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Perturbation analysis of rational Riccati equations

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In this paper, we consider the perturbation analyses of the continuous-time rational Riccati equations using the normwise, mixed and componentwise analyses, which arises from the stochastic H_∞ problems and the indefinite stochastic linear quadratic control problems. We derive sufficient conditions for the existence of stabilizing solutions of the perturbed rational Riccati equations. Moreover, we obtain the perturbation bounds for the relative errors with respect to the stabilizing solutions of the rational Riccati equations under three kinds of perturbation analyses. Numerical results are presented to illustrate sharper perturbation bounds under the normwise, mixed and componentwise perturbation analyses.

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1. Introduction

Consider the continuous-time rational Riccati equation (CRRE):

$$\begin{aligned} \mathcal{R}(X) \;\equiv\;& A^*X + XA + Q + \Pi_1(X) - [L + XB + \Pi_{12}(X)]\,[R + \Pi_2(X)]^\dagger \\ (1) \qquad \cdot\;& [L + XB + \Pi_{12}(X)]^* = 0, \end{aligned}$$

with

$$(2)\quad \Pi_1(X) \equiv \sum_{i=1}^{N} A_0^{i*} X A_0^i,\ \ \Pi_2(X) \equiv \sum_{i=1}^{N} B_0^{i*} X B_0^i,\ \ \Pi_{12}(X) \equiv \sum_{i=1}^{N} A_0^{i*} X B_0^i,$$

where $A, Q, A_0^i \in \mathbb{K}^{n\times n}$, $B, L, B_0^i \in \mathbb{K}^{n\times m}$ and $R \in \mathbb{K}^{m\times m}$, for $i = 1, 2, \ldots, N$. It arises in the linear time-invariant (LTI) systems such as stochastic H_∞ problem [21] and indefinite stochastic linear quadratic control problem [30]. The linear operator

$$\Pi(X) \equiv \begin{bmatrix} \Pi_1(X) & \Pi_{12}(X) \\ \Pi_{12}(X)^* & \Pi_2(X) \end{bmatrix}$$

is said to be *positive* if $\Pi(X) \geq 0$, for $X \geq 0$ and $\Pi_1(X)$, $\Pi_2(X) \geq 0$. The solution to the CRRE (1) $X \in \mathbb{K}^{n\times n}$, which is required to be maximal and stable, is difficult to solve. Only a few methods work, including Newton's method (NM) [10, 11], modified Newton's method (MNM) [19, 23] and homotopy method (HM) [33]. In this paper, we choose to apply an efficient method called the generalized Smith method (GSM) [14]. Let

$$
\begin{aligned}
\text{(3)} \qquad \mathcal{L}_A(X) &\equiv A^*X + XA, \qquad F_X(X) \equiv F(X)^{\dagger}E(X)^* \\
E(X) &\equiv L + XB + \Pi_{12}(X), F(X) \equiv R + \Pi_2(X),
\end{aligned}
$$

we apply NM to the CRRE (1) and get

$$
\text{(4)} \qquad \mathcal{L}_{A-BF_{X_k}(X_k)}(X_{k+1}) + \Pi_{X_k}(X_{k+1}) + T_{X_k} = 0,
$$

which is linear in X_{k+1}, with

$$
\begin{aligned}
\text{(5)} \qquad \Pi_Y(Z) &\equiv \begin{bmatrix} I \\ -F_Y(Y) \end{bmatrix}^* \Pi(Z) \begin{bmatrix} I \\ -F_Y(Y) \end{bmatrix}, \\
T_Y &\equiv \begin{bmatrix} I \\ -F_Y(Y) \end{bmatrix}^* T \begin{bmatrix} I \\ -F_Y(Y) \end{bmatrix}, \\
\text{(6)} \qquad T &\equiv \begin{bmatrix} Q & L \\ L^* & R \end{bmatrix} \geq 0.
\end{aligned}
$$

For the generalized inverse in (1) and (3), we need the null space requirement:

$$
\ker[F(X)] \subseteq \ker[E(X)], F(X) \geq 0.
$$

This came from the elimination of Lagrange multipliers from the optimal conditions, guaranteeing the solvability of the associated linear equation despite of the singularity of the corresponding matrix operator $F(X)$.

The stabilizing solution plays an important role in some applications of control theory and we introduce the stability definitions. We first represent $\sigma(T) \subset \mathbb{C}$ for the spectrum of a linear operator T and $\rho(T) = \max\{|\lambda||\lambda \in \sigma(T)\}$ for the spectral radius. An $n \times n$ matrix M is said to be c-stable if all of its eigenvalues lie in the open left-half complex plane s.t. $\sigma(M) \subset \mathbb{C}_-$, and M is said to be d-stable if its spectral radius satisfies $\rho(M) < 1$.

CRREs (1) can be degenerated into continuous-time algebraic Riccati equations called CAREs

$$
A^{\top}X + XA - XGX + H = 0,
$$

with $G = BR^{-1}B^\top$ and $H = CT^{-1}C^\top$, where $A, X \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{n\times p}$, $R \in \mathbb{R}^{m\times m}$ and $T \in \mathbb{R}^{p\times p}$, which arise in linear-quadratic optimal control problems [26, 29]. In the past 40 years, many efficient methods such as disk function method [1], matrix sign function method [4], structure-preserving doubling algorithm (SDA) [8, 9], NM [20], Schur method [27], and many others were developed.

The condition number, which is a measure of the sensitivity, is important in the numerical computation. Moreover, perturbation analysis is to study the sensitivity of solutions to the small perturbation in the input data, and the perturbation bounds are usually discussed. There were a number of references about perturbation analyses and perturbation bounds in [3, 5, 13, 16, 24, 25, 31, 32, 34], but only relatively few references discussed those of the CAREs with stochastic disturbances. Chiang et al. [6, 7] discussed the residual bound of the continuous-time stochastic algebraic Riccati equation (SARE) with one-dimensional Wiener process of white noise in the stochastic disturbance, including the normwise local and non-local residual bounds, derived from the appropriate solution of the SARE by NM. Furthermore, they derived the relative error and the condition number of SARE and provided a tight perturbation bound of the stabilizing solution to SARE. This paper, as an important extension of the previous research [7], derives new perturbation bounds for the relative errors with respect to the stabilizing solutions of CRREs with multi-dimensional disturbances, respectively. The CRREs are more complex than SARE, but the derivations of new perturbation bounds under normwise, mixed and componentwise perturbation analyses are simple. Moreover, we apply an efficient method called GSM to solve the perturbed CRREs and get the unique and stabilizing solutions.

The rest of this paper is organized as follows. We discuss the solvable results of CRREs (1) in Section 2. In Section 3, the perturbation equation of CRREs (1) is derived. We provide some sufficient conditions to present the existence of the solution of the perturbed equation (10) in the Supplementary Material S1, then recall some lemmas of stability analysis of the linear operator to discuss the uniqueness of the stabilizing solution shown in the Supplementary Material S2. Then, we compute the relative errors with respect to the unique and stabilizing solution of CRREs (1). By dropping the second and high-order terms in the perturbed CRREs (10), we derive new perturbation bounds under normwise, mixed and componentwise perturbation analyses, originated from Gohberg and Koltracht (1993). The algorithm about perturbation analyses of CRREs is provided and we select one representative numerical example to illustrate the sharpness of new perturbation bounds, corresponding to the relative errors of the stabilizing solutions of CRREs (1) in Section 4. Section 5 concludes the paper. Finally, we provide some proofs of several theorems in the Supplementary Material (http://intlpress.com/site/pub/files/_supp/amsa/2020/0005/0002/AMSA-2020-0005-0002-s001.pdf).

2. Solvable conditions

Before we discuss the perturbation analysis of CRREs, we introduce the concept of c-stability and some solvability results for stochastic control systems in continuous-time. This leads to the unique and stable solution of CRREs.

First we quote the theorem on c-stability.

Theorem 2.1. *Let $A \in \mathbb{R}^{n\times n}$, and consider linear operators $\mathcal{L}_A, \Pi : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$, where $\mathcal{L}_A$ is defined by $\mathcal{L}_A(X) = A^*X + XA$ and Π is nonnegative defined in (5). The following are equivalent:*

(a) For all $Y > 0$, $\exists X > 0$ such that $\mathcal{L}_A(X) + \Pi = -Y$;
(b) $\exists Y, X > 0$ such that $\mathcal{L}_A(X) + \Pi = -Y$;
(c) $\exists Y \geq 0$ with (A, Y) observable, $\exists X > 0$ such that $\mathcal{L}_A + \Pi = -Y$;
(d) $\sigma(\mathcal{L}_A + \Pi) \subset \mathbb{C}_-$;
(e) $\sigma(\mathcal{L}_A) \subset \mathbb{C}_-$ and $\rho(\mathcal{L}_A^{-1}\Pi) < 1$.

