Ark. Mat., **57** (2019), 157–189
DOI: 10.4310/ARKIV.2019.v57.n1.a9

Algebraic cycles and triple $K3$ burgers

Robert Laterveer

Abstract. We consider surfaces of geometric genus 3 with the property that their transcendental cohomology splits into 3 pieces, each piece coming from a $K3$ surface. For certain families of surfaces with this property, we can show there is a similar splitting on the level of Chow groups (and Chow motives).

1. Introduction

This note is about a class of surfaces which we propose to call *triple $K3$ burgers*. These are complex smooth projective surfaces S of general type of geometric genus 3, with the property that there exist 3 $K3$ surfaces X_j such that the transcendental cohomology $H^2_{tr}(S)$ splits

$$H^2_{tr}(S)\cong H^2_{tr}(X_0)\oplus H^2_{tr}(X_1)\oplus H^2_{tr}(X_2). \tag{1}$$

(The precise definition of triple $K3$ burgers is more restrictive, cf. Definition 3.1.)

The crystal ball of the Bloch–Beilinson–Murre conjectures [24], [25], [58], [35], [34] predicts that relation (1) also holds on the level of Chow groups (and provided the Hodge conjecture is true, the Chow motive of S should be of abelian type, in the sense of [49]). The main result of this note provides a verification of this prediction in certain cases:

Theorem. (=Theorem 5.1) *Let S be a triple $K3$ burger. Assume that either*
(i) $K_S^2=2$, *or*
(ii) $K_S^2=3$ *and the canonical map of S is base point free.*

Then there is an isomorphism (induced by a correspondence)

$$A^2_{hom}(S)\xrightarrow{\cong} A^2_{hom}(X_0)\oplus A^2_{hom}(X_1)\oplus A^2_{hom}(X_2),$$

where the X_j are the associated $K3$ surfaces.

(Here $A^2_{hom}()$ denotes the Chow group of 0–cycles of degree 0 with rational coefficients.)

In each of the cases of Theorem 5.1, these surfaces do exist (in case (i), they form a family of dimension at least 6; in case (ii) the moduli dimension is 4).

It is *not* a coincidence that the surfaces of Theorem 5.1 lie on or close to the Noether line $K^2=2p_g-4$. Indeed (as is known since the fundamental work of Horikawa [15], [16], [17], [18], [19]), the canonical model of a general type surface on or close to the Noether line admits a neat description as complete intersection in a certain weighted projective space. Thanks to such a description, surfaces as in Theorem 5.1 fit in nicely behaved universal families. Then, one can apply the alchemy of Voisin's method of "spread" [54], [57], [58] to transmute the base metal of the homological relation (1) into the pure gold of a rational equivalence.

We also prove (Subsection 6.1) that a triple $K3$ burger S as in Theorem 5.1 admits a canonical 0–cycle $\mathfrak{o}_S\in A^2(S)$, such that there is a splitting

$$A^2(S)=\mathbb{Q}[\mathfrak{o}_S]\oplus A^2_{hom}(S).$$

The cycle $\mathfrak{o}_S$ has the property that the intersection of certain divisors is proportional to $\mathfrak{o}_S$ (Proposition 6.8). Another characterization of $\mathfrak{o}_S$ is as follows (Proposition 6.4): for any positive integer k, the cycle $k\mathfrak{o}_S$ is the unique degree k 0–cycle z for which the effective orbit O_z has dimension $\geq k$. These results are based on similar results for the canonical 0–cycle of a $K3$ surface [21], [3], [58], [56].

In a sense, the present note is a sequel to [30], which dealt with certain surfaces of geometric genus $p_g=2$. The surfaces S of [30] are also studied in [14] and [37]; they have the property that their transcendental cohomology decomposes

$$H^2_{tr}(S)\cong H^2_{tr}(X_0)\oplus H^2_{tr}(X_1),$$

where X_0, X_1 are $K3$ surfaces. In [30], using arguments very similar to the present note, I proved there exists a similar splitting on the level of Chow groups.

Several open questions remain, which I hope someone will be able to answer (cf. Section 7).

Conventions. In this article, the word *variety* will refer to a reduced irreducible scheme of finite type over $\mathbb{C}$. A *subvariety* is a (possibly reducible) reduced subscheme which is equidimensional.

By default, all Chow groups will be with rational coefficients: we will denote by $A_j(X)$ the Chow group of j–dimensional cycles on X with $\mathbb{Q}$–coefficients; for X smooth of dimension n the notations $A_j(X)$ and $A^{n-j}(X)$ are used interchangeably. When dealing with Chow groups with integral coefficients, we will make this clear by writing $A_j(X)_{\mathbb{Z}}$.

The notations $A^j_{hom}(X)$, $A^j_{AJ}(X)$ will be used to indicate the subgroups of homologically trivial, resp. Abel–Jacobi trivial cycles. For a morphism $f\colon X\to Y$, we will write $\Gamma_f\in A_*(X\times Y)$ for the graph of f. The contravariant category of Chow motives (i.e., pure motives with respect to rational equivalence as in [43], [35]) will be denoted $\mathcal{M}_{\rm rat}$.

We use $H^j(X)$ to indicate singular cohomology $H^j(X,\mathbb{Q})$, and $H_j(X)$ to indicate Borel–Moore homology $H_j^{BM}(X,\mathbb{Q})$.

2. Preliminaries

2.1. Relative Künneth projectors

Lemma 2.1. *Let $\mathcal{S}\to B$ be as in Notation 3.13. There exist relative correspondences*

$$\pi_0^{\mathcal{S}},\quad \pi_2^{\mathcal{S}},\quad \pi_4^{\mathcal{S}}\quad \in A^2(\mathcal{S}\times_B\mathcal{S}),$$

with the property that for each $b\in B$, the restriction

$$\pi_i^{\mathcal{S}}|_b:=\pi_i^{\mathcal{S}}|_{S_b\times S_b}\quad \in H^4(S_b\times S_b)$$

is the ith Künneth component. Moreover,

$$(\pi_2^{\mathcal{S}}|_b)_*=\mathrm{id}\colon\quad A^2_{hom}(S_b)\ \longrightarrow\ A^2_{hom}(S_b).$$

Proof. This is well–known, and holds more generally for any family of surfaces with $H^1(S_b)=0$. Let $H\in A^1(\mathcal{S})$ be a relatively ample divisor, and let $d:=\deg(H^2|_{S_b})$. One defines

$$\begin{aligned}\pi_0^{\mathcal{S}}&:=\frac{1}{d}(p_1)^*(H^2),\\ \pi_4^{\mathcal{S}}&:=\frac{1}{d}(p_2)^*(H^2),\\ \pi_2^{\mathcal{S}}&:=\Delta_{\mathcal{S}}-\pi_0^{\mathcal{S}}-\pi_4^{\mathcal{S}}\quad\in\ A^2(\mathcal{S}\times_B\mathcal{S}).\end{aligned}$$

It is readily checked this does the job. ☐

2.2. Transcendental part of the motive

Theorem 2.2. (Kahn–Murre–Pedrini [26]) *Let S be any smooth projective surface, and let $h(S)\in\mathcal{M}_{\rm rat}$ denote the Chow motive of S. There exists a self–dual Chow–Künneth decomposition $\{\pi_i\}$ of S, with the property that there is a further splitting in orthogonal idempotents*

$$\pi_2=\pi_2^{alg}+\pi_2^{tr}\quad \textit{in } A^2(S\times S).$$

The action on cohomology is

$$(\pi_2^{alg})_* H^*(S) = N^1 H^2(S), \quad (\pi_2^{tr})_* H^*(S) = H^2_{tr}(S),$$

where the transcendental cohomology $H^2_{tr}(S) \subset H^2(S)$ is defined as the orthogonal complement of $N^1 H^2(S)$ with respect to the intersection pairing. The action on Chow groups is

$$(\pi_2^{alg})_* A^*(S) = N^1 H^2(S), \quad (\pi_2^{tr})_* A^*(S) = A^2_{AJ}(S).$$

This gives rise to a well–defined Chow motive

$$h_2^{tr}(S) := (S, \pi_2^{tr}, 0) \ \subset \ h(S) \quad \in \mathcal{M}_{\rm rat},$$

the so–called transcendental part of the motive of S.

Proof. Let $\{\pi_i\}$ be a Chow–Künneth decomposition as in [26, Proposition 7.2.1]. The assertion then follows from [26, Proposition 7.2.3]. □

3. Triple *K*3 burgers

3.1. Definition

Definition 3.1. A surface S is called a *triple* $K3$ *burger* if the following conditions are satisfied:
(0) S is minimal, of general type, with $q=0$ and $p_g=3$;
(i) there exist involutions $\sigma_j\colon S\to S$ $(j=0,1,2)$ that commute with one another, and such that the quotients

$$\overline{X}_j := S/ <\sigma_j> \quad (j=0,1,2)$$

are birational to a $K3$ surface X_j;
(ii) there is an isomorphism

$$\big((p_0)^*, (p_1)^*, (p_2)^*\big)\colon \quad H^2(\overline{X}_0,\mathcal{O})\oplus H^2(\overline{X}_1,\mathcal{O})\oplus H^2(\overline{X}_2,\mathcal{O}) \ \xrightarrow{\cong}\ H^2(S,\mathcal{O}),$$

where $p_j\colon S\to\overline{X}_j$ denotes the quotient morphism;
(iii) the involutions σ_j respect the canonical divisor:

$$(\sigma_j)^* K_S = K_S, \quad j=0,1,2.$$

Remark 3.2. Let $\Psi_j\in A^2(X_j\times S)$ $(j=0,1,2)$ be the correspondence defined by the diagram

$$\begin{array}{ccc} & & S \\ & & \downarrow \\ X_j & \to & \overline{X}_j \end{array}$$

where $X_j\to\overline{X}_j$ is a resolution of singularities and X_j is a $K3$ surface.

Since the $\overline{X}_j$ have only quotient singularities and quotient singularities are rational, condition (ii) of Definition 3.1 is equivalent to asking for an isomorphism

$$\big((\Psi_0)_*,(\Psi_1)_*,(\Psi_2)_*\big):\quad H^2(X_0,\mathcal{O})\oplus H^2(X_1,\mathcal{O})\oplus H^2(X_2,\mathcal{O})\xrightarrow{\cong} H^2(S,\mathcal{O}).$$

Also, since $(\Psi_j)_*$ is a homomorphism of Hodge structures, condition (ii) is equivalent to an isomorphism

$$\big((\Psi_0)_*,(\Psi_1)_*,(\Psi_2)_*\big):\quad H^2_{tr}(X_0)\oplus H^2_{tr}(X_1)\oplus H^2_{tr}(X_2)\xrightarrow{\cong} H^2_{tr}(S).$$

(Here, by definition $H^2_{tr}()\subset H^2()$ is the orthogonal complement of the Néron–Severi group under the cup product pairing.)

Also, since $(p_j)^*H^2(\overline{X}_j)$ is contained in the σ_j–invariant part of $H^2(S)$, condition (ii) is equivalent to the condition

$$H^2_{tr}(S)=H^2_{tr}(S)^{+--}\oplus H^2_{tr}(S)^{-+-}\oplus H^2_{tr}(S)^{--+}, \tag{2}$$

where $H^2_{tr}(S)^{+--}$ denotes the part of $H^2_{tr}(S)$ where σ_0 acts as the identity and σ_1,σ_2 act as minus the identity, and the other summands are defined similarly.

