A characterization of Herglotz-Nevanlinna functions in two variables via integral representations

Annemarie Luger and Mitja Nedic

Abstract. We derive an integral representation for Herglotz-Nevanlinna functions in two variables, which provides a complete characterization of this class in terms of a real number, two non-negative numbers and a positive measure satisfying certain conditions. Further properties of the class of representing measures are also discussed.

1. Introduction

Herglotz-Nevanlinna functions in one variable are functions analytic in the upper half-plane having non-negative imaginary part. This class has been very well studied during the last century and has proven very useful in many applications. It seems natural to consider corresponding functions in several variables, i.e. analytic functions that have non-negative imaginary part if all variables lie in the upper half-plane. From the point of view of applications, such functions are very interesting when considering linear passive systems with several parameters.

In the treatment of Herglotz-Nevanlinna functions in one variable, a very strong tool is the classical result that these functions can be characterized via an integral representation, cf. Theorem 2.1. It is hence a natural question to ask for a corresponding representation for functions of several variables.

Already almost 50 years ago, the form of such an integral representation was suggested by Vladimirov, see e.g. [8]. Using the very heavy machinery of classical

distribution theory, it is shown that every Herglotz-Nevanlinna function can be written in such a form, but it is a priory assumed that the measure appearing in the formula is the boundary measure of the function. Thus, the drawback in these results is that they do not specify the properties of the measure and hence cannot provide a characterization as in the case of only one variable. In [9], the authors use a different approach in order to find an integral characterization, however, the obtained representation becomes much more involved. In view of the present result, one can in fact say that both the representation and the conditions on the measure are *too* complicated since it turns out that many terms there actually vanish or simplify radically, see Remark 3.4 for more details.

Recently, the question of characterizations was taken up again by Agler, McCarthy and Young in [1], see also Agler, Tully-Doyle and Young in [2]. They found a characterization via operator representations, however, only for certain subclasses of functions satisfying an asymptotic condition.

In the present paper, we solve the characterization problem for the whole class of Herglotz-Nevanlinna functions in two variables by deriving an integral representation together with conditions on the representing measure. Even if these representations are of the same form as in [8], our result contains considerably more information, since we obtain a full (but simple) description of all representing measures. Moreover, the proof is shorter and more elementary in the sense that is uses only Theorems 2.1 and 2.5 which are, in their essence, built upon Cauchy's integral formula and Helly's selection principle. We use, as in the classical proof for the one dimensional case, a corresponding result for the polydisk and then a transformation of variables. However, unlike in the one variable case, the terms arising from the boundary of the area of integration are quite delicate and need very careful treatment in order to simplify the representation to the desired form.

It appears that the requirements on the representing measures have quite strong consequences, which are discussed in Section 4.

2. Notations and a brief recap of known results

As usual, $\mathbb{D}$ denotes the unit disk in the complex plane while $\mathbb{C}^+$ denotes the upper and $\mathbb{C}_+$ denotes the right half-plane. Throughout this paper, we will use the convention that z denotes the complex variable that lies in the upper half-plane while w denotes the variable that lies in the disk. We recall also the fact that the unit disk and the upper half-plane are biholomorphic. One map achieving this is $\varphi\colon \mathbb{C}^+ \to \mathbb{D}$ defined as $\varphi\colon z \mapsto \frac{z-\mathrm{i}}{z+\mathrm{i}}$. Its inverse is then given as $\varphi^{-1}\colon w \mapsto \mathrm{i}\frac{1+w}{1-w}$. Note also that φ is a bijection between the sets $\mathbb{R}$ and $S^1 \setminus \{1\}$.
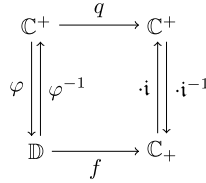
Figure 1. The relationship between q and f.

It is often convenient to consider Herglotz-Nevanlinna functions that do not attain real values. In particular, every such function $q\colon \mathbb{C}^+ \to \mathbb{C}^+$ then uniquely determines a function $f\colon \mathbb{D} \to \mathbb{C}_+$ with respect to the biholomorphisms φ and $\cdot\mathrm{i}$, as elaborated by the diagram in Figure 1. The converse also holds; a function f uniquely determines q with respect to the same biholomorphisms. It can be shown that the only Herglotz-Nevanlinna functions that are excluded from this correspondence are in fact real-constant functions.

We recall now the integral representation theorem due to Nevanlinna [7], which was presented in its current form by Cauer [3].

Theorem 2.1. (Nevanlinna) *A function $q\colon \mathbb{C}^+ \to \mathbb{C}$ is a Herglotz-Nevanlinna function if and only if q can be written as*

$$(2.1) \qquad q(z) = a + bz + \frac{1}{\pi} \int_{-\infty}^{\infty} \left(\frac{1}{t-z} - \frac{t}{1+t^2} \right) d\mu(t),$$

where $a \in \mathbb{R}$, $b \geq 0$ and μ is a positive Borel measure on $\mathbb{R}$ satisfying the growth condition

$$(2.2) \qquad \int_{-\infty}^{\infty} \frac{1}{1+t^2} \, d\mu(t) < \infty.$$

Remark 2.2. Moreover, the parameters a, b and μ are unique with these properties.

The importance and beauty of this theorem is that it gives a complete characterization of the function q in terms of the numbers a and b and the measure μ. But it also provides a tool for handling Herglotz-Nevanlinna functions. We mention the following property that will be of use to us further on.

Proposition 2.3. *Let q be a Herglotz-Nevanlinna function. Then the non-tangential limit*

$$(2.3) \qquad \lim_{z \to \infty} \frac{q(z)}{z} = b,$$

where $b \geq 0$ is the number that appears in representation (2.1).

Recall that $z\widehat{\rightarrow}\infty$ is a shorthand notation for $|z|{\rightarrow}\infty$ in the Stoltz domain $\{z{\in}\mathbb{C}^+ \mid \theta{\leq}\arg(z){\leq}\pi{-}\theta\}$ for any $\theta{\in}(0,\frac{\pi}{2}]$.

Let us denote by $\mathbb{C}^{+2}{:=}\{z{=}(z_1,z_2){\in}\mathbb{C}^2 \mid \mathrm{Im}[z_1]{>}0, \mathrm{Im}[z_2]{>}0\}$ the poly-upper half-plane in $\mathbb{C}^2$. Our main object of interest is the following class of functions in two variables.