If any of these conditions is fulfilled then A is called c-stable relative to Π.

Then we present some definitions associated with c-stability.

Definition 2.2 (c-Stabilizability)**.** *A matrix pair (A, B) ($A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$) is said to be c-stabilizable relative to Π if there is a matrix $\hat{F} \in \mathbb{R}^{m\times n}$ such that $A - B\hat{F}$ is c-stable relative to*

$$\hat{\Pi} \equiv \begin{bmatrix} I \\ -\hat{F} \end{bmatrix}^* \Pi \begin{bmatrix} I \\ -\hat{F} \end{bmatrix}. \tag{7}$$

Definition 2.3 (Stabilizing Solution)**.** *If X is a solution of $\mathcal{R}(X) = 0$ and if $\hat{F} = \hat{F}(X)$ denotes the corresponding feedback matrix then X is called stabilizing, if $A - B\hat{F}$ is c-stable relative to $\hat{\Pi}$ as defined in (7).*

Definition 2.4 (c-Detectability)**.** *A pair (C, A) of matrices $A \in \mathbb{R}^{n\times n}$ and $C \in \mathbb{R}^{l\times n}$ is said to be c-detectable relative to Π if there is a matrix $K \in \mathbb{R}^{n\times l}$ such that $A - KC$ is c-stable relative to Π.*

Next we quote some solvability results for CRREs in our notations. The importance of these results is to establish conditions under which a semidefinite or stabilizing solution is the unique maximal stabilizing solution we seek. This also makes the selection of the initial X_0 easier for the NM, as any stabilizing solution X_0 will do.

Theorem 2.5 ([15, Theorem 5.2])**.** *Assume that* (A, B) *is c-stabilizable relative to* Π *and that there exists a matrix* $\hat{X} \in D(\mathcal{R})$*, the domain of* $\mathcal{R}$ *(in which the null space requirement is satisfied), with* $\ker[R+\Pi_2(\hat{X})] \subseteq \ker(B)$ *for which* $\mathcal{R}(\hat{X}) \geq 0$*. Then there exists a solution* $X_+ \in D(\mathcal{R})$ *of* $\mathcal{R}(X) = 0$ *such that* $X_+ \geq X$ *for every solution of* $\mathcal{R}(X) \geq 0$ *with* $\ker[R + \Pi_2(X)] \subseteq \ker(B)$*. Moreover, all the eigenvalues of*

$$A_+ \equiv A - B[R + \Pi_2(X_+)]^\dagger [L + X_+ B + \Pi_{12}(X_+)]^* \tag{8}$$

lie in the closed left half-plane.

Corollary 2.6 ([15, Corollary 5.3])**.** *Assume that* $\ker(R) \subseteq \ker(B)$*,* (A, B) *is c-stabilizable relative to* Π *and* $T \geq 0$ *(c.f. (6)). Then* $\mathcal{R}(X) = 0$ *has a solution* $X_+ \geq 0$*, and all the eigenvalues of the matrix* A_+ *in (8) lie in the closed left half-plane.*

Corollary 2.7 ([15, Corollary 5.4])**.** *Assume that* (A, B) *is c-stabilizable relative to* Π *and that there exists a matrix* $\hat{X} \in D(\mathcal{R})$ *with* $\ker[R+\Pi_2(\hat{X})] \subseteq \ker(B)$ *for which* $\mathcal{R}(\hat{X}) > 0$*. Then there exists a solution* $X_+ \in D(\mathcal{R})$ *of* $\mathcal{R}(X) = 0$ *such that* $X_+ > X$ *for every solution* $\mathcal{R}(X) > 0$ *with* $\ker[R + \Pi_2(X)] \subseteq \ker(B)$*. Moreover, all the eigenvalues of* A_+ *in (8) lie in the open left half-plane.*

Lemma 2.8 ([15, Lemma 5.5])**.** *If* $\mathcal{R}(X) = 0$ *has a stabilizing solution* X_s*, then* $X_s \geq X$ *for every solution* X *of* $\mathcal{R}(X) \geq 0$*. In particular,* X_s *is the (unique) maximal solution of* $\mathcal{R}(X) = 0$*.*

Lemma 2.9 ([15, Lemma 5.6])**.** *Assume that* $R > 0$*,* $T \geq 0$ *and that* $(Q - LR^\dagger L^*, A - BR^\dagger L^*)$ *is c-detectable relative to*

$$\check{\Pi} \equiv \begin{bmatrix} I \\ -R^\dagger L^* \end{bmatrix}^* \Pi \begin{bmatrix} I \\ -R^\dagger L^* \end{bmatrix}. \tag{9}$$

Then every positive semidefinite solution of $\mathcal{R}(X) = 0$ *is stabilizing.*

Corollary 2.10 ([15, Lemma 5.8])**.** *Assume that* $R \geq 0$*,* $\ker(R) \subseteq \ker(L)$*,* $Q > LR^\dagger L^*$*. If* $X \geq 0$ *is a solution of* $\mathcal{R}(X) = 0$*, then* X *is stabilizing and positive definite.*

Theorem 2.11 ([15, Theorem 5.9])**.** *Assume that* $R > 0$*,* $T \geq 0$*,* (A, B) *is c-stabilizable relative to* Π *and* $(Q - LR^\dagger L^*, A - BR^\dagger L^*)$ *is c-detectable relative* *to $\check{\Pi}$ (defined in (9)). Then $\mathcal{R}(X) = 0$ has a unique positive semidefinite solution X_+. Moreover, X_+ is stabilizing and maximal among all solutions of $\mathcal{R}(X) = 0$.*

3. Perturbation analysis of CRREs

From (1), we represent the perturbed CRRE as

$$\tilde{\mathcal{R}}(\tilde{X}) \equiv \tilde{A}^*\tilde{X} + \tilde{X}\tilde{A} + \tilde{Q} + \tilde{\Pi}_1(\tilde{X}) - \tilde{E}(\tilde{X})\tilde{F}(\tilde{X})^\dagger \tilde{E}^*(\tilde{X}) = 0, \tag{10}$$

with

$$\tilde{E}(\tilde{X}) = \tilde{L} + \tilde{X}\tilde{B} + \tilde{\Pi}_{12}(\tilde{X}) \text{ and } \tilde{F}(\tilde{X}) = \tilde{R} + \tilde{\Pi}_2(\tilde{X}),$$

and express the perturbed disturbances $\tilde{\Pi}_1(\tilde{X})$, $\tilde{\Pi}_2(\tilde{X})$, $\tilde{\Pi}_{12}(\tilde{X})$ as

$$\tilde{\Pi}_1(\tilde{X}) \equiv \sum_{i=1}^N \tilde{A}_0^{i*}\tilde{X}\tilde{A}_0^i, \tilde{\Pi}_2(\tilde{X}) \equiv \sum_{i=1}^N \tilde{B}_0^{i*}\tilde{X}\tilde{B}_0^i, \tilde{\Pi}_{12}(\tilde{X}) \equiv \sum_{i=1}^N \tilde{A}_0^{i*}\tilde{X}\tilde{B}_0^i,$$

where

$$\begin{aligned} \tilde{A} &= A + \Delta A, \tilde{Q} = Q + \Delta Q, \tilde{L} = L + \Delta L, \tilde{B} = B + \Delta B, \\ \tilde{R} &= R + \Delta R, \tilde{A}_0^i = A_0^i + \Delta A_0^i, \tilde{B}_0^i = B_0^i + \Delta B_0^i, \tilde{X} = X + \Delta X, \end{aligned}$$

for $i = 1, 2, \ldots, N$ and ΔA, ΔQ, ΔL, ΔB, ΔR, ΔA_0^i, ΔB_0^i are small perturbation matrices. In order to compute the first-order perturbation matrices, we seperate $\tilde{E}(\tilde{X})$ and $\tilde{F}(\tilde{X})$ into

$$\begin{aligned} \tilde{E}(\tilde{X}) &= \tilde{E}(X) + E(\Delta X), \tilde{E}(X) = E(X) + \delta E, \\ \tilde{F}(\tilde{X}) &= \tilde{F}(X) + F(\Delta X), \tilde{F}(X) = F(X) + \delta F, \end{aligned}$$

with

$$\begin{aligned} E(\Delta X) &= E_1(\Delta X) + E_2(\Delta X), \delta E = \delta E_1 + \delta E_2, \\ F(\Delta X) &= F_1(\Delta X) + F_2(\Delta X), \delta F = \delta F_1 + \delta F_2, \end{aligned}$$

