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Exponential localization in 2D pure magnetic wells

Y. Guedes Bonthonneau, N. Raymond and S. Vũ Ngọc

Abstract. We establish a magnetic Agmon estimate in the case of a purely magnetic single non-degenerate well, by means of the Fourier-Bros-Iagolnitzer transform and microlocal exponential estimates *à la* Martinez-Sjöstrand.

1. Introduction

The question of proving the localization of a quantum state has many mathematical facets. In this article, we investigate the case of the magnetic Laplacian and prove, under a geometric confinement property on the magnetic intensity, an Agmon-type localization estimate for low-lying eigenfunctions of this operator.

The interest in the magnetic Laplacian has several origins. From a quantum mechanical viewpoint, this operator is a simplified model for describing the motion of an electron in a strong magnetic field, when the electrostatic interaction and the relativistic effects are ignored; its construction is explained for instance [7]. In the book [15], the authors recall that the same operator also appears in the linearization of the Ginzburg-Landau functional in the domain of superconductors. In Spectral Geometry, the magnetic Laplacian is often regarded as a natural variant of the Laplace-Beltrami operator when the symplectic form of the cotangent bundle is twisted by the pull-back of a closed 2-form from the base manifold, and has proved important in the study of magnetic geodesics; see for instance [25], and references therein. In the present study, we consider the magnetic Laplacian on the plane, which can be defined as follows.

When B is a real function on $\mathbb{R}^2$, a semiclassical magnetic Laplacian associated with B is a family of operators, depending on a parameter $h>0$, of the form

$$\mathscr{L}_h=(-ih\nabla-\mathbf{A})^2=(hD_1-A_1(x))^2+(hD_2-A_2(x))^2\,,\quad D=-i\partial\,. \tag{1.1}$$

Here, $\mathbf{A}=(A_1, A_2)$ is a potential vector associated with B, i.e $B=\partial_1 A_2 - \partial_2 A_1$. Notice that the semiclassical limit $h\to 0$ is related to the limit of strong magnetic field $(1/h)B$.

The spectral theory of $\mathscr{L}_h$ has received the attention of several authors; in particular, it follows from [19] that if B is smooth and admits a global non-degenerate minimum, uniquely attained at some $x\in\mathbb{R}^2$, the bottom of the spectrum of $\mathscr{L}_h$ (for h small enough) is comprised of multiplicity one eigenvalues $\lambda_0(h)<\lambda_1(h)\ldots$, with

$$\lambda_j(h) = b_0 h + (C_1 + C_2 j)h^2 + o(h^2)\,.$$

The corresponding eigenfunctions are concentrated around x, in the sense that their L^2 mass outside of a fixed neighborhood of x is $\mathscr{O}(h^\infty)$. The purpose of this article is to obtain a stronger concentration in the case when B is *real analytic*.

1.1. Statement of the result

From now on, we assume the following:

(i). The magnetic field B has a unique minimum b_0 at $x=0$. It is positive, non-degenerate, and not attained at infinity ($\liminf B > b_0$).

(ii). There exists a complex strip $\mathcal{S}=\mathbb{R}^2+i[-a,a]^2$ ($a>0$) to which B can be holomorphically extended as a bounded function.

(iii). The function $(x_1,x_2)\mapsto \int_0^{x_1} \frac{\partial B(u,x_2)}{\partial x_2}\,du$ is bounded on the strip $\mathcal{S}$.

For example, $B=2-e^{-|x|^2}$ satisfies our assumptions. We will say that a function $f:\mathbb{R}^n\to\mathbb{R}$ *goes linearly to infinity at infinity* if there is a constant $C>0$ such for $|x|>C$, $f>|x|/C$. Our main result is the following exponential localization estimate.

Theorem 1.1. *Consider a Lipschitz function $d:\mathbb{R}^2\to\mathbb{R}_+$ with a unique and non-degenerate minimum at* 0, $d(0)=0$, *and going linearly to infinity at infinity, and let $K>0$. Then there exist $C, h_0, \varepsilon>0$ such that, for all $h\in(0,h_0)$ and $u\in L^2(\mathbb{R}^2)$ such that*

$$\mathscr{L}_h u = h\mu u \quad \textit{with } \mu \le b_0 + Kh\,,$$

we have

$$\int_{\mathbb{R}^2} e^{\varepsilon d(x)/h} |u(x)|^2\,dx \le C\|u\|^2_{L^2(\mathbb{R}^2)}\,,$$

Observe that here, the third Assumption (iii) seems technical, and depends on a choice of a system of coordinates, but we have not been able to remove it. Also note that since we are not trying to optimize constants in our theorem, the value of $a>0$ in (ii) is not essential. As a consequence, to lighten notations, we will use $a>0$ as a generic constant throughout the paper. The size of the strip on which we are working will be reduced a finite number of times.

1.2. Eigenvalues asymptotics

Strong localization of eigenfunctions, such as the one claimed by Theorem 1.1, is often a footprint of discrete spectrum. Indeed, under Assumption (i), it follows from the usual theory (see [2]) that below $h \liminf B$, the spectrum of $\mathscr{L}_h$ is discrete. Let $\lambda_0(h) \leq \lambda_1(h) \leq \ldots \lambda_\ell(h) \leq \ldots \leq \liminf B$ be the (possibly finite) sequence of such eigenvalues, repeated according to their multiplicity.

The following theorem has been established via a dimensional reduction in [20] (see also [19] and the review paper [18]) and via a Birkhoff normal form in [37]. In fact, this theorem does not require the analyticity of B (i.e Assumptions (ii) and (iii)), but rather $\mathscr{C}^\infty$ bounds on B.

Theorem 1.2. ([19], [20] and [37])

$$\forall \ell \in \mathbb{N}\,, \quad \lambda_\ell(h) = b_0 h + \left(2\ell \frac{\sqrt{\det H}}{b_0} + \frac{(\operatorname{Tr} H^{\frac{1}{2}})^2}{2b_0}\right) h^2 + o(h^2)\,, \tag{1.2}$$

where $b_0 = \min_{\mathbb{R}^2} B$ *and* $H = \frac{1}{2}\operatorname{Hess}_{(0,0)} B$.

1.3. Complex WKB expansions

With Theorem 1.2 comes the question of describing the eigenfunctions. Inspired by the results about the semiclassical Schrödinger operator with an electric potential, we can wonder whether the complex version of the famous Wentzel-Kramers-Brillouin (WKB) Ansatz can be adapted to the magnetic case. Such constructions, solving formally the eigenvalue problem, are rather rare in the context of the pure magnetic Laplacian; see however [29, VI, $\mathcal{S}$2]. Their existence has been established for the first time in a multi-scale framework in [4] and then in non-degenerate magnetic wells (*i.e.*, under Assumption (i)) in [16]. Let us recall the latter result (which was generalized to the Riemannian setting in [35]).

Theorem 1.3. ([16]) *Under Assumption* (i), *and after a rotation, we can assume*

$$B(x_1, x_2) = b_0 + \alpha x_1^2 + \gamma x_2^2 + \mathscr{O}(\|x\|^3)\,, \qquad \text{with } 0 < \alpha \leq \gamma\,. \tag{1.3}$$

Let $\ell \in \mathbb{N}$. *There exist*

(i). *a neighborhood* $\mathcal{V}$ *of* $(0,0)$ *in* $\mathbb{R}^2$,

(ii). *an analytic function* S *on* $\mathcal{V}$ *satisfying*

$$\operatorname{Re} S(x) = \frac{b_0}{2}\left[\frac{\sqrt{\alpha}}{\sqrt{\alpha}+\sqrt{\gamma}} x_1^2 + \frac{\sqrt{\gamma}}{\sqrt{\alpha}+\sqrt{\gamma}} x_2^2\right] + \mathscr{O}(\|x\|^3)\,,$$

(iii). *a sequence of analytic functions* $(a_j)_{j\in\mathbb{N}}$ *on* $\mathcal{V}$,
(iv). *a sequence of real numbers* $(\mu_j)_{j\in\mathbb{N}}$ *satisfying*

$$\mu_0 = b_0\,, \quad \mu_1 = 2\ell\frac{\sqrt{\alpha\gamma}}{b_0} + \frac{(\sqrt{\alpha}+\sqrt{\gamma})^2}{2b_0}\,,$$

such that, for all $J\in\mathbb{N}$, *and uniformly in* $\mathcal{V}$,

$$e^{S/h}\left((-ih\nabla-\mathbf{A})^2 - h\sum_{j\geq 0}^{J}\mu_j h^j\right)\left(e^{-S/h}\sum_{j\geq 0}^{J} a_j h^j\right) = \mathscr{O}(h^{J+2})\,.$$

The WKB constructions in [4] and [16] give a positive answer to the open problem mentioned by Helffer in [17, Section 6.1]: in generic situations with pure magnetic field, WKB constructions corresponding to the low lying spectrum exist. Once the WKB analysis is done, we want to know to which extent the Ansätze are approximations of the exact eigenfunctions $u_\ell\in L^2(\mathbb{R}^2)$. It follows from Theorem 1.2 that, when h is small enough, the eigenvalues are simple and separated by a gap of order $\sim h^2$. Thanks to the Spectral Theorem, we deduce that the WKB Ansätze are approximations in the L^2-sense, and even in a weighted L^2-space thanks to Theorem 1.1 (up to taking a smaller ε).

Corollary 1.4. *Denote by* $u_{\ell,J} = \chi(x)e^{-S/h}\sum_{j\geq 0}^{J} a_j h^j$, *with* $\chi\in\mathscr{C}_0^\infty(\mathcal{V})$, *and constant around the origin. Then, for fixed* $\ell\in\mathbb{N}$ *and* $\varepsilon>0$ *small enough, we have for some* $\theta\in\mathbb{R}$

$$\|e^{\varepsilon d(x)/h}(e^{i\theta}u_\ell - u_{\ell,J})\|_{L^2(\mathbb{R}^2)} = \mathscr{O}(h^{\frac{J}{2}}).$$

(This will be proved at the end of Section 5.) In contrast with Theorem 1.1, the WKB Ansätze decay like $e^{-\mathrm{Re}\,S/h}$ away from the magnetic well; thus, the approximation should actually hold in a slight perturbation of the weighted space $L^2(e^{-2\mathrm{Re}\,S/h})$. Behind this question lies the tunneling effect problem: such exponential estimates are the heart of the analysis of the interaction between multiple magnetic wells. The present paper does not go that far,(1) but establishes that the eigenfunctions decay like $e^{-\varphi(x)/h}$ for some non-negative function φ. These types of estimates are well-known and proved in the *electric* Schrödinger operator $-h^2\Delta+V$, where they go by the name of Agmon (see [1], [21] and [38]). As we will see, the purely magnetic case seems to necessitate a significantly more advanced strategy, based on the Fourier-Bros-Iagolnitzer (FBI) transform. (In [21], the FBI transform does appear, but not for proving the exponential localization; it is used in a second step, to control the asymptotic expansion of eigenvectors and eigenvalues.)

(1) The only known (and optimal) result of pure magnetic tunnelling has recently been proved in a two-dimensional setting in [5] by means of microlocal dimensional reductions.

1.4. Failure of the naive Agmon estimates

Let us explain formally why the electric strategy fails in giving the optimal Agmon estimates in the pure magnetic case (see also [36, Proposition 4.23] for a slightly different presentation). This strategy is based on the following formula:

$$e^{\varphi/h}(-ih\nabla-\mathbf{A})^2e^{-\varphi/h}=(-ih\nabla-\mathbf{A}+\nabla\varphi)^2\,,$$

where φ is bounded and Lipschitz continuous, and on using the *coercivity* of the real part

$$\operatorname{Re}\langle e^{\varphi/h}(-ih\nabla-\mathbf{A})^2e^{-\varphi/h}u,u\rangle=\|(-ih\nabla-\mathbf{A})u\|^2-\|\nabla\varphi\, u\|^2\,,$$

where $u\in\mathscr{C}_0^\infty(\mathbb{R}^2)$. Then, we want to use the magnetic field, and we notice that

$$\|(-ih\nabla-\mathbf{A})u\|^2\geq h\int_{\mathbb{R}^2}B(x)|u|^2\,dx\,,$$

so that, for all $\lambda\in\mathbb{R}$,

$$\operatorname{Re}\left\langle\left(e^{\varphi/h}(-ih\nabla-\mathbf{A})^2-\lambda\right)e^{-\varphi/h}u,u\right\rangle\geq\int_{\mathbb{R}^2}\left(hB(x)-|\nabla\varphi|^2-\lambda\right)|u|^2\,dx\,.$$

From this last inequality, we see that the only possibility to control the gradient is that φ actually depends on h. With the choice $\varphi=h^{\frac{1}{2}}\Phi$, where Φ is the Agmon distance (to 0) associated with the metric $(B-b_0-|\nabla\Phi|^2)_+dx^2$, we can deduce that, for eigenvalues such that $\lambda=b_0h+\mathscr{O}(h^2)$, the corresponding eigenfunctions $\psi(=e^{-\varphi/h}u)$ satisfy, for h small enough,

$$\int_{\mathbb{R}^2}e^{2\Phi/h^{\frac{1}{2}}}|\psi|^2\,dx\leq C\|\psi\|^2\,. \tag{1.4}$$

Due to the non-degeneracy of the minimum of B, Φ may be chosen with a unique and non-degenerate minimum at 0. Thus, (1.4) tells us for instance that the ground state is *a priori* exponentially localized at the scale $h^{\frac{1}{4}}$ near the minimum. This is consistent with Theorem 1.3, but much worse than expected. In the analytic case, the construction of WKB quasi-modes made in [16] suggests that one should be able to do better; namely, to prove that the eigenfunctions are localized at the scale $h^{\frac{1}{2}}$ near the minimum. That it is indeed the case is the main result of this article.