Definition 2.4. A holomorphic function $q\colon \mathbb{C}^{+2}{\rightarrow}\mathbb{C}$ with non-negative imaginary part is called a *Herglotz-Nevanlinna function* (*in two variables*).

In the situation of one variable, a standard proof of Theorem 2.1 uses the Riesz-Herglotz theorem, see e.g. [3], that gives an integral representation for functions on the unit disk with positive real part. It is then possible to use the biholomorphisms discussed earlier to return to functions defined on the upper half-plane.

In order to apply the same strategy in several variables, we use a generalization of the Riesz-Herglotz theorem, presented first by Korányi and Pukánszky [6], that completely characterizes functions defined on the unit polydisk in $\mathbb{C}^n$ that have positive real part. It seems that Vladimirov has independently the same result in [4], which is used in [9]. Here, we present the theorem only for $n{=}2$ and with slightly different notation that is more inclined towards our purpose of giving a representation of Herglotz-Nevanlinna functions.

Theorem 2.5. *A function f on the unit polydisk $\mathbb{D}^2$ is holomorphic and has non-negative real part if and only if f can be written as*

$$
\begin{aligned}
f(w_1,w_2) = {}& \mathfrak{i}\,\mathrm{Im}[f(0,0)] \\
& + \frac{1}{4\pi^2}\iint_{[0,2\pi)^2}\left(\frac{2}{(1-w_1\mathrm{e}^{-\mathfrak{i}s_1})(1-w_2\mathrm{e}^{-\mathfrak{i}s_2})}-1\right)d\nu(s_1,s_2),
\end{aligned}
$$
(2.4)

where ν is a finite positive Borel measure on $[0,2\pi)^2$ satisfying the condition that

$$
\iint_{[0,2\pi)^2}\mathrm{e}^{\mathfrak{i}m_1s_1}\mathrm{e}^{\mathfrak{i}m_2s_2}\,d\nu(s_1,s_2)=0
$$
(2.5)

for every pair of indices $m_1,m_2{\in}\mathbb{Z}$ satisfying $m_1m_2{<}0$.

3. The theorem in two variables

Before presenting the main theorem, we introduce some notation. Denote by K the kernel function depending on $(z_1,z_2){\in}\mathbb{C}^{+2}$ and $(t_1,t_2){\in}\mathbb{R}^2$, defined as

$$
K\big((z_1,z_2),(t_1,t_2)\big):=-\frac{\mathfrak{i}}{2}\bigg(\frac{1}{t_1-z_1}-\frac{1}{t_1+\mathfrak{i}}\bigg)\bigg(\frac{1}{t_2-z_2}-\frac{1}{t_2+\mathfrak{i}}\bigg)+\frac{\mathfrak{i}}{(1+t_1^2)(1+t_2^2)}.
$$

We will also need the Poisson kernel of $\mathbb{C}^{+2}$, which we prefer to write using complex coordinates as

$$\mathcal{P}\big((z_1, z_2), (t_1, t_2)\big) := \frac{\operatorname{Im}[z_1]}{|t_1 - z_1|^2} \frac{\operatorname{Im}[z_2]}{|t_2 - z_2|^2}.$$

Note that $\mathcal{P} > 0$ for any $(z_1, z_2) \in \mathbb{C}^{+2}$ and any $(t_1, t_2) \in \mathbb{R}^2$.

The main result of this paper is as follows.

Theorem 3.1. *A function $q \colon \mathbb{C}^{+2} \to \mathbb{C}$ is a Herglotz-Nevanlinna function in two variables if and only if q can be written as*

$$(3.1) \qquad q(z_1, z_2) = a + b_1 z_1 + b_2 z_2 + \frac{1}{\pi^2} \iint_{\mathbb{R}^2} K\big((z_1, z_2), (t_1, t_2)\big) \, d\mu(t_1, t_2),$$

where $a \in \mathbb{R}$, $b_1, b_2 \geq 0$, and μ is a positive Borel measure on $\mathbb{R}^2$ satisfying the growth condition

$$(3.2) \qquad \iint_{\mathbb{R}^2} \frac{1}{(1+t_1^2)(1+t_2^2)} \, d\mu(t_1, t_2) < \infty$$

and the so-called Nevanlinna condition

$$(3.3) \qquad \iint_{\mathbb{R}^2} \operatorname{Re}\left[\left(\frac{1}{t_1 - \overline{z_1}} - \frac{1}{t_1 + \mathfrak{i}}\right)\left(\frac{1}{t_2 - z_2} - \frac{1}{t_2 - \mathfrak{i}}\right)\right] d\mu(t_1, t_2) = 0$$

for all $(z_1, z_2) \in \mathbb{C}^{+2}$.

Remark 3.2. Moreover, the numbers a, b_1, b_2 and the measure μ are uniquely determined, as it is shown in Corollary 3.5 and Proposition 4.1 respectively.

Remark 3.3. Observe that there is no analogue to the Nevanlinna condition (3.3) for functions of one variable.

Proof. Let us assume first that q is a Herglotz-Nevanlinna function and consider the possibility that q attains a real value. Then there exists a point $(\zeta_1, \zeta_2) \in \mathbb{C}^{+2}$ such that $\operatorname{Im}[q(\zeta_1, \zeta_2)] = 0$. Since q is a Herglotz-Nevanlinna function, it is holomorphic and its imaginary part $\operatorname{Im}[q] \geq 0$ is therefore pluriharmonic. It follows now from the maximum principle for pluriharmonic functions that $\operatorname{Im}[q] \equiv 0$ on $\mathbb{C}^{+2}$, and hence the function q admits a representation of the form (3.1) with $a = q(\zeta_1, \zeta_2)$, $b_1 = b_2 = 0$ and $\mu \equiv 0$. Thus, the theorem holds in this case.

We may now restrict ourselves to the case when q does not attain a real value. Then there exists a function f on $\mathbb{D}^2$ with positive real part, such that

$$q(z_1, z_2) = \mathfrak{i} f(\varphi(z_1), \varphi(z_2)),$$

Figure 2. The area of integration.

where φ is given as in Section 2. Using representation (2.4) of the function f yields

$$
\begin{aligned}
q(z_1, z_2) = {}&-\mathrm{Im}[f(0,0)] \\
&+\frac{\mathrm{i}}{4\pi^2} \iint_{[0,2\pi)^2} \left(\frac{2}{(1-\varphi(z_1)\mathrm{e}^{-\mathrm{i}s_1})(1-\varphi(z_2)\mathrm{e}^{-\mathrm{i}s_2})} - 1 \right) d\nu(s_1, s_2).
\end{aligned}
$$

We obtain the first term in representation (3.1) by setting $a := -\mathrm{Im}[f(0,0)] \in \mathbb{R}$.