where

$$E_1(\Delta X) \;=\; \Delta X B + \sum_{i=1}^N A_0^{i*}\Delta X B_0^i, F_1(\Delta X) = \sum_{i=1}^N B_0^{i*}\Delta X B_0^i,$$

$$\begin{aligned}
E_2(\Delta X) &= \Delta X\Delta B + \sum_{i=1}^{N}(A_0^{i*}\Delta X\Delta B_0^i + \Delta A_0^{i*}\Delta X B_0^i + \Delta A_0^{i*}\Delta X\Delta B_0^i),\\
F_2(\Delta X) &= \sum_{i=1}^{N}(B_0^{i*}\Delta X\Delta B_0^i + \Delta B_0^{i*}\Delta X B_0^i + \Delta B_0^{i*}\Delta X\Delta B_0^i),\\
\delta E_1 &= \Delta L + X\Delta B + \sum_{i=1}^{N}(A_0^{i*}X\Delta B_0^i + \Delta A_0^{i*}XB_0^i),\\
\delta E_2 &= \sum_{i=1}^{N}\Delta A_0^{i*}X\Delta B_0^i,\\
\delta F_1 &= \Delta R + \sum_{i=1}^{N}(B_0^{i*}X\Delta B_0^i + \Delta B_0^{i*}XB_0^i), \delta F_2 = \sum_{i=1}^{N}\Delta B_0^{i*}X\Delta B_0^i.
\end{aligned}$$

$E(\Delta X)$ and $F(\Delta X)$ are linear functions of ΔX, δE_1 and δF_1 are first-order perturbation matrices, δE_2 and δF_2 are second-order perturbation matrices. Set $\Phi_C = A - BF(X)^{\dagger}E(X)^*$ and $\Psi_C^i = A_0^i - B_0^i F(X)^{\dagger}E(X)^*$, for $i = 1, 2, \ldots, N$, then we can obtain $\tilde{\Phi}_C = \tilde{A} - \tilde{B}\tilde{F}(X)^{\dagger}\tilde{E}(X)^*$ and $\tilde{\Psi}_C^i = \tilde{A}_0^i - \tilde{B}_0^i\tilde{F}(X)^{\dagger}\tilde{E}(X)^*$, we obtain the following equation from (1) and (10) (11)

$$\tilde{\mathcal{R}}(\tilde{X}) - \mathcal{R}(X) = \tilde{\Phi}_C^*\Delta X + \Delta X\tilde{\Phi}_C + \sum_{i=1}^{N}\tilde{\Psi}_C^{i*}\Delta X\tilde{\Psi}_C^i - E_1 - E_2 - h_2(\Delta X) = 0,$$

using the generalization of the Sherman-Morrison-Woodbury formula (GSMWF) [12]

$$(A + UV^*)^{\dagger} = A^{\dagger} - A^{\dagger}U(I + V^*A^{\dagger}U)^{-1}V^*A^{\dagger},$$

on

$$\begin{aligned}
\tilde{F}(\tilde{X})^{\dagger} &= (\tilde{F}(X) + F(\Delta X))^{\dagger} = \tilde{F}(X)^{\dagger} - \tilde{F}(X)^{\dagger}F(\Delta X)F_1\tilde{F}(X)^{\dagger},\\
\tilde{F}(X)^{\dagger} &= (F(X) + \delta F)^{\dagger} = F(X)^{\dagger} - F(X)^{\dagger}\delta F F_2 F(X)^{\dagger},\\
F_1 &= (I_m + \tilde{F}(X)^{\dagger}F(\Delta X))^{-1}\\
&= I_m - \tilde{F}(X)^{\dagger}(I_m + F(\Delta X)\tilde{F}(X)^{\dagger})^{-1}F(\Delta X),\\
F_2 &= (I_m + F(X)^{\dagger}\delta F)^{-1} = I_m - F(X)^{\dagger}(I_m + \delta F F(X)^{\dagger})^{-1}\delta F,
\end{aligned}$$

with

$$\begin{aligned}
E_1 &= -\Delta A^* X - X\Delta A - \Delta Q - \sum_{i=1}^{N}(A_0^{i*} X \Delta A_0^i + \Delta A_0^{i*} X A_0^i) \\
&- E(X)F(X)^\dagger \delta F_1 F(X)^\dagger E(X)^* \\
&+ E(X)F(X)^\dagger \delta E_1^* + \delta E_1 F(X)^\dagger E^*(X), \\
E_2 &= -\sum_{i=1}^{N}(\Delta A_0^{i*} X \Delta A_0^i) - E(X)F(X)^\dagger \delta F_2 F(X)^\dagger E(X)^* \\
&+ E(X)F(X)^\dagger \delta F F(X)^\dagger F_2^* \delta F F(X)^\dagger E(X)^* + E(X)F(X)^\dagger \delta E_2^* \\
&- E(X)F(X)^\dagger \delta F F_2 F(X)^\dagger \delta E^* + \delta E_2 F(X)^\dagger E(X)^* \\
&- \delta E F(X)^\dagger \delta F F_2 F(X)^\dagger \tilde{E}(X)^* + \delta E F(X)^\dagger \delta E^*, \\
h_2(\Delta X) &= \tilde{E}(X)\tilde{F}(X)^\dagger F(\Delta X)\tilde{F}(X)^\dagger F_1^* F(\Delta X)\tilde{F}(X)^\dagger \tilde{E}(X)^* \\
&- \tilde{E}(X)\tilde{F}(X)^\dagger F(\Delta X) F_1 \tilde{F}(X)^\dagger E(\Delta X)^* \\
&- E(\Delta X)\tilde{F}(X)^\dagger F(\Delta X) F_1 \tilde{F}(X)^\dagger \tilde{E}(X)^* \\
&+ E(\Delta X)\tilde{F}(\tilde{X})^\dagger E(\Delta X)^*.
\end{aligned}$$

We rewrite (11) into

$$\tilde{\Phi}_C^* \Delta X + \Delta X \tilde{\Phi}_C + \sum_{i=1}^{N} \tilde{\Psi}_C^{i*} \Delta X \tilde{\Psi}_C^i = E_1 + E_2 + h_2(\Delta X). \tag{12}$$

From the definitions of $\tilde{\Phi}_C$ and $\tilde{\Psi}_C^i$, we obtain

$$\begin{aligned}
\tilde{\Phi}_C &= (A + \Delta A) - (B + \Delta B)(F(X)^\dagger \\
&- F(X)^\dagger \delta F F_2 F(X)^\dagger)(E(X) + \delta E)^* \\
&= \Phi_C + \Delta\Phi_C, \\
\tilde{\Psi}_C^i &= (A_0^i + \Delta A_0^i) - (B_0^i + \Delta B_0^i)(F(X)^\dagger \\
&- F(X)^\dagger \delta F F_2 F(X)^\dagger)(E(X) + \delta E)^* \\
&= \Psi_C^i + \Delta\Psi_C^i,
\end{aligned}$$

with

$$\begin{aligned}
\Delta\Phi_C &= \Delta A - BF(X)^\dagger \delta E^* + BF(X)^\dagger \delta F F_2 F(X)^\dagger E(X)^* \\
&+ BF(X)^\dagger \delta F F_2 F(X)^\dagger \delta E^* - \Delta B F(X)^\dagger E(X)^* - \Delta B F(X)^\dagger \delta E^* \\
&+ \Delta B F(X)^\dagger \delta F F_2 F(X)^\dagger E(X)^* + \Delta B F(X)^\dagger \delta F F_2 F(X)^\dagger \delta E^*,
\end{aligned}$$

$$\begin{aligned}\Delta\Psi_C^i &= \Delta A_0^i - B_0^i F(X)^\dagger \delta E^* + B_0^i F(X)^\dagger \delta F F_2 F(X)^\dagger E(X)^* \\ &+ B_0^i F(X)^\dagger \delta F F_2 F(X)^\dagger \delta E^* - \Delta B_0^i F(X)^\dagger E(X)^* - \Delta B_0^i F(X)^\dagger \delta E^* \\ &+ \Delta B_0^i F(X)^\dagger \delta F F_2 F(X)^\dagger E(X)^* + \Delta B_0^i F(X)^\dagger \delta F F_2 F(X)^\dagger \delta E^*.\end{aligned}$$