(This uses some Hodge theory. E.g., let us consider $H^2_{tr}(S)^{++-}$. This is a Hodge substructure of $H^2_{tr}(S)$, and so if it is non–trivial, it must have Gr^0_F of dimension ≥ 1. But then, as it is contained in the image of $H^2_{tr}(X_0)$, it must have Gr^0_F of dimension $=1$. This implies that

$$(\Psi_0)_*H^2_{tr}(X_0)=H^2_{tr}(S)^{++-},$$

as both sides are Hodge substructures of $H^2_{tr}(S)$ with $\dim\mathrm{Gr}^0_F=1$. But for the same reason, we have

$$(\Psi_1)_*H^2_{tr}(X_1)=H^2_{tr}(S)^{++-},$$

and so

$$(\Psi_0)_*H^2_{tr}(X_0)=(\Psi_1)_*H^2_{tr}(X_1)\quad\text{in } H^2_{tr}(S).$$

But this is absurd, because it contradicts the surjectivity in condition (ii). We conclude that $H^2_{tr}(S)^{++-}$ must be zero. Applying the same reasoning to the other eigenspaces, one arrives at the isomorphism (2).)

Remark 3.3. Definition 3.1 is directly inspired by the definition of Todorov surfaces [47], [28], [33], [41].

One could extend Definition 3.1 to surfaces of any geometric genus: a surface S is called an m–tuple $K3$ burger if $p_g(S)=m$ and there exist m involutions $\sigma_1, ..., \sigma_m$ such that the quotients $S/<\sigma_j>$ are birational to $K3$ surfaces and their transcendental cohomology generates $H^2_{tr}(S)$ as in condition (ii). For $m=1$ (i.e., "simple $K3$ burgers"), one obtains certain Todorov surfaces. (NB: There is a slight difference with the definition of Todorov surfaces; in the definition of a Todorov surface one merely asks, instead of (iii), that the involution σ is composed with the bicanonical map).

Surfaces similar to the case $m=2$ of Definition 3.1 (i.e., "double $K3$ burgers") have been studied in [14], [37], [30].

Remark 3.4. A closely related construction (which also inspired the present note) appears in recent work of Garbagnati [14, Section 6.1]. Let S be the minimal model of the surface U_{10} of [14, Section 6.1]. Then S satisfies conditions (0), (i) and (ii) of Definition 3.1 (and I am not sure about condition (iii)). Also, it follows from [14, Theorem 3.1] that $K_S^2=9$, and so S is not among the cases covered by Theorem 5.1.

The fact that $K_S^2=9$ means that S is quite far from the Noether line; hence there is (as far as I am aware) not a nice and simple, Horikawa–style description of the canonical model of S as a weighted complete intersection. Due to the lack of such a description, the method of "spread" does not seem to apply to S, and I do not know how to handle the Chow groups of S.

Remark 3.5. Condition (iii) in Definition 3.1 is admittedly somewhat ad hoc. The reason I have added condition (iii) is that otherwise, I am not able to prove Theorem 5.1.

(More precisely: condition (iii) ensures that the involutions σ_j come from involutions of the ambient space (which will be a weighted projective space); as such, the involutions exist family–wise, which will be crucial to the argument.)

Remark 3.6. Todorov surfaces have been classified: there are 11 irreducible families, each of dimension 12 [33]. Likewise, it is perhaps possible to classify triple $K3$ burgers. The next subsection provides a first step.

3.2. Structural results

Notation 3.7. Let $\mathbb{P}$ be some weighted projective space, with weighted homogeneous coordinates $[x_0{:}x_1{:}...{:}x_n]$. We define involutions $s_j\in\mathrm{Aut}(\mathbb{P})$, $j=0, ..., n$, by

$$s_j[x_0:\ldots:x_n]=[x_0:\ldots:-x_j:\ldots:x_n].$$

Similarly, for $0\leq i<j\leq n$ we define involutions $s_{ij}\in\mathrm{Aut}(\mathbb{P})$ by

$$s_{ij}[x_0:\ldots:x_n]=[x_0:\ldots:-x_i:x_{i+1}:\ldots:-x_j:x_{j+1}:\ldots:x_n].$$

Similarly, we define involutions s_{ijk} involving 3 minus signs.

Proposition 3.8. *Let S be a triple $K3$ burger with $K^2=2$. Then S is isomorphic to a smooth degree 8 hypersurface in $\mathbb{P}(1^3,4)$ invariant under $G=<\sigma_0,\sigma_1,\sigma_2>$, where $\{\sigma_0,\sigma_1,\sigma_2\}$ are one of the following*:

(*i*)

$$\{\sigma_0,\sigma_1,\sigma_2\}=\{s_0,s_1,s_2\}.$$

(*ii*)

$$\{\sigma_0,\sigma_1,\sigma_2\}=\{s_0,s_1,s_{01}\}.$$

(*iii*)

$$\{\sigma_0,\sigma_1,\sigma_2\}=\{s_{01},s_{02},s_0\}.$$

(*iv*)

$$\{\sigma_0,\sigma_1,\sigma_2\}=\{s_{01},s_{02},s_{12}\}.$$

Conversely, any such surface S is a triple $K3$ burger with $K^2=2$, and the associated $K3$ surfaces are obtained as $\overline{X_j}=S/<\sigma_j>$, where the σ_j are as in (*i*)*–*(*iv*).

Proof. Since S is minimal, of general type, with $K^2=2$ and $p_g=3$, we know that S is isomorphic to a smooth degree 8 hypersurface in $\mathbb{P}:=\mathbb{P}(1^3,4)$ [17]. Since the involutions σ_j $(j=0,1,2)$ preserve the polarization K_S, they are induced by involutions of $\mathbb{P}$. Let $[x_0:x_1:x_2:x_3]$ be weighted homogeneous coordinates for $\mathbb{P}$. After a projective transformation, we may suppose the involutions are defined by adding a minus sign in front of one or two or three of the x_i, i.e. the σ_j are of the form s_i,s_{ij},s_{012}, where $i,j\in\{0,1,2\}$.

Griffiths residue calculus (which also exists for weighted projective hypersurfaces, cf. [11], [2]) shows that $H^{0,2}(S)$ is generated by the image under the residue map of the holomorphic forms with poles

$$x_0\Omega/f,\quad x_1\Omega/f,\quad x_2\Omega/f. \tag{3}$$

Here, f is a defining equation for S and Ω is the standard 3–form

$$\Omega:=\sum_{i=0}^{2}(-1)^i x_i dx_0\wedge\ldots\widehat{dx_i}\ldots dx_3-4x_3dx_0\wedge dx_1\wedge dx_2$$

[11, 2.1.3], [2, Example 9.4].

The involution s_{012} acts as -1 on the form Ω. Hence, the involution s_{012} acts either as $(+1,+1,+1)$ or as $(-1,-1,-1)$ on the three generators (3) (depending on whether s_{012} acts as $+1$ or as -1 on f). As such, the quotient $S/{<}s_{012}{>}$ can not be a $K3$ surface, and so s_{012} is not among the σ_j.

Suppose now the σ_j are all of type s_i. The involution s_i acts on Ω as -1, and on f as ± 1. Considering the action on generators (3), clearly the only possibility is (i).

Suppose next that exactly one of the σ_j is of type s_{ij} (and so the others are of type s_i). Up to a coordinate change, we may suppose $\sigma_2 = s_{01}$. The involution s_{01} acts on Ω as $+1$, and on f as ± 1. Since the quotient $S/{<}s_{01}{>}$ is $K3$, the action on f has to be the identity, and so s_{01} acts on the generators (3) as $(-1,-1,+1)$. Clearly, the only possibility for $\{\sigma_0,\sigma_1\}$ is now $\{s_0,s_1\}$, and so we are in case (ii).

Next, let us suppose that exactly two of the σ_j are of type s_{ij}, say $\sigma_0 = s_{01}$ and $\sigma_1 = s_{02}$. As per above, the case $s_{ij}(f) = -f$ can be excluded. We conclude that σ_0 acts on the generators (3) as $(-1,-1,+1)$, and σ_1 acts as $(-1,+1,-1)$. The remaining involution $\sigma_2 = s_i$ should act as $(+1,-1,-1)$, and so $\sigma_2 = \sigma_0$, and we are in case (iii).

Finally, if all three σ_j are of type s_{ij}, they need to be different (for otherwise, there is a generator (3) not preserved by any of the σ_j). Hence, we are in case (iv).

The converse is clear from the above argument. (Note that the involutions σ_j commute because they commute as automorphisms of $\mathbb{P}$.) □

Remark 3.9. Triple $K3$ burgers as in Proposition 3.8(i) form a family of moduli dimension 6. Indeed, after a change of variables the equation defining S is of the form

$$(x_3)^2 = f(x_0,x_1,x_2),$$

i.e. S is a double cover of the plane branched along an octic f, where x_0,x_1,x_2 occur only in even degrees. This family has 6 moduli.

(The degree 8 equation

$$(x_3)^2 = f(x_0,x_1,x_2)$$

(with x_0,x_1,x_2 occurring in even degree) depends on 15 parameters, so smooth hypersurfaces of this type correspond to an open in $\mathbb{P}^{14}$. The group $PGL(3)$ acts on these hypersurfaces, and so we get $14-8=6$ moduli.)

One element in this family is the weighted Fermat hypersurface

$$x_0^8 + x_1^8 + x_2^8 + x_3^2 = 0.$$

The surfaces of Proposition 3.8(iii) and (iv) are the same family as that of (i); only the associated $K3$ surfaces are different, so there are different "burger structures" on elements of this family.

Proposition 3.10. *Let S be a triple $K3$ burger with $K^2{=}3$ and such that the canonical divisor is base–point free. Then S is isomorphic to a smooth degree 6 hypersurface in $\mathbb{P}(1^3,2)$ invariant under $G{=}{<}\sigma_0,\sigma_1,\sigma_2{>}$, where $\{\sigma_0,\sigma_1,\sigma_2\}$ are one of the following:*

(i)

$$\{\sigma_0,\sigma_1,\sigma_2\}=\{s_0,s_1,s_2\}.$$

(ii)

$$\{\sigma_0,\sigma_1,\sigma_2\}=\{s_0,s_1,s_{01}\}.$$

(iii)

$$\{\sigma_0,\sigma_1,\sigma_2\}=\{s_{01},s_{02},s_0\}.$$

(iv)

$$\{\sigma_0,\sigma_1,\sigma_2\}=\{s_{01},s_{02},s_{12}\}.$$

Conversely, any such surface S is a triple $K3$ burger with $K^2{=}3$, and the associated $K3$ surfaces are obtained as $\overline{X_j}{=}S/{<}\sigma_j{>}$, where the σ_j are as in (i)–(iv).

Proof. Since S is minimal, of general type, with $K^2{=}p_g{=}3$ and base point free canonical divisor, we know that S is isomorphic to a degree 6 hypersurface in $\mathbb{P}(1^3,2)$ [23].

