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A study on random differential equations of arbitrary order

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In this paper, the well-posedness of fractional random differential equations (FRDEs) involving Hilfer-Katugampola fractional derivative (HKFD) is discussed. The sufficient conditions to existence of solutions for FRDEs involving initial, nonlocal and impulsive conditions are generated using standard fixed point theorems. Further the stability of solution is verified by the concept proposed by Ulam. Uniqueness solutions of initial value problems for FRDEs using picards iterative techique and continuous dependence of data are also discussed.

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1. Preface

Fractional calculus is generalization of ordinary differentiation and integration to arbitrary non-integer order. The subject is as old as the differential calculus. Integral equations are one of the most useful mathematical tools in both pure and applied analysis. We can find numerous applications of differential and integral equation of fractional order in finance, hydrology, biophysics; thermodynamics control theory, statistical mechanics, astrophysics, cosmology and bioengineering [11, 12, 19]. For the significant development in fractional differential equations (FDEs) in recent years; see [2, 4, 5].

Evaluation of parameters of a dynamical system is not without uncertainties. When our knowledge about the parameters of a dynamic system is of statistical nature, that is, the information is probabilistic; the common approach in mathematical modeling of such systems is the use of random differential equations (RDEs) or stochastic differential equations. RDEs, as natural extensions of deterministic ones, arise in many applications and have been investigated by many mathematicians. We refer the reader to the papers. The analyses of FDEs with random parameters have been studied in, [15, 20, 23]. The existence results for a FRDEs is discussed here [3, 6, 16, 23]. As an extension of work [9], here we study the special case of kernel which represents Hilfer-Katugampola fractional derivative (HKFD). The generalization of HKFD and the properties are chiefly discussed by Oliveira and Capelas, in [18]. The detailed study and theoretical analysis of FDEs involving HKFD is discussed in [10].

2. Preliminary

Set (Ω, F, p) is a complete probability space. Define the Banach space of all continuous random functions space, $C([a,b] = J \times \Omega, \mathbb{R}) := \{\mathfrak{u} : J \times \Omega \to \mathbb{R}\}$ with the norm

$$\|\mathfrak{u}\|_C = \sup\left\{|\mathfrak{u}(t,\vartheta)| : t \in J,\ \vartheta \in \Omega\right\}.$$

We denote the weighted spaces of all continuous random functions space, defined by

$$C_{1-\gamma,\rho}(J,\mathbb{R}) = \left\{\mathfrak{u} : J \times \Omega \to \mathbb{R} : \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} \mathfrak{u}(t,\vartheta) \in C(J,\mathbb{R})\right\}, \quad 0 \le \gamma < 1,$$

with the norm

$$\|\mathfrak{u}\|_{C_{1-\gamma,\rho}} = \sup_{t\in J}\left|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} \mathfrak{u}(t,\vartheta)\right|.$$

Next, we introduce the piecewise continuous space

$$PC = \Big\{\mathfrak{u} : J \times \Omega \to \mathbb{R} : \mathfrak{u} \in C(t_k, t_{k+1}], k = 0, \dots, m; \text{ there exists } \mathfrak{u}_{(t_k^+)}(\vartheta) \text{ and } \mathfrak{u}_{(t_k^-)}(\vartheta)\Big\}.$$

Now, we give the weighted piecewise continuous space of the form $PC_{1-\gamma,\rho}(\vartheta)$,

$$PC_{1-\gamma,\rho} = \left\{\mathfrak{u} : \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} \mathfrak{u}(t,\vartheta)|_{t\in[t_k,t_{k+1}]} \in C[t_k, t_{k+1}], k = 0, \dots, m, \text{ where } 0 \le \gamma < 1\right\}.$$

Obviously, which is a Banach space with norm

$$\|\mathfrak{u}\|_{PC_{1-\gamma,\rho}} = \sup_{t\in(t_k,t_{k+1}]} \left\{ \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} \mathfrak{u}(t,\vartheta) \right\}, k = 0, \ldots, m.$$

Definition 2.1 ([13]). *The generalized left-sided fractional integral* ${}^\rho I^\alpha f$ *of order* $\alpha \in C(\Re(\alpha))$ *is defined by*

$$(1) \qquad \left({}^\rho I^\alpha\right) f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f(s) ds, \ t > a,$$

if the integral exists.

The generalized fractional derivative, corresponding to the generalized fractional integral (1), *is defined for* $0 \le a < t$, *by*

$$(2) \qquad \left({}^\rho D^\alpha f\right)(t) = \frac{\rho^{\alpha-n-1}}{\Gamma(n-\alpha)} \left(t^{1-\rho}\frac{d}{dt}\right)^n \int_a^t (t^\rho - s^\rho)^{n-\alpha+1} s^{\rho-1} f(s) ds,$$

if the integral exists.