We express the left-hand side of (12) using $\Delta\Phi_C$ and $\Delta\Psi_C^i$ as follows

$$\begin{aligned}\tilde{\Phi}_C^*\Delta X + \Delta X\tilde{\Phi}_C + \sum_{i=1}^N \tilde{\Psi}_C^{i*}\Delta X\tilde{\Psi}_C^i &= \Phi_C^*\Delta X + \Delta X\Phi_C \\ &+ \sum_{i=1}^N \Psi_C^{i*}\Delta X\Psi_C^i - h_1(\Delta X),\end{aligned} \tag{13}$$

with

$$\begin{aligned}h_1(\Delta X) &= -(\Delta\Phi_C^*\Delta X + \Delta X\Delta\Phi_C + \sum_{i=1}^N(\Psi_C^{i*}\Delta X\Delta\Psi_C^i \\ &+ \Delta\Psi_C^{i*}\Delta X\Psi_C^i + \Delta\Psi_C^{i*}\Delta X\Delta\Psi_C^i)).\end{aligned}$$

From (12) and (13), we get

$$\Phi_C^*\Delta X + \Delta X\Phi_C + \sum_{i=1}^N \Psi_C^{i*}\Delta X\Psi_C^i = E_1 + E_2 + h_1(\Delta X) + h_2(\Delta X).$$

Lemma 3.1. *Let X be the stabilizing solution of the CRRE (1) and $\tilde{X}$ be a symmetric solution of the perturbed CRRE (10), then ΔX satisfies the equation*

$$\Phi_C^*\Delta X + \Delta X\Phi_C + \sum_{i=1}^N \Psi_C^{i*}\Delta X\Psi_C^i = E_1 + E_2 + h_1(\Delta X) + h_2(\Delta X), \tag{14}$$

where

$$\begin{aligned}E_1 &= -\Delta A^* X - X\Delta A - \Delta Q - \sum_{i=1}^N (A_0^{i*} X\Delta A_0^i + \Delta A_0^{i*} X A_0^i) \\ &- E(X)F(X)^\dagger \delta F_1 F(X)^\dagger E(X)^* + E(X)F(X)^\dagger \delta E_1^* \\ &+ \delta E_1 F(X)^\dagger E^*(X); \\ E_2 &= -\sum_{i=1}^N (\Delta A_0^{i*} X\Delta A_0^i) - E(X)F(X)^\dagger \delta F_2 F(X)^\dagger E(X)^*\end{aligned}$$

$$
\begin{aligned}
&+ \quad E(X)F(X)^\dagger \delta F F(X)^\dagger F_2^* \delta F F(X)^\dagger E(X)^* + E(X)F(X)^\dagger \delta E_2^* \\
&- \quad E(X)F(X)^\dagger \delta F F_2 F(X)^\dagger \delta E^* + \delta E_2 F(X)^\dagger E(X)^* \\
&- \quad \delta E F(X)^\dagger \delta F F_2 F(X)^\dagger \tilde{E}(X)^* + \delta E F(X)^\dagger \delta E^*; \\
h_1(\Delta X) &= -(\Delta\Phi_C^* \Delta X + \Delta X \Delta \Phi_C + \sum_{i=1}^{N} (\Psi_C^{i*} \Delta X \Delta \Psi_C^i \\
&+ \quad \Delta\Psi_C^{i*} \Delta X \Psi_C^i + \Delta \Psi_C^{i*} \Delta X \Delta \Psi_C^i)); \\
h_2(\Delta X) &= \tilde{E}(X)\tilde{F}(X)^\dagger F(\Delta X)\tilde{F}(X)^\dagger F_1^* F(\Delta X)\tilde{F}(X)^\dagger \tilde{E}(X)^* \\
&- \quad \tilde{E}(X)\tilde{F}(X)^\dagger F(\Delta X) F_1 \tilde{F}(X)^\dagger E(\Delta X)^* \\
&- \quad E(\Delta X)\tilde{F}(X)^\dagger F(\Delta X) F_1 \tilde{F}(X)^\dagger \tilde{E}(X)^* \\
&+ \quad E(\Delta X)\tilde{F}(\tilde{X})^\dagger E(\Delta X)^*.
\end{aligned}
$$

E_1 and E_2 are first-order and high-order perturbation matrices respectively, and they do not depend on ΔX. $h_1(\Delta X)$ and $h_2(\Delta X)$ are the linear and high-order degree functions of ΔX, respectively.

3.1. Perturbation equation

We focus on the condition of the existence of a fixed point, that is, the perturbed CRRE (10) has some solution. From the definition of the linear operator L_c, we represent (14) as

$$L_c(\Delta X) = E_1 + E_2 + h_1(\Delta X) + h_2(\Delta X).$$

We provide a lemma to show the property of invertibility of the linear operator L_c.

Lemma 3.2 (Damm and Hinrichsen [11]). *Let $\mathcal{L} : \mathcal{H}^n \to \mathcal{H}^n$ be resolvent positive and $\Pi : \mathcal{H}^n \to \mathcal{H}^n$ be positive. Then the following are equivalent:*

- *(a)* $\mathcal{L} + \Pi$ *is stable, i.e.* $\sigma(\mathcal{L} + \Pi) \subset \mathbb{C}_-$.
- *(b)* $-(\mathcal{L} + \Pi)$ *is inverse positive.*
- *(c)* $\exists X > 0$*:* $(\mathcal{L} + \Pi)(X) < 0$.
- *(d)* $\sigma(\mathcal{L}) \subset \mathbb{C}_-$ *and* $\rho(\mathcal{L}^{-1}\Pi) < 1$.
- *(e)* $\sigma(\mathcal{L}) \subset \mathbb{C}_-$ *and* $\det(\mathcal{L} + \tau\Pi) \neq 0$ *for* $\tau \in [0, 1]$.

According to the definition of $\Pi_X(X)$ in (6), we can obtain that $\Pi_X = \sum_{i=1}^N \Psi_C^{i*} X \Psi_C^i$ is positive. Let X be the stabilizing solution to the CRREs (1), $\mathcal{L}_{\Phi_C}$ is resolvent positive since Φ_C is c-stable and $\mathcal{L}_A$ is a Lyapunov operator. By applying the Lemma 3.2, $L_c = \mathcal{L}_{\Phi_C} + \Pi_X$ is c-stable i.e. $\sigma(L_c) \subset \mathbb{C}_-$. In addition, L_c is invertible s.t. L_c^{-1} exists. Then, we define a function

$f(\Delta X)$ by

$$f(\Delta X) = L_c^{-1}E_1 + L_c^{-1}E_2 + L_c^{-1}h_1(\Delta X) + L_c^{-1}h_2(\Delta X). \tag{15}$$

$f(\Delta X)$ can be regarded as a continuous mapping $f : \mathcal{S}^{n\times n} \to \mathcal{S}^{n\times n}$, and any fixed point of the mapping f is a solution to the perturbed equation (14). We provide the existence and uniqueness of the fixed point in (15), that is, a unique and stabilizing solution to the perturbed CRREs (14). In conclusion, there exists a unique and stabilizing solution of (10) since X and ΔX are the unique stabilizing solutions of (1) and (14), respectively. Here, we sketch the outlines of proofs about the existence and uniqueness of the stabilizing solution to the perturbed CRREs (14):

[1] We apply the Brouwer fixed point theorem to show that it exists some fixed point to the mapping f in (15), that is, the existence of some solution to the perturbed equation (14) is existed.

[2] We apply the singular property of the linear operator to get the perturbation bound of perturbation matrices, and furthermore to show some conditions to get the stable perturbed linear operator. Therefore, we can get the uniqueness of the stabilizing solution to the perturbed equation (14).

For the details of proofs, please see the Supplementary Material S1 and S2.