Notice that estimate (1.4) does not require the analyticity of the magnetic field. We believe that it is optimal in the $\mathscr{C}^\infty$ category, where exponential estimates cannot be controlled in phase space and the techniques of the present paper don't apply. Even the construction of the WKB phase becomes problematic in the smooth case, see [6].

1.5. Related results

Some articles have been devoted to the Agmon estimates in the presence of a magnetic field, but almost always with an additional electric potential. For instance, in [22], the decay estimates are inherited from the electric potential and the magnetic field is considered as a perturbation (see in particular [22, p. 629]). In the same spirit, Agmon estimates are considered in [34, Theorem 1.1] (see also the closely related articles [14] and [32]) in the case of an electric well with constant magnetic field. It is proved that the magnetic field improves the decay of the eigenfunctions away from the electric well.

We will see in this paper that pure magnetic Agmon estimates at the "right" semiclassical scale can be obtained as projections of microlocal exponential estimates. Our strategy will be inspired by the ideas of Martinez [26] (see also [33], and [28] in relation with the corresponding WKB analysis). The fact that we are able to refine this point of view, which is based on the FBI transform, and to apply it to establish our new magnetic Agmon estimates, is reminiscent of Sjöstrand's pioneer work on analytic hypo-ellipticity [39].

Remark 1.5. Throughout our analysis, we will meet some known close links between magnetic and Toeplitz operators. These connections are described, for instance, in [11], or [24, Section 4]. In the context of Toeplitz operators, exponential decay estimates of eigenfunctions have been the subject of the recent works [12, Theorem C] and [24, Theorem 1.3]. In these papers, the semiclassical parameter is of the form $h=p^{-1}$, where $p\in\mathbb{N}$ is the degree of tensorization of a line bundle.

1.6. Organization and strategy

In Section 2, we perform various reductions to put the magnetic Laplacian in a "normal form". Section 3 is central in our analysis and is devoted to general properties of the Fourier-Bros-Iagolnitzer transform. Our presentation closely follows and, sometimes, completes the one exposed in the book by Martinez [27, Chapter 3]. This part of the investigation can also be considered an interpretation of the magnetic Laplacian as a Toeplitz operator. In Section 4, we prove that the FBI transform of an eigenfunction (with low energy) is exponentially localized at the scale $h^{\frac{1}{2}}$ near $0\in\mathbb{R}^2\times(\mathbb{R}^2)^*$. We proceed in two steps: firstly, we prove the exponential microlocalization near the characteristic manifold $\{(x,\xi)\in\mathbb{R}^2\times\mathbb{R}^2;\ \xi=\mathbf{A}(x)\}$, which is the zero energy level set of the classical Hamiltonian (Theorem 4.4); secondly, we establish an exponential localization inside the manifold (Theorem 4.5). In Section 5, we use the microlocal exponential estimates to deduce Theorem 1.1.

2. Normal form

In [37], the second and third author constructed a Fourier integral operator U_h that conjugates the magnetic Laplacian $\mathscr{L}_h$ to an operator of the form

$$\mathrm{Op}_h^{\mathrm{w}}(f(\mathcal{H}, x_2, \xi_2)) + \mathscr{O}(h^\infty),$$

microlocally near the characteristic set of $\mathscr{L}_h$, where $\mathcal{H}=h^2 D_{x_1}^2 + x_1^2$ and $\mathrm{Op}_h^{\mathrm{w}}$ denotes the Weyl quantization. If the symbol f were analytic, and the remainder $\mathscr{O}(h^\infty)$ improved to $\mathscr{O}(e^{-C/h})$, this would imply a natural (and probably optimal) exponential estimate on the bottom eigenfunctions of $\mathscr{L}_h$. However, the FIO U_h is constructed in a relatively non-explicit fashion, including a generically divergent Birkhoff normal form, and tracking those estimates down would require quite sophisticated tools of analytic microlocal analysis.

Since we "only" want to obtain decay of eigenfunctions and not the expansion of the bottom eigenvalues of $\mathscr{L}_h$ to any power of h, we will only need a rather crude normal form.

Lemma 2.1. *Under Assumptions* (i), (ii), (iii), *there exists* $a>0$ *and, for* $h>0$, *a unitary operator* U_h *acting on* $L^2(\mathbb{R}^2, dx)$ *such that*

$$U_h \mathscr{L}_h U_h^{-1} = \mathrm{Op}_h^{\mathrm{w}}(p_{\mathscr{L}}), \tag{2.1}$$

where $p_{\mathscr{L}}$ *is an* h*-dependent holomorphic function on* $\mathbb{R}^4 + i[-a,a]^4$, *such that*

$$p_{\mathscr{L}} = g^{11}\xi_1^2 + 2g^{12}\xi_1 x_1 + g^{22} x_1^2 + h^2 q, \tag{2.2}$$

where g^{11}, g^{12}, g^{22} *and* q *are holomorphic, bounded, and on* $\mathbb{R}^4$ *they are real valued. Additionally, the* g^{ij} *are critical at* 0, *and*

$$B(x,\xi) = \sqrt{g^{11}g^{22} - (g^{12})^2}, \tag{2.3}$$

when restricted to $\mathbb{R}^4$, *admits a positive non-degenerate minimum at* 0, *uniquely attained, and not attained at infinity.*

This type of operators, whose symbol is a quadratic form of some variables, with parameters, was studied by several authors in the context of hypo-ellipticity in the smooth category (see [8] and references therein), and in the analytic category by Sjöstrand in [39]. It would be interesting to obtain a global version of Sjöstrand's results in order to give a different proof of Theorem 1.1.

Observe that the exponential decay of eigenfunctions is not preserved by general unitary operators. However, we will see that U_h can be explicitly described, so this will not be an issue.

For a given magnetic field B, the choice of magnetic potential $\mathbf{A}$ is not unique. Any other choice $\mathbf{A}'$ differs from $\mathbf{A}$ by a gradient, *i.e.* $\mathbf{A}'=\mathbf{A}+\nabla f$. Then, the corresponding magnetic Laplacian is obtained by conjugating $\mathscr{L}_h$ by the multiplication operator $u\mapsto e^{if/h}u$, which is unitary, both pointwise and in L^2. Hence, it does not impact Theorem 1.1. Therefore we may, and we will, assume that

$$\mathbf{A}(x)=(0,A_2(x))\,,\quad A_2(x)=\int_0^{x_1} B(u,x_2)\,du\,, \tag{2.4}$$

Notice that A_2 is real-analytic, admits a holomorphic extension to the strip $\mathcal{S}$, and its derivatives are bounded on $\mathcal{S}$ according to Assumption (iii).

For $d=2$ or $d=4$ depending on the context and $a>0$, it will be convenient to set $\mathcal{S}_a:=\mathbb{R}^d+i[-a,a]^d$.

2.1. Normal form near the characteristic set

In this section, we prove Lemma 2.1. The operator U_h will be decomposed as the composition of a change of variables and a metaplectic operator. Let us start by constructing the change of variable.

The first idea, which is quite standard, is to choose coordinates in which the magnetic field is constant *as a 2 form.* In that case, the natural symplectic structure becomes canonical, and all the magnetic information is transferred to a variable Riemannian metric. The guiding model is the case of constant magnetic field and constant metric, where the magnetic Laplacian takes the form

$$\mathscr{L}_h^{\mathrm{ct}}=(hD_{x_1})^2+(hD_{x_2}-Bx_1)^2,$$

and its bottom eigenvalue is hB. The solutions, sometimes called *zero modes*, to

$$\left(\mathscr{L}_h^{\mathrm{ct}}-hB\right)u=0$$

are of the form $e^{-Bx_1^2/2h}f$, with f holomorphic, and they play an important role in the spectral analysis of the magnetic Dirac operator (see [3]).

Coming back to our problem, there are many diffeomorphisms $\varkappa$ of $\mathbb{R}^2$ such that $\varkappa_*B$ is the canonical 2 form (Darboux' lemma), so we pick the following

$$(x_1,x_2)=\varkappa(\tilde{x}_1,\tilde{x}_2),\quad \tilde{x}_1=\int_0^{x_1} B(x',x_2)\,dx',\ \ \tilde{x}_2=x_2.$$

That this defines indeed a global diffeomorphism of $\mathbb{R}^2$ is ensured by Assumption (i).

Lemma 2.2. *Under Assumptions* (i), (ii) *and* (iii), *$\varkappa$ is a bi-Lipschitz analytic diffeomorphism of $\mathbb{R}^2$ such that $\varkappa_*\mathbf{B}=d\tilde{x}_1\wedge d\tilde{x}_2$ and $\varkappa_*\mathbf{A}=\tilde{x}_1 d\tilde{x}_2$. Moreover, there exists $\lambda\geq 1$ and $a>0$ such that $\varkappa$ and $\varkappa^{-1}$ send $\mathcal{S}_{a'}$ to $\mathcal{S}_{\lambda a'}$ for all $a'\in(0,a/\lambda)$.*

It will be useful to let

$$\alpha(x_1,x_2)=\int_0^{x_1}\partial_{x_2}B(u,x_2)\,dx_1.$$

Proof. $\varkappa$ is a global diffeomorphism of $\mathbb{R}^2$ because B is positive, and $\varkappa^{-1}$ is well defined on $\mathcal{S}_a$. Next, there is a $C>0$ such that $|B|\le C$ and $|\alpha|\le C$, so that $\varkappa^{-1}$ maps $\mathcal{S}_{a'}$ into $\mathcal{S}_{2Ca'}$ for $0<a'<a$.

We can compute

$$d_x(\varkappa^{-1})=\begin{pmatrix}B(x) & \alpha(x)\\ 0 & 1\end{pmatrix}.$$

In particular,

$$(d_x(\varkappa^{-1}))^{-1}=\begin{pmatrix}\frac{1}{B(x)} & -\frac{\alpha(x)}{B(x)}\\ 0 & 1\end{pmatrix}.$$

Since $B\ge b_0>0$ on $\mathbb{R}^2$, and using Assumption (ii), there exists $0<a_0<a$ such that $|B|^{-1}\le(\operatorname{Re}B)^{-1}\le 1/(2b_0)$ on $\mathcal{S}_{a_0}$. In particular, on $\mathcal{S}_{a_0}$, $(d_x(\varkappa^{-1}))^{-1}$ is bounded.

Around each real point x, we can apply the holomorphic local inversion theorem and deduce that there are $\varepsilon_x,\varepsilon'_x>0$ such that $\varkappa^{-1}$ is a biholomorphism between the ball of radius ε_x centered at x and its image, which contains the ball of radius ε'_x around $\varkappa^{-1}(x)$. One can give lower bounds to the constants ε_x, ε'_x, expressed only in terms of the C^2 norms of $\varkappa^{-1}$, and an upper bound on $(d_x(\varkappa^{-1}))^{-1}$. In particular, we can choose them independent of x.

