

Hamiltonian Carleman approximation and the density property for coadjoint orbits

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Abstract. For a complex Lie group G with a real form $G_0 \subset G$, we prove that any Hamiltonian automorphism ϕ of a coadjoint orbit \mathcal{O}_0 of G_0 whose connected components are simply connected, may be approximated by holomorphic \mathcal{O}_0 -invariant symplectic automorphism of the corresponding coadjoint orbit of G in the sense of Carleman, provided that \mathcal{O} is closed. In the course of the proof, we establish the Hamiltonian density property for closed coadjoint orbits of all complex Lie groups.

1. Introduction

Let $\omega_0 = dx_1 \wedge dx_{n+1} + \dots + dx_n \wedge dx_{2n}$ be the standard symplectic form on \mathbb{R}^{2n} , and let $\omega = dz_1 \wedge dz_{n+1} + \dots + dz_n \wedge dz_{2n}$ be the standard holomorphic symplectic form on \mathbb{C}^{2n} . We denote by $\text{Symp}(\mathbb{R}^{2n}, \omega_0)$ the group of smooth symplectic automorphisms of $(\mathbb{R}^{2n}, \omega)$, and by $\text{Symp}(\mathbb{C}^{2n}, \omega)$ the group of holomorphic symplectic automorphisms of $(\mathbb{C}^{2n}, \omega)$ (throughout this paper, smooth will mean C^∞ -smooth). The following problem was proposed by N. Sibony (private communication):

Problem 1.1. *Can any element in $\text{Symp}(\mathbb{R}^{2n}, \omega_0)$ be approximated in the sense of Carleman by elements in $\text{Symp}(\mathbb{C}^{2n}, \omega)$ leaving \mathbb{R}^{2n} invariant?*

Motivated by this problem, and also connections to physics (see below), in this article we will consider the analogous problem in the more general setting of complexifications of coadjoint orbits of real Lie groups. For a real or complex manifold X with a symplectic form ω we will denote by $\text{Ham}(X, \omega)$ the smooth path-connected component of the identity in $\text{Symp}(X, \omega)$. We will prove the following.

Theorem 1.2. *Let G be a complex Lie group, and let $G_0 \subset G$ be a real form. Let $\mathcal{O}_0 \subset \mathfrak{g}_0^*$ be a coadjoint orbit whose connected components are simply connected, let $\mathcal{O} \subset \mathfrak{g}^*$ be the coadjoint orbit containing \mathcal{O}_0 . Assume that \mathcal{O} is closed, and let*

ω_0 (resp. ω) denote the canonical symplectic form on \mathcal{O}_0 (resp. \mathcal{O}). Then given $\phi \in \text{Ham}(\mathcal{O}_0, \omega_0)$, a positive continuous function $\varepsilon(x)$ on \mathcal{O}_0 , and $r \in \mathbb{N}$, there exists $\psi \in \text{Ham}(\mathcal{O}, \omega)$ such that $\psi(\mathcal{O}_0) = \mathcal{O}_0$ and

$$\|\psi(x) - \phi(x)\|_{C^r} < \varepsilon(x)$$

for all $x \in \mathbb{R}^{2n}$.

Here the C^r -approximation may be obtained with respect to any given Riemannian metric on the appropriate jet-space of \mathcal{O}_0 . The connection with Problem 1.1 is that if G (resp. G_0) is the complex (resp. real) Heisenberg group, then there are coadjoint orbits $(\mathcal{O}_0, \mathcal{O}) \approx (\mathbb{R}^2, \mathbb{C}^2)$ with $(\mathbb{R}^2, \mathbb{C}^2)$ equipped with the standard symplectic structures as above. The case of an arbitrary n is obtained by taking products. Note that $\text{Symp}(\mathbb{R}^{2n}, \omega_0)$ coincides with $\text{Ham}(\mathbb{R}^{2n}, \omega_0)$ and $\text{Symp}(\mathbb{C}^{2n}, \omega)$ coincides with $\text{Ham}(\mathbb{C}^{2n}, \omega)$.

For information about Carleman approximation by functions, please see the recent survey [6]. Related to Theorem 1.2, it was proved in [14] that any diffeomorphism of \mathbb{R}^k can be approximated by holomorphic automorphisms of \mathbb{C}^n in the Carleman sense, provided that $k < n$, however, in that case \mathbb{R}^k was not left invariant.

In the course of the proof we will also prove the following, which follows from quite standard arguments in Andersén-Lempert theory, as soon as one considers the right setup.

Theorem 1.3. *All closed coadjoint orbits of a complex Lie group have the Hamiltonian density property.*

The corresponding theorem for \mathbb{C}^{2n} was proved by F. Forstnerič [5]. The volume density property of closed coadjoint orbits of a complex Lie group G in the case that G is an algebraic group is a corollary of Theorem 1.3 in [9] due to Kaliman and Kutzschebauch.

Symplectic manifolds are very important objects in physics. In a recent program called “Quantization via Complexification” proposed by Gukov and Witten, the quantization of a symplectic manifold (M, ω_0) is studied through the quantization of its complexification (X, ω, τ) ([8][7][18]). In general, the physical symmetry on (M, ω_0) is given by $\text{Symp}(M, \omega_0)$ (resp. $\text{Ham}(M, \omega_0)$), while that of the complexification (X, ω, τ) is given by $\text{Symp}(X, \omega, \tau)$ (resp. $\text{Ham}(X, \omega, \tau)$). So in physical terminology, our result is transferred to the following: no symmetry is broken after complexification.

The article is organized as follows. In Section 2 we will give the relevant background on coadjoint orbits. In Section 3 we start by setting up a general framework for our approximation. Then we give some results on C^r regularity of flows of families of smooth vector fields and C^r -convergence of consistent algorithms,

following the unpublished Diplomarbeit [16] by B. Schär at the University of Bern, and we prove some results of Andersén-Lempert type, leaving invariant a totally real center. Finally we prove the main theorem. In Section 4 we include some standard examples of coadjoint orbits.

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2. Preliminaries on coadjoint orbits of Lie groups and complexifications

2.1. Canonical symplectic structures on coadjoint orbits

In this subsection we will collect some standard material on co-adjoint orbits (see [13] for more details). Let G be a real Lie group with Lie algebra \mathfrak{g} . As a vector space, we can identify \mathfrak{g} with $T_e G$, the tangent space of G at the identity. For $g \in G$, the conjugate map from G to itself given by $x \mapsto gxg^{-1}$ fixes e , and hence induces a linear isomorphism of \mathfrak{g} . In this manner, we get a linear representation Ad of G on \mathfrak{g} , which is called the adjoint representation of G . The adjoint representation of G induces a representation of G on the dual space \mathfrak{g}^* of \mathfrak{g} , which is called the coadjoint representation of \mathfrak{g} and will be denoted by Ad^* . More precisely, the coadjoint representation is defined as

$$(\text{Ad}^*(g)\xi, v) := (\xi, \text{Ad}(g^{-1})v), \quad g \in G, \quad \xi \in \mathfrak{g}^*, \quad v \in \mathfrak{g},$$

where (\cdot, \cdot) denotes the pairing between \mathfrak{g}^* and \mathfrak{g} . The orbits of elements in \mathfrak{g}^* under the action of G are called coadjoint orbits.