Before transforming the area of integration to $\mathbb{R}^2$, we divide the integral over $[0, 2\pi)^2$, which is shown in Figure 2, into four parts and investigate each part separately.

Considering first the integral over the open square $(0, 2\pi)^2$, we make a change of variables where $\mathrm{e}^{\mathrm{i}s_i} = \frac{t_i - \mathrm{i}}{t_i + \mathrm{i}}$ for $i = 1, 2$. The area of integration thus transforms into $\mathbb{R}^2$ and the measure ν transforms into a measure μ, related by the chosen change of variables as

$$
d\nu(s_1, s_2) = \frac{4}{(1+t_1^2)(1+t_2^2)}\, d\mu(t_1, t_2).
$$

As an immediate consequence of this transformation, we see that the measure μ satisfies condition (3.2) since ν is a finite measure and

$$
\iint_{\mathbb{R}^2} \frac{1}{(1+t_1^2)(1+t_2^2)}\, d\mu(t_1, t_2) = \frac{1}{4} \iint_{(0,2\pi)^2} d\nu(s_1, s_2) < \infty.
$$

The integral over the open square $(0, 2\pi)^2$ thus becomes

$$
\begin{aligned}
\frac{\mathrm{i}}{4\pi^2} \iint_{(0,2\pi)^2} &\left(\frac{2}{(1-\varphi(z_1)\mathrm{e}^{-\mathrm{i}s_1})(1-\varphi(z_2)\mathrm{e}^{-\mathrm{i}s_2})} - 1 \right) d\nu(s_1, s_2) \\
&= \frac{1}{\pi^2} \iint_{\mathbb{R}^2} K\big((z_1, z_2), (t_1, t_2)\big)\, d\mu(t_1, t_2),
\end{aligned}
$$

where the equality between the two expressions comes exclusively from symbolic manipulations of the left-hand side along with the discussed change of variables. This gives us the integral term of representation (3.1).

We now consider the part of the integral that runs over one side of the square, namely $\{0\}\times(0,2\pi)$. Let us denote $d\nu_1(s_2):=d\nu(0,s_2)$ and let μ_1 be a measure on $\mathbb{R}$ related to ν_1 as

$$d\nu_1(s_2) = \frac{2}{1+t_2^2}\,d\mu_1(t_2).$$

Thus

$$\frac{\mathrm{i}}{4\pi^2}\int_{\{0\}\times(0,2\pi)}\left(\frac{2}{(1-\varphi(z_1)\mathrm{e}^{-\mathrm{i}s_1})(1-\varphi(z_2)\mathrm{e}^{-\mathrm{i}s_2})}-1\right)d\nu(s_1,s_2)$$
$$= \frac{1}{2\pi^2}\int_{-\infty}^{\infty}\left(\frac{z_1+\mathrm{i}}{2\mathrm{i}}\frac{1+t_2z_2}{t_2-z_2}+\frac{z_1-\mathrm{i}}{2\mathrm{i}}\frac{1+t_2\mathrm{i}}{t_2-\mathrm{i}}\right)\frac{1}{1+t_2^2}\,d\mu_1(t_2).$$

For simplicity, we introduce the function q_1 defined as

$$q_1(\zeta):=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1+t\zeta}{t-\zeta}\frac{1}{1+t^2}\,d\mu_1(t).$$

The above integral can then be written as

$$\frac{1}{2\pi}\left(\frac{z_1+\mathrm{i}}{2\mathrm{i}}q_1(z_2)+\frac{z_1-\mathrm{i}}{2\mathrm{i}}q_1(\mathrm{i})\right).$$

We note that q_1 is a Herglotz-Nevanlinna function in one variable with the numbers a and b from representation (2.1) both equal to 0 and the measure equal to μ_1.

An analogous procedure for the other side of the square gives that

$$\frac{\mathrm{i}}{4\pi^2}\int_{(0,2\pi)\times\{0\}}\left(\frac{2}{(1-\varphi(z_1)\mathrm{e}^{-\mathrm{i}s_1})(1-\varphi(z_1)\mathrm{e}^{-\mathrm{i}s_2})}-1\right)d\nu(s_1,s_2)$$
$$= \frac{1}{2\pi}\left(\frac{z_2+\mathrm{i}}{2\mathrm{i}}q_2(z_1)+\frac{z_2-\mathrm{i}}{2\mathrm{i}}q_2(\mathrm{i})\right)$$

where the function q_2 are defined in an analogous way as in the previous case.

Finally, integration over the corner point $\{0\}\times\{0\}$ gives that

$$\frac{\mathrm{i}}{4\pi^2}\int_{\{0\}\times\{0\}}\left(\frac{2}{(1-\varphi(z_1)\mathrm{e}^{-\mathrm{i}s_1})(1-\varphi(z_2)\mathrm{e}^{-\mathrm{i}s_2})}-1\right)d\nu(s_1,s_2)$$
$$= \frac{\mathrm{i}}{4\pi^2}\left(\frac{2}{(2\mathrm{i})^2}(z_1+\mathrm{i})(z_2+\mathrm{i})-1\right)\nu(\{(0,0)\}).$$

Hence, we have so far arrived at a representation of the function q of the form

$$q(z_1, z_2) = a + \frac{i}{4\pi^2}\left(\frac{2}{(2i)^2}(z_1+i)(z_2+i)-1\right)\nu(\{(0,0)\})$$

$$+\frac{1}{2\pi}\left(\frac{z_2+i}{2i}q_2(z_1)+\frac{z_2-i}{2i}q_2(i)\right)$$

$$+\frac{1}{2\pi}\left(\frac{z_1+i}{2i}q_1(z_2)+\frac{z_1-i}{2i}q_1(i)\right)$$

$$(3.4) \qquad\qquad +\frac{1}{\pi^2}\iint_{\mathbb{R}^2}K\big((z_1,z_2),(t_1,t_2)\big)\,d\mu(t_1,t_2).$$

While the first and last part are already as desired, we still have to show that the middle three terms indeed give the two linear terms from representation (3.1). This will be done by showing that the functions q_1 and q_2 actually are of a very particular form.