Additionally, if $\varkappa^{-1}(x)=\varkappa^{-1}(y)$ for some $x,y\in\mathcal{S}_{a'}$ with $0<a'<a_1$, then $x_2=y_2$, and $\int_{x_1}^{y_1}B=0$. Observe that

$$0=\operatorname{Re}\int_{x_1}^{y_1}B=\int_{\operatorname{Re}x_1}^{\operatorname{Re}y_1}\operatorname{Re}B(t+i\operatorname{Im}x_1,x_2)\,dt-\int_{\operatorname{Im}x_1}^{\operatorname{Im}y_1}\operatorname{Im}B(\operatorname{Re}y_1+it,x_2)\,dt.$$

Since $\operatorname{Re}B\ge b_0/2$ on $\mathcal{S}_{a_1}$, we deduce that $|\operatorname{Re}(x_1-y_1)|\le C(a')^2$ for some $C>0$. In particular, according to the argument above, if a' is small enough, this implies that $x_1=y_1$. For such an $a'>0$, $\varkappa^{-1}$ is a biholomorphism between $\mathcal{S}_{a'}$ and $\varkappa^{-1}(\mathcal{S}_{a'})$, which satisfies

$$\mathcal{S}_{a''}\subset\varkappa^{-1}(\mathcal{S}_{a'})\subset\mathcal{S}_{Ca'},$$

for some $a''>0$. Further, $\varkappa^{-1}$ is uniformly Lipschitz, and so is its inverse. Taking $\min(a',a'')$ as the new value of a and λ the Lipschitz constant of $\varkappa$, $\varkappa^{-1}$ ends the proof. □

We can associate $\varkappa$ with a unitary operator $U_\varkappa$ by setting

$$U_\varkappa f(\tilde{x})=\operatorname{Jac}(\varkappa)^{1/2}f(\varkappa(\tilde{x})).$$

According to Lemma A.1 and keeping the same notation, we have

$$U_\varkappa \mathscr{L}_h U_\varkappa^{-1} = (-ih\nabla_{\tilde{x}} - \widetilde{\mathbf{A}}) g^* (-ih\nabla_{\tilde{x}} - \widetilde{\mathbf{A}}) - h^2 V\,. \tag{2.5}$$

Here, V is explicit in terms of $\varkappa$, and g^* is the dual Riemannian metric $(d\varkappa^T d\varkappa)^{-1}$. Note also that

$$(-ih\nabla_{\tilde{x}} - \widetilde{\mathbf{A}}) g^* (-ih\nabla_{\tilde{x}} - \widetilde{\mathbf{A}}) = \mathrm{Op}_h^{\mathrm{w}}(g^*(\tilde{\xi} - \widetilde{\mathbf{A}}(\tilde{x}), \tilde{\xi} - \widetilde{\mathbf{A}}(\tilde{x}))) + \mathscr{O}(h^2)\,, \tag{2.6}$$

where the $\mathscr{O}(h^2)$ comes from the explicit computation of the subprincipal term with the composition formula (the operator in the right hand side is symmetric).

From explicit expressions for the remainders, and dropping the tilde on the variable x, we deduce that

$$U_\varkappa \mathscr{L}_h U_\varkappa^{-1} = \mathrm{Op}_h^{\mathrm{w}}(\|(\xi_1, \xi_2 - x_1)\|_{g^*}^2 + \mathscr{O}(h^2))\,, \tag{2.7}$$

where the remainder symbol is of the form $h^2 q_1$, q_1 holomorphic and bounded on some $\mathcal{S}_a$ with $a > 0$. Moreover, letting $\widetilde{B} = B \circ \varkappa$ and $\widetilde{\alpha} = \alpha \circ \varkappa$, we get

$$\|(\xi_1, \xi_2 - x_1)\|_{g^*}^2 = \widetilde{B}^2 \xi_1^2 + (\xi_2 - x_1 + \widetilde{\alpha}\xi_1)^2\,. \tag{2.8}$$

We are now almost in the desired form. We consider the following symplectomorphism

$$\varkappa_{\mathcal{M}}(x,\xi) = (x + A\xi, \xi)\,, \quad \text{where } A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,,\ A^{-1} = A\,.$$

It is associated with the metaplectic operator $\mathcal{M}$, defined as

$$\mathcal{M}u(x_1, x_2) := \frac{1}{(2\pi h)^2} \int_{\mathbb{R}^4} e^{\frac{i}{h}\Phi(x,y,\xi)} u(y)\, dy\, d\xi\,, \tag{2.9}$$

the phase being given by

$$\Phi(x,y,\xi) = \varphi(x,\xi) - \langle y, \xi\rangle\,, \quad \varphi(x,\xi) = \left\langle x - \frac{1}{2}A\xi, \xi \right\rangle\,,$$

and φ being the generating function of $\varkappa_{\mathcal{M}}$. We observe that

$$\mathcal{M}u(x) = (2\pi h)^{-2} \int_{\mathbb{R}^2} e^{\frac{i}{h}\langle x,\xi\rangle} \mathcal{F}_h u(\xi) e^{-\frac{i}{2h}\langle A\xi,\xi\rangle}\, d\xi\,,$$

where we used the semiclassical Fourier transform

$$\mathcal{F}_h u(\xi) = \int_{\mathbb{R}^2} e^{-\frac{i}{h}\langle x,\xi\rangle} u(x)\, dx\,, \quad \mathcal{F}_h^{-1} v(x) = (2\pi h)^{-2} \int_{\mathbb{R}^2} e^{\frac{i}{h}\langle x,\xi\rangle} v(\xi)\, dx\,.$$

Recalling that

$$\mathcal{F}_h^{-1}(UV) = \mathcal{F}_h^{-1}(U) \star \mathcal{F}_h^{-1}(V)\,,$$

the operator $\mathcal{M}$ can be written as a convolution operator

$$\mathcal{M}u = K \star u\,, \quad K = \mathcal{F}_h^{-1} e^{-\frac{i}{2h}\langle A\xi,\xi\rangle} = \frac{1}{2\pi h} e^{\frac{i}{2h}\langle Ax,x\rangle}\,, \tag{2.10}$$

where we used the well-known result about the Fourier transform of a quadratic exponential.

For a symbol σ in $\mathcal{S}'(\mathbb{R}^4)$, which is surely the case of the symbols we are manipulating so far, we have the exact "Egorov" correspondence

$$\mathcal{M}^{-1}\operatorname{Op}_h^{\mathrm{w}}(\sigma)\mathcal{M} = \operatorname{Op}_h^{\mathrm{w}}(\sigma\circ\varkappa_{\mathcal{M}})\,. \tag{2.11}$$

It follows that $\mathcal{M}^{-1}U_\varkappa\mathscr{L}_h U_\varkappa^{-1}\mathcal{M}$ is in the form announced by Lemma 2.1. It remains to check the conditions on the coefficients. We find that

$$g^{11} = (\widetilde{B}^2 + \widetilde{\alpha}^2)\circ\varkappa_{\mathcal{M}}, \quad g^{12} = \widetilde{\alpha}\circ\varkappa_{\mathcal{M}}, \quad g^{22} = 1.$$

Then

$$B(x,\xi) = \sqrt{g^{11}g^{22} - (g^{12})^2} = B(\varkappa(\varkappa_{\mathcal{M}}(x,\xi))), \tag{2.12}$$

is suitably non-degenerate according to Assumption (i), and it remains to check that $\widetilde{\alpha}\circ\varkappa_{\mathcal{M}}$ is critical at 0. But this is true if α itself is critical at 0, and this holds since (B being critical at 0)

$$\alpha = x_1^2\partial^2_{x_2,x_1}B(0) + \mathscr{O}(x^3).$$

It is important to observe that since $\mathcal{M}$ somehow mixes x and ξ variables, it does not preserve exponential decay of functions. However, in a sense to be precised later, we will get decay in "x and ξ", which is preserved by $\mathcal{M}$.

In the sequel, it will be convenient to let

$$p_{\mathcal{M}} := g^{11}\xi_1^2 + 2g^{12}x_1\xi_1 + g^{22}x_1^2\,. \tag{2.13}$$

Additionally, we will distinguish variables by setting $X_1 = (x_1,\xi_1)$, $X_2 = (x_2,\xi_2)$. In these new variables, the characteristic set of $\mathscr{L}_h$ becomes $\{X_1 = 0\}$.

2.2. Reduction to a bounded symbol

Our strategy strongly relies on the presentation of the Fourier-Bros-Iagolnitzer (in short, FBI) transform given in Martinez' book [27]. There, many results require that operators have symbols in the class $S(1)$, which is the space of smooth functions on phase space that are uniformly bounded, together with all their derivatives. However, because the magnetic Laplacian is a differential operator of positive order, its symbol does not belong to that class. The statements we will use could probably be extended to the general case of symbols with more general order functions. It is to avoid this, and concentrate on the essential arguments, that we have decided to restrict ourselves to the case of a bounded magnetic field, a situation where we can reduce the problem to a problem in the $S(1)$ class, as follows.

Initially, the symbol of magnetic Laplacian is polynomial in ξ and hence belongs to a class with gains of powers of $\langle\xi\rangle$: locally in x,

$$|\partial_x^\alpha \partial_\xi^\beta p_\varkappa| \leq C_{\alpha,\beta}\langle\xi\rangle^{2-|\beta|}. \tag{2.14}$$

This still holds after the change of variables $\varkappa$. However, the metaplectic transform $\mathcal{M}$ mixes the x and ξ variables, so that we do not gain powers of ξ anymore. Recall that for a non-negative function m on $\mathbb{R}^d$, $S_{\mathbb{R}^d}(m)$ is defined as the set of functions σ that satisfy estimates

$$|\partial_x^\alpha \sigma| \leq C_\alpha m, \ \ \alpha \in \mathbb{N}^d.$$

If σ is holomorphic on a complex strip $\mathbb{R}^d \subset \mathcal{S} \subset \mathbb{C}^d$, we shall say that $\sigma \in S_{\mathcal{S}}(m)$ if

$$\forall z \in \mathcal{S}, \quad |\partial_x^\alpha \sigma(z)| \leq C_\alpha m(\operatorname{Re} z), \quad \alpha \in \mathbb{N}^d.$$

Also recall what it means for a non-vanishing smooth function m on $T^*\mathbb{R}^2 = \mathbb{R}^4$ to be an *admissible order function*. First, one requires that $m \in S_{\mathbb{R}^4}(m)$. Second, there is an $N>0$ such that for some $C>0$ and any $(x,\xi), (x',\xi') \in T^*\mathbb{R}^2$,

$$\frac{m(x,\xi)}{m(x',\xi')} \leq C\langle(x-x',\xi-\xi')\rangle^N. \tag{2.15}$$

Given two admissible order functions m and m', then $1/m$ and mm' also are admissible and we have the following result (see for instance [13, Proposition 7.7]): if $\sigma \in S(m)$ and $\sigma' \in S(m')$, then

$$\operatorname{Op}_h^{\mathrm{w}}(\sigma)\operatorname{Op}_h^{\mathrm{w}}(\sigma') = \operatorname{Op}_h^{\mathrm{w}}\left(\sigma\sigma' + \frac{h}{2i}\{\sigma,\sigma'\} + \mathscr{O}_{S(mm')}(h^2)\right), \tag{2.16}$$

with the usual sign convention $\{f,g\} = \partial_\xi f \partial_x g - \partial_x f \partial_\xi g$. Following the result of Boutet de Monvel-Krée [9], a refinement of estimate (2.16) shows that if σ, σ' had a holomorphic extension to a strip, with uniform estimates, then the symbol of the product also does, with uniform estimates. Consider now

$$m_{\mathcal{M}}(X_1,X_2) = 1 + p_{\mathcal{M}}(X_1,X_2).$$

Lemma 2.3. *Assume that $p_{\mathcal{M}}$ is in the form (2.13), with coefficients satisfying the conclusion of Lemma 2.1. Then $m_{\mathcal{M}}$ is an admissible order function, and $p_{\mathcal{M}} \in S_{\mathcal{S}_a}(m_{\mathcal{M}})$ for some $a>0$.*

If these assumptions are satisfied, we will introduce a bounded spectral parameter μ and work with

$$
\begin{aligned}
\mathscr{P} &= \mathrm{Op}_h^{\mathrm{w}}\left(\frac{1}{1+p_{\mathcal{M}}}\right)\mathrm{Op}_h^{\mathrm{w}}(p_{\mathscr{L}}-h\mu)\\
&= \mathrm{Op}_h^{\mathrm{w}}\left(\frac{p_{\mathcal{M}}-h\mu}{1+p_{\mathcal{M}}}+\mathscr{O}_{S_{\mathcal{S}_a}(1)}(h^2)\right) = \mathrm{Op}_h^{\mathrm{w}}(p_h),
\end{aligned}
\tag{2.17}
$$

where $p_{\mathscr{L}}=p_{\mathcal{M}}+h^2q$, see (2.2), so that $p_h\in S_{\mathcal{S}_a}(1)$, uniformly with respect to h and μ.