A fundamental fact about coadjoint orbits \mathcal{O} is that they carry canonical G -invariant symplectic structures, which are defined as follows. For a vector $v \in \mathfrak{g}$, we have that Ad^* induces an action of the one-parameter subgroup $\{\exp(tv); t \in \mathbb{R}\}$ of G generated by v on \mathcal{O} , where $\exp: \mathfrak{g} \rightarrow G$ is the exponential map. So we get a smooth vector field X_v on \mathcal{O} whose value at $\xi \in \mathcal{O}$ is given by

$$X_v(\xi) = \left. \frac{d}{dt} \right|_{t=0} \text{Ad}^*(\exp(tv))\xi.$$

For $\xi \in \mathcal{O}$, we have that $\{X_v(\xi); v \in \mathfrak{g}\} = T_\xi \mathcal{O}$ since the action of G on \mathcal{O} is transitive.

The kernel of the linear map $v \mapsto X_v(\xi)$ from \mathfrak{g} to $T_\xi \mathcal{O}$ is \mathfrak{g}_ξ , which is the Lie algebra of the isotropy subgroup $G_\xi := \{g \in G; g\xi = \xi\}$ of G at ξ . The symplectic form ω on \mathcal{O} is defined by

$$\omega(X_u(\xi), X_v(\xi)) := \xi([u, v]), \quad u, v \in \mathfrak{g}.$$

We check that ω is a well-defined symplectic form on \mathcal{O} :

• ω is well defined. It suffices to check that for $u \in \mathfrak{g}$ with $X_u(\xi)=0$ we have $\xi([u, v])=0$ for all $v \in \mathfrak{g}$. If $X_u(\xi)=0$, then $u \in \mathfrak{g}_\xi$, the Lie algebra of the isotropy subgroup G_ξ . Hence $\text{Ad}^*(\exp(tu))\xi=\xi$ for all $t \in \mathbb{R}$, which is equivalent to that $(\text{Ad}^*(\exp(tu))\xi, v)=(\xi, v)$ for all $t \in \mathbb{R}$ and $v \in \mathfrak{g}$. Note that $(\text{Ad}^*(\exp(tu))\xi, v)=(\xi, \text{Ad}(\exp(-tu))v)$, taking derivative with t at $t=0$ we get $\xi([u, v])=-(\xi, [-u, v])=0$.

• ω is nondegenerate. For $u \in \mathfrak{g}$, we need to show that $\omega(X_u(\xi), X_v(\xi))=0$ for all $v \in \mathfrak{g}$ implies $X_u(\xi)=0$. By definition,

$$\begin{aligned} \omega(X_u(\xi), X_v(\xi)) &= -\xi([-u, v]) \\ &= -\left(\xi, \frac{d}{dt}\Big|_{t=0} \text{Ad}(\exp(-tu))v\right) \\ &= -\left(\frac{d}{dt}\Big|_{t=0} \text{Ad}^*(\exp(tu))\xi, v\right). \end{aligned}$$

So $\omega(X_u(\xi), X_v(\xi))=0$ for all $v \in \mathfrak{g}$ implies $\frac{d}{dt}\Big|_{t=0} \text{Ad}^*(\exp(tu))\xi=X_u(\xi)=0$.

• ω is closed.

Since all $X_u(u \in \mathfrak{g})$ generate $T_\xi \mathcal{O}$ for all $\xi \in \mathcal{O}$, and $\omega(X_u, X_v)$ are smooth functions on \mathcal{O} , we have that ω is a smooth 2-form on \mathcal{O} . Letting $u, v, w \in \mathfrak{g}$, we have

$$\begin{aligned} &X_u \omega(X_v(\xi), X_w(\xi)) \\ &= \frac{d}{dt}\Big|_{t=0} \omega(\text{Ad}^*(\exp(tu))\xi, (X_v(\text{Ad}^*(\exp(tu))\xi), X_w(\text{Ad}^*(\exp(tu))\xi))) \\ &= \frac{d}{dt}\Big|_{t=0} (\text{Ad}^*(\exp(tu))\xi, [v, w]) \\ &= \frac{d}{dt}\Big|_{t=0} (\xi, \text{Ad}(\exp(-tu))[v, w]) \\ &= -(\xi, [u, [v, w]]). \end{aligned}$$

By a basic formula about exterior differentiation, we have

$$\begin{aligned} &d\omega(X_u, X_v, X_w) \\ &= X_u \omega(X_v, X_w) - X_v \omega(X_u, X_w) + X_w \omega(X_u, X_v) \\ &\quad - \omega([X_u, X_v], X_w) + \omega([X_u, X_w], X_v) - \omega([X_v, X_w], X_u), \end{aligned}$$

which vanishes for all $u, v, w \in \mathfrak{g}$ by the Jacobi identity and the above formula. Hence $d\omega=0$.

For $u \in \mathfrak{g}$, we can view u as a function on \mathfrak{g}^* and hence a smooth function on the orbit \mathcal{O} . A basic fact is the following.

Lemma 2.1. *For any $u \in \mathfrak{g}$, the vector field X_u on \mathcal{O} is Hamiltonian with respect to ω and its potential is u itself. In particular, the symplectic structure ω on \mathcal{O} is invariant under the action of identity component of G .*

Proof. It suffices to show that

$$du(X_v)(\xi) = i_{X_u}\omega(X_v)(\xi) = \xi([u, v])$$

for all $v \in \mathfrak{g}$. Note that

$$\begin{aligned} & du(X_v) \\ &= \frac{d}{dt}(\text{Ad}^*(\exp(tv))\xi, u)|_{t=0} \\ &= \left(\xi, \frac{d}{dt}\text{Ad}(\exp(-tv))u|_{t=0} \right) \\ &= \xi([u, v]). \quad \square \end{aligned}$$

The following lemma, which is a direct corollary of Theorem 2.13 in [15], is also useful.

Lemma 2.2. *For $\xi \in \mathfrak{g}^*$, if the coadjoint orbit $\mathcal{O} = G \cdot \xi$ is closed in \mathfrak{g}^* , then the canonical map $G/G_\xi \rightarrow \mathfrak{g}^*$ with image \mathcal{O}_ξ is a proper embedding and hence \mathcal{O}_ξ is a closed submanifold of \mathfrak{g}^* .*

The above construction starts from a real Lie group. In the same way, we can start from a complex Lie group and carry out the same procedure. Then all coadjoint orbits in \mathfrak{g}^* are complex manifolds with canonical G -invariant holomorphic symplectic forms, and the statements parallel to Lemma 2.1 and Lemma 2.2 hold.

2.2. Complexification of coadjoint orbits of real Lie groups

Let G be a connected complex Lie group with a real form G_0 , i.e., G_0 is a Lie subgroup of G (not necessarily closed) and $\mathfrak{g} = \mathfrak{g}_0 \oplus i\mathfrak{g}_0$, where \mathfrak{g} and \mathfrak{g}_0 are the Lie algebras of G and G_0 respectively. We want to show that coadjoint orbits of G are complexifications of the corresponding coadjoint orbits of G_0 . We start by introducing the following

Definition 2.3. A complexification of a symplectic manifold (M^{2n}, ω_0) is a triple (X^{2n}, ω, τ) :

- (X, ω) is a holomorphic symplectic manifold,
- τ is an anti-holomorphic involution of X ,
- $M \hookrightarrow X$ (proper embedding) and $\omega|_M = \omega_0$,
- $\tau|_M = Id$, $\tau^*\omega = \bar{\omega}$.