Let $\alpha\in\mathbb{C}^+$ and consider the function $\tilde{q}_1$ defined by fixing the second variable, i.e. $\tilde{q}_1(z):=q(z,\alpha)$ for $z\in\mathbb{C}^+$, which is a Herglotz-Nevanlinna function in one variable. In view of Proposition 2.3, we consider the non-tangential limit

$$\lim_{z\hat{\to}\infty}\frac{\tilde{q}_1(z)}{z} = \lim_{z\hat{\to}\infty}\frac{a}{z}$$

$$+\lim_{z\hat{\to}\infty}\frac{i}{(2\pi)^2}\frac{1}{z}\left(\frac{2}{(2i)^2}(z+i)(\alpha+i)-1\right)\nu(\{(0,0)\})$$

$$+\lim_{z\hat{\to}\infty}\frac{1}{2\pi}\left(\frac{\alpha+i}{2iz}q_2(z)+\frac{\alpha-i}{2iz}q_2(i)\right)$$

$$+\lim_{z\hat{\to}\infty}\frac{1}{2\pi}\left(\frac{z+i}{2iz}q_1(\alpha)+\frac{z-i}{2iz}q_1(i)\right)$$

$$+\lim_{z\hat{\to}\infty}\frac{1}{\pi^2}\iint_{\mathbb{R}^2}\frac{K\big((z,\alpha),(t_1,t_2)\big)}{z}\,d\mu(t_1,t_2)$$

$$(3.5) \qquad = -\frac{i}{4\pi^2}\frac{1}{2}(\alpha+i)\nu(\{(0,0)\})+\frac{1}{2\pi}\frac{q_1(\alpha)+q_1(i)}{2i}.$$

Here, we used that

$$\lim_{z\hat{\to}\infty}\frac{q_2(z)}{z}=0$$

by Proposition 2.3 and that the last term vanishes as the interchange of the limit and integral is valid as the assumptions of Lebesgue's dominated convergence theorem are satisfied.

Again, Proposition 2.3 now implies that the limit (3.5) is non-negative, i.e. for any $\alpha\in\mathbb{C}^+$ it holds

$$(3.6) \qquad \frac{1}{2\pi}\frac{q_1(\alpha)+q_1(i)}{2i}-\frac{i}{4\pi^2}\frac{1}{2}(\alpha+i)\nu(\{(0,0)\})\geq 0.$$

Choosing, in particular, $\alpha = i$, this implies that

$$\frac{1}{2\pi}\frac{2q_1(i)}{2i} + \frac{1}{4\pi^2}\nu(\{(0,0)\}) \geq 0.$$

But this is only possible if $q_1(i) = d_1 i$ for some $d_1 \geq 0$. The left hand side of inequality (3.6) now takes the form

$$\frac{1}{2\pi}\frac{q_1(\alpha) + d_1 i}{2i} - \frac{i}{4\pi^2}\frac{1}{2}(\alpha + i)\nu(\{(0,0)\}).$$

This is a holomorphic function in the variable $\alpha \in \mathbb{C}^+$ which is real valued and hence constant. This implies that

$$q_1(\alpha) = 2\pi i b_1 - \frac{1}{2\pi}(\alpha + i)\nu(\{(0,0)\})$$

for some $b_1 \in \mathbb{R}$. Recall that, by definition, the function q_1 is a Herglotz-Nevanlinna function without a linear term, i.e. the number b from representation (2.1) is equal to 0, hence, by Proposition 2.3, it holds that

$$\lim_{\alpha \hat{\to} \infty}\frac{q_1(\alpha)}{\alpha} = 0.$$

This implies that $\nu(\{(0,0)\}) = 0$ and $q_1(\alpha) = 2\pi i b_1$ where $b_1 = \frac{d_1}{2\pi} \geq 0$.

In the same way, we fix now the first variable $\beta \in \mathbb{C}^+$ and consider the function $\tilde{q}_2$ defined as $\tilde{q}_2(z) := q(\beta, z)$ for $z \in \mathbb{C}^+$. The same reasoning gives that $q_2(\beta) = 2\pi i b_2$ for some $b_2 \geq 0$.

Returning to representation (3.4), we see that the second term is equal to 0, the third term becomes $b_2 z_2$, while the fourth term becomes $b_1 z_1$. This completes representation (3.1).

It remains to show that the measure μ satisfies condition (3.3). We begin by recalling that ν satisfies condition (2.5), which implies that

$$\iint_{[0,2\pi)^2}\sum_{(n_1,n_2)\in\mathbb{N}^2}\overline{w_1}^{n_1}w_2^{n_2}e^{in_1 s_1}e^{-in_2 s_2}\,d\nu(s_1,s_1) = 0$$

for any $(w_1, w_2) \in \mathbb{D}^2$, since a geometric series permits the interchange of integration and summation. An analogous statement holds also for the conjugate of the above series. We thus conclude that

$$(3.7) \qquad \iint_{[0,2\pi)^2}\mathrm{Re}\left[\frac{\overline{w_1}w_2 e^{is_1}e^{-is_2}}{(1 - \overline{w_1}e^{is_1})(1 - w_2 e^{-is_2})}\right]d\nu(s_1,s_2) \equiv 0.$$

The proof of Theorem 3.1 so far will allow us to change the area of integration in (3.7) into the open square. To this end, we begin by splitting the area of integration

in formula (3.7) into four parts as previously. The integral over the set $\{0\}\times\{0\}$ vanishes since we have shown that $\nu(\{(0,0)\})=0$.

The integrals over the sets $\{0\}\times(0,2\pi)$ and $(0,2\pi)\times\{0\}$ are also equal to 0. To see this, recall that the functions q_1 and q_2, which were Herglotz-Nevanlinna functions in one variable defined via integrals over these lines, have been shown to be identically equal to $d_1\mathfrak{i}$ and $d_2\mathfrak{i}$ respectively, where $d_1,d_2\geq 0$. Since the measure appearing in the representation of a Herglotz-Nevanlinna function in one variable is unique, the measures μ_1 and μ_2 have to be equal to $d_1\lambda_\mathbb{R}$ and $d_2\lambda_\mathbb{R}$ respectively, where $\lambda_\mathbb{R}$ denotes the Lebesgue measure on $\mathbb{R}$. Integration over the sets $\{0\}\times(0,2\pi)$ and $(0,2\pi)\times\{0\}$ in formula (3.7) thus reduces to integrals of the from

$$\int_{[0,2\pi)} e^{\mathfrak{i}ns}\,ds,$$

which vanish for $n\in\mathbb{Z}\setminus\{0\}$.