To classify the possible involutions, one proceeds exactly as in the proof of Proposition 3.8. ☐

Remark 3.11. Triple $K3$ burgers with $K^2{=}3$ and K_S base–point free form a family of dimension 4. (Indeed, under the natural map

$$\mathbb{P}(1^3,2)\ \longrightarrow\ \mathbb{P}(2^4),$$

the hypersurfaces as in Proposition 3.10 correspond to degree 6 hypersurfaces in $\mathbb{P}(2^4)$. But under the natural isomorphism

$$\mathbb{P}(2^4)\ \xrightarrow{\cong}\ \mathbb{P}(1^4)=\mathbb{P}^3,$$

the degree 6 hypersurfaces in $\mathbb{P}(2^4)$ correspond to degree 3 hypersurfaces in $\mathbb{P}^3$, for which there are 4 moduli.)

We note that there is a subfamily given by triple covers of the plane, and this subfamily has moduli dimension 1.

(The degree 6 equation

$$(x_3)^3 = f(x_0,x_1,x_2)$$

with x_0,x_1,x_2 occurring in even degree depends on 10 parameters. We get $10-1-\dim PGL(3)=1$.)

One element in the family (which is also in the subfamily of triple planes) is given by the weighted Fermat hypersurface

$$x_0^6+x_1^6+x_2^6+x_3^3=0.$$

Remark 3.12. I have not been able to classify triple $K3$ burgers with $K^2=3$ without the assumption that K_S be base point free. When K_S is *not* base–point free, it is known [23] there is exactly one base–point, and the canonical model of S is isomorphic to a bidegree $(3,6)$ complete intersection in $\mathbb{P}(1^3,2,3)$. However, determining the possible involutions σ_j in this case seems to get messy.

Similarly, triple $K3$ burgers with $K^2=4$ and K_S base point free are complete intersections in a weighted projective space [40]. I have not been able to classify them.

3.3. Families

This section establishes some notation. The two cases in Notation 3.13 correspond to two cases of Propositions 3.8 and 3.10.

Notation 3.13. Let

$$\mathcal{S} \longrightarrow B$$

denote one of the following families:
(i) (Case (i) of Proposition 3.8) The family of all smooth hypersurfaces in $\mathbb{P}:=\mathbb{P}(1^3,4)$ of type

$$f_b(x_0,x_1,x_2,x_3)=0,$$

where f_b is weighted homogeneous of degree 8, and x_0,x_1,x_2 occur only in even degree. Let S_b denote the fibre of $\mathcal{S}$ over $b\in B$.
(ii) (Case (i) of Proposition 3.10) The family of all smooth hypersurfaces in $\mathbb{P}=\mathbb{P}(1^3,2)$ of type

$$f_b(x_0,x_1,x_2,x_3)=0,$$

where f_b is weighted homogeneous of degree 6, and x_0,x_1,x_2 occur only in even degree. Let S_b denote the fibre of $\mathcal{S}$ over $b\in B$.

Remark 3.14. Let $\mathcal{S}\to B$ be the family as in Notation 3.13(i) (resp. (ii)). Then any fibre S_b is a triple $K3$ burger with $K^2=2$ (resp. $K^2=3$). This is immediate from Proposition 3.8 (resp. Proposition 3.10).

Lemma 3.15. *Let $\mathcal{S}\to B$ be one of the two families of Notation 3.13. The variety $\mathcal{S}$ is a smooth quasi–projective variety.*

Proof. Let us treat case (i); the other case is similar. By construction, there are morphisms

$$\begin{array}{ccc} \mathcal{S} & \xrightarrow{\pi} & \mathbb{P} \\ \downarrow{\scriptstyle \nu} & & \\ B & & \end{array}$$

Let $\overline{\mathcal{S}}\to\overline{B}$ denote the universal family of all (not necessarily smooth) hypersurfaces in $\mathbb{P}$ of type

$$f_b(x_0,x_1,x_2,x_3)=0,$$

where f_b is weighted homogeneous of degree 8 and x_0,x_1,x_2 only occur in even degrees. Then $\overline{B}$ is a projective space containing B as a Zariski open.

Lemma 3.16. *For any $x\in\mathbb{P}(1^3,4)$, there exists $b\in\overline{B}$ such that $x\notin S_b$.*

Proof. There is a $(\mathbb{Z}/2\mathbb{Z})^3$ cover

$$\mathbb{P}(1^3,4) \longrightarrow \mathbb{P}(2^3,4)\cong\mathbb{P}(1^3,2)=:\mathbb{P}'.$$

The surfaces in $\overline{\mathcal{S}}\to\overline{B}$ correspond to the complete linear system $\mathbb{P}H^0(\mathbb{P}',\mathcal{O}_{\mathbb{P}'}(4))$ which is (ample hence) base point free. $\square$

Lemma 3.16 ensures that $\overline{\mathcal{S}}$ is a projective bundle over $\mathbb{P}(1^3,4)$, in particular it is a projective quotient variety. Any surface S_b with $b\in B$ avoids the singular point of $\mathbb{P}(1^3,4)$, and so $\mathcal{S}$ is Zariski open inside a projective bundle over the non–singular locus of $\mathbb{P}(1^3,4)$. It follows that $\mathcal{S}$ is smooth. $\square$

4. Trivial Chow groups

This intermediate section contains a result asserting the triviality of a certain Chow group. This result (Proposition 4.1) will be the most essential ingredient in the proof of our main result (Theorem 5.1 in the next section). The proof of Proposition 4.1 occupies Subsection 4.2, and uses a stratification argument borrowed from [29].

Proposition 4.1. *Let $\mathcal{S}\to B$ be a family of surfaces as in Notation 3.13. Let $B^0\subset B$ be a Zariski open, and let $\mathcal{S}^0\to B^0$ be the family obtained by restriction. Then*

$$A^2_{hom}(\mathcal{S}^0\times_{B^0}\mathcal{S}^0)=0.$$

4.1. Weak and strong property

Definition 4.2. (Totaro [48]) For any (not necessarily smooth) quasi–projective variety X, let $A_i(X,j)$ denote Bloch's higher Chow groups with rational coefficients (these groups are sometimes written $A^{n-i}(X,j)_{\mathbb{Q}}$ or $CH^{n-i}(X,j)_{\mathbb{Q}}$, where $n{=}\dim X$). As explained in [48, Section 4], the relation with algebraic K–theory ensures there are functorial cycle class maps

$$A_i(X,j) \longrightarrow \mathrm{Gr}^W_{-2i} H_{2i+j}(X),$$

compatible with long exact sequences (here W_* denotes Deligne's weight filtration on Borel–Moore homology [39]).

We say that X has the *weak property* if the cycle class maps induce isomorphisms

$$A_i(X) \xrightarrow{\cong} W_{-2i} H_{2i}(X)$$

for all i.

We say that X has the *strong property* if X has the weak property, and, in addition, the cycle class maps induce surjections

$$A_i(X,1) \twoheadrightarrow \mathrm{Gr}^W_{-2i} H_{2i+1}(X)$$

for all i.

Lemma 4.3. ([48]) *Let X be a quasi–projective variety, and $Y\subset X$ a closed subvariety with complement $U{=}X\setminus Y$. If X has the strong property and Y has the weak property, then U has the strong property.*

Proof. This is [48, Lemma 6]. □

Lemma 4.4. *Let X be a quasi–projective variety, and $Y\subset X$ a closed subvariety with complement $U{=}X\setminus Y$. If Y and U have the strong property, then so does X.*

Proof. This is the same argument as [48, Lemma 7], which is a slightly different statement. As in loc. cit., using the localization property of higher Chow groups [7], [31], one finds a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
A_i(U,1) & \to & A_i(Y) & \to & A_i(X) & \to & A_i(U) & \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\mathrm{Gr}^W_{-2i} H_{2i+1}(U) & \to & \mathrm{Gr}^W_{-2i} H_{2i}(Y) & \to & \mathrm{Gr}^W_{-2i} H_{2i}(X) & \to & \mathrm{Gr}^W_{-2i} H_{2i}(U) & \to 0
\end{array}$$

A diagram chase reveals that under the assumptions of the lemma, the one but last vertical arrow is an isomorphism.

Continuing these long exact sequences to the left, there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc} A_i(Y,1) & \to & A_i(X,1) & \to & A_i(U,1) & \to & A_i(Y) & \to \\ \downarrow & & \downarrow & & \downarrow & & \downarrow\cong & \\ \mathrm{Gr}^W_{-2i}H_{2i+1}(Y) & \to & \mathrm{Gr}^W_{-2i}H_{2i+1}(X) & \to & \mathrm{Gr}^W_{-2i}H_{2i+1}(U) & \to & \mathrm{Gr}^W_{-2i}H_{2i}(Y) & \to \end{array}$$

Chasing some more inside this diagram, one finds that the second vertical arrow is a surjection. □

Corollary 4.5. *Let X be a quasi–projective variety that admits a stratification such that each stratum is of the form $\mathbb{A}^k\setminus L$, where L is a finite union of linearly embedded affine subspaces. Then X has the strong property.*

Proof. Affine space has the strong property (this is the homotopy invariance for higher Chow groups). The subvariety L has the weak property. Doing a diagram chase as in Lemma 4.4 (or directly applying [48, Lemma 6]), it follows that the variety $\mathbb{A}^k\setminus L$ has the strong property. The corollary now follows from Lemma 4.4. □

Lemma 4.6. *Let X be a quasi–projective variety with the strong property. Let $Y\to X$ be a projective bundle. Then Y has the strong property.*

Proof. This follows from the projective bundle formula for higher Chow groups [6]. □

4.2. Proof of Proposition 4.1

Proof. (i) ($K^2=2$) Let us use the shorthand

$$\begin{aligned} &\mathbb{P}:=\mathbb{P}(1^3,4),\\ &M:=\mathbb{P}\times\mathbb{P},\\ &N:=\Big\{(f_b,p,p')\ \in\overline{B}\times\mathbb{P}\times\mathbb{P}\ |\ f_b(p)=f_b(p')=0\ \Big\}\quad\subset\ \overline{B}\times M. \end{aligned}$$

The goal is to prove that

$$A^2_{hom}(N)\overset{??}{=}0. \tag{4}$$

This implies Proposition 4.1 for case (i), because (4) implies triviality of A^2 of any open in N, and $\mathcal{S}\times_B\mathcal{S}$ is an open in N.

Inside M, we have various "partial diagonals"

$$\begin{aligned}
\Delta_M = \Delta_{+++} &:= \Big\{ (p,p') \in \mathbb{P}\times\mathbb{P} \mid p = p' \Big\},\\
\Delta_{+-+} &:= \Big\{ (p,p') \in \mathbb{P}\times\mathbb{P} \mid [p_0 : p_1 : p_2 : p_3] = [p'_0 : -p'_1 : p'_2 : p'_3] \Big\},\\
\Delta_{-++} &:= \Big\{ (p,p') \in \mathbb{P}\times\mathbb{P} \mid [p_0 : p_1 : p_2 : p_3] = [-p'_0 : p'_1 : p'_2 : p'_3] \Big\},\\
\Delta_{++-} &:= \Big\{ (p,p') \in \mathbb{P}\times\mathbb{P} \mid [p_0 : p_1 : p_2 : p_3] = [p'_0 : p'_1 : -p'_2 : p'_3] \Big\},\\
\Delta_{+--} &:= \Big\{ (p,p') \in \mathbb{P}\times\mathbb{P} \mid [p_0 : p_1 : p_2 : p_3] = [p'_0 : -p'_1 : -p'_2 : p'_3] \Big\},\\
\Delta_{-+-} &:= \Big\{ (p,p') \in \mathbb{P}\times\mathbb{P} \mid [p_0 : p_1 : p_2 : p_3] = [-p'_0 : p'_1 : -p'_2 : p'_3] \Big\},\\
\Delta_{--+} &:= \Big\{ (p,p') \in \mathbb{P}\times\mathbb{P} \mid [p_0 : p_1 : p_2 : p_3] = [-p'_0 : -p'_1 : p'_2 : p'_3] \Big\},\\
\Delta_{---} &:= \Big\{ (p,p') \in \mathbb{P}\times\mathbb{P} \mid [p_0 : p_1 : p_2 : p_3] = [p'_0 : p'_1 : p'_2 : -p'_3] \Big\},
\end{aligned}$$

(Here, we write $p=[p_0{:}p_1{:}p_2{:}p_3]$ and $p'=[p'_0{:}p'_1{:}p'_2{:}p'_3]$. We observe that the various $\Delta_{\pm\mp\pm}$ are just the graphs of the elements of the group $(\mathbb{Z}/2\mathbb{Z})^3 = <\sigma_0,\sigma_1,\sigma_2> \subset \mathrm{Aut}(\mathbb{P})$.)