For a point $\xi \in \mathfrak{g}_0^*$ (resp. \mathfrak{g}^*), the coadjoint orbit through ξ in \mathfrak{g}_0^* (resp. \mathfrak{g}^*) will be denoted by $\mathcal{O}_\xi^{\mathbb{R}}$ (resp. \mathcal{O}_ξ). The canonical symplectic form on $\mathcal{O}_\xi^{\mathbb{R}}$ (defined in §2.1) is denoted by ω_0 , and the canonical holomorphic symplectic form on \mathcal{O}_ξ is denoted by ω . For a point $\xi \in \mathfrak{g}_0^*$, we can also view ξ as an element in \mathfrak{g}^* . Then it is clear that $\mathcal{O}_\xi^{\mathbb{R}} \subset \mathcal{O}_\xi$ and $\omega|_{\mathcal{O}_\xi^{\mathbb{R}}} = \omega_0$.

We will focus on closed orbits. In the case that G_0 is reductive, the adjoint representation of G_0 (resp. G) on \mathfrak{g}_0 (resp. \mathfrak{g}) and the coadjoint representation G_0 (resp. G) on \mathfrak{g}_0^* (resp. \mathfrak{g}^*) are isomorphic, and hence the adjoint orbits and coadjoint orbits coincide. For $\xi \in \mathfrak{g}_0$, the closedness of $\mathcal{O}_\xi^{\mathbb{R}}$ in \mathfrak{g} is equivalent to the closedness of \mathcal{O}_ξ in \mathfrak{g} , and they are equivalent to that ξ is semisimple, that is Ad_ξ , viewed as an operator on both \mathfrak{g}_0 and \mathfrak{g} , is diagonalizable. In the general case, we will show that if \mathcal{O}_ξ is closed then $\mathcal{O}_\xi^{\mathbb{R}}$ is also closed. We will prove the following.

Theorem 2.4. *If $\xi \in \mathfrak{g}_0^*$ is such that $\mathcal{O}_\xi \subset \mathfrak{g}^*$ is closed, then the following holds.*

- (1) \mathcal{O}_ξ is a closed complex submanifold of \mathfrak{g}^* ;
- (2) $\mathcal{O}_\xi^{\mathbb{R}}$ consists of some connected components of $\mathcal{O}_\xi \cap \mathfrak{g}_0^*$ and is a closed submanifold of \mathcal{O}_ξ ;
- (3) $(\mathcal{O}_\xi, \omega, \tau)$ is a complexification of $(\mathcal{O}_\xi^{\mathbb{R}}, \omega_0)$ in the sense of Definition 2.3.

Note that (1) in the above theorem is a direct corollary of Theorem 2.13 in [15].

Proof. We start by giving a lemma.

Lemma 2.5. *Let $\xi \in \mathfrak{g}_0^*$ be such that $\mathcal{O}_\xi \subset \mathfrak{g}^*$ is closed. Then the following holds.*

- (1) $G_{0,\xi}$ is a real form of G_ξ , where $G_{0,\xi}$ and G_ξ are the isotropy subgroups of G_0 and G at ξ respectively.
- (2) $\mathcal{O}_\xi^{\mathbb{R}}$ is a totally real submanifold of \mathcal{O}_ξ of maximal dimension.

Proof. Let $\sigma_0: G_0 \rightarrow \mathcal{O}_\xi^{\mathbb{R}}$, $\sigma: G \rightarrow \mathcal{O}_\xi$ be the orbit maps given by $g \mapsto g\xi$ with differentials at the identity $d\sigma_0: \mathfrak{g}_0 \rightarrow T_\xi \mathcal{O}_\xi^{\mathbb{R}} \subset \mathfrak{g}_0^*$ and $d\sigma: \mathfrak{g} \rightarrow T_\xi \mathcal{O}_\xi \subset \mathfrak{g}^*$. We have $d\sigma_0(u) = X_u$ for $u \in \mathfrak{g}_0$ and $d\sigma(u) = X_u$ for $u \in \mathfrak{g}$. For $u = a + ib \in \mathfrak{g}$ with $a, b \in \mathfrak{g}_0$, $X_u = 0$ is equivalent to $\omega(X_u, X_v)(\xi) = \xi([u, v]) = 0$ for all $v \in \mathfrak{g}_0$. But this is equivalent to $\xi([a, v]) = \xi([b, v]) = 0$ for all $v \in \mathfrak{g}_0$. Hence $a, b \in \ker d\sigma_0$ and $\ker d\sigma = \ker d\sigma_0 \oplus i \ker d\sigma_0$ and so $G_{0,\xi}$ is a real form of G_ξ . This implies that $\mathcal{O}_\xi^{\mathbb{R}}$ is a totally real submanifold of \mathcal{O}_ξ of maximal dimension. \square

Let \mathcal{O}_ξ be as in the above lemma. We now define an anti-holomorphic involution on \mathcal{O}_ξ . Let $\tau: \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ be the conjugation map given by $\tau(a + ib) = a - ib$, $u, v \in \mathfrak{g}_0^*$. Then τ is anti-holomorphic and is an involution, i.e., $\tau^2 = Id$. It is clear that $\tau(\mathcal{O}_\xi)$ is also a complex submanifold of \mathfrak{g}^* . Note that by Lemma 2.5 we have that $\mathcal{O}_\xi^{\mathbb{R}}$ is

a totally real submanifold of both \mathcal{O}_ξ and $\tau(\mathcal{O}_\xi)$ of maximal dimension, by identity theorem for holomorphic functions we have $\mathcal{O}_\xi = \tau(\mathcal{O}_\xi)$, and so τ is an anti-holomorphic involution on \mathcal{O}_ξ .

We move forward to prove that $\mathcal{O}_\xi^{\mathbb{R}}$ is a closed submanifold of \mathcal{O}_ξ and hence a closed submanifold of \mathfrak{g}_0^* . By Theorem 2.13 in [15] and noting that \mathcal{O}_ξ is closed in \mathfrak{g}^* , it suffices to prove that $\mathcal{O}_\xi^{\mathbb{R}}$ is a closed subset of \mathcal{O}_ξ . Note that since a closed connected component of the set of fixed points of a compact Lie group acting on a smooth manifold is a closed submanifold (see Theorem 5.1 in [11]), any connected component of $\mathcal{O}_\xi \cap \mathfrak{g}_0^*$, which is the fixed point set of τ , is a closed submanifold of \mathcal{O}_ξ . Note also that $\mathcal{O}_\xi \cap \mathfrak{g}_0^*$ is a union of coadjoint orbits of G_0 and any orbit of G_0 contained in \mathcal{O}_ξ is a totally real submanifold of \mathcal{O}_ξ of maximal dimension by Lemma 2.5, so $\mathcal{O}_\xi^{\mathbb{R}}$ is closed in \mathcal{O}_ξ since its complement in \mathcal{O}_ξ , which is a union of some orbits of G_0 , is open in \mathcal{O}_ξ .