Definition 2.2 ([18]). *The Hilfer-Katugampola fractional operator with respect to* t, *of order* $\rho > 0$, *is defined by*

$$(3) \qquad \begin{aligned} \left({}^\rho D^{\alpha,\beta} f\right)(t) &= \left({}^\rho I^\alpha \left(t^{\rho-1}\frac{d}{dt}\right) {}^\rho I^{(1-\beta)(1-\alpha)}\right)(t) \\ &= \left({}^\rho I^\alpha \delta_\rho {}^\rho I^{(1-\beta)(1-\alpha)}\right)(t). \end{aligned}$$

Lemma 2.3. *[18] Let* $\alpha, \beta > 0$, *and*

$$\begin{gathered} {}^\rho I^\alpha \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\beta-1}(t) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\beta-1}, \\ {}^\rho D^\alpha \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-1}(t) = 0. \end{gathered}$$

Lemma 2.4. *[18] If* $\alpha > 0$ *and* $0 \le \mu < 1$, *then* ${}^\rho I^\alpha$ *is bounded from* $C_\mu(J,\mathbb{R})$ *into* $C_\mu(J,\mathbb{R})$. *In addition, if* $\mu \le \alpha$, *then* ${}^\rho I^\alpha$ *is bounded from* $C_\mu(J,\mathbb{R})$ *into* $C(J,\mathbb{R})$.

Lemma 2.5. *[27] Suppose* $\alpha > 0$, $a(t,\vartheta)$ *is a nonnegative function locally integrable on* $J \times \Omega$ *(some* $T \le \infty$*), and let* $g(t,\vartheta)$ *be a nonnegative, nondecreasing continuous function defined on* $J \times \Omega$, *such that* $g(t,\vartheta) \le K$ *for* *some constant K. Further let $\mathfrak{u}(t,\vartheta)$ be a nonnegative locally integrable on $J \times \Omega$ function with*

$$\mathfrak{u}(t,\vartheta) \leq a(t,\vartheta) + g(t,\vartheta) \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\mathfrak{u}(s,\vartheta)ds, \quad (t,\vartheta) \in J \times \Omega,$$

with some $\alpha > 0$. Then

$$\mathfrak{u}(t,\vartheta) \leq a(t,\vartheta) + \int_a^t \left[\sum_{n=1}^{\infty} \frac{(g(t,\vartheta)\Gamma(\alpha))^n}{\Gamma(n\alpha)} \left(\frac{t^\rho - s^\rho}{\rho}\right)^{n\alpha-1} s^{\rho-1}\right] a(s,\vartheta)ds,$$
$$(t,\vartheta) \in J \times \Omega.$$

Remark 2.6. *Under the hypothesis of Lemma 2.5 let $a(t,\vartheta)$ be a nondecreasing function on $[a,b]$. Then $\mathfrak{u}(t,\vartheta) \leq a(t,\vartheta)E_\alpha g(t,\vartheta)\Gamma(\alpha)\left(\frac{t^\rho - a^\rho}{\rho}\right)^\alpha$, where E_α is the Mittag-Leffler function defined by*

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)}, \ z \in C, \ Re(\alpha) > 0.$$

Lemma 2.7. *Let $\mathfrak{u} \in PC_{1-\gamma,\rho}$ satisfies the following inequality*

$$|\mathfrak{u}(t,\vartheta)| \leq a(t,\vartheta) + g(t,\vartheta) \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} |\mathfrak{u}(s,\vartheta)| \, ds + \sum_{0<t_k<t} |\mathfrak{u}_{t_k}(\vartheta)|,$$

where c_1 is a nonnegative, continuous and nondecreasing function and c_2, λ_i are constants. Then

$$|\mathfrak{u}(t,\vartheta)| \leq a(t,\vartheta) \left(1 + \lambda E_\alpha \left(g(t,\vartheta)\Gamma(\alpha)\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-1}\right)\right)^k$$
$$\times E_\alpha\left(g(t,\vartheta)\Gamma(\alpha)\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha-1}\right),$$

where $\lambda = \sup\{\lambda_k : k = 1,2,3,\ldots,m\}$.

