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On the locus of Prym curves where the Prym-canonical map is not an embedding

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Abstract. We prove that the locus of Prym curves (C, η) of genus $g \geq 5$ for which the Prym-canonical system $|\omega_C(\eta)|$ is base point free but the Prym-canonical map is not an embedding is irreducible and unirational of dimension $2g+1$.

1. Introduction

Let $g \geqslant 2$ and $\mathcal{R}_g$ be the moduli space of *Prym curves*, that is, of pairs (C, η), with C a smooth complex projective genus g curve and η a non-zero 2-torsion point of $\mathrm{Pic}^0(C)$. It is well-known that $\mathcal{R}_g$ is irreducible of dimension $3g-3$ and that the natural forgetful map $\mathcal{R}_g \to \mathcal{M}_g$, where $\mathcal{M}_g$ denotes the moduli space of smooth genus g curves, is finite of degree $2^{2g}-1$. The complete linear system $|\omega_C(\eta)|$ is of dimension $g-2$ and it is base point free unless C is hyperelliptic and $\eta \simeq \mathcal{O}_C(p-q)$, with p and q ramification points of the g^1_2 (cf. Lemma 2.1 below).

In this note we study the locally closed locus $\mathcal{R}^0_g$ in $\mathcal{R}_g$ of Prym curves (C, η) such that the *Prym-canonical system* $|\omega_C(\eta)|$ is base point free but the morphism $C \to \mathbb{P}^{g-2}$ it defines (the so-called *Prym-canonical map*) is not an embedding. Note that $\mathcal{R}^0_g$ is clearly dense in $\mathcal{R}_g$ for $g \leqslant 4$. Our main result is the following:

Theorem 1.1. *Let $g \geqslant 5$. The locus $\mathcal{R}^0_g$ is irreducible and unirational of dimension $2g+1$ and lies in the tetragonal locus.*

By the tetragonal locus $\mathcal{R}^1_{g,4}$ in $\mathcal{R}_g$ we mean the inverse image via $\mathcal{R}_g \to \mathcal{M}_g$ of the tetragonal locus $\mathcal{M}^1_{g,4}$ of $\mathcal{M}_g$.

We also show:

Proposition 1.2. *For general $(C, \eta) \in \mathcal{R}^0_g$, $g \geqslant 5$, the Prym-canonical map is birational onto its image, and its image has precisely two nodes.*

Although we believe that these results are of independent interest, our main motivation for studying the locus $\mathcal{R}_g^0$ is that it naturally contains pairs (C,η) where C is a smooth curve lying on an Enriques surface S such that

$$\phi(C)=\min\{E\cdot C\,|\,E\in\mathrm{Pic}(S), E>0, E^2=0\}=2,$$

and $\eta{=}\mathcal{O}_C(K_S)$, cf. Examples 5.1 and 5.2 and Remark 5.5, in which case the Prym-canonical map associated to η is the restriction to C of the map defined by the complete linear system $|C|$ on S. The locus $\mathcal{R}_g^0$ indeed naturally shows up in our recent work [6] concerning the moduli of smooth curves lying on an Enriques surface, in which we use the results in this note. Besides, we show in [6] that $\mathcal{R}_g^0$ is dominated by curves on Enriques surfaces for $5\leqslant g\leqslant 8$.

The paper is organized as follows. Section 2 is devoted to recalling some preliminary results. The irreducibility and unirationality of $\mathcal{R}_g^0$ is proved in Section 3, whereas its dimension is computed in Section 4. We conclude with the proof of Proposition 1.2 together with the mentioned examples on Enriques surfaces.

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2. Preliminary results

2.1. A basic lemma on Prym curves

The following is an immediate consequence of the Riemann-Roch theorem (see also [8, §0.1] or [12, Pf. of Lemma 2.1]). We include the proof for the reader's convenience.

Lemma 2.1. *Let (C,η) be any Prym curve of genus $g\geqslant 3$. Then*:

(i) *p is a base point of $|\omega_C(\eta)|$ if and only if $|p+\eta|\neq\varnothing$. This happens if and only if C is hyperelliptic and $\eta\sim\mathcal{O}_C(p-q)$, with p and q ramification points of the g^1_2. In particular, p and q are the only base points*;

(ii) *if $|\omega_C(\eta)|$ is base point free, then it does not separate p and q (possibly infinitely near) if and only if $|p+q+\eta|\neq\varnothing$. This happens if and only if C has a g^1_4 and $\eta\sim\mathcal{O}_C(p+q-x-y)$, where $2(p+q)$ and $2(x+y)$ are members of the g^1_4. In particular, also x and y are not separated by $|\omega_C(\eta)|$.*

Proof. We prove only (ii) and leave (i) to the reader. Assume that $|\omega_C(\eta)|$ is base point free. Then p and q are not separated by the linear system $|\omega_C(\eta)|$ if and only if $h^0(\omega_C(\eta)-p)=h^0(\omega_C(\eta)-p-q)$. By Riemann-Roch and Serre duality, this is equivalent to $h^0(\eta+p)+1=h^0(\eta+p+q)$. By (i), we have $h^0(\eta+p)=0$, whence the latter condition is $h^0(\eta+p+q)=1$. This is equivalent to $h^0(\eta+p+q)>0$, because if $h^0(\eta+p+q)>1$, then we would have $h^0(\eta+p)>0$, a contradiction. This proves the first assertion.

We have $|p+q+\eta|\neq\varnothing$ if and only if $p+q+\eta\sim x+y$, for $x,y\in C$. This implies $2(p+q)\sim 2(x+y)$, whence C has a g^1_4 with $2(p+q)$ and $2(x+y)$ as its members. Conversely, if $2(p+q)$ and $2(x+y)$ are distinct members of a g^1_4 on C, then $\eta:=\mathcal{O}_C(p+q-x-y)$ is a 2-torsion element of $\mathrm{Pic}^0(C)$ and satisfies the condition that $|p+q+\eta|\neq\varnothing$. $\square$

The lemma says in particular that the locus in $\mathcal{R}_g$ of pairs (C,η) for which the Prym-canonical system $|\omega_C(\eta)|$ is not base-point free dominates the hyperelliptic locus via the forgetful map $\mathcal{R}_g\to\mathcal{M}_g$.

Recall that the tetragonal locus $\mathcal{R}^1_{g,4}$ is irreducible of dimension $2g+3$ if $g\geqslant 7$ and coincides with $\mathcal{R}_g$ if $g\leqslant 6$. Lemma 2.1 implies that $\mathcal{R}^0_g\subseteq\mathcal{R}^1_{g,4}$, thus proving the last statement in Theorem 1.1.