Since $\omega|_{\mathcal{O}_\xi^{\mathbb{R}}} = \omega_0$ is real, we have $(\tau^*(\omega)) = \omega$ on $\mathcal{O}_\xi^{\mathbb{R}}$. By the identity theorem, we see that $\tau^*(\omega) = \bar{\omega}$. \square

3. The density property and Hamiltonian Carleman approximation

By the considerations in the previous section, to prove Theorem 1.2 we may consider the following framework. We let $Z \subset \mathbb{C}^N$ be a closed connected complex manifold equipped with a holomorphic symplectic form ω . Assume that Z_0 is the union of some smooth connected components of $Z \cap \mathbb{R}^N$ with $\dim_{\mathbb{R}} Z_0^j = \dim_{\mathbb{C}}(Z)$ for each component Z_0^j , and such that $\omega_0 := \omega|_{Z_0}$ is a real symplectic form on Z_0 . Letting $u_j, j=1, \dots, N$, denote the coordinates on \mathbb{C}^N , we assume further that for any linear function $u = \sum_{j=1}^N c_j \cdot u_j$ the vector field X_u defined on Z by $\iota_{X_u} \omega = \partial u$ is complete.

We let $\mathcal{V}_h(Z)$ denote the Lie algebra of holomorphic Hamiltonian vector fields on Z , and we let $\mathcal{V}_h^I(Z)$ denote the Lie sub-algebra of $\mathcal{V}_h(Z)$ generated by complete vector fields.

3.1. The density property for $\mathcal{V}_h(Z)$

The following lemma follows easily from the assumptions above and standard arguments in Andersén-Lempert theory.

Lemma 3.1. *In the setting above we have that $\mathcal{V}_h^I(Z)$ is dense in $\mathcal{V}_h(Z)$. Moreover, if $A \subset \mathbb{R}^n$ is compact, and if X_y is a continuous family of smooth vector fields on Z_0 with $X_y \in \mathcal{V}_h(Z_0)$ for all $y \in A$, then if $\varepsilon > 0$, $r \in \mathbb{N}$, and if $K \subset Z_0$*

is compact, there exists a continuous family of complete holomorphic vector fields $Y_{y,1}, \dots, Y_{y,n} \in \mathcal{V}_h^I(Z)$, all tangent to Z_0 , such that

$$\|X_y - \sum_{j=1}^n Y_{y,j}\|_{C^r(K)} < \varepsilon.$$

Proof. We write $u_j = x_j + iy_j$. We start with the case of $X_y \in \mathcal{V}_h(Z_0)$ for each fixed $y \in A$. Let u_y be a family of potentials for X_y , continuous in C^{r+1} -norm in the (x) -variables with respect to y . By Weierstrass' Approximation Theorem we may approximate u arbitrarily well in C^{r+1} -norm by polynomials P in x depending continuously on y , and so for the purpose of approximating X we will consider the vector fields \tilde{X}_y defined on Z_0 by $dP_y = \iota_{\tilde{X}_y} \omega_0$. First we let P_y^o denote the polynomials P_y extended in the obvious way to polynomials in the variables u_j , and we let \tilde{X}_y^o denote the holomorphic vector fields defined on Z by $\partial P_y^o = \iota_{\tilde{X}_y^o} \omega$.

We will first show that \tilde{X}_y^o is tangent to Z_0 . For any point $\zeta \in Z_0$, since Z_0 is real analytic, there exists a real holomorphic embedding $g: U \rightarrow Z$ with $g(\mathbb{R}^{2n} \cap U) = Z_0 \cap g(U)$, where U is a neighbourhood of the origin in \mathbb{C}^{2n} and $g(0) = \zeta$. Setting $\hat{\omega} = g^* \omega$ and $\hat{P}_y = g^* P_y^o$ it suffices to show that the vector fields \hat{X}_y defined by $\partial \hat{P}_y = \iota_{\hat{X}_y} \hat{\omega}$ is tangent to \mathbb{R}^{2n} . Note that \hat{P}_y and $\hat{\omega}$ are real. In particular

$$\omega(z) = \sum_{i < j} a_{i,j}(z) dz_i \wedge dz_j$$

where all $a_{i,j}$ are real holomorphic functions. Furthermore, we have that

$$\alpha_y(z) = \partial \hat{P}_y(z) = \sum_{j=1}^{2n} b_j(z) dz_j,$$

with $b_j(z)$ real holomorphic functions, so it is straight forward to see that \hat{X}_y are all real holomorphic vector fields.

We may now write

$$(1) \quad P_y^o(u) = \sum_{|\alpha| \leq N} a_{y,\alpha} u^\alpha,$$

where the $a_{y,\alpha}$'s are real valued continuous functions on A .

It is a fundamental result in Andersén-Lempert Theory that

$$(2) \quad u^\alpha = \sum_{k=1}^{M_\alpha} b_k^\alpha \cdot (c_{k,1}^\alpha \cdot u_1 + \dots + c_{k,N}^\alpha \cdot u_N)^{|\alpha|},$$

where all coefficients may be taken to be real. Hence, we have that $P_y(u)$ is a sum of real polynomials of the form

$$(3) \quad g_{y,k}^\alpha(u) = d_{y,k}^\alpha \cdot (c_{k,1}^\alpha \cdot u_1 + \cdots + c_{k,N}^\alpha \cdot u_N)^{|\alpha|} = d_{y,k}^\alpha \cdot (f_{y,k}^\alpha(u))^{|\alpha|},$$

where by assumption $X_{f_{y,k}^\alpha}$ is complete on Z . Since we have that

$$X_{(f_{y,k}^\alpha)^\alpha} = |\alpha| (f_{y,k}^\alpha)^{|\alpha|-1} X_{f_{y,k}^\alpha}$$

we see that $X_{(f_{y,k}^\alpha)^\alpha}$ is complete, and this concludes the proof of the lemma. \square

3.2. Convergence in C^r -norm

For the lack of a suitable reference we will in this subsection include a result on C^r -regularity of solutions of ODE's (the following lemma), and a result on C^r -approximation by consistent algorithms (see Theorem 3.4 below). We will need the results for vector fields on smooth manifolds, but since they are all local in nature, we state and prove them in \mathbb{R}^n .

Lemma 3.2. *Let D be an open set in \mathbb{R}^n and let $X^j, X: [0, 1] \times D \rightarrow \mathbb{R}^n$ be smooth maps, $j \geq 1$. We view $X_t^j := X^j(t, \cdot)$ and $X_t = X(t, \cdot)$ as time dependent smooth vector fields on D . Assume that $\phi^j, \phi: [0, 1] \times D \rightarrow D$ are flows on D generated by X^j and X with $\phi^j(0, x) = \phi(0, x) \equiv x$, namely*

$$(4) \quad \frac{d\phi^j(t, x)}{dt} = X^j(t, \phi^j(t, x)), \quad \frac{d\phi(t, x)}{dt} = X(t, \phi(t, x)).$$

Then if

$$\lim_{j \rightarrow \infty} \|X_t^j - X_t\|_{C^r(K)} \rightarrow 0$$

uniformly for all t on any compact subset K in D , then we have

$$\lim_{j \rightarrow \infty} \|\phi_t^j - \phi_t\|_{C^r(K)} \rightarrow 0$$

uniformly for all t on any compact subset K in D (for which ϕ_t exists). Here $r \geq 0$ is a fixed integer, $\phi_t^j = \phi^j(t, \cdot)$, and $\|f\|_{C^r(K)}$ denotes the C^r -norm of f on K , i.e., the maximum of the L^∞ norms of all partial derivatives of f up to order r on K with respect to the variables x .