Theorem 2.8 (Krasnoselskii's fixed point theorem [7])**.** *Let X be a Banach space, let B be a bounded closed convex subset of X and let $\mathfrak{P}_1, \mathfrak{P}_2$ be mapping from B into X such that $\mathfrak{P}_1\mathfrak{u} + \mathfrak{P}_2\mathfrak{v}, \in B$ for every pair $\mathfrak{u}, \mathfrak{v} \in B$. If $\mathfrak{P}_1$ is contraction and $\mathfrak{P}_2$ is completely continuous, then the equation $\mathfrak{P}_1\mathfrak{u} + \mathfrak{P}_2\mathfrak{u} = \mathfrak{u}$ has a solution on B.*

Theorem 2.9 (Schaefer's Fixed Point Theorem [7]). *Let K be a Banach space and let $\mathfrak{P} : K \to K$ be completely continuous operator. If the set $\{\mathfrak{u} \in K : \mathfrak{u} = \delta \mathfrak{P}\mathfrak{u}$ for some $\delta \in (0,1)\}$ is bounded, then $\mathfrak{P}$ has a fixed point.*

Theorem 2.10 (Banach Fixed Point Theorem [7]). *Suppose Q be a non-empty closed subset of a Banach space E. Then any contraction mapping $\mathfrak{P}$ from Q into itself has a unique fixed point.*

3. Initial value problem for FRDEs

Consider the FRDEs involving HKFD of the form

$$
\begin{cases} {}^{\rho}D^{\alpha,\beta}\mathfrak{u}(t,\vartheta) = \mathfrak{g}(t,\vartheta,\mathfrak{u}(t,\vartheta)), \quad t \in J := (a,b], \\ {}^{\rho}I^{1-\gamma}\mathfrak{u}(t,\vartheta)|_{t=a} = \mathfrak{u}_a(\vartheta), \end{cases} \tag{4}
$$

where $\vartheta \in \Omega$, ${}^{\rho}D^{\alpha,\beta}$ is the HKFD of order α $(0 < \alpha < 1)$ and type β $(0 \le \beta \le 1)$ and ${}^{\rho}I^{1-\gamma}$ is generalized fractional integral of order $1-\gamma$ $(\gamma = \alpha+\beta-\alpha\beta)$. Let $\mathfrak{g} : J \times \Omega \times \mathbb{R} \to \mathbb{R}$ is a given continuous function. Here, ϑ is random variable and $\mathfrak{u}_a(\vartheta)$ is random initial condition.

In this section, we prove existence and uniqueness of proposed Cauchy type problem (4). Before starting and proving this result, we list the following condition:

(H1) Lipschitz condition: There exist a constant $\ell(t,\vartheta) > 0$ such that

$$
|\mathfrak{g}(t,\vartheta,\mathfrak{u}) - \mathfrak{g}(t,\vartheta,\overline{\mathfrak{u}})| \le \ell(t,\vartheta)\, |\mathfrak{u} - \overline{\mathfrak{u}}|\,,
$$

for any $\mathfrak{u}, \overline{\mathfrak{u}} \in \mathbb{R}$, and $t \in J$.

Lemma 3.1. *Let $\gamma = \alpha + \beta - \alpha\beta$, where $0 < \alpha < 1$ and $0 \le \beta \le 1$. Let $\mathfrak{g} : J \times \Omega \times \mathbb{R} \to \mathbb{R}$ be a function such that $\mathfrak{g} \in C_{1-\gamma;\rho}(J,\mathbb{R})$ for any $\mathfrak{u} \in C_{1-\gamma;\rho}(J,\mathbb{R})$. If $\mathfrak{u}$ satisfies the problem*

$$
\begin{aligned} {}^{\rho}D^{\alpha,\beta}\mathfrak{u}(t,\vartheta) &= \mathfrak{g}(t,\vartheta,\mathfrak{u}(t,\vartheta)), \\ {}^{\rho}I^{1-\gamma}\mathfrak{u}(t,\vartheta)|_{t=a} &= \mathfrak{u}_a(\vartheta), \end{aligned}
$$

if and only if $\mathfrak{u}$ satisfies the Volterra integral equation of second kind

$$
\mathfrak{u}(t,\vartheta) = \frac{\mathfrak{u}_a(\vartheta)}{\Gamma(\gamma)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1} s^{\rho-1}\mathfrak{g}(s,\vartheta,\mathfrak{u}(s,\vartheta))ds. \tag{5}
$$