Let us define the Zariski opens

$$\begin{aligned}
M^0 &:= M \setminus (\cup \Delta_{\pm\mp\pm}),\\
N^0 &:= N \setminus \pi^{-1}(\cup \Delta_{\pm\mp\pm}).
\end{aligned}$$

Corollary 4.5 implies that the union $\cup\Delta_{\pm\mp\pm}$ has the strong property. Since $M = \mathbb{P}\times\mathbb{P}$ has the strong property, so does M^0 (Lemma 4.3). The morphism from N^0 to M^0 has constant dimension (Lemma 4.7), so it is a projective bundle and N^0 also has the strong property (Lemma 4.6).

Lemma 4.7. *Let*

$$(p,p') \quad \in \ M \setminus (\cup \Delta_{\pm\mp\pm}).$$

Then (p,p') imposes 2 independent conditions on $\overline{B}$, i.e. there exists $b \in \overline{B}$ such that S_b contains p but not p'.

Proof. Consider the map

$$r \times r: \quad M = \mathbb{P}\times\mathbb{P} \longrightarrow \mathbb{P}'\times\mathbb{P}',$$

where $\mathbb{P}'$ is as before $\mathbb{P}(2,2,2,4)$. The condition $(p,p') \notin (\cup\Delta_{\pm\mp\pm})$ implies that $r(p) \neq r(p')$. Since $\mathbb{P}'$ is isomorphic to $\mathbb{P}'' := \mathbb{P}(1,1,1,2)$ (and sections of $\mathcal{O}_{\mathbb{P}'}(8)$ correspond under this isomorphism to sections of $\mathcal{O}_{\mathbb{P}''}(4)$), Lemma 4.8 below shows there exists S_b separating the points p and p'.

Lemma 4.8. *Let $\mathbb{P}''$ be the weighted projective space $\mathbb{P}(1,1,1,2)$. Then the line bundle $\mathcal{O}_{\mathbb{P}''}(4)$ is very ample.*

Proof. The coherent sheaf $\mathcal{O}_{\mathbb{P}''}(4)$ is locally free, because 4 is a multiple of the weights [11]. To see that this line bundle is very ample, we use the following numerical criterion:

Proposition 4.9. (Delorme [10]) *Let $P=\mathbb{P}(q_0,q_1,\dots,q_n)$ be a weighted projective space. Let m be the least common multiple of the q_j. Suppose every monomial*

$$x_0^{b_0}x_1^{b_1}\dots x_n^{b_n}$$

of (weighted) degree km ($k\in\mathbb{N}^*$) is divisible by a monomial of (weighted) degree m. Then $\mathcal{O}_P(m)$ is very ample.

(This is the case $E(x)=0$ of [10, Proposition 2.3(iii)].)

Using Proposition 4.9, Lemma 4.8 is now easily established. □

Let us now finish the proof of Proposition 4.1 for case (i). Any point

$$(p,p') \quad \in \ M^1 := (\cup \Delta_{\pm\mp\pm}) \quad \subset M$$

imposes exactly one condition on $\overline{B}$; indeed p imposes one condition (Lemma 3.16), and since $r(p)=r(p')$ in $\mathbb{P}'=\mathbb{P}(2,2,2,4)$, any S_b containing p also contains p'. This means that N^1 has the structure of a projective bundle over M^1. We have seen above that M^1 has the strong property. It follows from Lemma 4.6 that

$$N^1 := \pi^{-1}(M^1) \quad \subset N$$

has the strong property. Lemma 4.4 now implies that N has the strong property, and so equality (4) is proven.
(ii) ($K^2=3$). Similar to case (i), except that $\mathbb{P}$ is now $\mathbb{P}(1^3,2)$ and the degree of the hypersurfaces is 6. Instead of Lemma 4.8, we now use that $\mathcal{O}_{\mathbb{P}^3}(3)$ is very ample. □

5. Main

Theorem 5.1. *Let S be a triple $K3$ burger, and let $X_j (j=0,1,2)$ be the associated $K3$ surfaces. Assume that either*
(i) $K_S^2=2$, or
(ii) $K_S^2=3$ and the canonical map is base point free.

Then there is an isomorphism

$$(\Psi_0)_*+(\Psi_1)_*+(\Psi_2)_*: \quad A^2_{hom}(X_0)\oplus A^2_{hom}(X_1)\oplus A^2_{hom}(X_2) \xrightarrow{\cong} A^2_{hom}(S).$$

Proof. First, a reduction step. Let us define eigenspaces

$$A^2(S)^{\pm\mp\pm} := \{a \in A^2(S) \mid (\sigma_0)^*(a) = \pm a,\ (\sigma_1)^*(a) = \mp a,\ (\sigma_2)^*(a) = \pm a\}.$$

We now make the following claim:

Claim 5.2. *Let S be as in Theorem 5.1. Any eigenspace with an odd number of minus signs is trivial, i.e.*

$$A^2(S)^{---} = A^2(S)^{-++} = A^2(S)^{+-+} = A^2(S)^{++-} = 0.$$

Moreover,

$$A^2_{hom}(S)^{+++} = 0.$$

Before proving the claim, let us verify that the claim suffices to prove the theorem: the claim implies there is a decomposition

$$A^2_{hom}(S) = A^2_{hom}(S)^{+--} \oplus A^2_{hom}(S)^{-+-} \oplus A^2_{hom}(S)^{--+}. \tag{5}$$

Also, since necessarily

$$(\Psi_0)_* A^2(S) \ \subset\ A^2(S)^{+\pm\pm},$$

the claim implies that

$$(\Psi_0)_* A^2_{hom}(S) \ \subset\ A^2(S)^{+--}.$$

What's more, since

$$(\Psi_0)_*(\Psi_0)^* = 2\,\mathrm{id}\colon \quad A^2(S)^{+\pm\pm} \longrightarrow A^2(S)^{+\pm\pm},$$

there is actually equality

$$(\Psi_0)_*(\Psi_0)^* A^2_{hom}(S) = A^2(S)^{+--}.$$

(And similarly, for reasons of symmetry,

$$(\Psi_1)_*(\Psi_1)^* A^2_{hom}(S) = A^2(S)^{-+-},$$
$$(\Psi_2)_*(\Psi_2)^* A^2_{hom}(S) = A^2(S)^{--+}\ .)$$

Therefore, the decomposition (5) is equivalent to the decomposition

$$A^2_{hom}(S) = (\Psi_0)_*(\Psi_0)^* A^2_{hom}(S) \oplus (\Psi_1)_*(\Psi_1)^* A^2_{hom}(S) \oplus (\Psi_2)_*(\Psi_2)^* A^2_{hom}(S).$$

This proves the surjectivity statement of the theorem

Again using the claim, one deduces that the composition

$$A^2_{hom}(X_0)\oplus A^2_{hom}(X_1)\oplus A^2_{hom}(X_2) \xrightarrow{(\Psi_0)_*+(\Psi_1)_*+(\Psi_2)_*} A^2_{hom}(S) \xrightarrow{((\Psi_0)^*,(\Psi_1)^*,(\Psi_2)^*)} A^2_{hom}(X_0)\oplus A^2_{hom}(X_1)\oplus A^2_{hom}(X_2)$$

equals twice the identity. This proves the injectivity statement of the theorem.

It remains to prove the claim. First, let us treat case (ii) of Propositions 3.8 and 3.10. In this case, $\sigma_2{=}\sigma_0\circ\sigma_1$ (i.e., $G{:=}{<}\sigma_0,\sigma_1,\sigma_2{>}\cong(\mathbb{Z}/2\mathbb{Z})^2$), and so the first part of the claim is trivially true. The second part of the claim is also true for these cases: indeed, there is equality

$$A^2_{hom}(S)^{+++}=A^2_{hom}(S/G).$$

But the surface S/G is a degree 8 hypersurface in $\mathbb{P}(1,2,2,4)$ (resp. a degree 6 hypersurface in $\mathbb{P}(1,2^3)$), and so S/G is a surface with quotient singularities and ample anticanonical bundle. Such surfaces are rational [59, Theorem 2.3], and hence $A^2_{hom}(S/G){=}0$.

Next, let us consider the cases (i), (iii) and (iv) of Propositions 3.8 and 3.10. In this case, the surfaces S_b are elements of the families of Notation 3.13. The argument, in a nutshell, is now as follows: the correspondences Ψ_j exist as relative correspondences for the whole family of triple $K3$ burgers. Using the trivial Chow groups result (Proposition 4.1), one can upgrade a vanishing in cohomology to a vanishing of Chow groups.

We now proceed to prove Claim 5.2 for surfaces as in Proposition 3.8(i), (iii) and (iv). (The cases of Proposition 3.10(i), (iii) and (iv) are mostly the same, modulo some mutatis mutandis which we will indicate below).

Cases (i), (iii), (iv) of Proposition 3.8: Let

$$\mathcal{S}\longrightarrow B$$

denote the universal family of surfaces as in Notation 3.13(i). Let $\{\sigma_0,\sigma_1,\sigma_1\}$ be either $\{s_0,s_1,s_2\}$ or $\{s_{01},s_{02},s_0\}$, and let

$$\mathcal{X}_j:=\mathcal{S}/\sigma_j \quad (j=0,1,2)$$

denote the universal families of associated $K3$ surfaces as in Notation 3.13. For any $b{\in}B$, we will write S_b for the fibre of $\mathcal{S}$ over b, and X_{0b} (resp. X_{1b} resp. X_{2b}) for the fibre of $\mathcal{X}_0$ (resp. $\mathcal{X}_1$ resp. $\mathcal{X}_2$) over b. Likewise, we will write σ_{0b}, σ_{1b}, σ_{2b} for the restriction of σ_0 (resp. σ_1 resp. σ_2) to S_b. For a relative correspondence $\Gamma{\in}A^*(\mathcal{S}\times_B\mathcal{S})$, we will use the shorthand

$$\Gamma|_b:=\Gamma|_{S_b\times S_b}\quad\in A^*(S_b\times S_b)$$

for the restriction (i.e., the image of Γ under the Gysin homomorphism induced by the inclusion $b \hookrightarrow B$).