Proof. It is a basic fact about differential equations (see e.g. Theorem 2.8 in [17]) that the lemma holds in the case $r=0$ since X_t and the X_t^j are assumed to be smooth (in particular Lipschitz). It is also a fact (however not as basic) that since

X_t (resp. X_t^j) is smooth, we have that $\phi(t, x)$ (resp. $\phi^j(t, x)$) is smooth (see e.g. Theorem 2.10 in [17]).

We will proceed by induction on r , and as induction hypothesis (I_r) we will assume that the theorem holds for some $r-1$ with $r \geq 1$. As just pointed out we have that (I_1) holds.

Letting $A(t, x)$ (resp. $A^j(t, x)$) denote the Jacobian of X_t (resp. X_t^j), and using the chain rule and the equality of mixed partials in (4), we see that $\frac{\partial \phi}{\partial x_i}(t, x)$ (resp. $\frac{\partial \phi^j}{\partial x_i}(t, x)$) is a solution to the initial value problem (variational equation)

$$(5) \quad \dot{y} = A(t, \phi(t, x)) \cdot y$$

(resp. $\dot{y} = A^j(t, \phi^j(t, x)) \cdot y$) with initial value $x_0 = (0, \dots, 1, \dots, 0)$ with the 1 at the i th spot. Now $A(t, \phi(t, x))$ (resp. $A^j(t, \phi^j(t, x))$) is smooth, and since

$$A^j(t, \phi^j(t, x)) \longrightarrow A(t, \phi(t, x))$$

in the $(r-1)$ -norm as $j \rightarrow \infty$, it follows by the induction hypothesis that $\frac{\partial \phi^j}{\partial x_i}(t, x) \rightarrow \frac{\partial \phi}{\partial x_i}(t, x)$ in $(r-1)$ -norm as $j \rightarrow \infty$. \square

3.3. Consistent algorithms and C^r -norms

Let $D \subset \mathbb{R}^n$ be a domain, let A be a compact subset of \mathbb{R}^m , and let $X(y, x): A \times D \rightarrow \mathbb{R}^n$ be a smooth map, which for each fixed y we interpret as a vector field on D . Let $\phi_{t,y}$ denote the phase flow of X_y . A *consistent algorithm* for X is a smooth map $\psi: I \times A \times D \rightarrow \mathbb{R}^n$ (here $I \subset \mathbb{R}$ is an unspecified interval containing the origin) such that

$$(6) \quad \frac{d}{dt} \Big|_{t=0} \psi_{y,t}(x) = X_y(x).$$

The following is a basic result on approximation of flows by means of consistent algorithms.

Theorem 3.3. *With notation as above, let $\psi_{y,t}(x)$ be a consistent algorithm for X . Let $K \subset D$ be a compact set, let $T > 0$, and assume that the flow $\phi_{y,t}(x)$ exists for all $x \in K, y \in A$ and all $0 \leq t \leq T$. Then for each $(t, x) \in I_T \times A \times K$ ($I_T = [0, T]$) we have that $\psi_{y,t/n}^n(x)$ exists for all sufficiently large n . Moreover $\psi_{y,t(y)/n}^n(x) \rightarrow \phi_{y,t(y)}(x)$ uniformly as $n \rightarrow \infty$, where $t: A \rightarrow I_T$ is a continuous function.*

In [1], Theorem 2.1.26, this was stated and proved without the uniformity in (t, x) , and without the parameter space A . Although the proof in our case is exactly the same, we include it for completeness.

Proof. Fix a constant $C > 0$ such that the following holds.

$$(1) \quad \|\psi_{y,t}(x) - \phi_{y,t}(x)\| \leq C \cdot t^2, \text{ and}$$

$$(2) \quad \|\phi_{y,t}(x) - \phi_{y,t}(x')\| \leq e^{C \cdot t} \cdot \|x - x'\|,$$

for all x, x' in an open neighbourhood Ω of the full $\phi(t, y, \cdot)$ -orbit of K , and for all t sufficiently small. We let $\bar{\Omega} \subset D$. Fix two points $x \in K, y \in A$, and for a fixed $n \in \mathbb{N}$ (large) we define

$$x_k := \psi_{y,t(y)/n}^k(x) \text{ and } y_k := \phi_{y,k \cdot t(y)/n}(x) = \phi_{y,t(y)/n}^k.$$

It is not a priori clear that x_k is well defined even for large n , this will follow from the following. Fix an initial $n \in \mathbb{N}$ such that $\psi_{y,T/n}$ exists on Ω for all $y \in A$. We claim that the following holds:

$$(7) \quad \|x_k - y_k\| \leq C \cdot \left(\frac{t(y)}{n}\right)^2 \cdot k \cdot e^{(k-1) \cdot C \cdot \frac{t(y)}{n}},$$

for all $k \leq n$. That this holds for $k=1$ follows immediately from (1), and to prove it for arbitrary $k \leq n$ we proceed by induction. By possibly having to increase n we see from (7) that $x_k \in \Omega$, and so x_{k+1} is well defined by the initial condition on n . We then get that

$$\begin{aligned} \|\psi_{y,t(y)/n}(x_k) - \phi_{y,t(y)/n}(y_k)\| &\leq \|\psi_{y,t(y)/n}(x_k) - \phi_{y,t(y)/n}(x_k)\| \\ &\quad + \|\phi_{y,t(y)/n}(x_k) - \phi_{y,t(y)/n}(y_k)\| \\ &\leq C \cdot \left(\frac{t(y)}{n}\right)^2 + e^{C \cdot \frac{t(y)}{n}} \cdot C \cdot \left(\frac{t(y)}{n}\right)^2 \cdot k \cdot e^{(k-1) \cdot C \cdot \frac{t(y)}{n}} \\ &\leq C \cdot \left(\frac{t(y)}{n}\right)^2 (1 + k \cdot e^{k \cdot C \cdot \frac{t(y)}{n}}) \\ &\leq C \cdot \left(\frac{t(y)}{n}\right)^2 \cdot (k+1) \cdot e^{k \cdot C \cdot \frac{t(y)}{n}}. \end{aligned}$$

This finishes the induction step, and we see that for sufficiently large n we have that x_k is well defined for $k \leq n$ independently of $x \in D$ and $y \in A$, and we have that $\|x_k - y_k\| \leq C \cdot \left(\frac{T}{n}\right)^2 \cdot (k+1) \cdot e^{k \cdot C \cdot \frac{T}{n}}$. \square

We will need the following generalisation of Theorem 3.3 which was proved in [16].

Theorem 3.4. *With the setup as in Theorem 3.3 we have that $\psi_{y,t/n}^n(x) \rightarrow \phi_t(x)$ uniformly in C^r -norm with respect to the variables (x) on $I_T \times A \times K$ as $n \rightarrow \infty$, for any fixed $r \in \mathbb{N}$.*

Since the proof of this theorem was given in the unpublished diplomarbeit [16] we will include it here.