Remark 2.4. Since the intensity of the magnetic field is given by

$$
\sqrt{\det \mathrm{Hess}_{X_1}(p_h)_{|X_1=0}}+\mathscr{O}(h)
$$

(see Equation (2.12)) the fact that it is globally bounded is actually necessary for obtaining $p_h\in S_{\mathcal{S}_a}(1)$.

Proof. Let us check that $p_{\mathcal{M}}\in S_{\mathcal{S}_a}(m_{\mathcal{M}})$. Since the coefficients g^{ij} are in $S_{\mathcal{S}_a}(1)$ for some $a>0$, one finds that

$$
|\partial^\alpha p_{\mathcal{M}}|\le C_\alpha(1+|X_1|^2) \quad \text{on } \mathcal{S}_a.
$$

Thus, it suffices to show that there exists $\lambda>0$ such that

$$
1+\mathrm{Re}\, p_{\mathcal{M}}\ge \lambda(1+|X_1|^2)\,. \tag{2.18}
$$

Let us start by proving this on $\mathbb{R}^4$. Note that (2.18) is satisfied for example if $\lambda\le 1$ and everywhere

$$
\lambda\le \frac{g^{11}+g^{22}-\sqrt{(g^{11}-g^{22})^2+4(g^{12})^2}}{2}\,.
$$

Let

$$
C=\sup\{|g^{11}|+|g^{22}|\}\,,\quad C'=\inf\{g^{11}g^{22}-(g^{12})^2\}\,.
$$

The quantity in the right hand side is larger than

$$
2(g^{11}g^{22}-(g^{12})^2)/(g^{11}+g^{22})\ge 2C'/C>0\,,
$$

uniformly on $T^*\mathbb{R}^2$. Now, we turn to the case that $(x,\xi)=(\mathrm{Re}\,x,\mathrm{Re}\,\xi)+i(u,v)$. Then, we can write

$$
\begin{aligned}
\mathrm{Re}\, p_{\mathcal{M}} &= \mathrm{Re}\,\left(g^{11}(\xi_1^2-v_1^2)+2g^{12}(\xi_1x_1-u_1v_1)+g^{22}(x_1^2-u_1^2)\right)\\
&\quad -2\mathrm{Im}\,\left(g^{11}\xi_1v_1+2g^{12}(\xi_1u_1+x_1v_1)+g^{22}x_1u_1\right)\\
&\ge \lambda'(\mathrm{Re}\,X_1)^2-Ca^2(1+|\mathrm{Re}\,X_1|),
\end{aligned}
$$

where λ' may be smaller than the λ from before, but is still non-negative if we assume that $\inf\{\operatorname{Re} g^{11}\operatorname{Re} g^{22}-(\operatorname{Re} g^{12})^2\}>0$ on $\mathbb{R}^4+i[-a,a]^4$. Up to taking a small enough, this holds.

Finally, we consider the temperance of the symbol. We already know that for some constants C, C',

$$\frac{1+p_{\mathcal{M}}(x,\xi)}{1+p_{\mathcal{M}}(x',\xi')}\leq\frac{1+C(X_1')^2}{1+C'X_1^2},$$

whence we find

$$\frac{1+p_{\mathcal{M}}(x,\xi)}{1+p_{\mathcal{M}}(x',\xi')}\leq\frac{C}{C'}(1+\lambda(X_1-X_1')^2),$$

for λ large enough. □

3. About the FBI transform

Our main tool in this section will be the Fourier-Bros-Iagolnitzer (FBI) transform. Several versions exist in the literature, see [23]; in this paper we follow [27, Chapter 3], and the FBI transform we use here is defined, for $u\in\mathscr{S}'(\mathbb{R}^2)$, by

$$Tu(x,\xi)=\alpha_h\int_{\mathbb{R}^2}e^{i(x-y)\xi/h}e^{-|x-y|^2/2h}u(y)\,dy\,,\quad \alpha_h=2^{-1}(\pi h)^{-\frac{3}{2}}\,.$$

The α_h is chosen so that T is isometric from $L^2(\mathbb{R}^2)$ to $L^2(\mathbb{R}^4)$. The knowledge of Tu implies the knowledge of u via the inversion formula:(2)

$$u(y)=\alpha_h\int_{\mathbb{R}^4}e^{-i(x-y)\xi/h-|x-y|^2/2h}Tu(x,\xi)\,dx\,d\xi=T^*Tu\,. \tag{3.1}$$

It will be essential later that

$$(h(\partial_x-i\partial_\xi)-i\xi)T=0. \tag{3.2}$$

In other words, T maps $L^2(\mathbb{R}^2)$ into the closed subspace of $L^2(\mathbb{R}^4)$ of functions of the form $e^{\frac{-\xi^2}{2h}}f(x-i\xi)$, where f is holomorphic on $\mathbb{C}^2$.

3.1. Towards a Toeplitz representation

Since the naive Agmon tactic fails, it seems natural to try and use weights in phase space that depend on both x and ξ. However, it is not easy to understand the behavior of an operator of the type $\operatorname{Op}_h^{\mathrm{w}}(e^{\psi(x,\xi)/h})$, all the more if ψ was not bounded. (Although, in the case of a quadratic ψ, see the recent article [10].) Following the strategy of [27, 3.5] and [31], we use the FBI transform to simplify

(2) sometimes called coherent state decomposition; in relation with the magnetic Laplacian, it has been used in [4, Section 2.3].

this, as $e^{\psi(x,\xi)/h}$ can be seen as an multiplication operator on $L^2(\mathbb{R}^4)$. Precisely, let us consider the following quantity

$$\langle m e^{\psi/h} T \mathscr{P} u, e^{\psi/h} T u\rangle_{L^2(\mathbb{R}^4)}\,, \tag{3.3}$$

where $\mathscr{P}$ is defined in (2.17), and $m \in S(1)$ is multiplier (it is not an order function!). In this section, $\psi \in S(1)$ and might depend on parameters uniformly with respect to the $S(1)$-topology, and all the bounds will depend on $d\psi$ only.

Since the FBI transform we are using has a quadratic phase, we have an exact formula

$$T\,\mathrm{Op}_h^{\mathrm{w}}(\sigma) = \mathrm{Op}_h^{\mathrm{w}}(\sigma_T) T\,,$$

where $\sigma_T(x,\xi,x^*,\xi^*) = \sigma(x-\xi^*, x^*)$, valid for $\sigma \in \mathscr{S}'(\mathbb{R}^4)$. From this, we get

$$\langle m e^{\psi/h} T \mathscr{P} u, e^{\psi/h} T u\rangle_{L^2(\mathbb{R}^4)} = \langle m e^{\psi/h} \mathscr{P}_T T u, e^{\psi/h} T u\rangle_{L^2(\mathbb{R}^4)}\,.$$

We set

$$\mathscr{P}_T^{\psi} = e^{\psi/h} \mathscr{P}_T e^{-\psi/h}\,, \quad u_\psi = e^{\psi/h} T u = T_\psi u\,,$$

so that

$$(3.3) = \langle m \mathscr{P}_T^{\psi} u_\psi, u_\psi\rangle_{L^2(\mathbb{R}^4)}\,.$$

Thanks to our analyticity assumption and [27, Lemma 3.5.4] or [31, Corollary 5], $\mathscr{P}_T^{\psi}$ is still a pseudo-differential operator with symbol in $S(1)$. Its symbol satisfies

$$p_h^{\psi} = p_h(x-\xi^* - i\partial_\xi \psi, x^* + i\partial_x \psi) + \mathscr{O}(h^2)\,. \tag{3.4}$$

Since we use the Weyl quantization, we have indeed $\mathscr{O}(h^2)$ and not only $\mathscr{O}(h)$. Now, we apply [27, Theorem 3.5.1] or [31, Theorem 1], which gives

$$\langle T \mathscr{P} u, m e^{2\psi/h} T u\rangle_{L^2(\mathbb{R}^4)} = \int_{\mathbb{R}^4} p_{h,m}^{\psi}(x,\xi;h) |u_\psi|^2 \, dx\, d\xi + \mathscr{O}(h^2) \|u_\psi\|^2\,, \tag{3.5}$$

with

$$p_{h,m}^{\psi}(x,\xi;h) := m(x,\xi) p_h(x + 2\partial_{\overline{z}}\psi, \xi - 2i\partial_{\overline{z}}\psi) + \mathscr{O}(h)\,. \tag{3.6}$$

Here, we have introduced the complex variable $z = x + i\xi$, and

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_\xi), \quad \partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_\xi).$$

We stress again that the all the constants in the estimates only involve ψ via semi-norms of $d\psi$ in $S(1)$.

3.2. Subprincipal term

In fact, we can even describe the term estimated by $\mathscr{O}(h)$ and we will actually need it. For that purpose, and also for the convenience of the reader, let us revisit and refine [27, Theorem 3.5.1].

3.2.1. General expression of the subprincipal term

Let us focus on the proof of (3.5) once we have (3.4). The following proposition shows how to explicitly write a pseudo-differential operator acting on the range of T_ψ (in the sense of quadratic forms) as a multiplication operator modulo $\mathscr{O}(h^2)$.

Proposition 3.1. *Consider a symbol* $q=q_0(x,\xi,x^*,\xi^*)\in S_{\mathbb{R}^8}(1)$. *We have*

$$\langle \mathrm{Op}_h^{\mathrm{w}}(q)\, u_\psi, u_\psi\rangle_{L^2(\mathbb{R}^4)} = \int_{\mathbb{R}^4} (\tilde{q}_0(x,\xi)+h\tilde{q}_1(x,\xi))|u_\psi|^2\, dx\, d\xi + \mathscr{O}(h^2)\|u_\psi\|^2_{L^2(\mathbb{R}^4)}\,,$$

where

$$\tilde{q}_0(x,\xi)=q_0(x,\xi,\xi-\partial_\xi\psi,\partial_x\psi)\,,\quad \tilde{q}_1(x,\xi)=\frac{1}{2}\left(\{\sigma_f,g\}+\{f,\sigma_g\}\right)_{f=g=0}\,,$$

with

$$\begin{aligned} f(x,\xi,x^*,\xi^*)&=x^*-\xi+\partial_\xi\psi\\ g(x,\xi,x^*,\xi^*)&=\xi^*-\partial_x\psi\\ \sigma_f(x,\xi,x^*,\xi^*)&=\int_0^1 \partial_{x^*}q_0(x,\xi,\xi-\partial_\xi\psi+tf,\xi^*)\,dt\\ \sigma_g(x,\xi,x^*,\xi^*)&=\int_0^1 \partial_{\xi^*}q_0(x,\xi,\xi-\partial_\xi\psi,\partial_x\psi+tg)\,dt\,. \end{aligned} \tag{3.7}$$

Proof. Let us follow the presentation by Martinez. The computations also appear in [31]. We consider

$$r_1(x,\xi,x^*,\xi^*)=q(x,\xi,x^*,\xi^*)-q(x,\xi,\xi-\partial_\xi\psi,\partial_x\psi)\,,$$

By the Taylor formula,

$$r_1=f\sigma_f+g\sigma_g\,.$$

We set $F=\mathrm{Op}_h^{\mathrm{w}} f$ and $G=\mathrm{Op}_h^{\mathrm{w}} g$. Since we use the Weyl quantization, we have

$$\mathrm{Op}_h^{\mathrm{w}}(f\sigma_f)=\frac{1}{2}(F\,\mathrm{Op}_h^{\mathrm{w}}(\sigma_f)+\mathrm{Op}_h^{\mathrm{w}}(\sigma_f)F)+\mathscr{O}(h^2)\,.$$

(Here, the symbol f is not in $S(1)$, however all its derivatives are, which is essential in the computation.) Next, we observe that Equation (3.2) implies that $FT_\psi=iGT_\psi$

and deduce

$$\frac{1}{2}\langle (F\,\mathrm{Op}_h^{\mathrm{w}}(\sigma)+\mathrm{Op}_h^{\mathrm{w}}(\sigma)F)u_\psi, u_\psi\rangle = \frac{i}{2}\langle [\mathrm{Op}_h^{\mathrm{w}}(\sigma), G]u_\psi, u_\psi\rangle\,.$$

Thus (again, since $dg\in S(1)$)

$$\langle \mathrm{Op}_h^{\mathrm{w}}(f\sigma_f)u_\psi, u_\psi\rangle = \frac{h}{2}\langle \mathrm{Op}_h^{\mathrm{w}}(\{\sigma_f, g\})u_\psi, u_\psi\rangle + \mathscr{O}(h^2)\|u_\psi\|^2\,.$$

