–108. MR3076694
35. Gabelli, S. and Roitman, M., On finitely stable domains I, *J. Comm. Algebra* **11** (2019), 49–67. MR3922425
36. Gao, W. and Geroldinger, A., On products of k atoms, *Monatsh. Math.* **156** (2009), 141–157. MR2488859
37. Geroldinger, A., Gotti, F. and Tringali, S., On strongly primary monoids with a focus on Puiseux monoids, *J. Algebra* **567** (2021), 310–345. MR4159257
38. Geroldinger, A. and Halter-Koch, F., *Non-Unique Factorizations. Algebraic, Combinatorial and Analytic Theory*, Pure and Applied Mathematics **278**, Chapman & Hall/CRC, 2006. MR2194494
39. Geroldinger, A. and Hassler, W., Local tameness of v-noetherian monoids, *J. Pure Appl. Algebra* **212** (2008), 1509–1524. MR2391663
40. Geroldinger, A., Kainrath, F. and Reinhart, A., Arithmetic of seminormal weakly Krull monoids and domains, *J. Algebra* **444** (2015), 201–245. MR3406175
41. Geroldinger, A., Ramacher, S. and Reinhart, A., On v-Marot Mori rings and C-rings, *J. Korean Math. Soc.* **52** (2015), 1–21. MR3299367
42. Geroldinger, A. and Reinhart, A., The monotone catenary degree of monoids of ideals, *Internat. J. Algebra Comput.* **29** (2019), 419–457. MR3955817
43. Geroldinger, A. and Roitman, M., On strongly primary monoids and domains, *Commun. Algebra* **48**, 4085–4099. MR4124682
44. Geroldinger, A., Schmid, W. A. and Zhong, Q., Systems of sets of lengths: transfer Krull monoids versus weakly Krull monoids, in *Rings, Polynomials, and Modules*, pp. 191–235, Springer, Cham, 2017. MR3751698
45. Geroldinger, A. and Zhong, Q., Long sets of lengths with maximal elasticity, *Can. J. Math.* **70** (2018), 1284–1318. MR3850544
46. Geroldinger, A. and Zhong, Q., Sets of arithmetical invariants in transfer Krull monoids, *J. Pure Appl. Algebra* **223** (2019), 3889–3918. MR3944459
47. Geroldinger, A. and Zhong, Q., Factorization theory in commutative monoids,

Semigroup Forum **100** (2020), 22–51. MR4061567

48. Gilmer, R., *Commutative Semigroup Rings*, The University of Chicago Press, 1984. MR0741678
49. Grynkiewicz, D. J., The characterization of finite elasticities. *Springer Lecture Notes in Math., to appear*
50. Halter-Koch, F., Divisor theories with primary elements and weakly Krull domains, *Boll. Un. Mat. Ital. B* **9** (1995), 417–441. MR1333970
51. Halter-Koch, F., *Ideal Systems. An Introduction to Multiplicative Ideal Theory*, Dekker, New York, 1998. MR1828371
52. Hetzel, A. J., Lawson, A. L. and Reinhart, A., On finite molecularization domains, *J. Commut. Algebra* **13** (2021), 69–87. MR4280184
53. Heubo-Kwegna, O. A., Olberding, B. and Reinhart, A., Group-theoretic and topological invariants of completely integrally closed Prüfer domains, *J. Pure Appl. Algebra* **220** (2016), 3927–3947. MR3517563
54. Juett, J., Generalized comaximal factorization of ideals, *J. Algebra* **352** (2012), 141–166. MR2862178
55. Juett, J. R., Mooney, C. P. and Ndungu, L. W., Unique factorization of ideals in commutative rings with zero divisors, *Commun. Algebra* **49** (2021), 2101–2125. MR4232488
56. Kainrath, F., On some arithmetical properties of noetherian domains, in *Advances in Rings, Modules and Factorizations*, Springer Proc. Math. Stat. **321**, pp. 217–222, Springer, Berlin, 2020. MR4113966
57. Klingler, L. and Omairi, A., Unique decomposition into ideals for commutative Noetherian rings, *J. Pure Appl. Algebra* **224**, no. 9, 106364, 12 (2020). MR4083790
58. Matsumura, H., *Commutative Ring Theory*, Cambridge University Press, Cambridge, 1997. MR0879273
59. Oh, J. S., On the algebraic and arithmetic structure of the monoid of product-one sequences, *J. Commut. Algebra* **12** (2020), 409–433. MR4146368
60. Oh, J. S. and Zhong, Q., On minimal product-one sequences of maximal length over dihedral and dicyclic groups, *Commun. Korean Math. Soc.* **35** (2020), 83–116. MR4060202
61. Olberding, B. and Reinhart, A., Radical factorization in commutative rings, monoids, and multiplicative lattices, *Algebra Universalis* **80** (2019). Article 24. MR3954364
62. Olberding, B. and Reinhart, A., Radical factorization in finitary ideal systems, *Commun. Algebra* **48** (2020), 228–253. MR4060027
63. Reinhart, A., Structure of general ideal semigroups of monoids and domains, *J. Commut. Algebra* **4** (2012), 413–444. MR3024263
64. Schmid, W. A., Some recent results and open problems on sets of lengths of Krull monoids with finite class group, in *Multiplicative Ideal Theory and Factorization Theory*, pp. 323–352, Springer, Berlin, 2016. MR3565815
65. Tringali, S., Structural properties of subadditive families with applications to factorization theory, *Israel J. Math.* **234** (2019), 1–35. MR4040819
66. Zhong, Q., On elasticities of locally finitely generated monoids, *J. Algebra* **534** (2019), 145–167. MR3979069

67. Zhong, Q., On the arithmetic of Mori monoids and domains, *Glasgow Math. Journal* **62** (2020), 313–322. MR4085045

Alfred Geroldinger
Institute of Mathematics and Scientific Computing
University of Graz
NAWI Graz, Heinrichstraße 36, AT-8010 Graz
Austria
alfred.geroldinger@uni-graz.at

M. Azeem Khadam
Institute of Mathematics and Scientific Computing
University of Graz
NAWI Graz, Heinrichstraße 36, AT-8010 Graz
Austria
azeem.khadam@uni-graz.at

Received June 1, 2021
in revised form November 21, 2021