Proof. We start by considering the C^1 -norm. For this we define a smooth vector field on $\mathbb{R}^n \times \mathbb{R}^{n^2}$ by

$$(8) \quad Y_y(x, z) = (X(x), \frac{\partial X_y}{\partial x}(x)z).$$

We claim first that

$$(9) \quad \Phi_y(t, x) = (\phi_y(t, x), \frac{\partial \phi_y}{\partial x}(t, x))$$

is the phase flow of Y with initial value (x, Id) . This follows by using the chain rule to see that

$$(10) \quad \frac{d}{dt} \frac{\partial \phi_y}{\partial x}(t, x) = \frac{\partial}{\partial x} \frac{d\phi_y}{dt}(t, x) = \frac{\partial (X_y \circ \phi_y)}{\partial x}(t, x) = \frac{\partial X_y}{\partial x}(\phi_y(t, x)) \frac{\partial \phi_y}{\partial x}(t, x).$$

The next step is to write down a convenient consistent algorithm for Y_y . We have that

$$(11) \quad \frac{d}{dt} \Big|_{t=0} \frac{\partial \psi_{y,t}(x)}{\partial x} = \frac{\partial}{\partial x} \frac{d}{dt} \Big|_{t=0} \psi_{y,t}(x) = \frac{\partial X_y}{\partial x}(x).$$

Therefore, the map

$$(12) \quad \Psi_y(t, x, y) = (\psi_y(t, x), \frac{\partial \psi_y}{\partial x}(t, x)z)$$

is a consistent algorithm for the vector field Y_y , and so we have that

$$\lim_{n \rightarrow \infty} \Psi_{y,t/n}^n \longrightarrow \Phi_{y,t}$$

uniformly as $n \rightarrow \infty$.

We will now show that the second component of $\Psi_{y,t/n}^n(x, \text{Id})$ is equal to $\frac{\partial \psi_{y,t/n}^n}{\partial x}(x)$, from which it will follow that

$$\frac{\partial \psi_{y,t/n}^n}{\partial x}(x) \longrightarrow \frac{\partial \phi_y}{\partial x}(t, x)$$

uniformly as $n \rightarrow \infty$ (by convergence of consistent algorithms in C^0 -norm). We will show this by induction.

Note first that

$$\Psi_{y,t/n}(x, y) = (\psi_y(t/n, x), \frac{\partial \psi_y}{\partial x}(t/n, x)y),$$

so if we set $y = \text{Id}$, we see that the second component of $\Psi_{y,t/n}^1(x, \text{Id})$ is equal to $\frac{\partial \psi_{y,t/n}^1}{\partial x}(x)$. As our induction hypothesis I_m ($1 \leq m < n$) we now assume that the

second component of $\Psi_{y,t/n}^m(x, \text{Id})$ is equal to $\frac{\partial \psi_{y,t/n}^m}{\partial x}(x)$ (as we have seen I_1 holds). We then have that

$$\Psi_{y,t/n}^m(x, \text{Id}) = (\psi_{y,t/n}^m(x), \frac{\partial \psi_{y,t/n}^m}{\partial x}(x)).$$

We get that

$$\Psi_{y,t,n}^{m+1}(x, \text{Id}) = (\psi_{y,t/n}^{m+1}(x), \frac{\partial \psi_{y,t/n}}{\partial x}(\psi_{y,t/n}^m(x)) \frac{\partial \psi_{y,t/n}^m}{\partial x}(x)).$$

And on the other hand, by the chain rule we have that

$$\frac{\partial}{\partial x} \psi_{y,t/n}(\psi_{y,t/n}^m(x)) = \frac{\partial \psi_{y,t/n}}{\partial x}(\psi_{y,t/n}^m(x)) \frac{\partial \psi_{y,t/n}^m}{\partial x}(x).$$

This completes the proof of convergence in C^1 -norm, and the general case follows by induction on r . \square

For the following corollary we introduce some notation. Let $X: I \times D \rightarrow \mathbb{R}^n$ be a smooth map which we interpret as a time dependent vector field on a domain $D \subset \mathbb{R}^n$. Fix an $n \in \mathbb{N}$. For each $j=0, 1, 2, \dots, n-1$ and $t \in I_T$ we let $X^{j t/n}(x)$ denote the time independent vector field $X_{(t_j)/n}(x)$, and we let $\psi_s^{j t/n}$ denote its flow. Finally, on any compact set $K \subset D$ and any for all t where the following is well defined, we set

$$(13) \quad \psi_t^n := \psi_{t/n}^{(n-1)t/n} \circ \psi_{t/n}^{(n-2)t/n} \circ \dots \circ \psi_{t/n}^{t/n} \circ \psi_{t/n}^0.$$

We have the following corollary to Theorem 3.4.

Corollary 3.5. *Let $X: I \times D \rightarrow \mathbb{R}^n$ be a smooth map which we interpret as a time dependent vector field on a domain $D \subset \mathbb{R}^n$, and let ϕ_t denote its phase flow. Let $K \subset D$ be a compact set and let I_T be an interval such that $\phi_t(x)$ exists on $I_T \times K$. Then $\psi_t^n(x) \rightarrow \phi_t(x)$ uniformly and in C^k -norm in the variables (x) on $I_T \times K$ as $n \rightarrow \infty$ (where ψ_t^n is defined as in (13)).*

Proof. We consider the time independent vector field Y on the orbit $(t, \phi_t(D))$ by

$$Y(t, x) := (1, X(t, x)).$$

Then the flow of Y is given by

$$\Phi_s(t, x) := (t+s, \phi_{t+s}(\phi_t^{-1}(x))).$$

Now for each t we let ψ_s^t denote the flow of the time independent vector field X_t where t is fixed, and we define

$$\Psi_s(t, x) = (t+s, \psi_s^t(x)).$$

Then Ψ_s is a consistent algorithm for Y , and so $\Psi_{s/n}^n \rightarrow \Phi_s$ uniformly on $I \times K$ in C^k -norm as $n \rightarrow \infty$. Now starting from $t=0$ and $x \in K$ it is easy to see by induction on j that

$$(14) \quad \Psi_{s/n}^j(x) = (js/n, \psi_{s/n}^{(j-1)s/n} \circ \dots \circ \psi_{s/n}^0(x)),$$

for $j \leq n$, and for $j=n$ we see that the second component gives (13) after substituting t for s . \square

3.4. An Andersén-Lempert type Theorem

With the setup from the beginning of this section, we now prove a theorem that will be the key ingredient in the proof of Theorem 1.2.

Theorem 3.6. *Assume that all connected components of Z_0 are simply connected and, and let $\psi: [0, 1] \times Z_0 \rightarrow Z_0$ be a smooth map such that $\psi_t \in \text{Symp}(Z_0, \omega_0)$ for each $t \in [0, 1]$. Assume further that $\psi_0 = \text{id}$, and that there exists an $R \geq 0$ such that $\psi_t(x) = x$ for all $\|x\| \leq R$, for all $t \in [0, 1]$. Then for any $r \in \mathbb{N}$ and $a > 0, 0 < b < R$, there exists a sequence $\Psi_{j,t}: Z \rightarrow Z$ with $\Psi_{j,t} \in \text{Symp}(Z, \omega)$ for all $j \in \mathbb{N}, t \in [0, 1]$, and with $\Psi_{j,t}$ Z_0 -invariant, such that*

$$(15) \quad \lim_{j \rightarrow \infty} \|\Psi_{j,t} - \psi_t\|_{C^r(a\overline{\mathbb{B}}_R^N \cap Z_0)} = 0$$

and

$$(16) \quad \lim_{j \rightarrow \infty} \|\Psi_{j,t} - \text{id}\|_{C^r(b\overline{\mathbb{B}}^N \cap Z)} = 0$$

uniformly in t .