The lemma also enables us to detect the locus $\mathcal{R}^{0,\mathrm{nb}}_g$ in $\mathcal{R}^0_g$ where the Prym-canonical morphism is not birational onto its image:

Corollary 2.2. *Let (C,η) be any Prym curve of genus $g\geqslant 4$ such that the Prym-canonical system $|\omega_C(\eta)|$ is base point free. If the Prym-canonical map is not birational onto its image, then it is of degree two onto a smooth elliptic curve.*

The locus $\mathcal{R}^{0,\mathrm{nb}}_g$ is irreducible of dimension $2g-2$ and dominates the bielliptic locus in $\mathcal{M}_g$. More precisely, $\mathcal{R}^{0,\mathrm{nb}}_g$ consists of pairs (C,η), with C bielliptic and $\eta:=\varphi^*\eta'$, where $\varphi:C\to E$ is a bielliptic map and η' is a nontrivial 2-torsion element in $\mathrm{Pic}^0(E)$.

Proof. Let (C,η) be as in the statement. Denote by C' the image of the Prym-canonical morphism $\varphi:C\to\mathbb{P}^{g-2}$. Let μ be the degree of φ and d the degree of C'. Then $d\mu=2g-2$ and, since C' is non-degenerate in $\mathbb{P}^{g-2}$, we must have $d\geqslant g-2$. Since $g\geqslant 4$, then $2\leqslant\mu\leqslant 3$; moreover $\mu=3$ implies that $g=4$ and φ maps C three-to-one to a conic. The latter case cannot happen: indeed, we would have $\omega_C(\eta)=2\mathcal{L}$, where $|\mathcal{L}|$ is a g^1_3. Then $4\mathcal{L}=2\omega_C$. Since $|2\omega_C|$ is cut out by quadrics on the canonical image of C in $\mathbb{P}^3$, it follows that the only quadric containing the canonical model is a cone. Then $|\mathcal{L}|$ is the unique g^1_3 on C and $2\mathcal{L}=\omega_C$, thus η is trivial, a contradiction.

Hence $\mu=2$, and then $d=g-1$, so that C' is a *curve of almost minimal degree.* It is easy to see, using the fact that $|\omega_C(\eta)|$ is complete, that C' is a smooth elliptic curve (alternatively, apply [5, Thm. 1.2]). Hence C is bielliptic and any pair of points p and q identified by φ satisfy $p+q\sim\varphi^*(r)$ for a point $r\in C'$. Thus $2p+2q\sim\varphi^*(2r)$ is a g^1_4. By Lemma 2.1(ii) we have $\eta\sim\mathcal{O}_C(p+q-x-y)$, where also $\varphi(x)=\varphi(y)$, whence $x+y\sim\varphi^*(z)$, for a $z\in C'$. Hence, again by Lemma 2.1(ii), we have $\eta\sim p+q-x-y\sim\varphi^*(r-z)$ and $r-z$ is a nontrivial 2-torsion element in $\mathrm{Pic}^0(C')$, because $\varphi^*:\mathrm{Pic}^0(C')\to\mathrm{Pic}^0(C)$ is injective.

Conversely, if C is a bielliptic curve, it admits at most finitely many double covers $\varphi:C\to E$ onto an elliptic curve (cf. e.g., [2]; in fact, for $g\geqslant 6$, it admits a unique such map), and for any such φ and any nontrivial 2-torsion element η' in $\mathrm{Pic}^0(E)$, we have $\eta'\sim r-z$, for $r,z\in E$. Letting $\varphi^*(r)=p+q$ and $\varphi^*(z)=x+y$, we see that $2(p+q)\sim 2(x+y)$ and $\eta=\varphi^*\eta'$ satisfies the conditions of Lemma 2.1(ii).

We have therefore proved that $\mathcal{R}_g^{0,\mathrm{nb}}$ consists of pairs (C,η), with C bielliptic and $\eta:=\varphi^*\eta'$, where $\varphi:C\to E$ is a bielliptic map and η' is a nontrivial 2-torsion element in $\mathrm{Pic}^0(E)$.

The statement about the dimension of $\mathcal{R}_g^{0,\mathrm{nb}}$ follows since the bielliptic locus has dimension $2g-2$. To prove its irreducibility, consider the map $f:\mathcal{R}_g^{0,\mathrm{nb}}\to\mathcal{R}_1$ associating to (C,η) the pair (E,η') as above. We study the fibres of this map. Consider the following obvious cartesian diagram defining $\mathcal{H}$, where $U\subset\mathrm{Sym}^{2g-2}(E)$ is the open subset consisting of reduced divisors:

$$\begin{array}{ccc} \mathcal{H} & \longrightarrow & \mathrm{Pic}^{g-1}(E) \\ \big\downarrow & & \big\downarrow{\scriptstyle \otimes 2} \\ U & \longrightarrow & \mathrm{Pic}^{2g-2}(E) \end{array}$$

By Riemann's existence theorem, $\mathcal{H}/\operatorname{Aut}(E)$ is in one-to-one correspondence with the two-to-one covers of E branched at $2g-2$ points. Then the fibre of f over (E,η') is isomorphic to $\mathcal{H}/\operatorname{Aut}(E)$ by what we said above. Now note that $\mathcal{H}$ is irreducible, since it fibres over (an open subset of) $\mathrm{Pic}^{g-1}(E)$ with fibres that are projective spaces of dimension $2g-3$. Hence also $\mathcal{H}/\operatorname{Aut}(E)$ is irreducible.

The irreducibility of $\mathcal{R}_g^{0,\mathrm{nb}}$ now follows from the irreducibility of $\mathcal{R}_1$. Actually $\mathcal{R}_1$ is irreducible and rational. To see this consider the irreducible family of elliptic curves $y^2=x(x-1)(x-\lambda)$, where $\lambda\in\mathbb{C}\setminus\{0,1\}$. The three non-trivial points of order two of the fibre $\mathcal{C}_\lambda$ over λ may be identified with the points $(0,0),(1,0)$ and $(\lambda,0)$. Moreover, the j-invariant of the fibres defines a six-to-one map $j:\mathbb{C}\setminus\{0,1\}\to\mathcal{M}_1$. Now consider on this family the two sections defined by the points $(0,0),(1,0)$ which stay fixed as λ varies. It is an exercise to prove that the irreducible family of two-marked elliptic curves we obtain in this way is isomorphic to the moduli space of pairs $(C,(\eta_1,\eta_2))$ where C is a smooth elliptic curve and (η_1,η_2) is an ordered pair of distinct non-trivial 2-torsion points of $\mathrm{Pic}^0(\mathrm{C})$. This moduli space is, in turn, isomorphic to the moduli space $\mathcal{M}_1^{(2)}$ of elliptic curves with a level 2 structure [9, Ex. 2.2.1]. Finally $\mathcal{M}_1^{(2)} \simeq \mathbb{C} \setminus \{0,1\}$ maps two-to-one dominantly to $\mathcal{R}_1$, via the map $(C,(\eta_1,\eta_2)) \mapsto (C,(\eta_1+\eta_2))$. This proves the statement. □