By definition (cf. Remark 3.2), we know that there is a fibrewise isomorphism

$$(6) \qquad \begin{aligned} H^2_{tr}(S_b) &\cong H^2_{tr}(S_b)^{+--} \oplus H^2_{tr}(S_b)^{-+-} \oplus H^2_{tr}(S_b)^{--+} \\ &\cong H^2_{tr}(X_{0b}) \oplus H^2_{tr}(X_{1b}) \oplus H^2_{tr}(X_{2b}) \quad \forall b \in B. \end{aligned}$$

That is, there are no eigenspaces with an odd number of minus signs:

$$(7) \qquad H^2_{tr}(S_b)^{---} = H^2_{tr}(S_b)^{-++} = H^2_{tr}(S_b)^{+-+} = H^2_{tr}(S_b)^{++-} = 0 \quad \forall b \in B.$$

Also, there is no eigenspace without minus signs:

$$(8) \qquad H^2_{tr}(S_b)^{+++} = 0 \quad \forall b \in B.$$

Let us define a relative correspondence

$$\Gamma^{---} := \frac{1}{8}(\Delta_{\mathcal{S}} - \Gamma_{\sigma_0}) \circ (\Delta_{\mathcal{S}} - \Gamma_{\sigma_1}) \circ (\Delta_{\mathcal{S}} - \Gamma_{\sigma_2}) \circ \pi_2^{\mathcal{S}} \quad \in A^2(\mathcal{S} \times_B \mathcal{S}).$$

(For details on the formalism of relative correspondences and their composition, cf. [35, Chapter 8] whose conventions are met with in our set–up.)

We observe that for any $b \in B$, the restriction

$$\Gamma^{---}|_b \quad \in A^2(S_b \times S_b)$$

is a projector on $H^2(S_b)^{---}$.

In terms of correspondences, the vanishing $H^2_{tr}(S_b)^{---} = 0$ in (7) is equivalent to the statement that

$$\big(\Gamma^{---}|_b\big) \circ \pi^{S_b}_{2,tr} = 0 \quad \text{in } H^4(S_b \times S_b) \quad \forall b \in B.$$

(Here, $\pi^{S_b}_{2,tr}$ is a projector defining the transcendental part of the motive as in Theorem 2.2.) This is in turn equivalent to the statement that for any $b \in B$, there exists a divisor $D_b \subset S_b$, and a cycle γ_b supported on $D_b \times D_b \subset S_b \times S_b$, such that

$$\big(\Gamma^{---}|_b\big) \circ \pi_2^{S_b} = \gamma_b \quad \text{in } H^4(S_b \times S_b).$$

Using a Baire category argument as in [54, Proposition 3.7] or [57, Lemma 1.4], these data can be "spread out" over the base B, i.e. one can find a divisor $\mathcal{D} \subset \mathcal{S}$ and a cycle γ supported on $\mathcal{D} \times_B \mathcal{D} \subset \mathcal{S} \times_B \mathcal{S}$ such that

$$\big(\Gamma^{---} \circ \pi_2^{\mathcal{S}}\big)|_b = \gamma|_b \quad \text{in } H^4(S_b \times S_b) \quad \forall b \in B.$$

In other words, the relative correspondence

$$\Gamma := \Gamma^{---} \circ \pi_2^{\mathcal{S}} - \gamma \quad \in A^2(\mathcal{S} \times_B \mathcal{S})$$

is fibrewise homologically trivial:

$$\Gamma|_b \quad \in A^2_{hom}(S_b \times S_b) \quad \forall b \in B.$$

The next step is to make Γ globally homologically trivial. Employing a Leray spectral sequence argument as in [54, Lemmas 3.11 and 3.12], this can be done by adding a cycle coming from the ambient space $\mathbb{P}$. More precisely, the argument of [54, Lemmas 3.11 and 3.12] proves the following: up to shrinking the base (i.e., after replacing B by a dense Zariski open $B' \subset B$, and writing $B := B'$ for simplicity), there exists $\delta \in A^2(\mathbb{P} \times \mathbb{P})$ such that

$$\Gamma + (\delta \times B)|_{\mathcal{S} \times_B \mathcal{S}} \quad \in A^2_{hom}(\mathcal{S} \times_B \mathcal{S}).$$

In view of the fact that $A^2_{hom}(\mathcal{S} \times_B \mathcal{S}) = 0$ (Proposition 4.1), it follows that

$$\Gamma + (\delta \times B)|_{\mathcal{S} \times_B \mathcal{S}} = 0 \quad \text{in } A^2(\mathcal{S} \times_B \mathcal{S}).$$

We know that for any $b \in B$, the restriction $\delta|_b$ acts trivially on $A^2_{hom}(S_b)$ (the action factors over $A^*_{hom}(\mathbb{P}) = 0$). The above thus implies in particular that

$$(\Gamma|_b)_* = 0 \colon \quad A^2_{hom}(S_b) \longrightarrow A^2_{hom}(S_b) \quad \forall b \in B.$$

By definition of Γ, this means that

$$\big(\Gamma^{---}|_b - \gamma|_b\big)_* = 0 \colon \quad A^2_{hom}(S_b) \longrightarrow A^2_{hom}(S_b) \quad \forall b \in B.$$

Since for $b \in B$ general, the restriction $\gamma|_b$ will still be supported on (divisor)×(divisor), we know that

$$(\gamma|_b)_* = 0 \colon \quad A^2_{hom}(S_b) \longrightarrow A^2_{hom}(S_b) \quad \text{for general } b \in B.$$

Thus, the above simplifies to

$$\big((\Gamma^{---} \circ \pi_2^{\mathcal{S}})|_b\big)_* = 0 \colon \quad A^2_{hom}(S_b) \longrightarrow A^2_{hom}(S_b) \quad \text{for general } b \in B.$$

Using a Baire category argument as in [12, Lemma 3.1], this can be extended to *all* elements of the base B: we actually have

$$\big((\Gamma^{---} \circ \pi_2^{\mathcal{S}}|_b\big)_* = 0 \colon \quad A^2_{hom}(S_b) \longrightarrow A^2_{hom}(S_b) \quad \forall b \in B,$$

where B is now once more (as in the beginning of the proof) the parameter space parameterizing all triple $K3$ burgers as in Notation 3.13.

By construction $\Gamma^{---}|_b$ acts on $A^2(S_b)^{---}$ as the identity, and

$$(\Gamma^{---}\circ\pi_2^{\mathcal{S}})|_b=\Gamma^{---}|_b\circ\pi_2^{S_b}$$

acts on $A^2_{hom}(S_b)^{---}$ as the identity. The above thus implies the vanishing

$$A^2_{hom}(S_b)^{---}=0 \quad \forall b\in B,$$

which proves the first part of the claim. The other parts of the claim are proven similarly, by choosing a different correspondence: e.g., for the second vanishing statement one considers the relative correspondence

$$\Gamma^{-++}:=\frac{1}{8}(\Delta_{\mathcal{S}}-\Gamma_{\sigma_0})\circ(\Delta_{\mathcal{S}}+\Gamma_{\sigma_1})\circ(\Delta_{\mathcal{S}}+\Gamma_{\sigma_2})\circ\pi_2^{\mathcal{S}} \quad \in A^2(\mathcal{S}\times_B\mathcal{S}).$$

Cases (i), (iii), (iv) of Proposition 3.10: The claim is proven by the same argument as in case (i), applied to the family $\mathcal{S}\to B$ as specified in Notation 3.13. The weighted projective space $\mathbb{P}$ now has different weights, and the defining equation has a different degree. The trivial Chow groups statement (Proposition 4.1) still holds for this family. □

6. Corollaries

6.1. The canonical 0–cycle

In this subsection, we work with integral Chow groups $A^i()_{\mathbb{Z}}$, instead of Chow groups with rational coefficients. Let S be a triple $K3$ burger as in Theorem 5.1. Thanks to Rojtman's theorem [42], Theorem 5.1 implies that

$$A^2(S)^{+++}_{\mathbb{Z}}\cong\mathbb{Z}.$$

Definition 6.1. Let S be a triple $K3$ burger as in Theorem 5.1. The canonical 0–cycle $\mathfrak{o}_S$ is defined as the unique degree 1 cycle such that

$$A^2(S)^{+++}_{\mathbb{Z}}=\mathbb{Z}[\mathfrak{o}_S]$$

(where $A^2(S)^{+++}_{\mathbb{Z}}$ denotes as before the subspace where σ_j acts as the identity for $j=0,1,2$).

Equivalently, $\mathfrak{o}_S$ is the unique degree 1 cycle z satisfying

$$(\Psi_j)^*(z)=\mathfrak{o}_{X_j} \quad \text{in } A^2(X_j)_{\mathbb{Z}} \quad (j=0,1,2),$$

where X_j are the associated $K3$ surfaces and the correspondences $\Psi_j\in A^2(X_j\times S)_{\mathbb{Z}}$ are as above.

Equivalently, $\mathfrak{o}_S$ is the unique degree 1 cycle z satisfying

$$(\Psi_j)_*(\Psi_j)^*(z)=2z \quad \text{in } A^2(S)_{\mathbb{Z}} \quad (j=0,1,2).$$

The equivalences in Definition 6.1 are valid because of the following lemma:

Lemma 6.2. *Let S be a triple $K3$ burger as in Theorem 5.1. Then*

$$(\Psi_0)_*(\mathfrak{o}_{X_0}) = (\Psi_1)_*(\mathfrak{o}_{X_1}) = (\Psi_2)_*(\mathfrak{o}_{X_2}) \quad \in \ A^2(S)_{\mathbb{Z}}.$$

Proof. The point is that there is a commutative diagram of surfaces

$$\begin{array}{ccccc} & & S & & \\ & \swarrow^{p_0} & \downarrow^{p_1} & \searrow^{p_2} & \\ \overline{X}_0 & & \overline{X}_1 & & \overline{X}_2 \\ & \searrow^{r_0} & \downarrow^{r_1} & \swarrow^{r_2} & \\ & & W & & \end{array}$$

where all arrows are degree 2 morphisms, and $A^2(W)_{\mathbb{Z}}=\mathbb{Z}$. (In case (i) of Theorem 5.1, the surface W is defined as the degree 8 hypersurface in $\mathbb{P}(2^3, 4)$ defined by the equation $f(t_0, t_1, t_2, x_3)=0$, where $f(x_0^2, x_1^2, x_2^2, x_3)=0$ is a defining equation for S. For cases (ii) and (iii), the construction is similar.)

Let us pick two divisors D, D' on W, and set

$$w := D\cdot D' \quad \in A^2(W).$$

The pullbacks to the various $\overline{X}_j$ are intersections of divisors, and so

$$(r_j)^*(w) = d\,\mathfrak{o}_{\overline{X}_j} \quad \text{in } A^2(\overline{X}_j) \quad (j=0,1,2).$$

(Here, $d=\deg(D\cdot D')$, and we define $\mathfrak{o}_{\overline{X}_j}$ to be $(q_j)_*(\mathfrak{o}_{X_j}.)$ This implies that

$$(\Psi_j)_*(d\,\mathfrak{o}_{X_j}) = (p_j)^*(d\,\mathfrak{o}_{\overline{X}_j}) = (r_j\circ p_j)^*(w) \quad \text{in } A^2(S)_{\mathbb{Z}} \quad (j=0,1,2),$$

and so

$$d(\Psi_0)_*(\mathfrak{o}_{X_0}) = d(\Psi_1)_*(\mathfrak{o}_{X_1}) = d(\Psi_2)_*(\mathfrak{o}_{X_2}) \quad \in \ A^2(S)_{\mathbb{Z}}.$$

Using Rojtman's theorem [42], this proves the lemma. □

We now recall the definition of the "effective orbit under rational equivalence" of a 0–cycle:

Definition 6.3. (Voisin [56]) Let S be any surface. Given a cycle $z\in A^2(S)_{\mathbb{Z}}$ of degree $k\geq 0$, we define the "effective orbit" O_z as

$$O_z := \bigcup_{z'\in X^{(k)}, z'\sim_{\text{rat}} z} \operatorname{supp}(z') \quad \subset X^{(k)}.$$

(Here, the union is taken over all k–tuples of points z' such that the 0–cycle associated to z' is rationally equivalent to the 0–cycle z in X.)