Lemma 3.2. *The Katugampola fractional integral operator* ${}^\rho I^\alpha$ *is bounded from* $C_{1-\gamma;\rho}(J,\mathbb{R})$ *to* $C_{1-\gamma;\rho}(J,\mathbb{R})$*:*

$$\|{}^\rho I^\alpha \mathfrak{g}\|_{C_{1-\gamma;\rho}} \leq M\frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^\alpha, \tag{6}$$

where, M *is the bound of bounded function* $\mathfrak{g}$*.*

Proof. From Lemma 2.4, the result follows. Now we prove the estimate (6), we have

$$\begin{aligned}\|{}^\rho I^\alpha \mathfrak{g}\|_{C_{1-\gamma;\rho}} &= \left\|\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} {}^\rho I^\alpha \mathfrak{g}\right\|_C \\ &\leq \|\mathfrak{g}\|_{C_{1-\gamma;\rho}} \frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^\alpha,\end{aligned}$$

using Lemma 2.3, we get

$$\|{}^\rho I^\alpha \mathfrak{g}\|_{C_{1-\gamma;\rho}} \leq M\frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^\alpha.$$

□

Theorem 3.3. *Here the assumption [H1] holds. There exists a unique solution* $\mathfrak{u}$ *for the Cauchy-type problem* (4) *in* $C_{1-\gamma;\rho}(J,\mathbb{R})$*.*

Proof. The integral Eq. (5) makes sense in any interval $[a,t_1] \subset [a,b]$. Choose t_1 such that

$$\ell(t,\vartheta)\frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left(\frac{t_1^\rho - a^\rho}{\rho}\right)^\alpha < 1 \tag{7}$$

holds and first we prove the existence of unique solution $\mathfrak{u} \in C_{1-\gamma;\rho}([a,t_1],\mathbb{R})$. We proceed as follows. Set Picard's sequence functions

$$\mathfrak{u}_0(t,\vartheta) = \frac{\mathfrak{u}_a(\vartheta)}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1}, \tag{8}$$

$$\mathfrak{u}_m(t,\vartheta) = \mathfrak{u}_0(t,\vartheta) + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\mathfrak{g}(s,\vartheta,\mathfrak{u}_{m-1}(s,\vartheta))ds, \quad m \in N. \tag{9}$$

We show that $\mathfrak{u}_m(t,\vartheta) \in C_{1-\gamma;\rho}(J,\mathbb{R})$. From Eq. (8), it follows that $\mathfrak{u}_0(t,\vartheta) \in C_{1-\gamma;\rho}(J,\mathbb{R})$. By Lemma 3.2, ${}^\rho I^\alpha$ is bounded from $C_{1-\gamma;\rho}(J,\mathbb{R})$ to

$C_{1-\gamma;\rho}(J,\mathbb{R})$, which gives $\mathfrak{u}_m(t,\vartheta) \in C_{1-\gamma;\rho}(J,\mathbb{R}),\ \ m \in N$. By Eq. (8) and Eq. (9), we have

$$\|\mathfrak{u}_1(t,\vartheta) - \mathfrak{u}_0(t,\vartheta)\|_{C_{1-\gamma;\rho}([a,t_1],\mathbb{R})} = \|{}^{\rho}I^{\alpha}\mathfrak{g}(t,\vartheta,\mathfrak{u}_0(t,\vartheta))\|_{C_{1-\gamma;\rho}([a,t_1],\mathbb{R})}$$

using Lemma 3.2

$$\|\mathfrak{u}_1(t,\vartheta) - \mathfrak{u}_0(t,\vartheta)\|_{C_{1-\gamma;\rho}([a,t_1],\mathbb{R})} \leq M\frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left(\frac{t_1^{\rho}-a^{\rho}}{\rho}\right)^{\alpha}. \tag{10}$$

Further we obtain

$$\begin{aligned}&\|\mathfrak{u}_2(t,\vartheta) - \mathfrak{u}_1(t,\vartheta)\|_{C_{1-\gamma;\rho}([a,t_1],\mathbb{R})}\\ &\quad\leq M\frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left(\frac{t_1^{\rho}-a^{\rho}}{\rho}\right)^{\alpha}\left(\ell(t,\vartheta)\frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left(\frac{t_1^{\rho}-a^{\rho}}{\rho}\right)^{\alpha}\right).\end{aligned} \tag{11}$$

Continuing in this way m-times, we obtain

$$\begin{aligned}&\|\mathfrak{u}_m(t,\vartheta) - \mathfrak{u}_{m-1}(t,\vartheta)\|_{C_{1-\gamma;\rho}([a,t_1],\mathbb{R})}\\ &\quad\leq M\frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left(\frac{t_1^{\rho}-a^{\rho}}{\rho}\right)^{\alpha}\left(\ell(t,\vartheta)\frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left(\frac{t_1^{\rho}-a^{\rho}}{\rho}\right)^{\alpha}\right)^{m-1}.\end{aligned} \tag{12}$$

By Eq. (7), we get

$$\|\mathfrak{u}_m(t,\vartheta) - \mathfrak{u}_{m-1}(t,\vartheta)\|_{C_{1-\gamma;\rho}([a,t_1],\mathbb{R})} \to 0,\ \ as\ m \to +\infty. \tag{13}$$