2.2. A result on linear systems on rational surfaces

We will need the following:

Theorem 2.3. (cf. [1, Cor. (4.6)]) *Let X be a smooth projective rational surface and δ a non-negative integer. Let $\mathcal{L}$ be a complete linear system on X such that*:

(i) *the general curve in $\mathcal{L}$ is smooth and irreducible*;

(ii) *the genus $p_a(\mathcal{L})$ of the general curve in $\mathcal{L}$ satisfies $p_a(\mathcal{L}) \geqslant \delta$*;

(iii) $\dim(\mathcal{L}) > 3\delta$;

(iv) *if $p_1, ..., p_\delta$ are general points of X, there is an element C of $\mathcal{L}$ singular at $p_1, ..., p_\delta$ such that for each irreducible component C' of C one has $K_X \cdot C' < 0$.*

Then, if $p_1, ..., p_\delta$ are general points of X and $\mathcal{L}(p_1^2, ..., p_\delta^2)$ is the subsystem of $\mathcal{L}$ formed by the curves singular at $p_1, ..., p_\delta$, one has:

(a) *the general curve in $\mathcal{L}(p_1^2, ..., p_\delta^2)$ is irreducible, has nodes at $p_1, ..., p_\delta$ and no other singularity*;

(b) $\dim(\mathcal{L}(p_1^2, ..., p_\delta^2)) = \dim(\mathcal{L}) - 3\delta$.

Proof. The proof of (a) is in [1]. As for (b), one has $\dim(\mathcal{L}(p_1^2, ..., p_\delta^2)) = \dim(\mathcal{L}) - 3\delta + \varepsilon$, with $\varepsilon \geqslant 0$. Consider the locally closed family of curves in $\mathcal{L}$ given by

$$\mathcal{F} := \bigcup_{p_1,\ldots,p_\delta} \mathcal{L}(p_1^2, ..., p_\delta^2),$$

where the union is made by varying $p_1, ..., p_\delta$ among all the δ-tuples of sufficiently general points of X. Of course

$$\dim(\mathcal{F}) = 2\delta + \dim(\mathcal{L}(p_1^2, ..., p_\delta^2)) = \dim(\mathcal{L}) - \delta + \varepsilon.$$

On the other hand, if C is a general element in $\mathcal{F}$, it has nodes at $p_1, ..., p_\delta$ and no other singularity by (a), hence the Zariski tangent space to $\mathcal{F}$ at C is the linear system $\mathcal{L}(p_1, ..., p_\delta)$ of curves in $\mathcal{L}$ containing $p_1, ..., p_\delta$. Since $p_1, ..., p_\delta$ are general, we have $\dim(\mathcal{L}(p_1, ..., p_\delta)) = \dim(\mathcal{L}) - \delta$, which proves that $\varepsilon = 0$. □

3. Irreducibility and unirationality of $\mathcal{R}_g^0$

In this section we prove a first part of Theorem 1.1, namely:

Proposition 3.1. *The locus $\mathcal{R}_g^0$ is irreducible and unirational for $g \geqslant 5$.*

The proof is inspired by the arguments in [1] and requires some preliminary considerations. In [1, Theorem (5.3)] the authors prove that some Hurwitz schemes $\mathcal{H}_{g,d}$ are unirational. Here we focus on the case $d=4$ and recall their construction.

Fix $g=2h+\varepsilon\geqslant 3$, with $0\leqslant\varepsilon\leqslant 1$. Then set $n=h+3+\varepsilon$ and

$$\delta=\binom{n-1}{2}-\binom{n-4}{2}-g=h+2\varepsilon.$$

Fix now $p,p_1,\ldots,p_\delta$ general points in the projective plane and consider the linear system $\mathcal{L}_n(p^{n-4},p_1^2,\ldots,p_\delta^2)$ of plane curves of degree n having multiplicity at least $n-4$ at p and multiplicity at least 2 at $p_1,\ldots,p_\delta$. As an application of Theorem 2.3, in [1, Cor. (4.7)] one proves that the dimension of $\mathcal{L}_n(p^{n-4},p_1^2,\ldots,p_\delta^2)$ is the expected one, i.e.,

$$\dim(\mathcal{L}_n(p^{n-4},p_1^2,\ldots,p_\delta^2))=\frac{n(n+3)}{2}-\frac{(n-4)(n-3)}{2}-3\delta=2h+9-\varepsilon,$$

and the general curve Γ in $\mathcal{L}_n(p^{n-4},p_1^2,\ldots,p_\delta^2)$ is irreducible, has an ordinary $(n-4)$-tuple point at p, nodes at $p_1,\ldots,p_\delta$, and no other singularity. The normalization C of Γ has genus g and it has a g^1_4, which is the pull-back to C of the linear series cut out on Γ by the pencil of lines through p.

Consider then the locally closed family of curves

$$\mathcal{H}:=\bigcup_{p_1,\ldots,p_\delta}\mathcal{L}_n(p^{n-4},p_1^2,\ldots,p_\delta^2),$$

where the union is made by varying $p_1,\ldots,p_\delta$ among all the δ-tuples of sufficiently general points of the plane. Then $\mathcal{H}$ is clearly irreducible, rational, of dimension $\dim(\mathcal{L}_n(p^{n-4},p_1^2,\ldots,p_\delta^2))+2\delta=4h+9+3\varepsilon$, and in [1] it is proved that the natural map $\mathcal{H}\dashrightarrow\mathcal{M}^1_{g,4}$ is dominant, so that $\mathcal{M}^1_{g,4}$ is unirational.

Proof of Proposition 3.1. To prove our result, we slightly modify the above argument from [1]. Let us fix $g\geqslant 5, n, \delta$ as above. Let $p,p_1,\ldots,p_\delta$ be general points in the plane.