In the same way, we get

$$\langle \mathrm{Op}_h^{\mathrm{w}}(g\sigma_g)u_\psi, u_\psi\rangle = \frac{h}{2}\langle \mathrm{Op}_h^{\mathrm{w}}(\{f, \sigma_g\})u_\psi, u_\psi\rangle + \mathscr{O}(h^2)\|u_\psi\|^2\,.$$

Therefore, iterating the argument,

$$\langle \mathrm{Op}_h^{\mathrm{w}}\, r_1 u_\psi, u_\psi\rangle = \frac{h}{2}\int_{\mathbb{R}^4} (\{\sigma_f, g\}+\{f,\sigma_g\})_{f=g=0}\, |u_\psi|^2\, dx\, d\xi + \mathscr{O}(h^2)\|u_\psi\|^2\,. \quad \square$$

Notation 3.2. *When $a\in S_{\mathbb{R}^4}(1)$, we let*

$$\hat{a}(x,\xi) = a(x+2\partial_{\overline{z}}\psi, \xi - 2i\partial_{\overline{z}}\psi)\,.$$

Corollary 3.3. *We have*

$$\langle T\mathscr{P}u, me^{2\psi/h}Tu\rangle_{L^2(\mathbb{R}^4)} = \int_{\mathbb{R}^4} p^\psi_{h,m}(x,\xi;h)|u_\psi|^2\, dx\, d\xi + \mathscr{O}(h^2)\|u_\psi\|^2\,, \tag{3.8}$$

where

$$p^\psi_{h,m}(x,\xi;h) = m(x,\xi)\hat{p}_h(x,\xi) + hp^\psi_{h,m,1}\,, \tag{3.9}$$

and

$$p^\psi_{h,m,1} = -2\partial_{\overline{z}} m\widehat{\partial_z p_h} + ms(x,\xi)\,,$$

with

$$s(x,\xi) = \frac{1}{2}\left(\{\sigma_f, g\}+\{f,\sigma_g\}\right)_{|f=g=0}\,,$$

where we used the notations of Proposition 3.1 with

$$q_0(x,\xi,x^*,\xi^*) = p_h(x-\xi^* - i\partial_\xi\psi, x^* + i\partial_x\psi)\,.$$

Proof. We apply Proposition 3.1 to the pseudo-differential operator $\mathrm{Op}_h^{\mathrm{w}} q = m\mathscr{P}_T^{\psi}$. By the composition formula,

$$q(x,\xi,x^*,\xi^*) = mq_0(x,\xi,x^*,\xi^*) + hq_1(x,\xi,x^*,\xi^*) + \mathscr{O}(h^2)\,,$$

where

$$\begin{aligned} q_1(x,\xi,x^*,\xi^*) &= (2i)^{-1}\{m(x,\xi), p_h(x-\xi^*-i\partial_\xi\psi, x^*+i\partial_x\psi)\} \\ &= -(2i)^{-1}\partial_x m\cdot\partial_\xi p_h + (2i)^{-1}\partial_\xi m\cdot\partial_x p_h\,. \end{aligned}$$

We deduce that

$$p_{h,m}^{\psi} = m(x,\xi)\hat{p}_h(x,\xi) + \frac{ih}{2}\left(\partial_x m\cdot\widehat{\partial_\xi p}_h - \partial_\xi m\cdot\widehat{\partial_x p}_h\right) + hs(x,\xi)\,,$$

with

$$\begin{aligned} s &= \frac{1}{2}(\{m\sigma_f, g\} + \{f, m\sigma_g\})_{|f=g=0} \\ &= \frac{m}{2}(\{\sigma_f, g\} + \{f, \sigma_g\})_{|f=g=0} + \frac{1}{2}(-\widehat{\partial_\xi p}_h\cdot\partial_\xi m - \widehat{\partial_x p}_h\cdot\partial_x m). \end{aligned}$$

But we have

$$\begin{aligned} &\frac{i}{2}\left(\partial_x m\cdot\widehat{\partial_\xi p}_h - \partial_\xi m\cdot\widehat{\partial_x p}_h\right) + \frac{1}{2}(-\widehat{\partial_\xi p}_h\cdot\partial_\xi m - \widehat{\partial_x p}_h\cdot\partial_x m) \\ &\qquad = \frac{1}{2}\partial_x m\cdot(-\widehat{\partial_x p_h + i\partial_\xi p_h}) + \frac{1}{2}\partial_\xi m\cdot(-\widehat{i\partial_x p_h - \partial_\xi p_h}) \\ &\qquad = -2\partial_{\overline{z}} m\cdot\widehat{\partial_z p}_h. \quad \square \end{aligned}$$

3.2.2. Rough estimate of the subprincipal terms

Here we use the variables introduced in Section 2, in order to describe $p_{h,m,1}^{\psi}$ in the case when $m = m(X_2)\in S(1)$, and $X_2 = (x_2,\xi_2)$. Recall that

$$p_h = \frac{g^{11}\xi_1^2 - 2g^{12}\xi_1 x_1 + g^{22}x_1^2 - h\mu}{1 + g^{11}\xi_1^2 - 2g^{12}\xi_1 x_1 + g^{22}x_1^2} + \mathscr{O}(h^2),$$

where the coefficients g^{ij} are in $S(1)$ on $\mathbb{R}^4 + i[-a,a]^4$, and $\mu\geq 0$. Then we notice that, since m only depends on z_2,

$$|\partial_{\overline{z}} m\cdot\widehat{\partial_z p}_h| = \left|(\partial_{\overline{z_2}} m)\widehat{\partial_{z_2} p}_h\right| \leq C(\min(|X_1|^2,1) + h^2)\,,$$

and that this term is zero when $m=1$. Also, we observe that a priori, $s\in S(1)$, so that

$$p_{h,m}^{\psi} = m\hat{p}_h + hm\mathscr{O}(1) + h\mathscr{O}(\min(|X_1|^2,1)) + \mathscr{O}(h^3)\,.$$

3.2.3. A more accurate description

When $\psi=\Psi(X_2)$, we can give a more explicit expression for s. It will be convenient to set

$$w(x,\xi,f,g):=p_h(x-2\partial_{\overline{z}}\psi-g,\xi+2i\partial_{\overline{z}}\psi+f)\,, \tag{3.10}$$

Then,

$$\sigma_f=\int_0^1\partial_f w(x,\xi,tf,g)\,dt\,,\quad \sigma_g=\int_0^1\partial_g w(x,\xi,0,tg)\,dt\,.$$

We have

$$\begin{aligned}\{\sigma_f,g\}_{f=g=0}=&\sum_{k,j}\{\xi_j,g_k\}\partial_{f_k}\partial_{\xi_j}w(x,\xi,0,0)\\&+\frac{1}{2}\{f_j,g_k\}\partial_{f_k}\partial_{f_j}w(x,\xi,0,0)+\{g_j,g_k\}\partial_{f_k}\partial_{g_j}w(x,\xi,0,0)\,,\end{aligned}$$

and

$$\{f,\sigma_g\}_{f=g=0}=\sum_{k,j}\{f_k,x_j\}\partial_{g_k}\partial_{x_j}w(x,\xi,0,0)+\frac{1}{2}\{f_k,g_j\}\partial_{g_k}\partial_{g_j}w(x,\xi,0,0)\,.$$

From the expressions of f and g, we notice that $\{\xi_j,g_k\}=-\delta_{jk}$, $\{f_k,x_j\}=\delta_{jk}$ and

$$\begin{aligned}\{g_k,g_j\}&=-\partial^2_{\xi_k,x_j}\psi-\partial^2_{\xi_j,x_k}\psi\,,\\\{g_k,f_j\}&=-\delta_{k,j}+\partial^2_{\xi_k,\xi_j}\psi+\partial^2_{x_k,x_j}\psi\,.\end{aligned}$$

Since $\psi=\Psi(X_2)$, the only non-zero terms involving ψ are obtained for $j=k=2$. Thus,

$$2s(x,\xi)=\left(\sum_k-\partial_{f_k}\partial_{\xi_k}+\frac{1}{2}\partial_{f_k}\partial_{f_k}+\partial_{g_k}\partial_{x_k}+\frac{1}{2}\partial_{g_k}\partial_{g_k}\right)w(x,\xi,0,0)+R_1\,, \tag{3.11}$$

where $R_1=\mathscr{O}(|\mathrm{d}^2\psi||\mathrm{d}^2_{X_2}p_h|)$. Let us look at the first term in the right-hand side of (3.11) and recall (3.10). Then, we can write it as

$$\left(\sum_k-\partial_{f_k}\partial_{\xi_k}+\frac{1}{2}\partial_{f_k}\partial_{f_k}+\partial_{g_k}\partial_{x_k}+\frac{1}{2}\partial_{g_k}\partial_{g_k}\right)w(x,\xi,0,0)=-\frac{1}{2}\widehat{\Delta p_h}+R_2\,,$$

where again, $R_2=\mathscr{O}(|\mathrm{d}^2\psi||\mathrm{d}^2_{X_2}p_h|)$, so that finally,

$$s=-\frac{1}{4}\widehat{\Delta p_h}+\mathscr{O}(\min(|X_1|^2,1))+\mathscr{O}(h^2).$$

We can summarize the discussion above in the following.

Scholium 3.4. *Under the conclusion of Lemma 2.1, consider* ψ *bounded with* $d\psi \in S(1)$ *and* $m = m(X_2) \in S(1)$. *Then*

$$\langle m e^{-\psi/h} T \mathscr{P} u, e^{-\psi/h} T u \rangle = \int_{\mathbb{R}^4} |u_\psi|^2 \left[m \hat{p}_h + hms + hr + \mathscr{O}(h^2) \right] dX_1 \, dX_2 \,,$$

where $r, s \in S(1)$ *and* $|r| \leq C(\min(|X_1|^2, 1))$. *Moreover, we have the following properties.*

(i). *When* $m=1$, $r=0$

(ii). *When* $\psi = \Psi(X_2)$,

$$s = -\frac{1}{4} \widehat{\Delta p_h}(x, \xi) + \breve{R} \,,$$

where $\breve{R} \in S(1)$ *and* $\breve{R} = \mathscr{O}(|\mathrm{d}^2 \Psi| \min(|X_1|^2 + h^2, 1))$.
Moreover, all estimates are uniform for h *small and* $d\psi$ *varying in a bounded subset of* $S(1)$.

Noticing that $\breve{R}$ is zero when $\psi = 0$, we get the following.

Proposition 3.5. *When* $\sigma \in S(1)$,

$$\begin{aligned} \langle \mathrm{Op}_h^{\mathrm{w}}(\sigma) u, u \rangle_{L^2(\mathbb{R}^2)} &= \langle T \, \mathrm{Op}_h^{\mathrm{w}}(\sigma) u, T u \rangle_{L^2(\mathbb{R}^4)} \\ &= \int_{\mathbb{R}^4} \left(\sigma(x,\xi) - \frac{h}{4} \Delta \sigma(x, \xi) + \mathscr{O}(h^2) \right) |Tu|^2 \, dx \, d\xi \,. \end{aligned} \tag{3.12}$$

Remark 3.6. This classical proposition (see [27, Corollary 3.5.7 & Section 3.6, Example 7] and consider also [40, Theorem 13.10]) is also true when σ is a quadratic form, and in this case the remainder $\mathscr{O}(h^2)$ is zero.

4. Microlocal Agmon estimates

In this section, we establish Agmon estimates with respect to X_1 in an exponentially weighted space with respect to X_2. These estimates are stated in Theorem 4.3 and 4.4. They imply Theorem 4.1. In this whole section we will consider $u \in L^2(\mathbb{R}^2)$ solving the equation

$$\mathscr{P} u = \mathrm{Op}_h^{\mathrm{w}} \left(\frac{1}{1 + p_{\mathcal{M}}} \right) \mathrm{Op}_h^{\mathrm{w}} (p_{\mathcal{M}} - h\mu + h^2 q) u = 0. \tag{4.1}$$

with $p_{\mathscr{L}} = p_{\mathcal{M}} + h^2 q$ satisfying the conclusion of Lemma 2.1, so that the conclusions of Scholium 3.4 applies.