One defines

$$\dim O_z := \sup_{V \subset O_z} \dim V,$$

where the supremum runs over all irreducible components $V \subset O_z$ (we note that O_z is known to be a countable union of closed subvarieties, so this is well–defined).

Inspired by [56], one can give a nice characterization of the canonical 0–cycle $\mathfrak{o}_S$:

Proposition 6.4. *Let S be a triple $K3$ burger as in Theorem 5.1. Let $k>0$ be an integer. Then $k\mathfrak{o}_S$ is the unique degree k 0–cycle $z \in A^2(S)_{\mathbb{Z}}$ satisfying* $\dim O_z \geq k$.

Proof. We actually prove a somewhat more general statement, which is based on Voisin's result [56, Theorem 1.4]. This result of Voisin's gives an alternative description of O'Grady's filtration $S^k_d()$ on the Chow group of 0–cycles of a $K3$ surface, in terms of effective orbits. We recall that for any $K3$ surface X, O'Grady's filtration [36] is defined as

$$S^k_d(X) := \bigl\{ z \in A^2(X)_{\mathbb{Z}} \mid z = z' + (k-d)\mathfrak{o}_X \bigr\}, \tag{9}$$

where z' is effective of degree d and $\mathfrak{o}_X$ is the canonical 0–cycle.

Voisin gives an interesting alternative description of the O'Grady filtration: for any $k>d\geq 0$, she proves [56, Theorem 1.4] that

$$S^k_d(X) = \bigl\{ z \in A^2(X)_{\mathbb{Z}} \mid O_z \subset X^{(k)} \neq \varnothing \text{ and } \dim O_z \geq k-d \bigr\}. \tag{10}$$

Let us now consider a triple $K3$ burger S as in Theorem 5.1. The canonical 0–cycle $\mathfrak{o}_S$ exists, and so definition (9) makes sense for S.

Step 1 (Unicity): Let $z \in A^2(S)_{\mathbb{Z}}$ of degree k, and let us assume that the orbit $O_z \subset S^{(k)}$ is non–empty of dimension $\geq k-d$, for some $k>d\geq 0$. According to (the proof of) Theorem 5.1, we can write z uniquely as

$$z = k\mathfrak{o}_S + z_0 + z_1 + z_2 \quad \text{in } A^2(S)_{\mathbb{Z}},$$

where $z_0 \in A^2_{hom}(S)^{+--}_{\mathbb{Z}}$ and z_1, z_2 are in $A^2_{hom}(S)^{-+-}_{\mathbb{Z}}$ resp. in $A^2_{hom}(S)^{--+}_{\mathbb{Z}}$.

The assumption on O_z implies that the cycles

$$(\Psi_j)^*(z) = k\mathfrak{o}_{X_j} + (\Psi_j)^*(z_j) \in A^2(X_j)_{\mathbb{Z}} \quad (j=0,1,2)$$

also have orbits O_{z_j} of dimension $\geq k-d$. Therefore, Voisin's result (10) implies that

$$(\Psi_j)^*(z) \quad \in S^k_d(X_j) \quad (j=0,1,2),$$

i.e. one can write

$$(\Psi_j)^*(z) = k\mathfrak{o}_{X_j} + (\Psi_j)^*(z_j) = z_j' + (k-d)\mathfrak{o}_{X_j} \quad \text{in } A^2(X_j)_{\mathbb{Z}} \quad (j=0,1,2),$$

where z_j' is effective of degree d. It follows that

$$(\Psi_j)^*(z_j) = z_j' - d\,\mathfrak{o}_{X_j} \quad \text{in } A^2(X_j)_{\mathbb{Z}} \quad (j=0,1,2).$$

Using the proof of Theorem 5.1, we find that

$$\begin{aligned} 2z &= 2k\,\mathfrak{o}_S + 2z_0 + 2z_1 + 2z_2 \\ &= 2k\,\mathfrak{o}_S + (\Psi_0)_*(\Psi_0)^*(z_0) + (\Psi_1)_*(\Psi_1)^*(z_1) + (\Psi_2)_*(\Psi_2)^*(z_2) \\ &= 2k\,\mathfrak{o}_S(\Psi_0)_*(z_0' - d\,\mathfrak{o}_{X_0}) + (\Psi_1)_*(z_1' - d\,\mathfrak{o}_{X_1}) + (\Psi_2)_*(z_2' - d\,\mathfrak{o}_{X_2}) \\ &= 2(k-3d)\mathfrak{o}_S + b_0 + b_1 + b_2 \qquad \text{in } A^2(S)_{\mathbb{Z}}, \end{aligned}$$

where $b_0+b_1+b_2$ is effective of degree $6d$. That is, we have

$$2z \quad \in \; S^{2k}_{6d}(S).$$

In particular, taking $d=0$ we obtain the following implication: if z is a degree k cycle with orbit O_z of dimension $\geq k$, then

$$2z = 2k\,\mathfrak{o}_S \quad \text{in } A^2(S)_{\mathbb{Z}}.$$

As $A^2_{hom}(S)_{\mathbb{Z}}$ is torsion free, it follows that

$$z = k\,\mathfrak{o}_S \quad \text{in } A^2(S)_{\mathbb{Z}}.$$

Step 2 (Existence): We now prove that the cycle $z=k\,\mathfrak{o}_S$ has orbit of dimension $\geq k$. This is the easier direction. Take $j\in\{0,1,2\}$, and let $\overline{C}\subset\overline{X}_j$ be any rational curve. Using Lemma 6.2, one finds that the curve $C:=(p_j)^{-1}(\overline{C})\subset S$ is a constant cycle curve, and any point $p\in C$ is such that $(\Psi_j)^*(p)=\mathfrak{o}_{X_j}$ and so p represents $\mathfrak{o}_S$. This proves the statement for $k=1$. For $k>1$, one notes that $C^{(k)}\subset S^{(k)}$ is contained in the orbit of $k\,\mathfrak{o}_S$. $\square$

Let Z be any smooth projective variety (say of dimension n), and let $z\in A^n_{hom}(Z)$ be a degree 0 0–cycle. It is known that z is *smash–nilpotent*, meaning that

$$z^{\times(N)} := \underbrace{z\times\ldots\times z}_{(N \text{ times})} = 0 \quad \text{in } A^{Nn}(Z^n)$$

for $N>>0$ [51], [52]. In the special case of the varieties under consideration in this note, one can give a precise estimate for the smash–nilpotence index N:

Proposition 6.5. *Let S be a triple $K3$ burger as in Theorem 5.1. Let $z\in A^2_{hom}(S)$ be a 0–cycle of the form*

$$z=z'-d\mathfrak{o}_S \quad \in A^2_{hom}(S),$$

where z' is an effective cycle of degree d. Then

$$z^{\times(3d+1)}:= \underbrace{z\times\ldots\times z}_{((3d+1)\ \ times)}=0 \ \ \textit{in}\ A^{6d+2}(S^{3d+1}).$$

Proof. The assumption means that z is in the subgroup $S^0_d(S)$ of the O'Grady filtration mentioned in the proof of Proposition 6.4 above. This implies that

$$(\Psi_j)^*(z) \quad \in S^0_d(X_j),\quad j=0,1,2.$$

For any positive integer r, Theorem 6.11 gives an isomorphism of Chow motives

$$t(S^r)\cong \bigoplus_{r_0+r_1+r_2=r} t((X_0)^{r_0})\otimes t((X_1)^{r_1})\otimes t((X_2)^{r_2}) \quad \text{in } \mathcal{M}_{\rm rat}$$

(induced by the Ψ_j), and so there is an isomorphism of Chow groups

$$\begin{aligned}&\sum_{r_0+r_1+r_2=r} \big(((\Psi_0)^{r_0})^*,((\Psi_1)^{r_1})^*,((\Psi_2)^{r_2})^*\big):\\ &A^{2r}(t(S)^{\otimes r}) \xrightarrow{\cong} \bigoplus_{r_0+r_1+r_2=r} A^{2r_0}(t(X_0)^{\otimes r_0})\otimes A^{2r_1}(t(X_1)^{\otimes r_1})\otimes A^{2r_2}(t(X_2)^{\otimes r_2}).\end{aligned}$$

In particular, this implies that there is an injection

$$\begin{aligned}&\sum_{r_0+r_1+r_2=r} \big(((\Psi_0)^{r_0})^*,((\Psi_1)^{r_1})^*,((\Psi_2)^{r_2})^*\big):\\ &A^{2r}(t(S)^{\otimes r}) \hookrightarrow \bigoplus_{r_0+r_1+r_2=r} A^{2r_0}((X_0)^{r_0})\otimes A^{2r_1}((X_1)^{r_1})\otimes A^{2r_2}((X_2)^{r_2}).\end{aligned} \tag{11}$$

Consider now the element $z^{\times r}$ for $r\geq 3d+1$. Since $z\in A^2_{hom}(S)=A^2(t(S))$, we have

$$z^{\times r} \quad \in \ A^{2r}(t(S)^{\otimes r}).$$

The image of $z^{\times r}$ in the right–hand side of the injection (11) is a sum of 0–cycles on the various products $(X_0)^{r_0}\times(X_1)^{r_1}\times(X_2)^{r_2}$. In each summand, one of the integers r_0,r_1,r_2 must be $\geq d+1$. The proposition now follows from the following lemma:

Lemma 6.6. (O'Grady [36]) *Let X be a $K3$ surface, and let $z\in S^0_d(X)$. Then*

$$z^{\times(d+1)}=0 \quad \textit{in}\ A^{2d+2}(X^{d+1}).$$

Proof. This is established in [36, (5.0.1)]. The reason is that z can be represented by a degree 0 0–cycle w on a curve $C\subset X$ of geometric genus d. This proves the lemma, for it is known since [52] that $w^{\times(d+1)}=0$ in $A^{d+1}(C^{d+1})$. $\square$

6.2. The canonical 0–cycle, bis

Definition 6.7. Let S be a triple $K3$ burger, and let X_j $(j=0,1,2)$ be the associated $K3$ surfaces. By definition, the subgroup of $K3$–type divisors $A^1_{K3}(S)_{\mathbb{Z}}\subset A^1(S)_{\mathbb{Z}}$ is defined as

$$A^1_{K3}(S)_{\mathbb{Z}} := \big((\Psi_0)_\ast A^1(X_0)_{\mathbb{Z}}\cap(\Psi_1)_\ast A^1(X_1)_{\mathbb{Z}}\big)+\big((\Psi_0)_\ast A^1(X_0)_{\mathbb{Z}}\cap(\Psi_2)_\ast A^1(X_2)_{\mathbb{Z}}\big) + \big((\Psi_1)_\ast A^1(X_1)_{\mathbb{Z}}\cap(\Psi_2)_\ast A^1(X_2)_{\mathbb{Z}}\big).$$

That is,

$$A^1_{K3}(S)_{\mathbb{Z}} = A^1(S)^{+++}_{\mathbb{Z}}\oplus A^1(S)^{++-}_{\mathbb{Z}}\oplus A^1(S)^{+-+}_{\mathbb{Z}}\oplus A^1(S)^{-++}_{\mathbb{Z}}.$$

Proposition 6.8. *Let S be a triple $K3$ burger as in Theorem 5.1. Let $D,D'\in A^1_{K3}(S)_{\mathbb{Z}}$. Then*

$$D\cdot D' = \deg(D\cdot D')\,\mathfrak{o}_S \quad \text{in } A^2(S)_{\mathbb{Z}}.$$

Proof. Since $A^2_{hom}(S)_{\mathbb{Z}}$ is torsion free [42], it suffices to prove the statement for Chow groups with $\mathbb{Q}$–coefficients. We have seen that

$$A^1_{K3}(S) = A^1(S)^{+++}\oplus A^1(S)^{++-}\oplus A^1(S)^{+-+}\oplus A^1(S)^{-++}.$$

Assuming that D and D' are in the same summand of this decomposition, we have

$$D\cdot D' \quad \in A^2(S)^{+++} = \mathbb{Q}[\mathfrak{o}_S],$$

and we are done.