Claim 3.2. *Consider the linear system $\mathcal{L}_{n-2}(p^{n-6},p_1^2,\ldots,p_\delta^2)$ of plane curves of degree $n-2$, having a point of multiplicity at least $n-6$ at p, and singular at $p_1,\ldots,p_\delta$. Then the dimension of $\mathcal{L}_{n-2}(p^{n-6},p_1^2,\ldots,p_\delta^2)$ is the expected one, i.e.,*

$$\dim(\mathcal{L}_{n-2}(p^{n-6},p_1^2,\ldots,p_\delta^2))=\frac{(n-2)(n+1)}{2}-\frac{(n-6)(n-5)}{2}-3\delta=2h-1-\varepsilon.$$

Proof of Claim 3.2. Assume first $g=5$, which implies $(h,\varepsilon,n,\delta)=(2,1,6,4)$. Then one has $\mathcal{L}_{n-2}(p^{n-6},p_1^2,\ldots,p_\delta^2)=\mathcal{L}_4(p_1^2,\ldots,p_4^2)$, which consists of all pairs of conics through $p_1,...,p_4$, and has dimension 2 as desired. We can assume next that $g\geqslant 6$, hence $h\geqslant 3$ and $n\geqslant 6$.

Let X be the blow-up of $\mathbb{P}^2$ at p. Note that the anticanonical system of X is very ample. Consider the linear system $\mathcal{L}$ proper transform on X of $\mathcal{L}_{n-2}(p^{n-6})$. One checks that X and $\mathcal{L}$ verify the hypotheses (i)–(iv) of Theorem 2.3. Indeed, (i) and (iv) are immediate, whereas (ii) and (iii) follow by standard computations and the fact that $h\geqslant 3$. Then the assertion follows by Theorem 2.3(b). □

Next fix two distinct lines r_1, r_2 through p and, for $1\leqslant i\leqslant 2$, two distinct points q_{ij}, both different from p, on the line r_i, with $1\leqslant j\leqslant 2$. Consider then the linear system $\mathcal{L}_n(p^{n-4},p_1^2,...,p_\delta^2;[q_{11},q_{12},q_{21},q_{22}])$ consisting of all curves in $\mathcal{L}_n(p^{n-4},p_1^2,...,p_\delta^2)$ whose intersection multiplicity with r_i at q_{ij} is at least 2, for $1\leqslant i,j\leqslant 2$.

Claim 3.3. *The linear system $\mathcal{L}_n(p^{n-4},p_1^2,...,p_\delta^2;[q_{11},q_{12},q_{21},q_{22}])$ has the expected dimension, i.e.,*

$$\begin{aligned}\dim(\mathcal{L}_n(p^{n-4},p_1^2,...,p_\delta^2;[q_{11},q_{12},q_{21},q_{22}])) &= \frac{n(n+3)}{2}-\frac{(n-4)(n-3)}{2}-3\delta-8\\ &= 2h+1-\varepsilon,\end{aligned}$$

and the general curve in $\mathcal{L}_n(p^{n-4},p_1^2,...,p_\delta^2;[q_{11},q_{12},q_{21},q_{22}])$ is irreducible, has a point of multiplicity $n-4$ at p, has nodes at $p_1,...,p_\delta$ and no other singularity, and is tangent at r_i in q_{ij}, for $1\leqslant i,j\leqslant 2$.

Proof of Claim 3.3. Let X be the blow-up of the plane at p, at the points $q_{i,j}$ and at the infinitely near points to q_{ij} along the line r_i, for $1\leqslant i,j\leqslant 2$. Note that the anticanonical system of X has a fixed part consisting of the strict transforms R_1, R_2 of r_1, r_2 plus the exceptional divisor E over p, and a movable part consisting of the pull back to X of the linear system of the lines in the plane.

Let $\mathcal{L}$ be the strict transform on X of $\mathcal{L}_n(p^{n-4};[q_{11},q_{12},q_{21},q_{22}])$, the linear system of curves of degree n with multiplicity at least $n-4$ at p and whose intersection multiplicity with r_i at q_{ij} is at least 2, for $1\leqslant i,j\leqslant 2$. One has

$$\dim(\mathcal{L})=\frac{n(n+3)}{2}-\frac{(n-4)(n-3)}{2}-8$$

and an application of Bertini's theorem shows that the general curve in $\mathcal{L}$ is smooth and irreducible and its genus is

$$p_a(\mathcal{L})=\binom{n-1}{2}-\binom{n-4}{2}\geqslant \delta.$$

Moreover

$$\dim(\mathcal{L})-3\delta=2h+1-\varepsilon>0.$$

Hence the linear system $\mathcal{L}(p_1^2,\ldots,p_\delta^2)$ of curves in $\mathcal{L}$ singular at $p_1,\ldots,p_\delta$ has dimension

$$\dim(\mathcal{L}(p_1^2,\ldots,p_\delta^2)) \geqslant 2h+1-\varepsilon.$$

We claim that $\mathcal{L}(p_1^2,\ldots,p_\delta^2)$ does not have R_1, R_2 or E in its fixed locus. Indeed, if E is in this fixed locus, then clearly also R_1 and R_2 split off $\mathcal{L}(p_1^2,\ldots,p_\delta^2)$. If R_1 is in the fixed locus, then by symmetry, also R_2 is in the fixed locus. So, suppose by contradiction that R_1, R_2 are in the fixed locus. Then, after removing them from $\mathcal{L}(p_1^2,\ldots,p_\delta^2)$ we would remain with $\mathcal{L}'$, the pull-back to X of $\mathcal{L}_{n-2}(p^{n-6},p_1^2,\ldots,p_\delta^2)$, which, by Claim 3.2, has dimension $2h-1-\varepsilon$. Hence we would have

$$2h-1-\varepsilon = \dim(\mathcal{L}_{n-2}(p^{n-6},p_1^2,\ldots,p_\delta^2)) = \dim(\mathcal{L}(p_1^2,\ldots,p_\delta^2)) \geqslant 2h+1-\varepsilon,$$

a contradiction.

Let now C be a general curve in $\mathcal{L}(p_1^2,\ldots,p_\delta^2)$. The above argument implies that no component of C is a fixed curve of the anticanonical system of X. Then for any irreducible component C' of C one has $K_X \cdot C' < 0$. In conclusion, $\mathcal{L}$ verifies the hypotheses (i)–(iv) of Theorem 2.3, and Claim 3.3 follows by the latter theorem. □

We now end the proof of Proposition 3.1. Consider the locally closed family of curves

$$\mathcal{G} := \bigcup_{p_1,\ldots,p_\delta,r_1,r_2,q_{11},q_{12},q_{21},q_{22}} \mathcal{L}_n(p^{n-4},p_1^2,\ldots,p_\delta^2;[q_{11},q_{12},q_{21},q_{22}])$$

where the union is made by varying $p_1,\ldots,p_\delta$ among all δ-tuples of general distinct points of X, r_1, r_2 among all pairs of distinct lines through p and $q_{ij} \neq p$ among all pairs of distinct points of r_i, for $1 \leqslant i,j \leqslant 2$.