Theorem 4.1. *Let* Ψ_1, Ψ_2 *be non-negative Lipschitz functions with a unique and non-degenerate minimum at* 0 *with minimum value* 0. *We also assume that they go linearly to infinity at infinity. We set* $\psi_0(x, \xi) = \Psi_1(X_1) + \Psi_2(X_2)$. *Given* $K > 0$, *there exist* $\varepsilon, h_0, C>0$*, such that, for all* $h\in(0,h_0)$*,* $\mu\leq b_0+Kh$ *and* u *solving* (4.1)*, we have*

$$\int_{\mathbb{R}^4} e^{2\varepsilon\psi_0(x,\xi)/h}|Tu|^2\,dx\,d\xi \leq C\int_{\mathbb{R}^4}|Tu|^2\,dx\,d\xi \quad (=C\,\|u\|^2).$$

4.1. Decay away from the characteristic manifold

In this section, we establish the exponential decay of Tu with respect to X_1; we use the notation from Section 3 with $\psi=\varepsilon\psi_0$.

4.1.1. First estimate

One will need the following elementary lemma.

Lemma 4.2. *Recall that* $p_{\mathcal{M}}(x,\xi)=g^{11}\xi_1^2-2g^{12}\xi_1x_1+g^{22}x_1^2$*. Then, there exist non-negative numbers* γ, c_1, c_2, c_3 *such that*

(i). *for all* $|X_1|\geq\gamma$, $\frac{p_{\mathcal{M}}}{1+p_{\mathcal{M}}}\geq c_1$,

(ii). *for all* $|X_1|\leq\gamma$, $\frac{p_{\mathcal{M}}}{1+p_{\mathcal{M}}}\geq c_2|X_1|^2$.

If, moreover, ε *is small enough,*

(i). *for all* $|X_1|\geq\gamma$, $\operatorname{Re}\frac{\hat{p}_{\mathcal{M}}}{1+\hat{p}_{\mathcal{M}}}\geq c_1$,

(ii). *for all* $|X_1|\leq\gamma$, $\operatorname{Re}\frac{\hat{p}_{\mathcal{M}}}{1+\hat{p}_{\mathcal{M}}}\geq \operatorname{Re}\hat{p}_{\mathcal{M}}-c_3|X_1|^4$*, and* $\operatorname{Re}\hat{p}_{\mathcal{M}}\geq c_2|X_1|^2$,

where we used Notation 3.2.

Theorem 4.3. *Given* $K>0$*, there exist* $\varepsilon, h_0, C>0$ *such that, for all* $h\in(0,h_0)$*,* $\mu\leq K$ *and* u *solving* (4.1)*, we have*

$$\int_{\mathbb{R}^4} e^{2\varepsilon(\Psi_1(X_1)+\Psi_2(X_2))/h}|Tu|^2\,dx\,d\xi \leq C\int_{\mathbb{R}^4} e^{2\varepsilon\Psi_2(X_2)/h}|Tu|^2\,dx\,d\xi\,.$$

Proof. Assume temporarily that Ψ_1 is bounded. Let us use Scholium 3.4 with $m=1$. Then, taking the real part, we get

$$\int_{\mathbb{R}^4}(\operatorname{Re}\hat{p}_h-Ch)|u_\psi|^2\,dx\,d\xi\leq 0\,.$$

Recall

$$p_h(x,\xi)=\frac{p_{\mathcal{M}}(x,\xi)-h\mu}{1+p_{\mathcal{M}}(x,\xi)}+\mathscr{O}(h^2)\,.$$

Since $p_{\mathcal{M}}\geq 0$,

$$\int_{\mathbb{R}^4}\left(\operatorname{Re}\frac{\hat{p}_{\mathcal{M}}}{1+\hat{p}_{\mathcal{M}}}-C(1+K)h\right)|u_\psi|^2\,dx\,d\xi\leq 0\,.$$

Consider $R>0$ and the set

$$J_R=\{X\in\mathbb{R}^4:|X_1|\geq Rh^{\frac{1}{2}}\}\,.$$

We write

$$\begin{aligned}\int_{J_R}\left(\operatorname{Re}\frac{\hat p_{\mathcal M}}{1+\hat p_{\mathcal M}}-C(1+K)h\right)|u_\psi|^2\,dx\,d\xi\\ \leq -\int_{\complement J_R}\left(\operatorname{Re}\frac{\hat p_{\mathcal M}}{1+\hat p_{\mathcal M}}-C(1+K)h\right)|u_\psi|^2\,dx\,d\xi\,,\end{aligned}$$

and notice

$$\left|\int_{\complement J_R}\left(\operatorname{Re}\frac{\hat p_{\mathcal M}}{1+\hat p_{\mathcal M}}-C(1+K)h\right)|u_\psi|^2\,dx\,d\xi\right|\leq C_Rh\int_{\mathbb{R}^4}e^{2\varepsilon\Psi_2(X_2)/h}|Tu|^2\,dx\,d\xi\,.$$

From Lemma 4.2, we get $\tilde c_2>0$ such that on J_R,

$$\operatorname{Re}\frac{\hat p_{\mathcal M}}{1+\hat p_{\mathcal M}}-C(1+K)h\geq\tilde c_2R^2h-C(1+K)h\,.$$

Choosing R large enough, we get

$$\int_{J_R}|u_\psi|^2\,dx\,d\xi\leq C_R\int_{\mathbb{R}^4}e^{2\varepsilon\Psi_2(X_2)/h}|Tu|^2\,dx\,d\xi\,,$$

and then

$$\int_{\mathbb{R}^4}|u_\psi|^2\,dx\,d\xi\leq C\int_{\mathbb{R}^4}e^{2\varepsilon\Psi_2(X_2)/h}|Tu|^2\,dx\,d\xi\,.$$

If Ψ_1 is not bounded, we introduce an appropriate cutoff function. For example, we apply the previous estimates to $\Psi_{1,k}:=\chi(k^{-1}\varepsilon\Psi_1(X_1))\varepsilon\Psi_1(X_1)$ and send k to $+\infty$. The estimates are independent of k because $d\Psi_{1,k}$ is uniformly bounded in $S(1)$. Then, we conclude with the Fatou lemma. □

4.1.2. Agmon estimate with multiplier

Let us now add a *multiplier* in the previous estimate. This can be done modulo $\mathscr{O}(h)$.

Theorem 4.4. *Consider $m=m(X_2)$ non-negative with $m\in S(1)$. Then, for $M>0$, there exist $\varepsilon,h_0,C>0$ such that, for all $h\in(0,h_0)$, $\mu\leq M$ and u solving (4.1), we have*

$$\int_{\mathbb{R}^4}me^{2\varepsilon(\Psi_1(X_1)+\Psi_2(X_2))/h}|Tu|^2\,dx\,d\xi\leq C\int_{\mathbb{R}^4}(m+h)e^{2\varepsilon\Psi_2(X_2)/h}|Tu|^2\,dx\,d\xi\,.$$

Proof. We use again Scholium 3.4, this time without assuming that $m=1$. We have

$$\operatorname{Re} \int_{\mathbb{R}^4} p^{\psi}_{h,m}(x,\xi;h)|u_\psi|^2 \, dx \, d\xi = \mathscr{O}(h^2)\|u_\psi\|^2 ,$$

so that, with Theorem 4.3,

$$\int_{\mathbb{R}^4} \operatorname{Re} p^{\psi}_{h,m}(x,\xi;h)|u_\psi|^2 \, dx \, d\xi \leq Ch^2 \int_{\mathbb{R}^4} e^{2\varepsilon\Psi_2(X_2)/h}|Tu|^2 \, dx \, d\xi \, .$$

Then, by Scholium 3.4,

$$\int_{\mathbb{R}^4} (m\operatorname{Re} \hat{p}_h(x,\xi;h) - Chm) \, |u_\psi|^2 \, dx \, d\xi \leq Ch \int_{\mathbb{R}^4} |X_1|^2|u_\psi|^2 \, dx \, d\xi \\ + Ch^2 \int_{\mathbb{R}^4} e^{2\varepsilon\Psi_2(X_2)/h}|Tu|^2 \, dx \, d\xi \, .$$

Using again Theorem 4.3 with a smaller ε to absorb the $|X_1|^2$ term,

$$\int_{\mathbb{R}^4} (m\operatorname{Re} \hat{p}_h(x,\xi;h) - Chm) \, |u_\psi|^2 \, dx \, d\xi \leq Ch^2 \int_{\mathbb{R}^4} e^{2\varepsilon\Psi_2(X_2)/h}|Tu|^2 \, dx \, d\xi \, .$$

Then, the analysis follows the same lines as in the proof of Theorem 4.4. The same splitting of the integral in the left-hand-side gives the conclusion. ☐

4.2. Subprincipal decay estimates

Let us now prove an exponential estimate with respect to all the phase space variables. In the previous section, we essentially used the ellipticity of the operator outside of the characteristic set. The results, while new in this precision as far as we know, are not surprising. However, in this section, we have to understand what is happening directly on the characteristic set, i.e understand in detail the *subprincipal* terms. This is a much finer analysis. At the microlocal level, the computations are similar to the ones in [39]; however, instead of using the Boutet de Monvel calculus for polynomial operators, we directly use the invertibility of an effective harmonic oscillator.

Theorem 4.5. *For $M>0$, there exist $\varepsilon, h_0, C>0$ such that, for all $h\in(0,h_0)$, $\mu\leq b_0+Mh$ and u solving (4.1), we have*

$$\int_{\mathbb{R}^4} e^{2\varepsilon\Psi_2(X_2)/h}|Tu|^2 \, dx \, d\xi \leq C \int_{\mathbb{R}^4} |Tu|^2 \, dx \, d\xi \, .$$

Proof. This time, $\psi=\Psi(X_2)$. Let us use Scholium 3.4 again with $m=1$. We get

$$\int_{\mathbb{R}^4} \operatorname{Re} \left(\hat{p}_h - \frac{h}{4}\widehat{\Delta p_h} - \breve{R} - Ch^2 \right) |u_\psi|^2 \, dx \, d\xi \leq 0 \, .$$

With Theorem 4.3, we can estimate $\breve{R}$ and get

$$\int_{\mathbb{R}^4} \mathrm{Re}\left(\hat{p}_h - \frac{h}{4}\widehat{\Delta p_h} - Ch^2\right)|u_\psi|^2\, dx\, d\xi \leq 0\,.$$

Observe that

$$\hat{p}_h + \mathscr{O}(h^2) = \frac{\hat{p}_{\mathcal{M}} - h\mu}{1+\hat{p}_{\mathcal{M}}} = \hat{p}_{\mathcal{M}} - h\mu + h\mu\frac{\hat{p}_{\mathcal{M}}}{1+\hat{p}_{\mathcal{M}}} - \frac{\hat{p}_{\mathcal{M}}^2}{1+\hat{p}_{\mathcal{M}}}\,. \tag{4.2}$$

The fourth term in the right-hand side is $\mathscr{O}(\min(|X_1|^4, 1))$, and can be absorbed using Theorem 4.3, and replaced by a $\mathscr{O}(h^2)$. The third term can also be absorbed in the same fashion, and replaced by $\mathscr{O}(h^2\mu)$. We deduce that

$$\int_{\mathbb{R}^4}\left(\mathrm{Re}\,\hat{p}_{\mathcal{M}} - b_0 h - \frac{h}{4}\mathrm{Re}\,\widehat{\Delta p_h} - C(1+K)h^2\right)|u_\psi|^2\, dx\, d\xi \leq 0\,.$$

Using Equation (4.2) to estimate the contribution from $\widehat{\Delta p_h}$, and using the same arguments,

$$\int_{\mathbb{R}^4}\left(\mathrm{Re}\,\hat{p}_{\mathcal{M}} - b_0 h - \frac{h}{4}\mathrm{Re}\,\widehat{\Delta_{X_1} p_{\mathcal{M}}} - C(1+K)h^2\right)|u_\psi|^2\, dx\, d\xi \leq 0\,,$$

Now, we will approximate $\hat{p}_M$ by a quadratic form in X_1, with coefficients depending only on X_2. To this end, let

$$\mathcal{Q}_{X_2}(X_1) := \left[\widehat{g^{11}}_{|X_1=0}\right]\xi_1^2 - 2\left[\widehat{g^{12}}_{|X_1=0}\right]\xi_1 x_1 + \left[\widehat{g^{22}}_{|X_1=0}\right]x_1^2.$$

(Observe that since ψ does not depend on X_1, $\hat{}$ and differentiation in X_1 commute.) Since the coefficients g^{ij} are assumed to be critical at 0, and $d\psi(0)=0$, we find

$$\begin{aligned}\mathrm{Re}\,\hat{p}_{\mathcal{M}} - \frac{h}{4}\mathrm{Re}\,\widehat{\Delta_{X_1} p_{\mathcal{M}}} &= \mathrm{Re}\,\mathcal{Q}_{X_2}(X_1) - \frac{h}{2}\,\mathrm{Tr}\,\mathrm{Re}\,\mathcal{Q}_{X_2}\\ &\quad + \mathscr{O}(|X_1|^2(\min(|X_1|^2+|X_2|^2, 1) + h|X_1|^2 + h|X_1|\min(|X_2|, 1)))\end{aligned}$$