Again by Lemma 3.2, it follows that

$$\begin{aligned}&\|{}^{\rho}I^{\alpha}\mathfrak{g}(t,\vartheta,\mathfrak{u}_m(t,\vartheta)) - {}^{\rho}I^{\alpha}\mathfrak{g}(t,\vartheta,\mathfrak{u}(t,\vartheta))\|_{C_{1-\gamma;\rho}([a,t_1],\mathbb{R})}\\ &\quad\leq \ell(t,\vartheta)\frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left(\frac{t_1^{\rho}-a^{\rho}}{\rho}\right)^{\alpha}\|\mathfrak{u}_m(t)-\mathfrak{u}(t)\|_{C_{1-\gamma;\rho}([a,t_1],\mathbb{R})},\end{aligned}$$

and hence by Eq. 13,

$$\|{}^{\rho}I^{\alpha}\mathfrak{g}(t,\vartheta,\mathfrak{u}_m(t,\vartheta)) - {}^{\rho}I^{\alpha}\mathfrak{g}(t,\vartheta,\mathfrak{u}(t,\vartheta))\|_{C_{1-\gamma;\rho}([a,t_1],\mathbb{R})} \to 0,\ \ as\ m \to +\infty. \tag{14}$$

From Eq. (13) and Eq. (14), it follows that $\mathfrak{u}(t,\vartheta)$ is the solution of integral Eq. (5) in $C_{1-\gamma;\rho}([a,t_1],\mathbb{R})$.

Now to show that the solution $\mathfrak{u}(t,\vartheta)$ is unique, consider there exists two solutions $\mathfrak{u}(t,\vartheta)$ and $\mathfrak{v}(t,\vartheta)$ of the integral equation (5) on $[a,t_1]$. Substituting them into Eq. (5) and using Lemma 2.4 with condition [H1], we get

$$
\begin{aligned}
(15)\qquad & \|\mathfrak{u}(t,\vartheta)-\mathfrak{v}(t,\vartheta)\|_{C_{1-\gamma;\rho}([a,t_1],\mathbb{R})} \\
& \leq \|{}^{\rho}I^{\alpha}\mathfrak{g}(t,\vartheta,\mathfrak{u}(t,\vartheta)) - {}^{\rho}I^{\alpha}\mathfrak{g}(t,\vartheta,\mathfrak{v}(t,\vartheta))\|_{C_{1-\gamma;\rho}([a,t_1],\mathbb{R})} \\
& \leq \ell(t,\vartheta)\frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left(\frac{t_1^{\rho}-a^{\rho}}{\rho}\right)^{\alpha}\|\mathfrak{u}(t)-\mathfrak{v}(t)\|_{C_{1-\gamma;\rho}([a,t_1],\mathbb{R})}.
\end{aligned}
$$

This yields $\ell(t,\vartheta)\frac{B(\gamma,\alpha)}{\Gamma(\alpha)}\left(\frac{t_1^{\rho}-a^{\rho}}{\rho}\right)^{\alpha} \geq 1$, which contradicts to condition (7). Thus there exists $\mathfrak{u}(t,\vartheta)=\mathfrak{u}_1(t,\vartheta)\in C_{1-\gamma;\rho}([a,t_1],\mathbb{R})$ as a unique solution on $[a,t_1]$.

Next, consider the interval $[t_1,t_2]$, where $t_2=t_1+h_1$, $h_1>0$ such that $t_2<b$. Now the integral Eq. (5) takes the form

(16)

$$
\begin{aligned}
\mathfrak{u}(t,\vartheta) = {} & \frac{\mathfrak{u}_a(\vartheta)}{\Gamma(\gamma)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}s^{\rho-1}\mathfrak{g}(s,\vartheta,\mathfrak{u}(s,\vartheta))ds \\
& + \frac{1}{\Gamma(\alpha)}\int_{a}^{t_1}\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}s^{\rho-1}\mathfrak{g}(s,\vartheta,\mathfrak{u}(s,\vartheta))ds, \quad t\in[t_1,t_2].
\end{aligned}
$$