3.2. Normwise condition number

We first discuss the perturbation analysis of the CRRE (1) and derive its normwise condition numbers. From (14), we drop the second and high-order terms and it yields

$$\begin{aligned}
&\Phi_C^*\Delta X + \Delta X\Phi_C + \sum_{i=1}^{N}\Psi_C^{i*}\Delta X\Psi_C^i \\
&\quad = -\Delta A^*X - X\Delta A - \Delta Q - \sum_{i=1}^{N}(A_0^{i*}X\Delta A_0^i \\
&\quad + \Delta A_0^{i*}XA_0^i) - E(X)F(X)^\dagger\Delta RF(X)^\dagger E(X)^* \\
&\quad - E(X)F(X)^\dagger\sum_{i=1}^{N}(B_0^{i*}X\Delta B_0^i \\
&\quad + \Delta B_0^{i*}XB_0^i)F(X)^\dagger E(X)^* \\
&\quad + E(X)F(X)^\dagger\Delta L^* + E(X)F(X)^\dagger\Delta B^*X
\end{aligned}$$

$$
\begin{aligned}
&+ E(X)F(X)^\dagger \sum_{i=1}^{N} (\Delta B_0^{i*} X A_0^i + B_0^{i*} X \Delta A_0^i) \\
&+ \Delta L F(X)^\dagger E(X)^* + X \Delta B F(X)^\dagger E(X)^* \\
&+ \sum_{i=1}^{N} (A_0^{i*} X \Delta B_0^i + \Delta A_0^{i*} X B_0^i) \\
&\cdot F(X)^\dagger E(X)^*. \qquad (16)
\end{aligned}
$$

We recall some notations about vectorization. For a matrix $A = [a_{ij}] \in \mathbb{R}^{n\times n}$, we define $\mathrm{vec}(A) = [a_1^\top, a_2^\top, \ldots, a_n^\top]^\top \in \mathbb{R}^{n^2}$, where $A = [a_1, a_2, \ldots, a_n]$ with $a_i \in \mathbb{R}^n$, $i = 1, 2, \ldots, n$. Some useful properties of Kronecker products found in [18] are listed below:

$$
\begin{aligned}
\|\mathrm{vec}(A)\|_2 &= \|A\|_F, \\
\|\mathrm{vec}(A)\|_\infty &= \|A\|_{\max}, \\
\mathrm{vec}(AXB) &= (B^\top \otimes A)\mathrm{vec}(X), \qquad (17)
\end{aligned}
$$

where $\|A\|_{\max} = \max_{i,j} |a_{ij}|$, for $i, j = 1, 2, \ldots, n$ and $B,\ X \in \mathbb{R}^{n\times n}$. By applying the operator "vec" to both sides of the equation (16), we reset the index i into j such as A_0^j and B_0^j in order to differentiate $\Pi_{12}(X)$ and $\Pi_2(X)$ inside of $E(X)$ and $F(X)$ and set $E(X)F(X)^\dagger \equiv e_f$, $A_0^{j*} X \equiv a_x^j$ and $E(X)F(X)^\dagger B_0^{j*} X \equiv e_b^j$, then obtain

$$
Z\mathrm{vec}(\Delta X) = Ps, \qquad (18)
$$

with

$$
\begin{aligned}
Z &= I_n \otimes \Phi_C^* + \Phi_C^* \otimes I_n + \sum_{i=1}^{N} \Psi_C^{i*} \otimes \Psi_C^{i*}, \\
P &= [P_1, P_2], P_2 = [P_2^1, P_2^2, \ldots, P_2^j], \\
s &= [s_1, s_2]^\top, s_2 = [s_2^1, s_2^2, \ldots, s_2^j],
\end{aligned}
$$

where

$$
\begin{aligned}
P_1 &= [-(X \otimes I_n), -(I_n \otimes X), -(I_n \otimes I_n), -(e_f \otimes e_f), (I_n \otimes e_f), \\
&\quad (X \otimes e_f), (e_f \otimes I_n), (e_f \otimes X)], \\
P_2^j &= [-(I_n \otimes a_x^j), -(a_x^j \otimes I_n), -(e_f \otimes e_b^j), -(e_b^j \otimes e_f), (a_x^j \otimes e_f), \\
&\quad (I_n \otimes e_b^j), (e_f \otimes a_x^j), (e_b^j \otimes I_n)],
\end{aligned}
$$

$$
\begin{aligned}
s_1 &= [\mathrm{vec}(\Delta A^*)^\top, \mathrm{vec}(\Delta A)^\top, \mathrm{vec}(\Delta Q)^\top, \mathrm{vec}(\Delta R)^\top, \mathrm{vec}(\Delta L^*)^\top,\\
&\quad \mathrm{vec}(\Delta B^*)^\top, \mathrm{vec}(\Delta L)^\top, \mathrm{vec}(\Delta B)^\top],\\
s_2^j &= [\mathrm{vec}(\Delta A_0^j)^\top, \mathrm{vec}(\Delta A_0^{j*})^\top, \mathrm{vec}(\Delta B_0^j)^\top, \mathrm{vec}(\Delta B_0^{j*})^\top, \mathrm{vec}(\Delta B_0^{j*})^\top,\\
&\quad \mathrm{vec}(\Delta A_0^j)^\top, \mathrm{vec}(\Delta B_0^j)^\top, \mathrm{vec}(\Delta A_0^{j*})^\top],
\end{aligned}
$$

for $j = 1, 2, \ldots, N$. Before we solve the equation (18) and derive the explicit expression and its upper bound of the normwise condition number of CRRE (1), we state a lemma about the nonsingular property of Z.

Lemma 3.3 (Horn and Johnson [22])**.** *Given A, $B \in \mathbb{R}^{n\times n}$ and let α_1, α_2, $\ldots$, α_n and β_1, β_2, $\ldots$, β_n be eigenvalues of A and B, respectively. Then the eigenvalues of $I_n \otimes A^\top + A^\top \otimes I_n + B^\top \otimes B^\top$ and $A^\top \otimes A^\top + B^\top \otimes B^\top - I_n \otimes I_n$ can be represented as $2\alpha_i + \beta_i^2$ and $\alpha_i^2 + \beta_i^2 - 1$, respectively.*

Let X be the maximal and stabilizing solution of (1), then $\sigma(L_c) \subset \mathbb{C}_-$. Furthermore,

$$
L_c(I_n) = \Phi_C^* + \Phi_C + \sum_{i=1}^{N} \Psi_C^{i*}\Psi_C \text{ and } \sigma(L_c(I_n)) \subset \mathbb{C}_-.
$$

From the Lemma 3.3,

$$
\sigma(Z) \subset \mathbb{C}_- \text{ and } \det(Z) \neq 0.
$$

Then, we can solve this equation (18)

$$
\mathrm{vec}(\Delta X) = Z^{-1}Ps. \tag{19}
$$

Define a mapping

$$
\varphi : (A, B, Q, L, R, A_0^j, B_0^j) \mapsto \mathrm{vec}(X),
$$

where X is the maximal and stabilizing solution of CRRE (1). The normwise condition number of CRRE (1) is defined as follows:

$$
n(\varphi, u) = \lim_{\epsilon\to 0} \sup_{\mathcal{E}} \frac{\|\mathrm{vec}(\Delta X)\|}{\delta(u + \Delta u, u)\|X\|_F}, \tag{20}
$$

where

$$
\delta(u + \Delta u, u) \quad = \quad \max_{\substack{i=1,2,\ldots,p\\ u_i \neq, 0}} \left\{ \frac{|\Delta u_i|}{|u_i|} \right\},
$$

$$\begin{aligned}\mathcal{E} &= \{\Delta E | \|\Delta E\|_F \le \epsilon \|E\|_F, E : A, B, Q, L, R, A_0^j, B_0^j, \epsilon > 0\},\\ u &\equiv [u_i] \equiv (u_1, u_2)^\top, u_2 = [u_2^1, u_2^2, \ldots, u_2^j], \text{for } j = 1, 2, \ldots, N,\end{aligned}$$

with

$$\begin{aligned}u_1 &= (\mathrm{vec}(A^*)^\top, \mathrm{vec}(A)^\top, \mathrm{vec}(Q)^\top, \mathrm{vec}(R)^\top, \mathrm{vec}(L^*)^\top, \mathrm{vec}(B^*)^\top,\\ &\quad \mathrm{vec}(L)^\top, \mathrm{vec}(B)^\top);\\ u_2^j &= (\mathrm{vec}(A_0^j)^\top, \mathrm{vec}(A_0^{j*})^\top, \mathrm{vec}(B_0^j)^\top, \mathrm{vec}(B_0^{j*})^\top, \mathrm{vec}(B_0^{j*})^\top, \mathrm{vec}(A_0^j)^\top,\\ &\quad \mathrm{vec}(B_0^j)^\top, \mathrm{vec}(A_0^{j*})^\top).\end{aligned}$$