Finally, the integral over the open square $(0,2\pi)^2$ remains and gives

$$(3.8) \qquad \iint_{(0,2\pi)^2} \mathrm{Re}\left[\frac{\overline{w_1}w_2e^{\mathfrak{i}s_1}e^{-\mathfrak{i}s_2}}{(1-\overline{w_1}e^{\mathfrak{i}s_1})(1-w_2e^{-\mathfrak{i}s_2})}\right]d\nu(s_1,s_2)\equiv 0.$$

We can now change the area of integration in formula (3.8) to $\mathbb{R}^2$ with the same change of coordinates used throughout the proof and expressed with the function φ from Figure 1. The integrand then transforms as

$$\frac{\overline{w_1}w_2e^{\mathfrak{i}s_1}e^{-\mathfrak{i}s_2}}{(1-\overline{w_1}e^{\mathfrak{i}s_1})(1-w_2e^{-\mathfrak{i}s_2})}=\frac{(\overline{z_1}+\mathfrak{i})(z_2-\mathfrak{i})(t_1-\mathfrak{i})(t_2+\mathfrak{i})}{4(t_1-\overline{z_1})(t_2-z_2)}.$$

We also get a factor

$$\frac{4}{(1+t_1^2)(1+t_2^2)}$$

that comes from $d\varphi$. Formula (3.8) thus transforms into

$$\iint_{\mathbb{R}^2}\mathrm{Re}\left[\frac{(\overline{z_1}+\mathfrak{i})(z_2-\mathfrak{i})(t_1-\mathfrak{i})(t_2+\mathfrak{i})}{4(t_1-\overline{z_1})(t_2-z_2)}\right]\frac{4}{(1+t_1^2)(1+t_2^2)}\,d\mu(t_1,t_2)\equiv 0.$$

Since

$$\frac{(\overline{z_1}+\mathfrak{i})(z_2-\mathfrak{i})(t_1-\mathfrak{i})(t_2+\mathfrak{i})}{4(t_1-\overline{z_1})(t_2-z_2)}\cdot\frac{4}{(1+t_1^2)(1+t_2^2)}=\left(\frac{1}{t_1-\overline{z_1}}-\frac{1}{t_1+\mathfrak{i}}\right)\left(\frac{1}{t_2-z_2}-\frac{1}{t_2-\mathfrak{i}}\right),$$

this implies that the measure μ does indeed satisfy condition (3.3). We have thus proven that every Herglotz-Nevanlinna function in two variables admits a representation of the form (3.1).

Conversely, let q be a function defined on $\mathbb{C}^{+2}$ by representation (3.1) with the number a, b_1, b_2 and the measure μ satisfying all the listed properties. The integral that appears in representation (3.1) is a well-defined expression since the measure μ satisfies condition (3.2). It is then easy to see that a function q defined in this way is holomorphic on $\mathbb{C}^{+2}$ since the kernel K is holomorphic and locally uniformly bounded on compact subsets of $\mathbb{C}^{+2}$.

To see also that $\mathrm{Im}[q] \geq 0$, consider the imaginary part of q given by representation (3.1), which is

$$\mathrm{Im}[q(z_1, z_2)] = b_1 \mathrm{Im}[z_1] + b_2 \mathrm{Im}[z_2] + \frac{1}{\pi^2} \iint_{\mathbb{R}^2} \mathrm{Im}\big[K\big((z_1, z_2), (t_1, t_2)\big)\big]\, d\mu(t_1, t_2).$$

Note that we are allowed to move the imaginary part into the integral due to μ being a real measure. It is now obvious that the first two terms are non-negative.

To see that the third term is also non-negative, observe that

$$\mathrm{Im}[K\big((z_1, z_2), (t_1, t_2)\big)] = \mathcal{P}\big((z_1, z_2), (t_1, t_2)\big)$$
$$- \frac{1}{2}\mathrm{Re}\left[\left(\frac{1}{t_1 - \overline{z_1}} - \frac{1}{t_1 + \mathrm{i}}\right)\left(\frac{1}{t_2 - z_2} - \frac{1}{t_2 - \mathrm{i}}\right)\right].$$

Since the measure μ satisfies property (3.8), we have that

$$\iint_{\mathbb{R}^2} \mathrm{Im}\big[K\big((z_1, z_2), (t_1, t_2)\big)\big]\, d\mu(t_1, t_2) = \iint_{\mathbb{R}^2} \mathcal{P}\big((z_1, z_2), (t_1, t_2)\big)\, d\mu(t_1, t_2) \geq 0,$$

where the last inequality follows from the positivity of the Poisson kernel. This finishes the proof. $\square$

Remark 3.4. As mentioned in the introduction, an integral representation of the form (3.1) already appears in [8], even for the case of $n \geq 2$ variables. However, it is only shown, by a completely different method, that every Herglotz-Nevanlinna function admits such a representation, but it is a priori assumed that the measure is the boundary measure of the represented function. It is not discussed which measures can actually appear there.

In [9], the authors use also a similar change of variables as in the present paper in order to find a characterization of Herglotz-Nevanlinna functions. However, this representation is not as simple as (3.1). Basically, all the integrals that come from the boundary of the area of integration are still present and hence also the corresponding Nevanlinna condition is much more involved.

For convenience, we highlight some minor results that appeared within the proof of Theorem 3.1.

Corollary 3.5. *Let q be a Herglotz-Nevanlinna function in two variables. Then the following four statements hold.*

(i) *If there exists a point $(\zeta_1, \zeta_2) \in \mathbb{C}^{+2}$ such that $\mathrm{Im}[q(\zeta_1, \zeta_2)] = 0$, then $q(z_1, z_2) = q(\zeta_1, \zeta_2)$ for all $(z_1, z_2) \in \mathbb{C}^{+2}$.*

(ii) *The imaginary part of the function q can be represented as*

$$\mathrm{Im}[q(z_1, z_2)] = b_1 \mathrm{Im}[z_1] + b_2 \mathrm{Im}[z_2] + \frac{1}{\pi^2} \iint_{\mathbb{R}^2} \mathcal{P}\big((z_1, z_2), (t_1, t_2)\big) \, d\mu(t_1, t_2),$$

where b_1, b_2 and μ are as in Theorem 3.1.