Proof. We may assume that $a > b$. Let $K \subset Z_0$ be a compact set containing the entire ψ_t -orbit of $a\overline{\mathbb{B}}_R^N \cap Z_0$ in its interior.

Step 1: Define the time dependent vector field

$$X_t(\psi_t(x)) := \frac{d}{dt} \psi_t(x).$$

Then ψ_t is the phase flow of X . Since $\psi_t(x) = x$ for all $\|x\| \leq R$ and $t \in [0, 1]$ we may extend X_t to be identically zero on an open neighbourhood of $b\overline{\mathbb{B}}^{2n}$. Consider ψ_t^n as in (13). By using Cartan's formula one sees that each $X_{jt/n}$ is Hamiltonian, hence each $\psi_s^{jt/n}$ is symplectic. By Corollary 3.5 we have that $\psi_t^n \rightarrow \psi_t$ uniformly in C^r -norm with respect to the variables (x) uniformly in t . Hence, fixing a large enough n it remains to prove that each t -parameter family of flows $\psi_s^{jt/n}$, is approximable

in C^r -norm in the variables (x) by a t -parameter family of flows $\tilde{\psi}_s^{j,t}$ uniformly in (t, s) , where $\tilde{\psi}_s^{j,t} \in \text{Symp}(Z, \omega)$ for each fixed t , and leaving Z_0 invariant. By Lemma 3.2 it suffices to approximate the t -parameter family $X_{jt/n}$ uniformly in C^r -norm in the variables (x) uniformly in t , by complete holomorphic vector fields which are tangent to Z_0 .

Step 2: The approximation of the family $X_{jt/n}$ on K is now immediate from Lemma 3.1, we just have to take some extra care to achieve that the approximation is uniformly close to the identity on $b\overline{\mathbb{B}^N} \cap Z$. Let $P_{j,t}$ denote the real potential for $X_{jt/n}$ on Z_0 extended to a smooth function on \mathbb{R}^N . Then $P_{j,t}$ may be chosen to be zero for $\|x\| \leq R$, and so $P_{j,t}$ may be extended to be zero on $R \cdot \mathbb{B}_\mathbb{C}^N$. Hence (by possibly having to decrease R slightly) since $R \cdot \mathbb{B}_\mathbb{C}^N \cup \mathbb{R}^N$ is polynomially convex, we may approximate $P_{j,t}$ to arbitrary precision on $R \cdot \mathbb{B}_\mathbb{C}^N \cup K$ by a parameter family $Q_{j,t}$ of holomorphic polynomials, and by setting $\tilde{Q}_{j,t}(u) := \frac{1}{2}(Q_{j,t}(u) + \overline{Q_{j,t}(\overline{u})})$ we obtain a real holomorphic polynomial approximating $Q_{j,t}$, and we get that $X_{\tilde{Q}_{j,t}}$ approximates $X_{jt/n}$ on K and the identity on $R \cdot \mathbb{B}_\mathbb{C}^N$. Following the proof of Lemma 3.1 we obtain complete Hamiltonian vector fields $Y_{t,k}$ on Z such that $\sum_{k=1}^M Y_{t,k} = X_{\tilde{Q}_{j,t}}$.

Step 3: Let $\sigma_s^{t,k}$ denote the flow of $Y_{t,k}$ for $k=1, \dots, M$, and set

$$(17) \quad \Psi_s^{t,j} := \sigma_s^{t,M} \circ \dots \circ \sigma_s^{t,1}.$$

Then $\Psi_s^{t,j}$ is a consistent algorithm for $X_{\tilde{Q}_{j,t}}$, and so by Theorem 3.4 we have that $(\Psi_{t/nm}^{t,j})^m \rightarrow \psi_{t/n}^{j,t}$ as $m \rightarrow \infty$. \square

3.5. Proof of Theorem 1.2

As explained in the previous section, we may prove the Carleman approximation in the context presented in the beginning of this section, *i.e.*, for the pair (Z, Z_0) . By assumption we have that ϕ is smoothly isotopic to the identity map, and so there exists a smooth isotopy $\psi_t, t \in [0, 1]$, of symplectomorphisms of Z_0 such that $\psi_0 = \text{id}$ and $\psi_1 = \phi$. For $R > 0$ we set $Z_R = R \cdot \overline{\mathbb{B}^N} \cap Z$ and $Z_{0,R} = R \cdot \overline{\mathbb{B}^N} \cap Z_0$. We will construct the approximating automorphism inductively.

Assume that we have constructed $\phi_j \in \text{Symp}(Z, \Omega)$ leaving Z_0 invariant, constructed an isotopy ψ_t^j with $\psi_t^j \in \text{Ham}(Z_0, \omega)$ for each $t \in [0, 1]$, and found $R_1^j, R_2^j \in \mathbb{N}$ with $R_1^j \geq j, R_2^j \geq R_1^j + 1$, such that the following hold:

- (1_j) $\phi_j(R_1^j \cdot \overline{\mathbb{B}^N}) \subset R_2^j \cdot \overline{\mathbb{B}^N}$,
- (2_j) $\|\phi_j - \phi_{j-1}\|_{R_1^{j-1} \overline{\mathbb{B}^N}} < \varepsilon_j$ (for $j \geq 2$),
- (3_j) $\psi_0^j = \text{id}$,
- (4_j) $\psi_t^j(x) = x$ for $\|x\| \leq R_2^j + \varepsilon_j$ for all $t \in [0, 1]$, and

(5_j) $\|\psi_1^j \circ \phi_j - \phi\|_{C^r(x)} < \varepsilon(x)$ for all $x \in Z_0$.

The inductive step is the following

(18) We may achieve (1_{j+1})–(5_{j+1}) with ε_{j+1} arbitrarily small.

Set $R_1^{j+1} = R_2^j + 1$ and fix S_1 such that Z_{0,S_1} contains the entire ψ_t^j -orbit and $(\psi_t^j)^{-1}$ -orbit of $Z_{0,R_1^{j+1}+1}$ in its interior. Fix a cutoff function $\chi(z)$ such that $\chi(z) = 0$ for all $\|z\| \leq S_1$ which is identically equal to 1 for $\|z\| \geq S_1 + 1$. By Theorem 3.6 we may approximate ψ_t^j to arbitrary precision near Z_{0,S_1+2} by Z_0 -invariant symplectomorphisms Ψ_t^j , which also approximates the identity to arbitrary precision near $R_2^j \cdot \overline{\mathbb{B}^N}$. Next we set

$$(19) \quad \sigma_t^j = (\Psi_t^j)^{-1} \circ \psi_t^j,$$

and note that we now may assume that this is as close as we like to the identity on $Z_{0,S+2}$.

Letting P_t denote the Hamiltonian potential for σ_t^j we may then consider the isotopy $\tilde{\sigma}_t^j$ whose potential is the function $\chi \cdot P_t$. Then we may still assume that $\tilde{\sigma}_t^j$ is close we like to the identity on Z_{0,S_1+2} , it is equal the identity on Z_{0,S_1} , and it is equal to σ_t^j outside of Z_{0,S_1+1} . Note that we may now assume that $\Psi_t^j \circ \tilde{\sigma}_t^j$ approximates ψ_t^j to arbitrary precision on Z_0 and also that $(\Psi_t^j \circ \tilde{\sigma}_t^j)^{-1}$ approximates $(\psi_t^j)^{-1}$ to arbitrary precision on Z_0 .