Of course $\mathcal{G}$ is irreducible and rational, and we have a map $\alpha: \mathcal{G} \dashrightarrow \mathcal{R}_g^0$ which sends a general curve $\Gamma \in \mathcal{G}$ to (C,η), where C is the normalization of Γ, and $\eta = \mathcal{O}_C(q_{11}+q_{12}-q_{21}-q_{22})$, where, by abusing notation, we denote by q_{ij} their inverse images in C, for $1 \leqslant i,j \leqslant 2$. Since $\mathcal{H} \dashrightarrow \mathcal{M}_{g,4}^1$ is dominant by [1, §5], then α is also dominant by Lemma 2.1. This proves the proposition. □

4. Dimension of $\mathcal{R}_g^0$

In this section we finish the proof of Theorem 1.1 with the:

Proposition 4.1. *The irreducible locus $\mathcal{R}_g^0$ has dimension $2g+1$ if $g \geqslant 5$.*

Proof. Let $\mathcal{H}_{g,4}$ denote the Hurwitz scheme parametrizing isomorphism classes of genus g degree 4 covers of $\mathbb{P}^1$. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{G} & \overset{\varphi}{\dashrightarrow} & \mathcal{H}_{g,4} \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathcal{R}_g^0 & \xrightarrow{\psi} & \mathcal{M}_{g,4}^1, \end{array}$$

where π and ψ are the forgetful maps, α is the dominant map from the last part of the proof of Proposition 3.1 and φ maps a general curve $\Gamma\in\mathcal{G}$ to the degree 4 cover defined by $2(q_{11}+q_{12})\sim 2(q_{21}+q_{22})$, using the notation of the proof of Proposition 3.1. Note that ψ is finite, whence the dimension of $\mathcal{R}_g^0$ equals the dimension of the image of $\pi\circ\varphi$.

The image of φ coincides with the locus $\mathcal{D}\subset\mathcal{H}_{g,4}$ parametrizing covers with two pairs of distinct ramification points each over the same branch point. By Riemann's existence theorem, $\mathcal{D}$ has codimension 2 in $\mathcal{H}_{g,4}$ (whence $\dim(\mathcal{D})=2g+1$). Since $\mathcal{G}$ is irreducible (cf. the proof of Proposition 3.1), so is $\mathcal{D}$. Moreover, as the bielliptic locus in $\mathcal{M}_g$ has dimension $2g-2$ and each bielliptic curve has a one-dimensional family of g_4^1s, the locus in $\mathcal{H}_{g,4}$ with bielliptic domain curve has dimension $2g-1$. Thus, the general element in the image of $\pi\circ\varphi$ is not bielliptic, whence the general element (C,η) in the image of α has Prym-canonical image birational to C, by Corollary 2.2 (and necessarily singular, by Lemma 2.1(ii)). It follows that the fibre over C of the restriction of π to $\mathcal{D}$ is finite. Indeed, C has finitely many preimages (C,η) in the image of α, and the Prym-canonical model of each of those has finitely many singular points, determining by Lemma 2.1(ii) only finitely many covers in $\mathcal{D}\subset\mathcal{H}_{g,4}$ mapping to C by π. Thus, the restriction of π to $\mathcal{D}$ is generically finite, whence the image of $\pi\circ\varphi$ has dimension $2g+1$. $\square$

5. Proof of Proposition 1.2 and some examples

Consider again the locus $\mathcal{D}\subset\mathcal{H}_{g,4}$ from the proof of Proposition 4.1 parametrizing isomorphism classes of covers with two pairs of distinct ramification points each over a single branch point. By Riemann's existence theorem again, the general point in $\mathcal{D}$ corresponds to a cover with only two such branch points. By Lemma 2.1(ii), if the domain curve has only one g_4^1, which is automatic if $g\geqslant 10$, then the Prym-canonical model of such a curve has precisely two nodes. It cannot have fewer singularities by Lemma 2.1. Thus, Proposition 1.2 is proved for $g\geqslant 10$.

Instead of embarking in a more refined treatment for $g\leqslant 9$, we note that certain curves on Enriques surfaces provide examples, for any genus $g\geqslant 5$, of curves with two-nodal Prym-canonical models, thus finishing the proof of Proposition 1.2:

Example 5.1. The general Enriques surface S contains no smooth rational curves [3] and contains smooth elliptic curves E_1, E_2, E_3 with $E_i \cdot E_j = 1$ for $i \neq j$ (and $E_i^2 = 0$ by adjunction), for $1 \leqslant i, j \leqslant 3$, cf. e.g. [7, Thm. 3.2] or [8, IV.9.E, p. 273]. It also contains a smooth elliptic curve $E_{1,2}$ such that $E_{1,2} \cdot E_1 = E_{1,2} \cdot E_2 = 2$, and $E_{1,2} \cdot E_3 = 1$, cf. e.g. [7, Thm. 3.2] or [8, IV.9.B, p. 270]. In particular, none of the numerical equivalence classes of $E_1, E_2, E_3, E_{1,2}$ are divisible in $\mathrm{Num}(S)$.

Consider, for any $g \geqslant 5$, the line bundle

$$H_g := \begin{cases} \mathcal{O}_S(\frac{g-2}{2}E_1 + E_2 + E_3), & g \text{ even} \\ \mathcal{O}_S(\frac{g-1}{2}E_1 + E_{1,2}), & g \text{ odd.} \end{cases}$$

The absence of smooth rational curves yields that H_g is nef. As $H_g^2 = 2g-2$, all curves in $|H_g|$ have arithmetic genus g. Moreover, we claim that $\phi(H_g) = E_1 \cdot H_g = 2$ (see the introduction for the definition of ϕ) and that the only numerical class computing $\phi(H_g)$ is E_1. Indeed, if g is even (resp., odd), then $E_1 \cdot H_g = 2$, $E_2 \cdot H_g = E_3 \cdot H_g = \frac{g}{2} \geqslant 3$ (resp., $E_1 \cdot H_g = 2$, $E_{1,2} \cdot H_g = g-1 \geqslant 4$), and if E is any nonzero effective divisor not numerically equivalent to any of E_1, E_2, E_3 (resp., $E_1, E_{1,2}$), then $E \cdot E_1 > 0$, $E \cdot E_2 > 0$ and $E \cdot E_3 > 0$ (resp., $E \cdot E_1 > 0$ and $E \cdot E_{1,2} > 0$) by [10, Lemma 2.1], so that $E \cdot H_g \geqslant \frac{g-2}{2} + 2 = \frac{g}{2} + 1 \geqslant 4$ (resp., $E \cdot H_g \geqslant \frac{g-1}{2} + 1 = \frac{g+1}{2} \geqslant 3$).