Theorem 3.4. *The explicit expression and its upper bound of the normwise condition number $n(\varphi, u)$ are*

$$\begin{aligned}n(\varphi, u) &= \|t\| / \|X\|_F,\\ n_u(\varphi) &= \|Z^{-1}\| w_1 / \|X\|_F,\end{aligned} \tag{21}$$

where

$$\begin{aligned}t &= |Z^{-1}(X \otimes I_n)| \mathrm{vec}(|A^*|) + |Z^{-1}(I_n \otimes X)| \mathrm{vec}(|A|)\\ &+ |Z^{-1}(I_n \otimes I_n)| \mathrm{vec}(|Q|) + |Z^{-1}(e_f \otimes e_f)| \mathrm{vec}(|R|)\\ &+ |Z^{-1}(I_n \otimes e_f)| \mathrm{vec}(|L^*|) + |Z^{-1}(X \otimes e_f)| \mathrm{vec}(|B^*|)\\ &+ |Z^{-1}(e_f \otimes I_n)| \mathrm{vec}(|L|) + |Z^{-1}(e_f \otimes X)| \mathrm{vec}(|B|)\\ &+ \sum_{j=1}^{N} (|Z^{-1}(I_n \otimes a_x^j)| \mathrm{vec}(|A_0^j|) + |Z^{-1}(a_x^j \otimes I_n)| \mathrm{vec}(|A_0^{j*}|)\\ &+ |Z^{-1}(e_f \otimes e_b^j)| \mathrm{vec}(|B_0^j|) + |Z^{-1}(e_b^j \otimes e_f)| \mathrm{vec}(|B_0^{j*}|)\\ &+ |Z^{-1}(a_x^j \otimes e_f)| \mathrm{vec}(|B_0^{j*}|) + |Z^{-1}(I_n \otimes e_b^j)| \mathrm{vec}(|A_0^j|)\\ &+ |Z^{-1}(e_f \otimes a_x^j)| \mathrm{vec}(|B_0^j|) + |Z^{-1}(e_b^j \otimes I_n)| \mathrm{vec}(|A_0^{j*}|));\\ w_1 &= 2\|A\|_F \|X\|_F + \|Q\|_F + \|e_{fa}\|_F^2 \|R\|_F + 2\|e_{fa}\|_F \|L\|_F\\ &+ 2\|e_{fa}\|_F \|B\|_F \|X\|_F + \sum_{j=1}^{N} (2\|a_{xa}^j\|_F \|A_0^j\|_F + 2\|e_{fa}\|_F \|B_0^j\|_F \|e_{ba}^j\|_F\\ &+ 2\|e_{fa}\|_F \|B_0^j\|_F \|a_{xa}^j\|_F + 2\|e_{ba}^j\|_F \|A_0^j\|_F).\end{aligned}$$

Proof. From the formula of the normwise condition number of CRRE (1), we compute Δu and $\mathrm{vec}(\Delta X)$ via (19). Adding some small perturbations to the vector u and we get the equality $s = \Delta u$. From the property of the absolute value, we have

$$\begin{aligned}
|e_f| &\leq |E(X)||F(X)^\dagger| \equiv e_{fa}, \\
|a_x^j| &\leq |A_0^{j*}||X| \equiv a_{xa}^j, \\
|e_b^j| &\leq |E(X)||F(X)^\dagger||B_0^{j*}||X| \equiv e_{ba}^j.
\end{aligned}$$

Hence, we derive the explicit expression and its perturbation bound of $n(\varphi, u)$ using the formula (20)

$$\begin{aligned}
n(\varphi, u) &= \frac{\||Z^{-1}P||u|\|}{\|X\|_F}; \\
&= \||Z^{-1}(X \otimes I_n)|\mathrm{vec}(|A^*|) + |Z^{-1}(I_n \otimes X)|\mathrm{vec}(|A|) \\
&+ |Z^{-1}(I_n \otimes I_n)|\mathrm{vec}(|Q|) + |Z^{-1}(e_f \otimes e_f)|\mathrm{vec}(|R|) \\
&+ |Z^{-1}(I_n \otimes e_f)|\mathrm{vec}(|L^*|) + |Z^{-1}(X \otimes e_f)|\mathrm{vec}(|B^*|) \\
&+ |Z^{-1}(e_f \otimes I_n)|\mathrm{vec}(|L|) + |Z^{-1}(e_f \otimes X)|\mathrm{vec}(|B|) \\
&+ \sum_{j=1}^{N}(|Z^{-1}(I_n \otimes a_x^j)|\mathrm{vec}(|A_0^j|) + |Z^{-1}(a_x^j \otimes I_n)|\mathrm{vec}(|A_0^{j*}|) \\
&+ |Z^{-1}(e_f \otimes e_b^j)|\mathrm{vec}(|B_0^j|) + |Z^{-1}(e_b^j \otimes e_f)|\mathrm{vec}(|B_0^{j*}|) \\
&+ |Z^{-1}(a_x^j \otimes e_f)|\mathrm{vec}(|B_0^{j*}|) + |Z^{-1}(I_n \otimes e_b^j)|\mathrm{vec}(|A_0^j|) \\
&+ |Z^{-1}(e_f \otimes a_x^j)|\mathrm{vec}(|B_0^j|) + |Z^{-1}(e_b^j \otimes I_n)|\mathrm{vec}(|A_0^{j*}|))\|/\|X\|_F; \\
&\equiv \|t\|/\|X\|_F; \\
&\leq \|Z^{-1}\|\||A^*||X| + |X||A| + |Q| + |e_f||R||e_f^*| + |e_f||L^*| \\
&+ |e_f||B^*||X| + |L||e_f^*| + |X||B||e_f^*| + \sum_{j=1}^{N}(|a_x^j||A_0^j| + |A_0^{j*}||a_x^{j*}| \\
&+ |e_b^j||B_0^j||e_f^*| + |e_f||B_0^{j*}||e_b^{j*}| + |e_f||B_0^{j*}||a_x^{j*}| + |e_b^j||A_0^j| \\
&+ |a_x^j||B_0^j||e_f^*| + |A_0^{j*}||e_b^{j*}|)\|_F/\|X\|_F; \\
&\leq \|Z^{-1}\|\||A^*||X| + |X||A| + |Q| + e_{fa}|R|e_{fa}^* + e_{fa}|L^*| \\
&+ e_{fa}|B^*||X| + |L|e_{fa}^* + |X||B|e_{fa}^* + \sum_{j=1}^{N}(a_{xa}^j|A_0^j| + |A_0^{j*}|a_{xa}^{j*} \\
&+ e_{ba}^j|B_0^j|e_{fa}^* + e_{fa}|B_0^{j*}|e_{ba}^{j*} + e_{fa}|B_0^{j*}|a_{xa}^{j*} + e_{ba}^j|A_0^j| + a_{xa}^j|B_0^j|e_{fa}^* \\
&+ |A_0^{j*}|e_{ba}^{j*})\|_F/\|X\|_F;
\end{aligned}$$

$$
\begin{aligned}
&\leq \|Z^{-1}\|(2\|A\|_F\|X\|_F + \|Q\|_F + \|e_{fa}\|_F^2\|R\|_F + 2\|e_{fa}\|_F\|L\|_F \\
&+ 2\|e_{fa}\|_F\|B\|_F\|X\|_F + \sum_{j=1}^{N}(2\|a_{xa}^j\|_F\|A_0^j\|_F \\
&+ 2\|e_{fa}\|_F\|B_0^j\|_F\|e_{ba}^j\|_F + 2\|e_{fa}\|_F\|B_0^j\|_F\|a_{xa}^j\|_F \\
&+ 2\|e_{ba}^j\|_F\|A_0^j\|_F))/\|X\|_F; \\
&\equiv \|Z^{-1}\|w_1/\|X\|_F.
\end{aligned}
$$

□

3.3. Mixed and componentwise condition numbers

By considering the matrix structure, we introduce a lemma about mixed and componentwise perturbation anlayses to compute condition numbers of CRREs (1) efficiently.

Lemma 3.5 (Gohberg and Koltracht [17]). *Let $F : \mathbb{R}^p \to \mathbb{R}^q$ be a continuous map defined on an open set* $\mathrm{Dom}(F) \subset \mathbb{R}^p$*. For a given $a \neq 0 \in$* $\mathrm{Dom}(F)$*, such that $F(a) \neq 0$, where* $\mathrm{Dom}(F)$ *denotes the domain of the function F. Let $B^0(a,\epsilon) = \{x : |x_i - a_i| \leq \epsilon|a_i|, i = 1,2,\ldots,p\}$, where $a = (a_1, a_2, \ldots, a_p)^\top, x = (x_1, x_2, \ldots, x_p)^\top \in \mathbb{R}^p$ and $\epsilon > 0$.*

(a) The mixed condition number of the map F at the point a is defined

$$m(F,a) = \lim_{\epsilon \to 0} \sup_{\substack{x \in B^0(a,\epsilon) \\ x \neq a}} \frac{\|F(x) - F(a)\|_\infty}{\|F(a)\|_\infty} \frac{1}{\delta(x,a)},$$

where $\delta(x,a) = \max_{\substack{i=1,2,\ldots,p \\ a_i \neq 0}} \left\{ \frac{|x_i - a_i|}{|a_i|} \right\}$.