Next we fix R_2^{j+1} such that $\Psi_1^j(R_1^{j+1}\mathbb{B}^N) \subset \subset R_2^{j+1}\mathbb{B}^N$. Choose T such that $Z_{0,T}$ contains the entire $\tilde{\sigma}_t^j$ -orbit and $(\tilde{\sigma}_t^j)^{-1}$ -orbit of $Z_{0,R_2^{j+1}+1}$ in its interior, let $\tilde{\chi}$ be a cutoff function such that $\tilde{\chi}(z) = 0$ for all $\|z\| \leq T$ which is identically equal to 1 for $\|z\| \geq T + 1$. Note that $\tilde{\sigma}_t^j$ may be extended continuously to the identity map near $R_1^{j+1}\mathbb{B}^N$, thus by arguing additionally as in the proof of Theorem 3.6, we may get an approximating isotopy $\tilde{\Psi}_t^j$ as in Theorem 3.6, approximating $\tilde{\sigma}_t^j$ to arbitrary precision near $Z_{0,T+2}$, and the identity to arbitrary precision on $R_1^{j+1}\mathbb{B}^N$.

Next, we set

$$(20) \quad \hat{\sigma}_t^j = \psi_t^j \circ (\Psi_t^j \circ \tilde{\Psi}_t^j)^{-1}$$

We let \hat{P}_t denote the potential of $\hat{\sigma}_t^j$, and we finally set ψ_t^{j+1} be the Hamiltonian flow on Z_0 whose potential is $\tilde{\chi}\hat{P}_t$, and note that ψ_t^{j+1} is the identity near $R_2^{j+1}\mathbb{B}^N$. Finally, setting $\phi_{j+1} := \Psi_1^j \circ \tilde{\Psi}_1^j \circ \phi_j$, we see that we have established (1_{j+1})–(5_{j+1}).

Finally, it is standard to construct the sequence ϕ_j , choosing each ε_j sufficiently small, such that ϕ_j converges to the desired approximating automorphism of Z , we leave the details to the reader.

4. Some examples

In this section we present some typical example of coadjoint orbits of real Lie groups and their complexifications.

Example 4.1. (The flat space) Let G_0 be the Heisenberg group given by

$$G_0 = \left\{ g = \begin{pmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} : a, b, c \in \mathbb{R} \right\},$$

whose complexification G is defined by the same form by requiring $a, b, c \in \mathbb{C}$.

- Coajoint orbits in \mathfrak{g}_0^* .

We can identify \mathfrak{g}_0^* with \mathbb{R}^3 , with the coadjoint action given by

$$g \cdot (x_1, x_2, x_3) = (x_1 + bx_3, x_2 - ax_3, x_3).$$

We consider the coadjoint orbit $\mathcal{O}_x^{\mathbb{R}}$ through $x = (x_1, x_2, x_3)$. If $x_3 = 0$, $\mathcal{O}_x^{\mathbb{R}}$ is a single point; if $x_3 \neq 0$, we can identify $\mathcal{O}_x^{\mathbb{R}}$ with (\mathbb{R}^2, ω_0) by the map

$$(x_1 + bx_3, x_2 - ax_3) \mapsto (x, y) \in \mathbb{R}^2,$$

where $\omega_0 = \frac{1}{x_3} dx_1 \wedge dx_2$. In particular, if $x_3 = 1$, $\mathcal{O}_x^{\mathbb{R}}$ is isomorphic as symplectic manifolds to \mathbb{R}^2 with the standard symplectic structure.

- Coajoint orbits in \mathfrak{g}_0^* .

We can identify \mathfrak{g}^* with \mathbb{C}^3 , with the adjoint action of G on \mathfrak{g}^* given by the same way. The coadjoint orbit \mathcal{O}_z with $z = (z_1, z_2, z_3) \in \mathbb{C}^3$ is a single point if $z_3 = 0$, and is isomorphic to (\mathbb{C}^2, Ω_0) with $\Omega_0 = z_3^{-1} dz_1 \wedge dz_2$ if $z_3 \neq 0$.

When $z = x \in \mathbb{R}^3$, $(\mathcal{O}_z, \Omega_0, \tau)$ is a complexification of $(\mathcal{O}_x, \omega_0)$, with the anti-holomorphic convolution τ given by the complex conjugate.

Example 4.2. (The parabolic space) Let $G_0 = SO(3) \approx SU(2)$ be the special orthogonal group. The complexification G of G_0 is $SO(3, \mathbb{C})$.

- Coajoint orbits in \mathfrak{g}_0^* .

We can identify \mathfrak{g}_0^* with \mathbb{R}^3 , with the coadjoint action given by rotations. The coadjoint orbits are parametrized by $\mathbb{R}_{\geq 0}$ and given by:

$$(21) \quad x_1^2 + x_2^2 + x_3^2 = R^2,$$

with the canonical symplectic structure given by

$$\omega = \frac{dx_2 \wedge dx_3}{x_1}.$$

When $R=0$, the orbit is a single point, and when $R=1$, the orbit is isomorphic to the Riemann sphere, with the symplectic structure given by the Fubini-Study metric.

- Coajoint orbits in \mathfrak{g}^* .

We identify $\mathfrak{g}^* = \mathfrak{g}_0^* \otimes \mathbb{C}$ with \mathbb{C}^3 . The origin is a closed coadjoint orbit, and other closed coadjoint orbits are parametrized by \mathbb{C}^* and have the form:

$$(22) \quad z_1^2 + z_2^2 + z_3^2 = R^2, \quad (R \in \mathbb{C}^*),$$

with the canonical holomorphic symplectic structure given by

$$\Omega = \frac{dz_2 \wedge dz_3}{z_1}.$$

When $R \in \mathbb{R}_{>0}$, the coajoint orbit of G given by (22) is a complexification of the coadjoint orbit of G_0 given by (21), with the antiholomorphic convolution τ given by the complex conjugate.

Example 4.3. (The hyperbolic space) Let $G_0 = SO(2, 1) \approx SL(2, \mathbb{R})$, whose complexification is $G = SO(2, 1, \mathbb{C}) \approx SL(2, \mathbb{C})$.

The coadjoint action is equivalent to the standard action of $SO(2, 1)$ on \mathbb{R}^3 . The orbits are parametrized by \mathbb{R} and have the following form:

$$x_3^2 - x_1^2 - x_2^2 = R.$$

They are divided into 3 types corresponding to $R > 0$, $R = 0$ and $R < 0$. Here we just consider the case that $R > 0$, $x_3 > 0$. For this case, the isotropy group of the coadjoint action at $(0, 0, R)$ is S^1 , and the coadjoint orbit is given by $SL(2, \mathbb{R})/S^1$, which is isomorphic to the upper half plane \mathbb{H} with the symplectic structure given by the Poincaré metric.

The coadjoint orbits in $\mathfrak{g}^* \approx \mathbb{C}^3$ are given by the same equations. They are complexifications of coadjoint orbits of G_0 , with the antiholomorphic convolution τ given by the complex conjugate.

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