By [8, Prop. 4.5.1, Thm. 4.6.3, Prop. 4.7.1, Thm. 4.7.1] the complete linear system $|H_g|$ is therefore base point free and defines a morphism φ_{H_g} that is birational onto a surface with only double lines as singularities; the double lines are the images of curves computing $\phi(H_g)$, which, by what we said above, are E_1 and E_1', the only member of $|E_1 + K_S|$. Thus, the image of φ_{H_g} is a surface with precisely two double lines $\varphi_{H_g}(E_1)$ and $\varphi_{H_g}(E_1')$ as singularities. Therefore, φ_{H_g} maps a general smooth $C \in |H|$ to a curve with precisely two nodes. Since φ_{H_g} restricted to C is the Prym-canonical map associated to $\eta := \mathcal{O}_C(K_S)$ by [8, Cor. 4.1.2], a general smooth curve C in $|H_g|$ together with η is an example of a Prym curve of any genus $g \geqslant 5$ with two-nodal Prym-canonical model.

We prove in [6, Thm. 2] that the general element in $\mathcal{R}_g^0$ is obtained in this way precisely for $5 \leqslant g \leqslant 8$.

Similar examples for odd $g \geqslant 7$ are obtained from the line bundle $H_g := \mathcal{O}_S(\frac{g-1}{2}E_1 + 2E_2)$ or $H_g := \mathcal{O}_S(\frac{g-1}{2}E_1 + 2E_2 + K_S)$, but (again by [6, Thm. 2]) the general element in $\mathcal{R}_g^0$ is not obtained in this way.

We conclude with an example of curves of genus 5 on an Enriques surface with 4-nodal Prym-canonical models and a result that will be used in [6]:

Example 5.2. With the same notation as in the previous example, set $H := \mathcal{O}_S(2E_1 + 2E_2 + K_S)$. Then $H^2 = 8$, so that any curve in $|H|$ has arithmetic genus 5. Moreover, $\phi(H) = 2$ and one easily checks that E_1 and E_2 are the only numerical equivalence classes computing $\phi(H)$. As in the previous example, the complete linear system $|H|$ is base point free and defines a morphism φ_H that is birational onto a surface with precisely four double lines as singularities, namely the images of E_1, E_2, E_1' and E_2', where E_i' is the only member of $|E_i+K_S|$, $i=1,2$. Thus φ_H maps a general smooth $C\in|H|$ to a curve with precisely four nodes, so that, again by [8, Cor. 4.1.2], the pairs $(C,\mathcal{O}_C(K_S))$ are genus 5 Prym curves with 4-nodal Prym-canonical models.

Also note that for any smooth $C\in|H|$, we have

$$\omega_C\simeq\mathcal{O}_C(E_1+E_2)^{\otimes 2}\simeq\mathcal{O}_C(E_1+E_2+K_S)^{\otimes 2},$$

whence C has two autoresidual g^1_4s, namely $|\mathcal{O}_C(E_1+E_2)|$ and $|\mathcal{O}_C(E_1+E_2+K_S)|$, and their difference is $\mathcal{O}_C(K_S)$. (A complete linear system $|D|$ is called *autoresidual* if D is a theta-characteristic, that is, $2D\sim\omega_D$.) Thus, $(C,\mathcal{O}_C(K_S))$ belongs to the locus in $\mathcal{R}_5$ consisting of Prym curves (C,η) carrying a theta-characteristic θ such that $h^0(\theta)=h^0(\theta+\eta)=2$. The next result shows that this is a general phenomenon in $\mathcal{R}^0_5$.

Proposition 5.3. *The locus in $\mathcal{R}^0_5$ of curves with 4-nodal Prym-canonical model is an irreducible unirational divisor whose closure in $\mathcal{R}_5$ coincides with the closure of the locus of Prym curves (C,η) carrying a theta-characteristic θ with $h^0(\theta)=h^0(\theta+\eta)=2$.*

Proof. Let us denote by $\mathcal{D}^0_5$ the locus of curves in $\mathcal{R}^0_5$ with 4-nodal Prym-canonical model, which is nonempty by the previous example. Let $\mathcal{V}$ denote the locus of curves of type $(4,4)$ on $\mathbb{P}^1\times\mathbb{P}^1$ with 4 nodes lying on the 4 nodes of a "square" configuration of two fibres of each projection to $\mathbb{P}^1$. We will prove that $\mathcal{V}$ is irreducible of dimension 16 and that there is a birational morphism

$$f:\mathcal{D}^0_5\longrightarrow\mathcal{V}':=\mathcal{V}/\operatorname{Aut}(\mathbb{P}^1\times\mathbb{P}^1).$$

To define f, let $(C,\eta)\in\mathcal{D}^0_5$. By Lemma 2.1 there are four pairs of distinct points (p,q), (x,y), (p',q') and (x',y') on C, each identified by the Prym-canonical map $\varphi:C\to\mathbb{P}^3$, such that

$$2(p+q)\sim 2(x+y),\quad 2(p'+q')\sim 2(x'+y')\quad\text{and}\tag{1}$$

$$\eta\sim p+q-x-y\sim x'+y'-p'-q'.\tag{2}$$

In particular, we get that

$$p+q+p'+q'\sim x+y+x'+y',\tag{3}$$

thus defining a base point free g^1_4 on C, which we call ℓ_1. We let $\mathcal{L}_1$ on C be the corresponding line bundle. Since there exists a pencil of hyperplanes in $\mathbb{P}^3$ through any two of the four nodes of $\Gamma := \varphi(C)$, we see that

$$h^0(\omega_C(\eta) - \mathcal{L}_1) = h^0\big(\omega_C(\eta)(-p-q-p'-q')\big) = 2. \tag{4}$$