(b) Suppose $F(a) = (f_1(a), f_2(a), \ldots, f_q(a))$ such that $f_j(a) \neq 0$, for $j = 1,2,\ldots,q$. The componentwise condition number of F at the point a is

$$c(F,a) = \lim_{\epsilon \to 0} \sup_{\substack{x \in B^0(a,\epsilon) \\ x \neq a}} \frac{\delta(F(x), F(a))}{\delta(x,a)},$$

where $\delta(F(x), F(a)) = \max_{j=1,2,\ldots,q} \left\{ \frac{|f_j(x) - f_j(a)|}{|f_j(a)|} \right\}$.

Theorem 3.6. *Based on Lemma 3.5 and the definitions of φ and u, we express the mixed and componentwise condition numbers of CRRE (1) as*

$$\begin{aligned} m(\varphi,u) &= \|t\|_\infty/\|X\|_{\max}, \\ c(\varphi,u) &= \|t/\mathrm{vec}(X)\|_\infty. \end{aligned}$$

The upper bounds of $m(\varphi,u)$ and $c(\varphi,u)$ are respectively

$$m_u(\varphi) = \|Z^{-1}\|_\infty w_2/\|X\|_{\max}, \tag{22}$$

$$c_u(\varphi) = \|\mathrm{Diag}^{-1}(\mathrm{vec}(|X|))Z^{-1}\|_\infty w_2, \tag{23}$$

where

$$\begin{aligned} w_2 &= 2\|A\|_{\max}\|X\|_{\max} + \|Q\|_{\max} + \|e_{fa}\|_{\max}^2\|R\|_{\max} + 2\|e_{fa}\|_{\max}\|L\|_{\max} \\ &+ 2\|e_{fa}\|_{\max}\|B\|_{\max}\|X\|_{\max} + \sum_{j=1}^{N}(2\|a_{xa}^j\|_{\max}\|A_0^j\|_{\max} \\ &+ 2\|e_{fa}\|_{\max}\|B_0^j\|_{\max}\|e_{ba}^j\|_{\max} + 2\|e_{fa}\|_{\max}\|B_0^j\|_{\max}\|a_{xa}^j\|_{\max} \\ &+ 2\|e_{ba}^j\|_{\max}\|A_0^j\|_{\max}). \end{aligned}$$

If $x = (x_1, x_2, \ldots, x_p)^\top \in \mathbb{R}^p$, then $\mathrm{Diag}(x)$ denotes the $p\times p$ diagonal matrix with $x_1, x_2, \ldots, x_p$ on its diagonal.

Proof. Let $x \equiv u + \Delta u$, $a \equiv u$ and $F \equiv \varphi$. Based on (a) of Lemma 3.5, the definition of the mapping φ and the properties of Kronecker products (17), we can get

$$\begin{aligned} \frac{\|F(x)-F(a)\|_\infty}{\|F(a)\|_\infty} &= \frac{\|\varphi(u+\Delta u)-\varphi(u)\|_\infty}{\|\varphi(u)\|_\infty} \\ &= \frac{\|\mathrm{vec}(X+\Delta X)-\mathrm{vec}(X)\|_\infty}{\|\mathrm{vec}(X)\|_\infty} \\ &= \frac{\|\mathrm{vec}(\Delta X)\|_\infty}{\|\mathrm{vec}(X)\|_\infty} = \frac{\|\mathrm{vec}(\Delta X)\|_\infty}{\|X\|_{\max}}. \end{aligned}$$

The formula of mixed condition number of CRRE (1) can be rewritten

$$m(\varphi,u) = \lim_{\epsilon\to 0}\sup_{\mathcal{F}} \frac{\|\mathrm{vec}(\Delta X)\|_\infty}{\delta(u+\Delta u,u)\|X\|_{\max}}, \tag{24}$$

where $\mathcal{F} = \{\Delta F | |\Delta F| \le \epsilon|F|, F : A, B, Q, L, R, A_0^j, B_0^j, \epsilon > 0\}$. Similarly, we can also rewrite the formula of componentwise condition number of

CRRE (1)

$$c(\varphi,u)=\lim_{\epsilon\to 0}\sup_{\mathcal{F}}\frac{1}{\delta(u+\Delta u,u)}\left\|\frac{\mathrm{vec}(\Delta X)}{\mathrm{vec}(X)}\right\|_\infty. \tag{25}$$

From (19) and (24), we can get the explicit expression and its perturbation bound of the mixed condition number of CRRE (1)

$$\begin{aligned} m(\varphi,u) &= \frac{\||Z^{-1}P||u|\|_\infty}{\|X\|_{\max}} \\ &= \|t\|_\infty/\|X\|_{\max} \\ &\le \|Z^{-1}\|_\infty w_2/\|X\|_{\max}. \end{aligned}$$

Analogously, the explicit expression and its upper bound of the componentwise condition number are derived using (25)

$$\begin{aligned} c(\varphi,u) &= \left\|\frac{|Z^{-1}P||u|}{\mathrm{vec}(X)}\right\|_\infty \\ &= \|t/\mathrm{vec}(X)\|_\infty \\ &\le \|\mathrm{Diag}^{-1}(\mathrm{vec}(|X|))Z^{-1}\|_\infty w_2. \end{aligned}$$

$\square$

4. Numerical experiments

In this section, we have shown relative errors with respect to the solutions of CREs (1) under normwise, mixed and componentwise perturbation analyses such as $\frac{\|\Delta X\|_F}{\|X\|_F}$, $\frac{\|\Delta X\|_{\max}}{\|X\|_{\max}}$ and $\left\|\frac{\Delta X}{X}\right\|_{\max}$ and their sharper upper bounds in (21), (22), (23). The numerical algorithm is described in Algorithm 1 for condition numbers of rational Riccati equations. We have chosen one representative example:

(1) The CRREs in Example 4.1 are quoted from [2].

All the numerical experiments were conducted using MATLAB [28] Version R2018a on a MacBook Pro with a 2.30 GHz Intel Core 8 Duo processor and 64 GB RAM, with machine accuracy $eps = 2.22\times 10^{-16}$.

For the numerical results, we express r_n^{CRRE}, r_m^{CRRE}, r_c^{CRRE} for relative errors; n_u^{CRRE}, m_u^{CRRE}, c_u^{CRRE} for their upper bounds in the normwise, mixed and componentwise perturbation analyses.

Algorithm 1 (Condition numbers of rational Riccati equations)

Input: $A, Q, A_0^i \in \mathbb{K}^{n\times n}$, $B, L, B_0^i \in \mathbb{K}^{n\times m}$, and $R \in \mathbb{K}^{m\times m}$, for $i = 1, 2, \ldots, N$;
Output: Normwise, mixed and componentwise condition numbers r_n^{CRRE}, r_m^{CRRE} and r_c^{CRRE}, upper bounds n_u^{CRRE}, m_u^{CRRE} and c_u^{CRRE};

Let ΔA, ΔQ, $\Delta A_0^i \in \mathbb{K}^{n\times n}$, ΔB, ΔL, $\Delta B_0^i \in \mathbb{K}^{n\times m}$, and $\Delta R \in \mathbb{K}^{m\times m}$ be selected from normal distribution and the weighted coefficients be 10^{-k} such as $\Delta E = 10^{-k} \times \mathrm{randn}(n, m)$, for $E \in \mathbb{K}^{n\times m}$;

Set $\tilde{A} = A + \Delta A$, $\tilde{Q} = Q + \Delta Q$, $\tilde{B} = B + \Delta B$, $\tilde{L} = L + \Delta L$, $\tilde{R} = R + \Delta R$, $\tilde{A}_0^i = A_0^i + \Delta A_0^i$ and $\tilde{B}_0^i = B_0^i + \Delta B_0^i$;

Solve the CRREs (1) and its perturbation equation (10) by the GSM and get the solutions X and $\tilde{X}$, respectively;

Compute the relative errors with respect to the solution X under the normwise, mixed and componentwise perturbation analyses,
$r_n^{CRRE} = \frac{\|\tilde{X}-X\|_F}{\|X\|_F}$, $r_m^{CRRE} = \frac{\|\tilde{X}-X\|_{\max}}{\|X\|_{\max}}$, and $r_c^{CRRE} = \left\| \frac{\mathrm{vec}(\tilde{X}-X)}{\mathrm{vec}(X)} \right\|_\infty$,
where $\frac{A}{B} = \frac{(a_{ij})}{(b_{ij})}$, with $A = (a_{ij})$ and $B = (b_{ij})$;

Estimate the upper bounds n_u^{CRRE}, m_u^{CRRE} and c_u^{CRRE} in (21), (22) and (23);

End Do

Example 1 (CRREs [2])

This example 1 is an application of a mathematical model about an L-1011 aircraft, quoted from [2, Example 3] with the addition of stochastic disturbances in Π. Consider the CRREs (1) with $n = 4$, $m = 2$ and $N = 1$,