(iii) *The number a from Theorem 3.1 is equal to*

$$a = \mathrm{Re}[q(\mathfrak{i}, \mathfrak{i})].$$

(iv) *For every $\alpha, \beta \in \mathbb{C}^+$ it holds that*

$$b_1 = \lim_{z \widehat{\to} \infty} \frac{q(z, \alpha)}{z} \quad and \quad b_2 = \lim_{z \widehat{\to} \infty} \frac{q(\beta, z)}{z},$$

where b_1, b_2 are as in Theorem 3.1. In particular, the limits are independent of α and β, respectively.

A further implication of Corollary 3.5 is given by the following statement.

Corollary 3.6. *Let q be a Herglotz-Nevanlinna function in two variables and let $\alpha, \beta \in \mathbb{C}^+$ be arbitrary. Then there exist constants $c_1, c_2 \leq 0$, independent of α and β, such that*

$$c_1 = \lim_{z \widehat{\to} 0} z q(z, \alpha) \quad and \quad c_2 = \lim_{z \widehat{\to} 0} z q(\beta, z).$$

Proof. Applying the change of variables $z \mapsto -\frac{1}{z}$ leads to Herglotz-Nevanlinna functions $z \mapsto q(-\frac{1}{z}, \alpha)$ and $z \mapsto q(\beta, -\frac{1}{z})$, for which Corollary 3.5(iv) implies the claim. $\square$

The proof of Theorem 3.1 has also given us additional information about measures satisfying condition (2.5).

Corollary 3.7. *Let ν be a finite positive Borel measure on $[0, 2\pi)^2$ satisfying condition (2.5). Then $\nu(\{(0,0)\}) = 0$ and there exist constants $e_1, e_2 \geq 0$ such that $\nu|_{\{0\} \times (0, 2\pi)} = e_1 \lambda_{(0, 2\pi)}$ and $\nu|_{(0, 2\pi) \times \{0\}} = e_2 \lambda_{(0, 2\pi)}$. In particular, it holds that*

$$\iint_{(0, 2\pi)^2} \mathrm{e}^{\mathrm{i} m_1 s_1} \mathrm{e}^{\mathrm{i} m_2 s_2} \, d\nu(s_1, s_2) = 0$$

for every pair of indices $m_1, m_2 \in \mathbb{Z}$ satisfying $m_1 m_2 < 0$.

We finish this section with some examples of representations of Herglotz-Nevanlinna functions in two variables.

Example 3.8. Let
$$q(z_1, z_2) = -\frac{1}{z_2}.$$
Then q is a Herglotz-Nevanlinna function in two variables, which can easily be shown by a direct computation of its imaginary part. Corollary 3.5 now says that
$$a = \mathrm{Re}[q(\mathfrak{i}, \mathfrak{i})] = 0,$$
while choosing $\alpha = \beta = \mathfrak{i}$ we get that
$$b_1 = \lim_{z \to \infty} \frac{q(z, \mathfrak{i})}{z} = 0 \quad \text{and} \quad b_2 = \lim_{z \to \infty} \frac{q(\mathfrak{i}, z)}{z} = 0.$$
The measure μ can also be reconstructed using Proposition 4.1 and is equal to
$$\mu = \lambda_{\mathbb{R}} \otimes \pi \delta_0.$$
Note that if q is regarded as a function of just one variable, the representing measure (in accordance with Theorem 2.1) is just a multiple of the Dirac measure, namely $\pi \delta_0$.

Example 3.9. Let
$$q(z_1, z_2) = 2 + z_1 + \frac{z_1 z_2 + z_2 - z_1 - 1}{z_1 + z_2}.$$
Then q is a Herglotz-Nevanlinna function in two variables with
$$a = \mathrm{Re}[q(\mathfrak{i}, \mathfrak{i})] = 2, \quad b_1 = \lim_{z \to \infty} \frac{q(z, \mathfrak{i})}{z} = 1 \quad \text{and} \quad b_2 = \lim_{z \to \infty} \frac{q(\mathfrak{i}, z)}{z} = 0$$
and the μ-measure of a Borel measurable set $U \subseteq \mathbb{R}^2$ equalling
$$\mu(U) = \pi \iint_U \chi_{\{t_1 = -t_2\}}(t_1, t_2) g(t_1, t_2) dt_1 dt_2,$$
where the function g is defined as $g(t_1, t_1) = -t_1 t_2 - t_2 + t_1 + 1$.

Example 3.10. Let
$$q(z_1, z_2) = 1 + (2 + \sqrt{z_1})(3 + \sqrt{z_2}),$$
where the branch cut of the square root function is taken along the negative real line. Then q is a Herglotz-Nevanlinna function in two variables with
$$a = \mathrm{Re}[q(\mathfrak{i}, \mathfrak{i})] = 7 + \frac{5}{\sqrt{2}}, \quad b_1 = \lim_{z \to \infty} \frac{q(z, \mathfrak{i})}{z} = 0 \quad \text{and} \quad b_2 = \lim_{z \to \infty} \frac{q(\mathfrak{i}, z)}{z} = 0,$$

and the measure μ equalling

$$\mu = 3\pi h_1 \chi_{\{t_1 < 0\}} \lambda_{\mathbb{R}} \otimes \lambda_{\mathbb{R}} + \lambda_{\mathbb{R}} \otimes 2\pi h_1 \chi_{\{t_2 < 0\}} \lambda_{\mathbb{R}} + h_2 \chi_{\{t_1 t_2 < 0\}} \lambda_{\mathbb{R}^2},$$

where the function h_1 is defined as $h_1(t) = \sqrt{-t}$ and the function h_2 is defined as $h_2(t_1, t_2) = \sqrt{-t_1 t_2}$.

4. Properties of the class of representing measures

We now return to the question of describing the measure μ in terms of the function q. In the one-variable case, this is done via the classic Stieltjes inversion formula [5]. Here, we present an elementary two-dimensional analogue to this formula.