We claim that

$$h^0(\omega_C(\eta) - 2\mathcal{L}_1) = 0. \tag{5}$$

Indeed, if not, we would have $\omega_C(\eta) \simeq 2\mathcal{L}_1$, which together with (4) would yield that $\Gamma \subset \mathbb{P}^3$ is contained in a quadric cone Q, with the pullback of the ruling of the cone cutting ℓ_1 on C. Let $\widetilde{Q}$ be the desingularization of Q. Then $\widetilde{Q} \simeq \mathbb{F}_2$. Since ℓ_1 is base point free, Γ does not pass through the vertex of Q, so that we may consider Γ as a curve in $\widetilde{Q}$. Denote by σ the minimal section of $\mathbb{F}_2$ (thus, $\sigma^2 = -2$), which is contracted to the vertex of Q, and by $\mathfrak{f}$ the class of the fibre of the ruling. Then, since $\Gamma \cdot \mathfrak{f} = 4$ and $\Gamma \cdot \sigma = 0$, we get that $\Gamma \sim 4\sigma + 8\mathfrak{f}$. In particular, $\omega_\Gamma \simeq \mathcal{O}_\Gamma(K_{\widetilde{Q}} + \Gamma)) \simeq \mathcal{O}_\Gamma(2\sigma + 4\mathfrak{f}) \simeq \mathcal{O}_\Gamma(4\mathfrak{f})$. Thus, from (3) we obtain

$$\omega_C \simeq \varphi^*(\omega_\Gamma)(-p-q-x-y-p'-q'-x'-y') \simeq \mathcal{O}_C(4\mathcal{L}_1 - 2\mathcal{L}_1) \simeq \mathcal{O}_C(2\mathcal{L}_1),$$

yielding $\eta = 0$, a contradiction. This proves (5).

The relations (4) and (5) imply that $\Gamma \subset \mathbb{P}^3$ is contained in a smooth quadric surface $Q \simeq \mathbb{P}^1 \times \mathbb{P}^1$. The first ruling is defined by the pencil ℓ_1, whereas the second is defined by the pencil $\ell_2 = |\mathcal{L}_2|$, where $\mathcal{L}_2 := \omega_C(\eta) - \mathcal{L}_1 = \omega_C(\eta)(-p-q-p'-q')$ by (4). The curve Γ is of type $(4,4)$ on Q, with four nodes. Since $\omega_\Gamma \simeq \omega_{\mathbb{P}^1 \times \mathbb{P}^1}(C) \simeq \mathcal{O}_\Gamma(2,2)$, we see that $\varphi^*(\omega_\Gamma) \simeq (\omega_C(\eta))^{\otimes 2} \simeq \omega_C^{\otimes 2}$. Thus,

$$\omega_C \simeq \omega_C^{\otimes 2}(-p-q-x-y-p'-q'-x'-y'),$$

whence

$$\omega_C \simeq \mathcal{O}_C(p+q+x+y+p'+q'+x'+y'). \tag{6}$$

Combining with (2), we find that

$$\mathcal{L}_2 \simeq \omega_C(\eta)(-p-q-p'-q') \simeq \mathcal{O}_C(p+q+x'+y') \simeq \mathcal{O}_C(p'+q'+x+y). \tag{7}$$

The relations (3) and (7) tell us that the four nodes of Γ lie on two pairs of fibres of each ruling of $\mathbb{P}^1 \times \mathbb{P}^1$, thus showing that $\Gamma \in \mathcal{V}$. Of course this is all well-defined up to automorphisms of $\mathbb{P}^1 \times \mathbb{P}^1$, so we see that the construction associates to (C, η) an element in $\mathcal{V}'$, which we define to be the image of (C, η) by f.

This defines the map f, and in particular shows that $\mathcal{V}$ is nonempty. We also note for later use that $\omega_C \simeq 2\mathcal{L}_1 \simeq 2\mathcal{L}_2$, so that $\mathcal{D}_5^0$ is contained in the locus of Prym curves (C, η) carrying a theta-characteristic θ with $h^0(\theta) = h^0(\theta + \eta) = 2$, which we henceforth call T_5. Moreover, via the forgetful map $\mathcal{R}_5 \to \mathcal{M}_5$, the locus T_5 maps to the locus of curves with two (complete) autoresidual g^1_4s, which we call $\mathcal{B}_5$.

We next prove that $\mathcal{V}$ is irreducible rational of dimension 16.

For any $X\in\mathcal{V}$, let $\nu:C\to X$ be the normalization; C has genus 5. If z_i, $i=1,2,3,4$, are the nodes of X, then the complete linear system

$$|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(X)\otimes\mathcal{J}_{z_1}^2\otimes\mathcal{J}_{z_2}^2\otimes\mathcal{J}_{z_3}^2\otimes\mathcal{J}_{z_4}^2|$$

has dimension 12, as expected. Indeed, letting r be its dimension, we clearly have $r\geqslant 12$; on the other hand, this complete linear system induces a g_{16}^{r-1} on C, whence $r-1\leqslant 11$ by Riemann-Roch. It follows that $\mathcal{V}$ is birational to $\mathbb{P}^{12}\times(\mathrm{Sym}^2(\mathbb{P}^1))^2$ (because of the freedom of varying the four lines in the square configuration), in particular it is irreducible rational of dimension $12+4=16$.

We now define the inverse of f. Given a curve $X\in\mathcal{V}$, let $\mathcal{L}_1$ and $\mathcal{L}_2$ be the line bundles of degree 4 on C defined by the pullbacks of the two rulings on $\mathbb{P}^1\times\mathbb{P}^1$. By the special position of the 4 nodes of X, the four pairs of points C lying above the four nodes of X, say (p,q), (x,y), (p',q') and (x',y'), satisfy

$$\mathcal{L}_1\simeq\mathcal{O}_C(p+q+p'+q')\simeq\mathcal{O}_C(x+y+x'+y'),$$
$$\mathcal{L}_2\simeq\mathcal{O}_C(p+q+x'+y')\simeq\mathcal{O}_C(x+y+p'+q'),$$

in particular, $\eta:=\mathcal{L}_1-\mathcal{L}_2$ is 2-torsion. Moreover, one can easily verify that $\omega_C(\eta)\simeq\mathcal{L}_1+\mathcal{L}_2$. Thus, the normalization $\nu:C\to X\subset\mathbb{P}^1\times\mathbb{P}^1$ followed by the embedding of $\mathbb{P}^1\times\mathbb{P}^1$ as a quadric in $\mathbb{P}^3$ induces the Prym-canonical map associated to $\omega_C(\eta)$, so that (C,η) has a 4-nodal Prym-canonical image. One readily checks that this map is the inverse of the map f defined above. Thus, we have proved that $\mathcal{D}_5^0$ is irreducible of dimension $\dim\mathcal{V}/(\mathrm{Aut}(\mathbb{P}^1\times\mathbb{P}^1))=16-6=10$.