Since the function $\mathfrak{u}(t,\vartheta)$ is uniquely defined on $[a,t_1]$, the last integral is known function and therefore the integral Eq. (16) can be written in the form

$$
(17)\qquad \mathfrak{u}(t,\vartheta) = \mathfrak{u}^{*}(t,\vartheta) + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t}\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}s^{\rho-1}\mathfrak{g}(s,\vartheta,\mathfrak{u}(s,\vartheta))ds,
$$

where

$$
\begin{aligned}
(18)\qquad \mathfrak{u}^{*}(t,\vartheta) = {} & \frac{\mathfrak{u}_a(\vartheta)}{\Gamma(\gamma)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\gamma-1} \\
& + \frac{1}{\Gamma(\alpha)}\int_{a}^{t_1}\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}s^{\rho-1}\mathfrak{g}(s,\vartheta,\mathfrak{u}(s,\vartheta))ds,
\end{aligned}
$$

is the known function. Using the same argument as above, we deduce that there exist a unique solution $\mathfrak{u}(t,\vartheta)=\mathfrak{u}_2(t,\vartheta)\in C_{1-\gamma;\rho}([t_1,t_2],\mathbb{R})$ on $[t_1,t_2]$. Taking interval $[t_2,t_3]$, where $t_3=t_2+h_2$, $h_2>a$ such that $t_3<b$, and repeating the above process, we obtain a unique solution $\mathfrak{u}(t)\in C_{1-\gamma;\rho}(J,\mathbb{R})$

of integral equation (5) such that $\mathfrak{u}(t) = \mathfrak{u}_j(t) \in C_{1-\gamma;\rho}([t_{j-1}, t_j], \mathbb{R})$ for $j = 1, 2, \ldots, l$, and $a = \mathfrak{u}_0 < \mathfrak{u}_1 < \ldots < \mathfrak{u}_l = b$. Using differential Eq. (4) and Lipschitz condition [H1], we obtain

(19)
$$\left\| {}^{\rho}D^{\alpha,\beta}\mathfrak{u}_m(t,\vartheta) - {}^{\rho}D^{\alpha,\beta}\mathfrak{u}(t,\vartheta) \right\|_{C_{1-\gamma;\rho}} = \|\mathfrak{g}(t, \mathfrak{u}_m(t,\vartheta)) - \mathfrak{g}(t, \mathfrak{u}(t,\vartheta))\|_{C_{1-\gamma;\rho}} \leq \ell(t,\vartheta) \|\mathfrak{u}_m(t,\vartheta) - \mathfrak{u}(t,\vartheta)\|_{C_{1-\gamma;\rho}}.$$

Clearly, (13) and (19) implies that ${}^{\rho}D^{\alpha,\beta}\mathfrak{u}(t,\vartheta) \in C_{1-\gamma;\rho}(J, \mathbb{R})$.

Thus, the proof of the theorem is complete. □

4. Solution of nonlocal initial value problem for FRDEs

First, we discuss the existence, uniqueness and stability of solutions of FRDEs with nonlocal condition of the form

(20)
$$\begin{cases} {}^{\rho}D^{\alpha,\beta}\mathfrak{u}(t,\vartheta) = \mathfrak{g}(t,\vartheta,\mathfrak{u}(t,\vartheta)), & t \in J := (a,b], \\ {}^{\rho}I^{1-\gamma}\mathfrak{u}(t,\vartheta)|_{t=a} = \mathfrak{u}(\vartheta) + h(\mathfrak{u},\vartheta). \end{cases}$$

We establish by an integral equation is as follows

(21)
$$\mathfrak{u}(t,\vartheta) = \frac{\mathfrak{u}_a(\vartheta) + h(\mathfrak{u},\vartheta)}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\mathfrak{g}(s,\vartheta,\mathfrak{u}(s,\vartheta))ds.$$

(H2) There exist $p,\ q : J \times \Omega \to \mathbb{R}$ with

$$|\mathfrak{g}(t,\vartheta,\mathfrak{u})| \leq p(t,\vartheta) + q(t,\vartheta)|\mathfrak{u}|,$$

for all $\mathfrak{u} \in \mathbb{R}$ and $P(\vartheta) = \sup_{t\in J} p(t,\vartheta)$, $Q(\vartheta) = \sup_{t\in J} q(t,\vartheta)$, for $t \in J$.

(H3) There exist a constant $\ell_{\mathfrak{h}}$, such that

$$|\mathfrak{h}(\mathfrak{u},\vartheta) - \mathfrak{h}(\mathfrak{v},\vartheta)| \leq \ell_{\mathfrak{h}}(t,\vartheta)|\mathfrak{u} - \mathfrak{v}|.$$

(H4) Let the functions $\mathfrak{u}_{t_k}\mathfrak{v}_{t_k} : J \times \Omega \to \mathbb{R}$ are continuous and there exists a constant $\ell^* > 0$, such that

$$|\mathfrak{u}_{t_k}(\vartheta) - \mathfrak{v}_{t_k}(\vartheta)| \leq \ell^*(t,\vartheta)|\mathfrak{u} - \mathfrak{v}|,$$

and

$$|\mathfrak{u}_{t_k}(\vartheta)| \leq r(t,\vartheta), \text{ for all } k = 1, 2, \ldots, m,$$

and we denote $R(\vartheta) = \sup_{t\in J} r(t,\vartheta)$.