Using Theorem 4.4, we can absorb $\mathscr{O}(h^k|X_1|^{2\ell}\min(|X_2|^2, 1))$ and replace it by

$$\mathscr{O}(h^{k+\ell}(\min(|X_2|^2, 1) + h)).$$

Therefore, using also $|X_1||X_2| \leq \varepsilon^{-1}|X_1|^2 + \varepsilon|X_2|^2$, we get

$$\begin{aligned}\int\left(\mathrm{Re}\,\mathcal{Q}_{X_2}(X_1) - b_0 h - \frac{h}{2}\,\mathrm{Tr}\,\mathrm{Re}\,\mathcal{Q}_{X_2} - C\varepsilon h\min(|X_2|^2, 1)\right)|u_\psi|^2\, dX_1\, dX_2 \leq\\ (1+\varepsilon^{-1}+K)h^2\int|u_\psi|^2\, dX_1\, dX_2.\end{aligned}$$

For fixed X_2, we recognize the Bargmann symbol of the "harmonic oscillator" in X_1 (see Remark 3.6) and thus

$$\int_{\mathbb{R}^2}\left(\operatorname{Re}\mathcal{Q}_{X_2}(X_1)-\frac{h}{2}\operatorname{Tr}\operatorname{Re}\mathcal{Q}_{X_2}\right)|u_\psi|^2\,dX_1\geq h\sqrt{\det\operatorname{Re}\mathcal{Q}_{X_2}}\int|u_\psi|^2\,dX_1.$$

So that

$$\int_{\mathbb{R}^4}\left(\sqrt{\det\operatorname{Re}\mathcal{Q}_{X_2}}-b_0-C\varepsilon\min(|X_2|^2,1)-C(1+\varepsilon^{-1}+K)h\right)|u_\psi|^2\,dx\,d\xi\leq 0\,.$$

Recall now that $B=\sqrt{\det\mathcal{Q}}$, so that

$$\sqrt{\det\operatorname{Re}\mathcal{Q}}=B(1+\mathscr{O}(\operatorname{Tr}\operatorname{Re}\mathcal{Q}^{-1}\operatorname{Im}\mathcal{Q}))=B(1+\varepsilon\mathscr{O}(\min(|X_2|^2,1))).$$

Under the conclusion on Lemma 2.1, we get the estimate

$$\int_{\mathbb{R}^4}\left(\min(|X_2|^2,1)(1-C\varepsilon)-C(1+\varepsilon^{-1}+K)h\right)|u_\psi|^2\,dx\,d\xi\leq 0\,.$$

The conclusion follows from the usual Agmon arguments, and again the fact that the constant only depend on derivatives of ψ. □

5. Space exponential decay

We are now in position to prove Theorem 1.1. Let $\hat{u}\in L^2(\mathbb{R}^2)$ such that $\mathscr{L}_h\hat{u}=h\mu\hat{u}$ and let $u=\mathcal{M}^{-1}U_\varkappa\hat{u}=\mathcal{M}^*U_\varkappa\hat{u}$. We have $\mathscr{P}u=0$, see Equation (4.1).

Remark 5.1. Following [27, Theorem 4.1.2], with Theorem 4.1, one could deduce (up to technicalities) that, if K is a compact set away from 0, we have

$$\|\hat{u}\|_{L^2(K)}=\mathscr{O}(e^{-c/h})$$

for some $c>0$. Below, one will get a more explicit result. Already observe that we can drop the factor $U_\varkappa$. Indeed, since $\varkappa$ is uniformly bi-Lipschitz, it preserves spatial exponential decay. So we can concentrate on $\tilde{u}:=\mathcal{M}u=U_\varkappa\hat{u}$.

With the notation of Theorem 4.1, we have, for ε small enough,

$$Tu\in L^2_{\varepsilon\psi_0}(\mathbb{R}^4),\qquad\text{where } L^2_{\varepsilon\psi_0}(\mathbb{R}^4):=L^2(\mathbb{R}^4;e^{\varepsilon\psi_0}dxd\xi)\,,$$

with a uniform bound: $\|Tu\|_{L^2_{\varepsilon\psi_0}(\mathbb{R}^4)}\leq C\|u\|_{L^2(\mathbb{R}^2)}$, where C does not depend on h. From this exponential decay in phase space, we wish to obtain exponential decay in the position variable x for $\tilde{u}$. We start with the inversion formula (3.1):

$$\hat{u}=\mathcal{M}T^*(Tu).$$

Let φ be a non-negative Lipschitz function, going linearly to infinity at infinity, having a unique and non-degenerate minimum at the origin, with minimal value 0 (let us call these functions *admissible weights*). We would like to obtain a uniform bound $\|e^{\varepsilon'\varphi}\hat{u}\|_{L^2(\mathbb{R}^2)} \leq C\|\hat{u}\|_{L^2(\mathbb{R}^2)}$, for some $\varepsilon'>0$ small enough. Thus, it is enough to prove that the operator

$$\mathcal{M}T^* : L^2_{\varepsilon\psi_0}(\mathbb{R}^4) \longrightarrow L^2_{\varepsilon'\varphi}(\mathbb{R}^2) \tag{5.1}$$

is uniformly bounded with respect to $h\in]0,h_0]$.

Lemma 5.2. *Let φ_1, φ_2 be admissible weights on $\mathbb{R}^d$. Then there exists $C>0$ such that*

$$C^{-1}\varphi_1 \leq \varphi_2 \leq C\varphi_1 \,. \tag{5.2}$$

Proof. By the Taylor formula at the origin, the estimate (5.2) is valid in a neighborhood of 0. By the linear behavior at infinity, it is valid outside of a compact set. On the remaining compact subset of $\mathbb{R}^d\setminus\{0\}$, it is enough to use that the range of φ_j is a compact interval of $]0,+\infty[$. □

A consequence of the lemma is that the choice of φ and ψ_0 in (5.1) is not relevant, as long as we don't seek the optimal constants and are allowed to play with $\varepsilon, \varepsilon'$.

Proposition 5.3. *Given an admissible weight φ on $\mathbb{R}^2$, there exists an admissible weight ψ_0 on $\mathbb{R}^4$ of the form required by Theorem 4.1, and a constant $C>0$ independent of h, such that for all $\varepsilon'\leq\varepsilon/C$, $\varepsilon\leq 1$, the operator $\mathcal{M}T^*$ defined in (5.1) is bounded by $\mathscr{O}(1)$.*

As a consequence, there exists ε'_0 such that if $\varepsilon'\leq\varepsilon'_0$, then there exists $C>0$ such that

$$\int_{\mathbb{R}^2} e^{\varepsilon'\varphi(x)/h}\,|\hat{u}(x)|^2\,dx \leq C\,\|\hat{u}\|^2_{L^2(\mathbb{R}^2)}\,.$$

Proof. By Lemma 5.2, we can always change the function φ, so we will pick a convenient one. First, consider the $\mathscr{C}^1$ Lipschitz function f defined by $f(\rho)=\rho^2$ if $\rho\in[0,1]$ and $f(\rho)=2\rho-1$ if $\rho\geq 1$. Notice that

$$\forall\rho\geq 0\,, \quad f(2\rho)\leq 4f(\rho)\,. \tag{5.3}$$

We define now the admissible weight $\varphi(x):=f(|x|)$, $x\in\mathbb{R}^2$.

A formula for $\mathcal{M}T^*$ can be obtained from the action of the FBI transform T on arbitrary metaplectic operators, see [27, 3.4]; here we derive it explicitly. We have

$$Tu(x,\xi) = \alpha_h e^{\frac{-\xi^2}{2h}}(L\star u)(z), \qquad \text{with } L(x)=e^{\frac{-x^2}{2h}},\ z:=x-i\xi\in\mathbb{C}^2\,.$$

From (2.10) we obtain $T\mathcal{M}^*u(x,\xi)=\alpha_h e^{\frac{-\xi^2}{2h}}((L\star\overline{K})\star u)(z)$, where $L\star\overline{K}$ is a complex Gaussian that can be computed explicitly, using in particular that $(I+iA)^{-1}=$

$\frac{1}{2}(I-iA)$:

$$L\star\overline{K}(y)=\sqrt{2}\pi h e^{\frac{-1}{4h}\langle (I+iA)y,y\rangle}\,,\quad A=\begin{pmatrix}0&1\\1&0\end{pmatrix}\,.$$

Hence

$$(T\mathcal{M}^*u)(x,\xi)=\widetilde{\alpha}_h e^{\frac{-\xi^2}{2h}}\int_{\mathbb{R}^2} e^{\frac{-1}{4h}\langle (I+iA)(z-y),z-y\rangle}u(y)\,dy\,,\quad \widetilde{\alpha}_h=\frac{\alpha_h}{\sqrt{2}}\,,$$

and therefore, taking the adjoint, we have for $v\in L^2_{\varepsilon\psi_0}(\mathbb{R}^4)$

$$(\mathcal{M}T^*)v(y)=\widetilde{\alpha}_h\int_{\mathbb{R}^4} e^{\frac{-\xi^2}{2h}}e^{\frac{-1}{4h}\langle (I-iA)(\bar{z}-y),\bar{z}-y\rangle}v(x,\xi)\,dx\,d\xi\,,\quad \bar{z}=x+i\xi\,.$$

Let $K_{\mathcal{M}T^*}(y,x,\xi)$ be the Schwartz kernel of $e^{\frac{\varepsilon'\varphi}{h}}\mathcal{M}T^*e^{-\frac{\varepsilon\psi_0}{h}}$, viewed as an operator $L^2(\mathbb{R}^4)\to L^2(\mathbb{R}^2)$, *i.e.*

$$K_{\mathcal{M}T^*}(y,x,\xi)=\widetilde{\alpha}_h e^{\frac{\varepsilon'\varphi(y)}{h}-\frac{|\xi|^2}{2h}-\frac{1}{4h}\langle (I-iA)(\bar{z}-y),\bar{z}-y\rangle-\frac{\varepsilon\psi_0(x,\xi)}{h}}\,.$$

We have

$$\begin{aligned}\operatorname{Re}\langle (I-iA)(\bar{z}-y),\bar{z}-y\rangle &= |x-y|^2-|\xi|^2-2\langle A\xi,x-y\rangle\\ &= |A\xi-(x-y)|^2-2\,|\xi|^2\,,\end{aligned}$$

where in the second line we used $|A\xi|^2=|\xi|^2$. Therefore,

$$|K_{\mathcal{M}T^*}(y,x,\xi)|\le\widetilde{\alpha}_h e^{\frac{\varepsilon'\varphi(y)}{h}-\frac{|A\xi-(x-y)|^2}{2h}-\frac{\varepsilon\psi_0(x,\xi)}{h}}\,.$$

Let us choose now, as we may, $\psi_0(x,\xi):=\varphi(x)+|\xi|^2/\langle\xi\rangle$. Indeed, ψ_0 is not of the form $\Psi_1+\Psi_2$, but is bounded from below by a function of this form (this can be written explicitly, or by invoking Lemma 5.2). By convexity of φ,

$$\varphi(y)\le\tfrac{1}{2}\varphi(2x)+\tfrac{1}{2}\varphi(2(y-x))\,,$$

and hence

$$|K_{\mathcal{M}T^*}(y,x,\xi)|\le\widetilde{\alpha}_h e^{\frac{\varepsilon'\varphi(2x)}{2h}-\frac{\varepsilon\varphi(x)}{h}+\frac{\varepsilon'\varphi(2(y-x))}{2h}-\frac{|x-y-A\xi|^2}{2h}-\frac{\varepsilon|\xi|^2}{h\langle\xi\rangle}}\,.$$

If $\varepsilon'\le\varepsilon/2$ then, by (5.3), $\varepsilon'\varphi(2x)/2\le\varepsilon\varphi(x)$ for all $x\in\mathbb{R}^2$ so that

$$|K_{\mathcal{M}T^*}(y,x,\xi)|\le\widetilde{\alpha}_h e^{\frac{\varepsilon'\varphi(2(y-x))}{2h}-\frac{|x-y-A\xi|^2}{2h}-\frac{\varepsilon|\xi|^2}{h\langle\xi\rangle}}\,. \tag{5.4}$$

We wish to conclude on the L^2 continuity of $\mathcal{M}T^*$ by applying the Schur lemma. For given (x,ξ), we make a change of variables to get

$$\int_{\mathbb{R}^2}|K_{\mathcal{M}T^*}(y,x,\xi)|\,dy=\int_{\mathbb{R}^2}|K_{\mathcal{M}T^*}(y+x+A\xi,x,\xi)|\,dy.$$