Proposition 4.1. *Let q be a Herglotz-Nevanlinna function in two variables. Then the measure μ in representation* (3.1) *is unique and can be determined from the boundary values of q. More precisely, let $\psi \colon \mathbb{R}^2 \to \mathbb{R}$ be a $\mathcal{C}^1$-function such that*

$$|\psi(x_1, x_2)| \leq \frac{C}{(1+x_1^2)(1+x_2^2)}$$

for some constant $C \geq 0$ and all $(x_1, x_2) \in \mathbb{R}^2$. Then
(4.1)
$$\iint_{\mathbb{R}^2} \psi(t_1, t_2) \, d\mu(t_1, t_2) = \lim_{\substack{y_1 \to 0^+ \\ y_2 \to 0^+}} \iint_{\mathbb{R}^2} \psi(x_1, x_2) \mathrm{Im}[q(x_1 + \mathrm{i}y_1, x_2 + \mathrm{i}y_2)] \, dx_1 \, dx_2.$$

Proof. We begin by using statement (ii) of Corollary 3.5 to rewrite the right-hand side of equality (4.1) as

$$\lim_{\substack{y_1 \to 0^+ \\ y_2 \to 0^+}} \iint_{\mathbb{R}^2} \psi(x_1, x_2) \bigg(b_1 y_1 + b_2 y_2$$

$$+ \frac{1}{\pi^2} \iint_{\mathbb{R}^2} \mathcal{P}\big((x_1 + \mathrm{i}y_1, x_2 + \mathrm{i}y_2), (t_1, t_2)\big) \, d\mu(t_1, t_2) \bigg) \, dx_1 \, dx_2.$$

It is readily seen that the part involving the term $b_1 y_1 + b_2 y_2$ tends to 0. What remains is the part involving the Poisson kernel, where we can use Fubini's theorem to change the order of integration. Another application of Lebesgue's dominated convergence theorem allows us to change the order of the limit and the first integral. We thus arrive at

$$\frac{1}{\pi^2} \iint_{\mathbb{R}^2} \lim_{\substack{y_1 \to 0^+ \\ y_2 \to 0^+}} \iint_{\mathbb{R}^2} \psi(x_1, x_2) \mathcal{P}\big((x_1 + \mathrm{i}y_1, x_2 + \mathrm{i}y_2), (t_1, t_2)\big) \, dx_1 \, dx_2 \, d\mu(t_1, t_2).$$

It remains to observe that, by a well known property of the Poisson kernel, the inner integral equals

$$\lim_{\substack{y_1 \to 0^+ \\ y_2 \to 0^+}} \iint_{\mathbb{R}^2} \psi(x_1, x_2) \mathcal{P}\big((x_1 + \mathrm{i}y_1, x_2 + \mathrm{i}y_2), (t_1, t_2)\big) \, dx_1 \, dx_2 = \pi^2 \psi(t_1, t_2).$$

In order to show the uniqueness of the representing measure, suppose that representation (3.1) of the function q holds for some measures μ_1 and μ_2. Recall that Corollary 3.5 shows that the numbers a, b_1, b_2 are uniquely determined by q. Using equality (4.1), we see that the left-hand side in this formula is the same for both μ_1 and μ_2. This implies that

$$\iint_{\mathbb{R}^2} \psi(t_1, t_2) \, d\mu_1(t_1, t_2) = \iint_{\mathbb{R}^2} \psi(t_1, t_2) \, d\mu_2(t_1, t_2)$$

for all functions ψ as above, which is possible only if $\mu_1 \equiv \mu_2$. $\quad\square$

Recall that in the second part of the proof of Theorem 3.1, we only required the growth condition to show that the integral involving the kernel function K is well-defined, while the Nevanlinna condition is needed only show that the integral of $\mathrm{Im}[K]$ is non-negative. We illustrate this by the following example of a finite measure.

Example 4.2. Considering the function defined by representation (3.1) with $a = 0$, $b_1 = 0$, $b_2 = 0$ and $\mu = \pi^2 \delta_{(0,0)}$, it is given by

$$\iint_{\mathbb{R}^2} K\big((z_1, z_2), (t_1, t_2)\big) \, d\delta_{(0,0)}(t_1, t_2) = \frac{\mathrm{i}(z_1 + \mathrm{i})(z_2 + \mathrm{i})}{2z_1 z_2} - \mathrm{i}.$$

Note that the measure $\pi^2 \delta_{(0,0)}$ does not satisfy the Nevanlinna condition (3.3). Hence, the above function is not a Herglotz-Nevanlinna function.

We show now that finite measures actually cannot satisfy the Nevanlinna condition.

Proposition 4.3. *Let q be a Herglotz-Nevanlinna function in two variables and let μ be its representing measure. Then μ cannot be a finite measure unless it is identically equal to 0.*

Proof. Recall first that for a Herglotz-Nevanlinna function $\tilde{q}$ of one variable with $b = 0$, where the number b is as in Theorem 2.1, and representing measure $\tilde{\mu}$, it holds that

$$\lim_{y \to \infty} y \mathrm{Im}[\tilde{q}(\mathrm{i}y)] = \frac{1}{\pi} \int_{-\infty}^{\infty} d\tilde{\mu}(t).$$

Observe that this identity holds even if one (and thus both) sides are equal to $+\infty$.

For a Herglotz-Nevanlinna function q in two variables that has $b_1 = b_2 = 0$, where the numbers b_1, b_2 are as in Theorem 3.1, Corollary 3.5(ii) and Lebesgue's monotone convergence theorem imply that

$$(4.2) \qquad \lim_{y \to \infty} y^2 \operatorname{Im}[q(iy, iy)] = \frac{1}{\pi^2} \iint_{\mathbb{R}^2} d\mu(t_1, t_2).$$

As in the one-variable case, the identity remains valid if one (and thus both) sides are equal to $+\infty$.

Suppose now that q is a Herglotz-Nevanlinna function in two variables with a finite representing measure μ. Note that μ is also a representing measure for the Herglotz-Nevanlinna function $\hat{q}$ defined as $\hat{q}(z_1, z_2) := q(z_1, z_2) - a - b_1 z_1 - b_2 z_2$, where a, b_1, b_2 correspond to the function q in the sense of Theorem 3.1.