We have left to prove that the closure of $\mathcal{D}_5^0$ in $\mathcal{R}_5$ coincides with the closure of T_5. We proved above that $\mathcal{D}_5^0$ is contained in T_5 and that the latter maps, via the finite forgetful map $\mathcal{R}_5\to\mathcal{M}_5$, to the locus $\mathcal{B}_5$ of curves with two autoresidual g_4^1s, which is irreducible of dimension 10 by [11, Thm. 2.10]. Below we give a direct proof of the latter fact, which also proves that the general member of $\mathcal{B}_5$ carries exactly two g_4^1s, equivalently two theta characteristics θ and θ' such that $h^0(\theta)=h^0(\theta')=2$. It will follow that there is an inverse rational map $\mathcal{B}_5\dashrightarrow T_5$ mapping C to $(C,\theta-\theta')$, proving that also T_5 is irreducible of dimension 10. Its closure must therefore coincide with the closure of $\mathcal{D}_5^0$, finishing the proof of the proposition.

So let C be a smooth, irreducible curve of genus 5 and consider its canonical embedding $C\subset\mathbb{P}^4$. Given $\xi=|D|$ a (complete) g_4^1 on C, the divisors in ξ span planes which sweep out a quadric Q_ξ of rank $r<5$. If ξ is not autoresidual, then Q_ξ has rank $r=4$ and it has another 1-dimensional system of planes which cut out on C the divisors of $\xi'=|K_C-D|$. In this case $Q_\xi=Q_{\xi'}$. Hence ξ is autoresidual if and only if Q_ξ has rank 3, and therefore it possesses only one 1-dimensional family of planes. This means that the homogeneous ideal of a curve in $\mathcal{B}_5$ in its canonical embedding contains two distinct rank 3 quadrics. Hence the general curve C in $\mathcal{B}_5$ is obtained by intersecting two general rank 3 quadrics in $\mathbb{P}^4$ with another general quadric. Note that the two rank 3 quadrics cut out a Del Pezzo surface S with 4 nodes, hence C is a general quadric section on S. The two autoresidual g^1_4 on C are cut out on C by the conics of the two pencils on S with base points two of the nodes.

From this description it follows that $\mathcal{B}_5$ is irreducible, 10-dimensional and that its general member contains precisely two autoresidual g^1_4s. Indeed, consider the $\mathbb{P}^{14}$ parametrizing all quadrics in $\mathbb{P}^4$. The locus $\mathcal{X}$ of quadrics of rank $r\leqslant 3$ is non-degenerate and has dimension 11. The net of quadrics defining a general curve C in $\mathcal{B}_5$ corresponds to a plane in $\mathbb{P}^{14}$ containing a general secant line to $\mathcal{X}$ (which, by its generality, contains only two points in $\mathcal{X}$), and an easy count of parameters shows that these planes clearly fill up a variety of dimension 34. Modding out by the 24-dimensional group of projective transformations of $\mathbb{P}^4$, we get dimension 10 for $\mathcal{B}_5$. $\square$

Remark 5.4. Denote, as in the last proof, by $\mathcal{D}^5_0$ the locus of Prym curves (C,η) carrying a theta-characteristic θ with $h^0(\theta)=h^0(\theta+\eta)=2$. By [4, Prop. 7.3 and Thm. 7.4] the locus $\mathcal{D}^5_0$ maps, via the Prym map $\mathcal{P}_5:\mathcal{R}_5\to\mathcal{A}_4$, to the irreducible divisor θ_{null} of principally polarized abelian varieties whose theta-divisor has a singular point at a 2-torsion point, and moreover the general member of $\mathcal{P}_5(\mathcal{D}^5_0)$ has precisely one ordinary double point, cf. [4, Pf. of Prop. 7.5]. It would be interesting to know if $\mathcal{D}^5_0$ dominates θ_{null}.

By [4, Prop. 7.3] one knows that the closure of $\mathcal{P}_5^{-1}(\theta_{\mathrm{null}})$ is the closure of the locus of Prym curves (C,η) carrying a theta-characteristic θ such that $h^0(\theta)+h^0(\theta+\eta)$ is even, which *properly* contains $\mathcal{D}^5_0$.

Remark 5.5. By contrast, if we consider the adjoint line bundle of the one in Example 5.2, that is, $H':=\mathcal{O}_S(2E_1+2E_2)$, then by [8, Prop. 4.1.2, Thm. 4.7.1, (F) p. 277] the morphism $\varphi_{H'}$ defined by $|H'|$ is of degree 2 onto a quartic Del Pezzo surface. In particular, $\varphi_{H'}$ maps any smooth $C\in|H|$ doubly onto an elliptic quartic curve in $\mathbb{P}^3$. Hence, the Prym curve $(C,\mathcal{O}_C(K_S))$ belongs to the locus $\mathcal{R}_5^{0,\mathrm{nb}}$ described in Corollary 2.2.

References

1. Arbarello, E. and Cornalba, M., Footnotes to a paper of Beniamino Segre, *Math. Ann.* **256** (1981), 341–362. MR0626954
2. Ballico, E. and Del Centina, A., Ramification points of double coverings of curves and Weierstrass points, *Ann. Mat. Pura Appl.* **177** (1999), 293–313. MR1747636

3. Barth, W. and Peters, C., Automorphisms of Enriques surfaces, *Invent. Math.* **73** (1983), 383–412. MR0718937
4. Beauville, A., Prym varieties and the Schottky problem, *Invent. Math.* **41** (1977), 149–196. MR0572974
5. Brodmann, M. and Schenzel, P., Arithmetic properties of projective varieties of almost minimal degree, *J. Algebraic Geom.* **16** (2007), 347–400. MR2274517
6. Ciliberto, C., Dedieu, T., Galati, C. and Knutsen, A. L., Moduli of curves on Enriques surfaces, *Adv. Math.* **365** (2020), 10.1016/j.aim.2020.107010. MR4065262
7. Cossec, F. R., On the Picard group of Enriques surfaces, *Math. Ann.* **271** (1985), 577–600. MR0790116
8. Cossec, F. R. and Dolgachev, I. V., *Enriques Surfaces. I*, Progress in Mathematics **76**, Birkhäuser Boston, Inc., Boston, MA, 1989. MR0986969
9. Harris, J. and Morrison, I., *Moduli of Curves*, Graduate Text in Mathematics **187**, Springer, 1998. MR1631825
10. Knutsen, A. L. and Lopez, A. F., A sharp vanishing theorem for line bundles on $K3$ or Enriques surfaces, *Proc. Amer. Math. Soc.* **135** (2007), 3495–3498. MR2336562
11. Knutsen, A. L., Lelli-Chiesa, M. and Verra, A., Moduli of non-standard Nikulin surfaces in low genus, *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, to appear. 10.2422/2036-2145.201802_018. arXiv:1802.01201.
12. Lange, H. and Sernesi, E., Quadrics containing a Prym-canonical curve, *J. Algebraic Geom.* **5** (1996), 387–399. MR1374713

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