(H5) There exists an increasing function $\varphi : J \times \Omega \to \mathbb{R}^+$ and there exists $\lambda_\varphi > 0$ such that for any $t \in J$

$$^{\rho}I^{\alpha}\varphi(t,\vartheta) \leq \lambda_\varphi \varphi(t,\vartheta).$$

Theorem 4.1. *Assume that hypotheses [H1]–[H3] are satisfied. Then, Eq. (20) has at least one solution.*

Proof. Consider the operator $P(\vartheta) : \Omega \times C_{1-\gamma,\rho} \to C_{1-\gamma,\rho}$.

Hence, $\mathfrak{u}$ is a solution for the problem (20) if and only if

$$\mathfrak{u}(t,\vartheta) = (P\mathfrak{u})(t,\vartheta),$$

where the equivalent integral equation which can be written in the operator form

$$(P\mathfrak{u})(t,\vartheta) = \frac{\mathfrak{u}_a(\vartheta) + h(\mathfrak{u},\vartheta)}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\mathfrak{g}(s,\vartheta,\mathfrak{u}(s,\vartheta))ds. \tag{22}$$

Consider the ball $B_r = \left\{\mathfrak{u} \in C_{1-\gamma,\rho} : \|\mathfrak{u}\|_{C_{1-\gamma,\rho}} \leq r\right\}$. Set $h(0,\vartheta) = H(\vartheta)$ Now we subdivide the operator P into two operator P_1 and P_2 on B_r as follows

$$P_1\mathfrak{u}(t,\vartheta) = \frac{\mathfrak{u}_a(\vartheta)}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \frac{h(\mathfrak{u},\vartheta)}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1},$$

and

$$P_2\mathfrak{u}(t,\vartheta) = \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\mathfrak{g}(s,\vartheta,\mathfrak{u}(s,\vartheta))ds.$$

The proof is separated into some steps.

Step 1. $P_1\mathfrak{u} + P_2\mathfrak{v} \in B_r$ for every $\mathfrak{u},\mathfrak{v} \in B_r$.

$$|P_1\mathfrak{u}(t,\vartheta)| = \left|\frac{\mathfrak{u}_a(\vartheta)}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \frac{h(t,\vartheta)}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1}\right|$$

$$\left| P_1\mathfrak{u}(t,\vartheta)\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} \right| \leq \frac{|\mathfrak{u}_a(\vartheta)|}{\Gamma(\gamma)} + \frac{|h(\mathfrak{u},\vartheta) - h(0,\vartheta)|}{\Gamma(\gamma)} + \frac{|h(0,\vartheta)|}{\Gamma(\gamma)}$$
$$\leq \frac{|\mathfrak{u}_a(\vartheta)|}{\Gamma(\gamma)} + \frac{\ell_h}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} \|\mathfrak{u}\|_{C_{1-\gamma,\rho}} + \frac{H(\vartheta)}{\Gamma(\gamma)}.$$

This gives

$$\|P_1\mathfrak{u}\|_{C_{1-\gamma,\rho}} \leq \frac{|\mathfrak{u}_a(\vartheta)|}{\Gamma(\gamma)} + \frac{\ell_h}{\Gamma(\gamma)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\gamma-1} \|\mathfrak{u}\|_{C_{1-\gamma,\rho}} + \frac{H(\vartheta)}{\Gamma(\gamma)}. \tag{23}$$

For the operator P_2

$$\left| P_2(t,\vartheta)\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} \right|$$
$$\leq \frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} |\mathfrak{g}(s,\vartheta,\mathfrak{u}(s,\vartheta))|\, ds$$
$$\leq \frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} |p(t,\vartheta)|\, ds$$
$$+ \frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} |q(t,\vartheta)|\, |\mathfrak{u}(s)|\, ds$$
$$\leq \frac{1}{\Gamma(\alpha+1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha} P(\vartheta)$$
$$+ \frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} B(\gamma,\alpha)\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\gamma-1} Q(\vartheta)\, \|\mathfrak{u}\|_{C_{1-\gamma,\rho}}\, ds$$

Thus, we obtain

$$\|P_2\mathfrak{u}\|_{C_{1-\gamma,\rho}(\vartheta)} \leq \frac{P(\vartheta)}{\Gamma(\alpha+1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha-\gamma+1} + \frac{Q(\vartheta)}{\Gamma(\alpha)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha} B(\gamma,\alpha)\, \|\mathfrak{u}\|_{C_{1-\gamma,\rho}}. \tag{24}$$

Linking (23) and (24), for every $\mathfrak{u}, \mathfrak{v} \in B_r$,

$$\|P_1\mathfrak{u} + P_2\mathfrak{v}\|_{C_{1-\gamma,\rho}} \leq \|P_1\mathfrak{u}\|_{C_{1-\gamma,\rho}} + \|P_2\mathfrak{v}\|_{C_{1-\gamma,\rho}} \leq r.$$

Step 2. P_1 is a contration mapping.