Next, let us assume D is in the first summand and D' is in another summand (say the second). Then

$$D\cdot D' \quad \in A^2(S)^{++-}.$$

But $A^2(S)^{++-}=0$ (proof of Theorem 5.1), and so $D\cdot D'=0$.

Finally, let us assume D and D' are in two different summands and neither is in the first summand (say $D\in A^1(S)^{+-+}$ and $D'\in A^1(S)^{-++}$). Then

$$D\cdot D' \quad \in A^2(S)^{--+}.$$

We have seen (proof of Theorem 5.1) that $A^2(S)^{--+}$ is mapped isomorphically (under $(\Psi_2)^\ast$) to $A^2_{hom}(X_2)$, and so to prove that $D\cdot D'=0$, it suffices to prove that

$$(\Psi_2)^\ast(D\cdot D') \overset{??}{=} 0 \quad \text{in } A^2_{hom}(X_2).$$

To this end, recall that (by construction) $(\Psi_2)^* = (q_2)^*(p_2)_*$ (where $p_2\colon S \to \overline{X}_2$ is projection to the $K3$ surface with double points, and $q_2\colon X_2 \to \overline{X}_2$ is a resolution of singularities). Hence,

$$\begin{aligned}(\Psi_2)^*(D\cdot D') &= (q_2)^*(p_2)_*(D\cdot D')\\ &= (q_2)^*\big(\overline{F}\cdot(p_2)_*(D')\big)\\ &= (q_2)^*(\overline{F})\cdot(q_2)^*(p_2)_*(D')\\ &= 0 \qquad \text{in } A^2_{hom}(X_2).\end{aligned}$$

Here, $\overline{F} \in A^1(\overline{X}_2)$ is a divisor such that $D = (p_2)^*(\overline{F})$. The last line follows from the celebrated Beauville–Voisin result that

$$\big(A^1(X_2)\cdot A^1(X_2)\big) \cap A^2_{hom}(X_2) = 0$$

for any $K3$ surface X_2 [3]. □

Remark 6.9. The behavior displayed in Proposition 6.8 is remarkable, because the dimension of $A^1_{K3}(S)$ tends to be large. For example, let S be a triple $K3$ burger with $K^2=2$. Then $A^1(S)^{+++}$ coincides with $A^1(T)$, where

$$T := S/ < \sigma_0, \sigma_1, \sigma_2 > .$$

The surface T can be identified with a degree 4 hypersurface in $\mathbb{P}(1^3, 2)$. Hence, T is isomorphic to the double cover of $\mathbb{P}^2$ branched along a quartic curve. In case the quartic curve is smooth, one has $\dim A^1(T) = \dim H^2(T) = 8$ [46], and so

$$\dim A^1_{K3}(S) \geq \dim A^1(S)^{+++} = 8.$$

6.3. Bloch conjecture

Corollary 6.10. *Let S be a triple $K3$ burger as in Theorem 5.1, and let $\sigma_0, \sigma_1, \sigma_2$ be the three covering involutions. Let $f \in \mathrm{Aut}(S)$ be a finite-order automorphism that commutes with the σ_j, and such that*

$$f^* = \mathrm{id}\colon \quad H^{2,0}(S) \longrightarrow H^{2,0}(S).$$

Then also

$$f^* = \mathrm{id}\colon \quad A^2(S) \longrightarrow A^2(S).$$

Proof. Since f commutes with the σ_j, f induces finite–order automorphisms $f_j \in \mathrm{Aut}(X_j), j=0,1,2$ that are symplectic. Huybrechts has proven [20] that one has

$$(f_j)^* = \mathrm{id}\colon \quad A^2(X_j) \longrightarrow A^2(X_j) \quad (j=0,1,2).$$

Theorem 5.1, combined with the commutative diagram

$$\begin{array}{ccc} A^2_{hom}(S) & \xrightarrow{f^*} & A^2_{hom}(S) \\ \uparrow{\scriptstyle (\Psi_j)^*} & & \uparrow{\scriptstyle (\Psi_j)^*} \\ A^2_{hom}(X_j) & \xrightarrow{(f_j)^*} & A^2_{hom}(X_j) \end{array} \qquad (j=0,1,2)$$

implies that

$$f^* = \mathrm{id}\colon \quad A^2_{hom}(S) \longrightarrow A^2_{hom}(S).$$

Since the 1–dimensional subspace $A^2(S)^{+++}$ is fixed by f, this proves the corollary. $\square$

6.4. Finite–dimensionality

Corollary 6.11. *Let S be a triple burger as in Theorem 5.1, and let X_j be the associated $K3$ surfaces. The morphism of Chow motives*

$$(\Psi_0, \Psi_1, \Psi_2)\colon \quad t(X_0) \oplus t(X_1) \oplus t(X_2) \longrightarrow t(S) \quad \textit{in } \mathcal{M}_{\mathrm{rat}}$$

is an isomorphism. (Here, $t()$ denotes the transcendental part of the motive, as in Theorem 2.2.)

Proof. We may suppose S and the X_j are defined over some subfield $k \subset \mathbb{C}$ which is finitely generated over $\mathbb{Q}$. To prove the isomorphism of motives, it suffices to prove there is an isomorphism

$$\begin{aligned} &\big((\Psi_0)_*, (\Psi_1)_*, (\Psi_2)_*\big) \\ &\quad : \quad A^2_{hom}((X_0)_K) \oplus A^2_{hom}((X_1)_K) \oplus A^2_{hom}((X_2)_K) \xrightarrow{\cong} A^2_{hom}(S_K) \end{aligned}$$

for all function fields $K = k(Z)$ of varieties Z defined over k [22, Lemma 1.1]. This is equivalent to proving Claim 5.2 for the surface S_K. Since $\mathbb{C}$ is a universal domain, one can choose an embedding $K \subset \mathbb{C}$. As is well–known (cf. [5, Appendix to Lecture 1]), this induces an injection

$$A^2(S_K) \hookrightarrow A^2(S_{\mathbb{C}}),$$

and so Claim 5.2 for S_K follows from Claim 5.2 for $S_{\mathbb{C}}$. $\square$

Corollary 6.12. *Let S be as in Theorem 5.1, and assume*

$$\dim H^2_{tr}(S) \le 7.$$

Then S has finite–dimensional motive (in the sense of Kimura [27]). What's more, S has motive of abelian type (in the sense of [49]).

Proof. Let X_0, X_1, X_2 be the associated $K3$ surfaces. Recall (Proposition 3.2) that there is an isomorphism

$$H^2_{tr}(S) \cong H^2_{tr}(X_0) \oplus H^2_{tr}(X_1) \oplus H^2_{tr}(X_2).$$

The X_j being $K3$ surfaces, the dimension of $H^2_{tr}(X_j)$ is at least 2, and so the assumption on $H^2_{tr}(S)$ implies that

$$\dim H^2_{tr}(X_j) \le 3 \quad (j = 0, 1, 2).$$

It follows from [38] that the X_j have finite-dimensional motive. In view of Corollary 6.11, the motive $t(S)$ is isomorphic to $t(X_0) \oplus t(X_1) \oplus t(X_2)$, and so this implies the corollary.

To see that S has motive of abelian type, one remarks that the $K3$ surfaces X_j either have a Shioda–Inose structure, or are rationally dominated by a Kummer surface [45], [32]. This implies that their motive is actually a submotive of the motive of an abelian surface. □

Remark 6.13. In fairness, I hasten to add that I am not sure whether surfaces S as in Corollary 6.12 exist. Indeed, one might naively expect that inside the families

$$\mathcal{X}_j \longrightarrow B \quad (j = 0, 1, 2)$$

of $K3$ surfaces associated to the family $\mathcal{S} \to B$ (cf. Notation 3.13), ρ–maximal surfaces lie analytically dense (and so ρ–maximal triple $K3$ burgers would also be analytically dense). But to prove this, one would need to know a Torelli result for this type of $K3$ surfaces.

For this reason, Corollary 6.12 is only a conditional result.

7. Open questions

Question 7.1. Can one prove Torelli type theorems for families of triple $K3$ burgers as in Theorem 5.1? As noted in Remark 6.13, this would have interesting consequences for the distribution of Picard numbers, and for the existence of certain finite–dimensional motives.

Question 7.2. Let S be a triple $K3$ burger as in Theorem 5.1. I wonder whether a stronger version of Proposition 6.8 might be true: is it the case that (as for $K3$ surfaces)

$$A^1(S)_{\mathbb{Z}} \cdot A^1(S)_{\mathbb{Z}} = \mathbb{Z}[\mathfrak{o}_S] \quad \subset A^2(S)_{\mathbb{Z}}??$$

On a related note, does S have a multiplicative Chow–Künneth decomposition, in the sense of [44]?

Question 7.3. Let S be a triple $K3$ burger as in Theorem 5.1. Is it the case that (as for $K3$ surfaces) the second Chern class $c_2(T_S) \in A^2(S)$ lies in the subgroup $\mathbb{Q}[\mathfrak{o}_S]$?

Question 7.4. Let X be a $K3$ surface, and let F be a simple rigid vector bundle on X. Voisin has proven [56, Theorem 1.9] that $c_2(F) \in A^2(X)$ lies in the subgroup $\mathbb{Q}[\mathfrak{o}_X]$. Can one prove a similar statement for triple $K3$ burgers?

(Presumably, Voisin's argument for $K3$ surfaces can be adapted to triple $K3$ burgers? At least the "dimension of orbit" part goes through unchanged (Proposition 6.4). However, Voisin's argument also involves Riemann–Roch calculations, which rely on having trivial canonical bundle. I have not pursued this.)

Question 7.5. Let $\pi\colon \mathcal{S} \to B$ be a family of surfaces (i.e., a smooth projective morphism with 2–dimensional fibres). According to Deligne [8], there is a decomposition isomorphism

$$R\pi_*\mathbb{Q} \cong \bigoplus_i R^i\pi_*\mathbb{Q}[-i]$$

in the derived category of sheaves of $\mathbb{Q}$–vector spaces on B. If the fibres of π are $K3$ surfaces, then according to Voisin [53], one can choose an isomorphism that becomes *multiplicative* after shrinking the base B. Can one do the same for a family of triple $K3$ burgers?