From (5.4) we have

$$|K_{\mathcal{M}T^*}(y+x+A\xi,x,\xi)|\leq\widetilde{\alpha}_h e^{\frac{\varepsilon'\varphi(2(y+A\xi))}{2h}-\frac{|y|^2}{2h}-\frac{\varepsilon\xi^2}{h\langle\xi\rangle}}\,.$$

Using again the convexity of φ,

$$\frac{\varepsilon'\varphi(2(y+A\xi))}{2h}-\frac{|y|^2}{2h}-\frac{\varepsilon\xi^2}{h\langle\xi\rangle}\leq\frac{\varepsilon'\varphi(4y)}{4h}+\frac{\varepsilon'\varphi(4A\xi)}{4h}-\frac{|y|^2}{2h}-\frac{\varepsilon\xi^2}{h\langle\xi\rangle}. \tag{5.5}$$

From Lemma 5.2, there exists $C_\varphi>0$ such that, $\varphi(4A\xi)\leq C_\varphi|\xi|^2/\langle\xi\rangle$. Hence, if $\varepsilon'\leq 4C_\varphi\varepsilon$, we get $\frac{\varepsilon'\varphi(4A\xi)}{4h}-\frac{\varepsilon\xi^2}{h\langle\xi\rangle}\leq 0$, and

$$\int_{\mathbb{R}^2}|K_{\mathcal{M}T^*}(y+x+A\xi,x,\xi)|\,dy\leq\widetilde{\alpha}_h\int_{\mathbb{R}^2}e^{\frac{\varepsilon'\varphi(4y)}{2h}-\frac{|y|^2}{2h}}\,dy\,.$$

Using Laplace's method, the integral on the right-hand side is $\mathscr{O}(h)$ provided $\varepsilon'<1/16$. Hence,

$$\int_{\mathbb{R}^2}|K_{\mathcal{M}T^*}(y,x,\xi)|\,dy\leq Ch\widetilde{\alpha}_h\,.$$

On the other hand, for a given y, we make an analogous change of variables:

$$\int_{\mathbb{R}^4}|K_{\mathcal{M}T^*}(y,x,\xi)|\,dx\,d\xi=\int_{\mathbb{R}^4}|K_{\mathcal{M}T^*}(y,x+y+A\xi,\xi)|\,dx\,d\xi$$

and write (5.4) as

$$|K_{\mathcal{M}T^*}(y,x+y+A\xi,\xi)|\leq\widetilde{\alpha}_h e^{\frac{\varepsilon'\varphi(2(x+A\xi))}{2h}-\frac{|x|^2}{2h}-\frac{\varepsilon\xi^2}{h\langle\xi\rangle}}$$

Applying Equation (5.5) with y replaced by x, and choosing $\varepsilon'\leq 2C_\varphi\varepsilon$, gives

$$\int_{\mathbb{R}^4}|K_{\mathcal{M}T^*}(y,x+y+A\xi,\xi)|\,dx\,d\xi\leq\widetilde{\alpha}_h\int_{\mathbb{R}^2}e^{-\frac{\varepsilon\xi^2}{2h\langle\xi\rangle}}\,d\xi\int_{\mathbb{R}^2}e^{\frac{\varepsilon'\varphi(4x)}{4h}-\frac{|x|^2}{2h}}\,dx\,.$$

Using that both integrals are $\mathscr{O}(h)$, we have

$$\int_{\mathbb{R}^4}|K_{\mathcal{M}T^*}(y,x+y+A\xi,\xi)|\,dx\,d\xi\leq Ch^2\widetilde{\alpha}_h\,.$$

Hence, the Schur lemma gives $\mathcal{M}T^*=\mathscr{O}(\widetilde{\alpha}_h h^{3/2})=\mathscr{O}(1):L^2_{\varepsilon\psi_0}(\mathbb{R}^4)\to L^2_{\varepsilon'\varphi}(\mathbb{R}^2)$. □

With this Proposition 5.3, the proof of Theorem 1.1 is complete. It would be very interesting to investigate the optimality of (φ,ε') for which Proposition 5.3 holds, in particular by relating $\varphi''(0)$ to the behavior of the magnetic field at the origin.

Proof of Corollary 1.4. In $L^2(\mathbb{R}^2)$, we can decompose $u_{\ell,J}=\alpha e^{i\theta}u_\ell+w$ with w orthogonal to u_ℓ, $\theta\in\mathbb{R}$ and $\alpha\geq 0$. Since the eigenvalues of $\mathscr{L}_h$ are h^2 separated, the Spectral Theorem implies that $\|(\mathscr{L}_h-\lambda_\ell(h))u_{\ell,J}\|\geq Ch^2\|w\|$, so $\|w\|\leq Ch^J$. In particular, $1-\alpha\leq (Ch^J)^2/2$. We then get

$$\|u_{\ell,J}-e^{i\theta}u_\ell\|^2_{L^2}=2(1-\alpha)\leq (Ch^J)^2.$$

Now, we turn to exponentially weighted spaces. We consider $\varepsilon>0$ small enough so that Theorem 1.1 applies to 2ε. Then we observe that

$$\|u_{\ell,J}-e^{i\theta}u_\ell\|_{L^2(e^{\varepsilon d(x)/h}dx)}\leq \|u_{\ell,J}-e^{i\theta}u_\ell\|^{1/2}_{L^2}\|u_{\ell,J}-e^{i\theta}u_\ell\|^{1/2}_{L^2(e^{2\varepsilon d(x)/h}dx)}. \qquad \square$$

A. Change of variables

Since $\mathscr{L}_h$ is invariantly defined by the 2-form $\mathbf{B}$ and the Riemannian metric on M, its principal and subprincipal Weyl symbols are well defined, which implies that a change of variables like the one defined in Lemma 2.2 and used in Lemma 2.1 acts naturally on the Weyl symbol modulo terms of order $\mathscr{O}(h^2)$ (see also [30]). Here we give a direct proof of this and compute explicitly the $\mathscr{O}(h^2)$ remainder.

Lemma A.1. *Consider a change of variable $\varkappa:\mathbb{R}^2_y\to\mathbb{R}^2_x$. We let*

$$U\psi=|g|^{\frac14}\psi\circ\varkappa=\mathrm{Jac}(\varkappa)^{\frac12}\psi\circ\varkappa\,.$$

We have

$$U\mathscr{L}_hU^{-1}=(-ih\nabla_y-\widetilde{\mathbf{A}})g^*(-ih\nabla_y-\widetilde{\mathbf{A}})-h^2V\,,$$

with

$$V=|g|^{-\frac12}\left(\mathrm{div}(|g|^{\frac14}g^*\nabla(|g|^{\frac14}))+\|g^*\nabla(|g|^{\frac14})\|^2\right)\,,$$

and

$$g^*=(g^{-1})^{\mathrm{T}}\,,\quad g=(d\varkappa)^{\mathrm{T}}(d\varkappa)\,,\quad \widetilde{\mathbf{A}}=(d\varkappa)^{\mathrm{T}}\circ\mathbf{A}\circ\varkappa\,.$$

Proof. Considering the quadratic form $\mathscr{Q}_h$ of $\mathscr{L}_h$ on $L^2(\mathbb{R}^2_x,dx)$, we have

$$\mathscr{Q}_h(\psi)=\int_{\mathbb{R}^2}\langle(-ih\nabla_y-\widetilde{\mathbf{A}}(y))\tilde\psi,(-ih\nabla_y-\widetilde{\mathbf{A}}(y))\tilde\psi\rangle_{g^*}|g|^{\frac12}\,dy\,,$$

where

$$g^*=(g^{-1})^{\mathrm{T}}\,,\quad g=(d\varkappa)^{\mathrm{T}}(d\varkappa)\,,\quad \tilde\psi=\psi\circ\varkappa\,,\quad \widetilde{\mathbf{A}}=(d\varkappa)^{\mathrm{T}}\circ\mathbf{A}\circ\varkappa\,.$$

In terms of forms, this means that

$$\varkappa^*g_0=g\,,\quad \varkappa^*\psi=\tilde\psi\,,\quad \varkappa^*(A_1dx_1+A_2dx_2)=\tilde A_1dy_1+\tilde A_2dy_2\,.$$

We let $P=-ih\nabla_y-\widetilde{\mathbf{A}}(y)$ and notice that

$$\begin{aligned}\mathscr{Q}_h(\psi)&=\int_{\mathbb{R}^2}\langle|g|^{\frac14}P\tilde\psi,|g|^{\frac14}P\tilde\psi\rangle_{g^*}\,dy\\&=\int_{\mathbb{R}^2}\langle P|g|^{\frac14}\tilde\psi,|g|^{\frac14}P\tilde\psi\rangle_{g^*}\,dy+\int_{\mathbb{R}^2}\langle[|g|^{\frac14},P]\tilde\psi,|g|^{\frac14}P\tilde\psi\rangle_{g^*}\,dy\,,\end{aligned}$$

and then

$$\begin{aligned}\mathscr{Q}_h(\psi)=&\int_{\mathbb{R}^2}\|P|g|^{\frac14}\tilde\psi\|^2_{g^*}\,dy\\&+\int_{\mathbb{R}^2}\langle[|g|^{\frac14},P]\tilde\psi,|g|^{\frac14}P\tilde\psi\rangle_{g^*}\,dy+\int_{\mathbb{R}^2}\langle P|g|^{\frac14}\tilde\psi,[|g|^{\frac14},P]\tilde\psi\rangle_{g^*}\,dy\,,\end{aligned}$$

so that

$$\begin{aligned}\mathscr{Q}_h(\psi)=&\int_{\mathbb{R}^2}\|P|g|^{\frac14}\tilde\psi\|^2_{g^*}\,dy\\&+2\mathrm{Re}\int_{\mathbb{R}^2}\langle[|g|^{\frac14},P]\tilde\psi,|g|^{\frac14}P\tilde\psi\rangle_{g^*}dy-\int_{\mathbb{R}^2}\|[|g|^{\frac14},P]\tilde\psi\|^2_{g^*}\,dy\,.\end{aligned}$$

Since $[P,|g|^{\frac14}]=-ih\nabla(|g|^{\frac14})$ and $\widetilde{\mathbf{A}}$ is real-valued, we deduce that

$$\begin{aligned}2\mathrm{Re}\int_{\mathbb{R}^2}\langle[|g|^{\frac14},P]\tilde\psi,|g|^{\frac14}P\tilde\psi\rangle_{g^*}\,dy&=2h\mathrm{Im}\int_{\mathbb{R}^2}\langle\tilde\psi\nabla(|g|^{\frac14}),|g|^{\frac14}(-ih\nabla_y)\tilde\psi\rangle_{g^*}\,dy\\&=2h^2\mathrm{Re}\int_{\mathbb{R}^2}\langle\tilde\psi\nabla(|g|^{\frac14}),|g|^{\frac14}\nabla_y\tilde\psi\rangle_{g^*}\,dy\\&=2h^2\mathrm{Re}\int_{\mathbb{R}^2}\tilde\psi(\mathbf{F}\cdot\nabla_y)\overline{\tilde\psi}\,dy\\&=h^2\int_{\mathbb{R}^2}\mathbf{F}\cdot(\nabla_y|\tilde\psi|^2)\,dy\\&=-h^2\int_{\mathbb{R}^2}\mathrm{div}\mathbf{F}\,|\tilde\psi|^2\,dy\,.\end{aligned}$$

where $\mathbf{F}=|g|^{\frac14}g^*\nabla(|g|^{\frac14})$. Therefore,

$$\mathscr{Q}_h(\psi)=\int_{\mathbb{R}^2}\|PU\psi\|^2_{g^*}\,dy-h^2\int_{\mathbb{R}^2}V(y)|U\psi|^2\,dy\,,$$

and the conclusion follows. □

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Y. Guedes Bonthonneau
CNRS, IRMAR – UMR 6625
Univ Rennes
F-35000 Rennes
France
yannick.bonthonneau@univ-rennes1.fr

S. Vũ Ngọc
CNRS, IRMAR – UMR 6625
Univ Rennes
F-35000 Rennes
France
san.vu-ngoc@univ-rennes1.fr

N. Raymond
Laboratoire Angevin de Recherche en Mathématiques, LAREMA, UMR 6093
Univ Angers
SFR Math-STIC, 2 boulevard Lavoisier
F-49045 Angers Cedex 01
France
nicolas.raymond@univ-angers.fr

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