The finiteness of μ and (4.2) imply $\lim_{y \to \infty} y^2 \operatorname{Im}[\hat{q}(iy, iy)] < \infty$ and hence $\lim_{y \to \infty} y \operatorname{Im}[\hat{q}(iy, iy)] = 0$. In terms of the Herglotz-Nevanlinna function $\tilde{q}$ defined as $\tilde{q}(z) := \hat{q}(z, z)$, this translates to

$$\frac{1}{\pi} \int_{-\infty}^{\infty} d\tilde{\mu}(t) = 0.$$

In follows that $\tilde{q} \equiv 0$ and so $\hat{q} \equiv 0$ on the diagonal in $\mathbb{C}^{+2}$. In particular $\hat{q}(i, i) = 0$ which together with statement (i) of Corollary 3.5 implies $\hat{q} \equiv 0$ and thus $\operatorname{Im}[\hat{q}] \equiv 0$. Statement (ii) of Corollary 3.5 now show that $\mu \equiv 0$ is the only possibility. $\quad \square$

We can now in fact say even more about the measures that are allowed in representation (3.1), namely that points in $\mathbb{R}^2$ are always zero sets.

Proposition 4.4. *Let q be a Herglotz-Nevanlinna function in two variables and let μ be its representing measure. Then $\mu(\{(\tau_1, \tau_2)\}) = 0$ for any point $(\tau_1, \tau_2) \in \mathbb{R}^2$.*

Proof. We begin by observing that

$$\lim_{\substack{z_1 \overset{\wedge}{\to} \tau_1 \\ z_2 \overset{\wedge}{\to} \tau_2}} (z_1 - \tau_1)(z_2 - \tau_2) K\big((z_1, z_2), (t_1, t_2)\big) = -\frac{i}{2} \chi_{\{(\tau_1, \tau_2)\}}(t_1, t_2)$$

for any point $(\tau_1, \tau_2) \in \mathbb{R}^2$.

For any fixed $\beta \in \mathbb{C}^+$ we calculate also that

$$\lim_{z_2 \overset{\wedge}{\to} \tau_2} (z_2 - \tau_2) q(\beta, z_2) = -\lim_{\omega_2 \overset{\wedge}{\to} \infty} \frac{1}{\omega_2} \widetilde{q_2}(\beta, \omega_2) = -\widetilde{b_2}(\tau_2).$$

Here, we used the change of variables $z_2 - \tau_2 = -\frac{1}{\omega_2}$ along with statement (iv) of Corollary 3.5 for the Herglotz-Nevanlinna function

$$\tilde{q}_2 \colon (\omega_1, \omega_2) \longmapsto q(\omega_1, -\tfrac{1}{\omega_2} + \tau_2).$$

Note that the number $\widetilde{b}_2(\tau_2)$ does of course depend on τ_2, but it does not depend on $\beta \in \mathbb{C}^+$. This implies that

$$\lim_{\substack{z_1 \overset{\triangleleft}{\to} \tau_1 \\ z_2 \overset{\triangleleft}{\to} \tau_2}} (z_1 - \tau_1)(z_2 - \tau_2) q(z_1, z_2) = - \lim_{z_1 \overset{\triangleleft}{\to} \tau_1} (z_1 - \tau_1) \widetilde{b}_2(\tau_2) = 0.$$

On the other hand, we can use Theorem 3.1, Lebesgue's dominated convergence theorem and our starting observation to show that

$$\lim_{\substack{z_1 \overset{\triangleleft}{\to} \tau_1 \\ z_2 \overset{\triangleleft}{\to} \tau_2}} (z_1 - \tau_1)(z_2 - \tau_2) q(z_1, z_2) = -\frac{i}{2\pi^2} \mu(\{(\tau_1, \tau_2)\}).$$

This finishes the proof. $\quad\square$

Conclusion

Theorem 3.1 provides the anticipated generalization of Theorem 2.1 to the case of two variables, improving the previous results of Vladimirov [8] and [9] as discussed in the introduction and in Remark 3.4. The main difference between the cases $n=1$ and $n=2$ exists first and foremost in the class of measures that can represent a Herglotz-Nevanlinna function in the respective dimension. While in the one dimensional case it is well known that all measures (satisfying the growth condition (2.2)) appear, we have now seen, for example, that all representing measures of Herglotz-Nevanlinna functions in two variables are atomless.

It is natural to ask what happens when $n > 2$. Here, a corresponding integral representation, as in [8], of the form (3.1) is expected. The proof, however, will definitely require more steps. The main part of the current result, namely the characterizing property of the representing measures, seems to be more complicated to generalize. This will be considered in a forthcoming paper.

References

1. AGLER, J., McCARTHY, J. E. and YOUNG, N. J., Operator monotone functions and Löwner functions of several variables, *Ann. of Math. (2)* **176** (2012), 1783–1826.
2. AGLER, J., TULLY-DOYLE, R. and YOUNG, N. J., Nevanlinna representations in several variables. arXiv:1203.2261v2.
3. CAUER, W., The Poisson integral for functions with positive real part, *Bull. Amer. Math. Soc.* **38** (1932), 713–717.

4. DROZHZHINOV, YU. N. and VLADIMIROV, V. S., Holomorphic functions in a poly-disc with non-negative imaginary part, *Mat. Zametki* **15** (1974), 55–61. English transl.: Math. Notes **15** (1974), 31–34.

5. KAC, I. S. and KREIN, M. G., R-functions–analytic functions mapping the upper half-plane into itself, *Amer. Math. Soc. Transl. Ser. 2* **103** (1974), 1–18.

6. KORÁNYI, A. and PUKÁNSZKY, L., Holomorphic function with positive real part on polycylinders, *Trans. Amer. Math. Soc.* **108** (1963), 449–456.

7. NEVANLINNA, R., Asymptotische Entwicklungen beschränkter Funktionen und das Stieltjessche Momentenproblem, *Ann. Acad. Sci. Fenn. (A)* **18** (1922), 1–53.

8. VLADIMIROV, V. S., *Generalized Function in Mathematical Physics*, Nauka, 1979. English transl.: Mir publishers, Moscow, 1979.

9. VLADIMIROV, V. S., Holomorphic functions with non-negative imaginary part in a tubular domain over a cone, *Mat. Sb.* **79** (1969), 182–152. English transl.: Math. USSR Sb. **8** (1969), 125–146.

Annemarie Luger
Department of Mathematics
Stockholm University
SE-106 91 Stockholm
Sweden
luger@math.su.se

Mitja Nedic
Department of Mathematics
Stockholm University
SE-106 91 Stockholm
Sweden
mitja@math.su.se