(This is closely related to the existence of a multiplicative Chow–Künneth decomposition, cf. [50, Section 4].)

Question 7.6. What are the generic and maximal Picard numbers for the families of triple $K3$ burgers of Theorem 5.1?

Question 7.7. Constructing quadruple $K3$ burgers (i.e., surfaces satisfying the $m=4$ analogon of Definition 3.1) seems a daunting task.

(For example: if we suppose S is a canonical surface of general type with $p_g=4$ and $K^2=5$, then we know [15] that S is isomorphic to a quintic in $\mathbb{P}^3$ with rational double points. Consider the involutions

$$\begin{aligned}\sigma_0[x_0:x_1:x_2:x_3] &= [-x_0:x_1:x_2:x_3],\\ \sigma_1[x_0:x_1:x_2:x_3] &= [x_0:-x_1:x_2:x_3],\\ \sigma_2[x_0:x_1:x_2:x_3] &= [x_0:x_1:-x_2:x_3],\\ \sigma_3[x_0:x_1:x_2:x_3] &= [x_0:x_1:x_2:-x_3],\end{aligned}$$

If S is a hypersurface invariant under σ_j (i.e., the defining equation of S contains only even powers of x_j), the quotient $S/{<}\sigma_j{>}$ is a $K3$ surface with double points. However, clearly there is no quintic hypersurface invariant under all 4 involutions σ_j!)

The following is a weaker question: can one at least find general type surfaces S with $p_g(S){=}4$ such that the transcendental cohomology of S splits in 4 pieces of $K3$ type? And what about $p_g{>}4$?

Acknowledgments. Thanks to the editor for his patience. Thanks to the referee for constructive remarks and for spotting an important oversight in a prior version. Thanks to Kai and Len, for enjoying Sesamstraat as much as I do.

References

1. Barth, W., Hulek, K., Peters, C. and van de Ven, A., *Compact complex surfaces*, 2nd ed., Springer, Berlin, 2004.
2. Batyrev, V. and Cox, D., On the Hodge structure of projective hypersurfaces in toric varieties, *Duke Math. J.* **75** (1994), 293–338.
3. Beauville, A. and Voisin, C., On the Chow ring of a $K3$ surface, *J. Algebraic Geom.* **13** (2004), 417–426.
4. Beauville, A., On the splitting of the Bloch–Beilinson filtration, *Algebraic cycles and motives*, London Math. Soc. Lecture Notes **344**, Cambridge University Press, Cambridge, 2007.
5. Bloch, S., *Lectures on algebraic cycles*, Duke Univ. Press, Durham, 1980.
6. Bloch, S., Algebraic cycles and higher K–theory, *Adv. Math.* **61** (1986), 267–304.
7. Bloch, S., The moving lemma for higher Chow groups, *J. Algebraic Geom.* **3** (1994), 537–568.
8. Deligne, P., Théorème de Lefschetz et critères de dégénerescence de suites spectrales, *Publ. Math. Inst. Hautes Études Sci.* **35** (1968), 107–126.
9. Deligne, P., La conjecture de Weil pour les surfaces $K3$, *Invent. Math.* **15** (1972), 206–226.
10. Delorme, C., Espaces projectifs anisotropes, *Bull. Soc. Math. France* **103** (1975), 203–223.
11. Dolgachev, I., Weighted projective varieties, in *Group actions and vector fields, Vancouver 1981*, Springer Lecture Notes in Mathematics **956**, Springer, Berlin, Heidelberg, New York, 1982.

12. FU, L., On the action of symplectic automorphisms on the CH_0–groups of some hyper-Kähler fourfolds, *Math. Z.* **280** (2015), 307–334.
13. FULTON, W., *Intersection theory*, Springer, Berlin, Heidelberg, New York, Tokyo, 1984.
14. GARBAGNATI, A., Smooth double covers of $K3$ surfaces. arXiv:1605.03438.
15. HORIKAWA, E., On deformation of quintic surfaces, *Invent. Math.* **31** (1975), 43–85.
16. HORIKAWA, E., Algebraic surfaces of general type with small c_1^2, I, *Ann. Math.* **104** (1976), 357–387.
17. HORIKAWA, E., Algebraic surfaces of general type with small c_1^2, II, *Invent. Math.* **37** (1976), 121–155.
18. HORIKAWA, E., Algebraic surfaces of general type with small c_1^2, III, *Invent. Math.* **47** (1978), 209–248.
19. HORIKAWA, E., Algebraic surfaces of general type with small c_1^2, IV, *Invent. Math.* **50** (1979), 103–128.
20. HUYBRECHTS, D., Symplectic automorphisms of K3 surfaces of arbitrary finite order, *Math. Res. Lett.* **19** (2012), 947–951.
21. HUYBRECHTS, D., *Lectures on $K3$ surfaces*, Cambridge University Press, Cambridge, 2016.
22. HUYBRECHTS, D., Motives of derived equivalent $K3$ surfaces, *Abh. Math. Sem. Hamburg.*
23. ILIEV, V., A note on certain surfaces, *Bull. Lond. Math. Soc.* **16** (1984), 135–138.
24. JANNSEN, U., *Motivic sheaves and filtrations on Chow groups*, Motives, Proceedings of Symposia in Pure Mathematics, Part 1 **55**, 1994.
25. JANNSEN, U., On finite–dimensional motives and Murre's conjecture, *Algebraic cycles and motives*, Cambridge University Press, Cambridge, 2007.
26. KAHN, B., MURRE, J. and PEDRINI, C., On the transcendental part of the motive of a surface, *Algebraic cycles and motives*, Cambridge University Press, Cambridge, 2007.
27. KIMURA, S., Chow groups are finite dimensional, in some sense, *Math. Ann.* **331** (2005), 173–201.
28. KYNEV, V., An example of a simply–connected surface of general type for which the local Torelli theorem does not hold, *C. R. Acad. Bulgare Sci.* **30** (1977), 323–325.
29. LATERVEER, R., Algebraic cycles and Todorov surfaces, *Kyoto J. Math.* (to appear). arXiv:1609.09629.
30. LATERVEER, R., Algebraic cycles and special Horikawa surfaces, preprint.
31. LEVINE, M., Techniques of localization in the theory of algebraic cycles, *J. Algebraic Geom.* **10** (2001), 299–363.
32. MORRISON, D., On $K3$ surfaces with large Picard number, *Invent. Math.* **75** (1984), 105–121.
33. MORRISON, D., On the moduli of Todorov surfaces, *Algebraic Geometry and Commutative Algebra in Honor of Masayoshi Nagata* **1**, Kinokuniya, Tokyo, 1988.
34. MURRE, J., On a conjectural filtration on the Chow groups of an algebraic variety, parts I and II, *Indag. Math. (N.S.)* **4** (1993), 177–201.
35. MURRE, J., NAGEL, J. and PETERS, C., *Lectures on the theory of pure motives*, Amer. Math. Soc. University Lecture Series **61**, Amer. Math. Soc., Providence, 2013.

36. O'Grady, K., Moduli of sheaves and the Chow groups of $K3$ surfaces, *J. Math. Pures Appl.* **100** (2013), 701–718.
37. Pearlstein, G. and Zhang, Z., A generic global Torelli theorem for certain Horikawa surfaces. arXiv:1702.06204v1.
38. Pedrini, C., On the finite dimensionality of a $K3$ surface, *Manuscripta Math.* **138** (2012), 59–72.
39. Peters, C. and Steenbrink, J., *Mixed Hodge structures*, Springer, Berlin, Heidelberg, New York, 2008.
40. Reid, M., Surfaces with $p_g=3$, $K^2=4$ according to E. Horikawa and D. Dicks. Available from homepages: warwick.ac.uk/masda/surf/more/Dicks.pdf.
41. Rito, C., A note on Todorov surfaces, *Osaka J. Math.* **46** (2009), 685–693.
42. Rojtman, A., The torsion of the group of 0–cycles modulo rational equivalence, *Ann. Math.* **111** (1980), 553–569.
43. Scholl, T., *Classical motives*, Motives, Proceedings of Symposia in Pure Mathematics, Part 1 **55**, Springer, Berlin, 1994.
44. Shen, M. and Vial, C., The Fourier transform for certain hyperKähler fourfolds, *Mem. Amer. Math. Soc.* **240** (2016), 1139.
45. Shioda, T. and Inose, H., On singular $K3$ surfaces, in *Complex Analysis and Algebraic Geometry*, pp. 119–136, Iwanami Shouten, Tokyo, 1977.
46. Steenbrink, J., On the Picard group of certain smooth surfaces in weighted projective spaces, in *Algebraic Geometry, Proceedings, La Rábida 1981*, Springer Lecture Notes in Mathematics **961**, pp. 302–313, Springer, Berlin, 1982.
47. Todorov, A., Surfaces of general type with $p_g=1$ and $(K,K)=1$, *Ann. Sci. Éc. Norm. Supér.* **13** (1980), 1–21.
48. Totaro, B., Chow groups, Chow cohomology, and linear varieties, *Forum Math. Sigma* **1** (2014), e1.
49. Vial, C., Remarks on motives of abelian type, *Tohoku Math. J.* **69** (2017), 195–220.
50. Vial, C., On the motive of some hyperkähler varieties, *J. Reine Angew. Math.* **725** (2017), 235–247.
51. Voevodsky, V., A nilpotence theorem for cycles algebraically equivalent to zero, *Int. Math. Res. Not.* **4** (1995), 187–198.
52. Voisin, C., Remarks on zero–cycles of self–products of varieties, *Moduli of vector bundles*, Proceedings of the Taniguchi Congress, Marcel Dekker, New York, Basel, Hong Kong, 1994.
53. Voisin, C., Chow rings and decomposition theorems for $K3$ surfaces and Calabi–Yau hypersurfaces, *Geom. Topol.* **16** (2012), 433–473.
54. Voisin, C., The generalized Hodge and Bloch conjectures are equivalent for general complete intersections, *Ann. Sci. Éc. Norm. Supér.* **46** (2013), 449–475.
55. Voisin, C., Bloch's conjecture for Catanese and Barlow surfaces, *J. Differential Geom.* **97** (2014), 149–175.
56. Voisin, C., Rational equivalence of 0–cycles on $K3$ surfaces and conjectures of Huybrechts and O'Grady, *Recent advances in algebraic geometry*, Cambridge University Press, Cambridge, 2014.
57. Voisin, C., The generalized Hodge and Bloch conjectures are equivalent for general complete intersections, II, *J. Math. Sci. Univ. Tokyo* **22** (2015), 491–517.
58. Voisin, C., *Chow Rings, Decomposition of the Diagonal, and the Topology of Families*,

Princeton University Press, Princeton and Oxford, 2014.
59. ZHANG, D.-Q., Algebraic surfaces with log–canonical singularities and the fundamental groups of their smooth parts, *Trans. Amer. Math. Soc.* **348** (1996), 4175–4184.

Robert Laterveer
Institut de Recherche Mathématique Avancée
CNRS – Université de Strasbourg
7 Rue René Descartes
67084 Strasbourg Cedex
France
robert.laterveer@math.unistra.fr

Received April 25, 2017
in revised form April 19, 2018