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1.89 & 0.39 & -5.53 \\ 0 & -0.034 & -2.98 & 2.43 \\ 0.034 & -0.0011 & -0.99 & -0.21 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 \\ 0.36 & -1.6 \\ -0.95 & -0.032 \\ 0.03 & 0 \end{bmatrix},$$

$$Q = \begin{bmatrix} 2.313 & 2.727 & 0.688 & 0.023 \\ 2.727 & 4.271 & 1.148 & 0.323 \\ 0.688 & 1.148 & 0.313 & 0.102 \\ 0.023 & 0.323 & 0.102 & 0.083 \end{bmatrix}, \quad L = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$X_0 = \begin{bmatrix} 1.3239 & 0.9015 & 0.5466 & -1.7672 \\ 0.9015 & 0.9607 & 0.4334 & -1.1989 \\ 0.5466 & 0.4334 & 0.4605 & -1.3633 \\ -1.7672 & -1.1989 & -1.3633 & 4.4612 \end{bmatrix}.$$

Table 1: Example 4.1 (Condition Numbers of CRREs; $n = 4$, $m = 2$, $N = 1$)

k	r_n^{CRRE}	$\epsilon_n n_u^{CRRE}$	r_m^{CRRE}	$\epsilon_m m_u^{CRRE}$	r_c^{CRRE}	$\epsilon_m c_u^{CRRE}$
5	4.5270e−05	2.7967e−03	3.7340e−05	5.4646e−04	3.7340e−05	1.6244e−03
6	4.5986e−06	4.2636e−04	3.8116e−06	1.6425e−04	3.8116e−06	4.8825e−04
7	5.3009e−07	2.4990e−05	5.3509e−07	1.3955e−05	5.3509e−07	4.1483e−05
8	1.1444e−07	2.6815e−06	1.1903e−07	2.0314e−06	1.1903e−07	6.0385e−06
9	1.5689e−09	2.4698e−07	1.9287e−09	2.9215e−07	1.9287e−09	8.6845e−07
10	1.1968e−09	2.4907e−08	1.3123e−09	2.0833e−08	1.3123e−09	6.1927e−08

X_0 is the initial solution of (4) to the CRREs solved by NM. Let the perturbed coefficient matrices ΔA, ΔB, ΔR, ΔL, ΔQ, ΔA_0 and ΔB_0 be generated using the MATLAB command "randn" associated with the weighted coefficient 10^{-k}, then we add the perturbation matrices to the original ones and get $(\tilde{A}, \tilde{B}, \tilde{R}, \tilde{L}, \tilde{Q}, \tilde{A}_0, \tilde{B}_0) = (A + \Delta A, B + \Delta B, R + \Delta R, L + \Delta L, Q + \Delta Q, A_0 + \Delta A_0, B_0 + \Delta B_0)$, which are coefficient matrices of the perturbed CRREs (10). We apply the efficient method called GSM [14] associated with NM to solve the CRREs (1) and perturbed CRREs (10) and get the unique stabilizing and maximal solutions X and $\tilde{X}$ respectively, then compute the relative errors under three kinds of perturbation analyses such as r_n^{CRRE}, r_m^{CRRE} and r_c^{CRRE} with different weighted coefficients 10^{-k}, for $k = 5, \ldots, 10$.

From Theorems 3.4 and 3.6, we can obtain their upper bounds n_u^{CRRE}, m_u^{CRRE} and c_u^{CRRE}. Set

$$\begin{aligned}
\epsilon_n : &= \max\left\{ \frac{\|\Delta A\|_F}{\|A\|_F}, \frac{\|\Delta B\|_F}{\|B\|_F}, \frac{\|\Delta R\|_F}{\|R\|_F}, \frac{\|\Delta L\|_F}{\|L\|_F}, \frac{\|\Delta Q\|_F}{\|Q\|_F}, \frac{\|\Delta A_0\|_F}{\|A_0\|_F}, \right. \\
& \qquad \left. \frac{\|\Delta B_0\|_F}{\|B_0\|_F} \right\}, \\
\epsilon_m : &= \min\{\epsilon : |\Delta A| \le \epsilon|A|, |\Delta B| \le \epsilon|B|, |\Delta Q| \le \epsilon|Q|, |\Delta L| \le \epsilon|L|, \\
& \qquad |\Delta R| \le \epsilon|R|, |\Delta A_0| \le \epsilon|A_0|, |\Delta B_0| \le \epsilon|B_0|, \epsilon > 0\},
\end{aligned}$$

it can be found from Table 1 that the perturbation bounds are close to relative errors under three kinds of perturbation analyses such as $r_n^{CRRE} \lesssim \epsilon_n n_u^{CRRE}$, $r_m^{CRRE} \lesssim \epsilon_m m_u^{CRRE}$ and $r_c^{CRRE} \lesssim \epsilon_m c_u^{CRRE}$, that is, (21), (22) and (23) give sharper upper bounds of the relative errors with respect to the unique stabilizing solution X.

Example 2 (CRREs [27])

The example 2 involves circulant matrices, modified from [27, Example 5] with the addition of stochastic disturbances A_0 and B_0. Consider the CRREs (1) with $n = m = 100$ and $N = 1$,

$$A = \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 & 1 \\ 1 & \ddots & \ddots & & & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & & & 0 \\ 0 & & & & & 1 \\ 1 & 0 & \cdots & 0 & 1 & -2 \end{bmatrix},$$

$$Q = \begin{bmatrix} 4.595 & -2 & 0 & \cdots & 0 & -2 \\ -2 & 4.99 & \ddots & & & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & & & 0 \\ 0 & & & & & -2 \\ -2 & 0 & \cdots & 0 & -2 & 4.99 \end{bmatrix},$$

$$L = 0_{100}, \quad BR^{-1}B^{\top} = I_{100}, \quad X_0 = I_{100},$$

$$A_0 = 0.1 * I_{100}, \quad B_0 = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & \cdots & \cdots & 0 \end{bmatrix}.$$

We generate one random sample (ΔA, ΔB, ΔR, ΔL, ΔQ, ΔA_0, ΔB_0) and obtain coefficient matrices of the perturbed CRREs (10), then the relative errors under the normwise, mixed and componentwise perturbation analysis are computed according to different weighted coefficients 10^{-k}, for $k = 9, 10, 11, 12, 13, 14$, respectively. Moreover, upper bounds under three kinds of perturbation analysis are following derived.

It can be seen from Table 2 that the values of the relative errors are closely bounded by our perturbation bounds of (21), (22) and (23). In other

Table 2: Example 4.2 (Condition Numbers of CRREs; $n = 100$, $m = 100$, $N = 1$)

k	r_n^{CRRE}	$\epsilon_n n_u^{CRRE}$	r_m^{CRRE}	$\epsilon_m m_u^{CRRE}$	r_c^{CRRE}	$\epsilon_m c_u^{CRRE}$
9	5.6622e−09	1.8864e−05	2.0797e−09	4.7837e−09	2.0797e−09	4.7837e−09
10	5.6475e−10	1.8865e−06	1.6699e−10	2.1944e−10	1.6699e−10	2.1944e−10
11	5.5968e−11	1.9095e−07	2.5674e−11	4.8917e−11	2.5674e−11	4.8917e−11
12	5.6183e−12	1.8865e−08	2.0756e−12	4.8413e−12	2.0756e−12	4.8413e−12
13	5.7175e−13	1.8679e−09	2.4425e−13	4.8040e−13	2.4425e−13	4.8040e−13
14	5.6072e−14	1.8882e−10	2.2871e−14	4.4806e−14	2.2871e−14	4.4806e−14

words, (21), (22) and (23) does provide a sharp upper bound of the relative errors of the stabilizing solution X. Table 2 shows that our estimates are tight.

5. Conclusions

In this paper, we present condition numbers and upper bounds for the rational Riccati equations (1) under small perturbation in the coefficient matrices. Furthermore, some sufficient conditions are presented for the existence of the stabilizing solution to the CRRE (1) and perturbed CRRE (10), respectively. We highlight and compare the practical performance of the derived condition numbers and perturbation bounds under the normwise, mixed and componentwise perturbation analyses through two numerical examples. Numerical results show that we provide tight and sharp perturbation bounds of the stabilizing solution to CRRE, that is, our perturbation bound is very sensitive to the condition numbers of the stabilizing solution in small- and medium-sized problems about applications of a mathematical model [2] and a circulant matrix [27], respectively. For the large-scale problem, we can also get the condition estimates and upper bounds by modifying our algorithm and only need to consider the numerically low-rank and sparsity structures in solving CRREs. This will be our future work.

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