For any $\mathfrak{u}, \mathfrak{v} \in B_r$,

$$\begin{aligned}
&\left| \left(\frac{t^\rho - a^\rho}{\rho} \right)^{1-\gamma} (P_1 \mathfrak{u}(t, \vartheta) - P_1 \mathfrak{v}(t, \vartheta)) \right| \\
&\leq \frac{1}{\Gamma(\gamma)} \left| h(\mathfrak{u}, \vartheta) - h(\mathfrak{v}, \vartheta) \right|, \\
&\leq \frac{1}{\Gamma(\gamma)} \left| h(\mathfrak{u}, \vartheta) - h(\mathfrak{v}, \vartheta) \right|, \\
&\leq \frac{\ell_h(t, \vartheta)}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\gamma-1} \|\mathfrak{u} - \mathfrak{v}\|_{C_{1-\gamma,\rho}}
\end{aligned}$$

This gives

$$\|(P_1 \mathfrak{u} - P_1 \mathfrak{v})\|_{C_{1-\gamma,\rho}} \leq \frac{\ell_h(t, \vartheta)}{\Gamma(\nu)} \left(\frac{b^\rho - a^\rho}{\rho} \right)^{\gamma-1} \|\mathfrak{u} - \mathfrak{v}\|_{C_{1-\gamma,\rho}}.$$

Thus, P_1 is a contraction mapping.

Step 3. The operator P_2 is compact and continuous.

According to Step 1, we know that

$$\begin{aligned}
\|P_2 \mathfrak{u}\|_{C_{1-\gamma,\rho}(\vartheta)} \leq{}& \frac{P(\vartheta)}{\Gamma(\alpha+1)} \left(\frac{b^\rho - a^\rho}{\rho} \right)^{\alpha-\gamma+1} \\
&+ \frac{Q(\vartheta)}{\Gamma(\alpha)} \left(\frac{b^\rho - a^\rho}{\rho} \right)^{\alpha} B(\gamma, \alpha) \|\mathfrak{u}\|_{C_{1-\gamma,\rho}}.
\end{aligned}$$

So operator P_2 is uniformly bounded.

Now, we confirm the compactness of operator P_2.

For $0 < t_l < t_m < T$, we have

$$\begin{aligned}
&\left| \left(\left(\frac{t_m^\rho - a^\rho}{\rho} \right)^{1-\gamma} P_2 \mathfrak{u}(t_m, \vartheta) - \left(\frac{t_l^\rho - a^\rho}{\rho} \right)^{1-\gamma} P_2 \mathfrak{v}(t_l, \vartheta) \right) \right| \\
&= \left| \frac{1}{\Gamma(\alpha)} \left(\frac{t_m^\rho - a^\rho}{\rho} \right)^{1-\gamma} \int_a^{t_m} \left(\frac{t_m^\rho - s^\rho}{\rho} \right)^{\alpha-1} s^{\rho-1} \mathfrak{g}(s, \vartheta, \mathfrak{u}(s, \vartheta)) ds \right. \\
&\qquad \left. - \frac{1}{\Gamma(\alpha)} \left(\frac{t_l^\rho - a^\rho}{\rho} \right)^{1-\gamma} \int_a^{t_l} \left(\frac{t_l^\rho - s^\rho}{\rho} \right)^{\alpha-1} s^{\rho-1} \mathfrak{g}(s, \vartheta, \mathfrak{u}(s, \vartheta)) ds \right|
\end{aligned}$$

tending to zero as $t_l \to t_m$. Thus P_2 is equicontinuous. Hence, the operator P_2 is compact on B_r by the Arzela-Ascoli Theorem. It follows from Theorem 2.8 that the problem (20) has at least one solution. □

Lemma 4.2. *Assume that the hypothesis* $(H1)$ *is satisfied. If*

$$\left(\frac{\ell_h(t,\vartheta)}{\Gamma(\gamma)} \left(\frac{b^\rho - a^\rho}{\rho} \right)^{\gamma-1} + \frac{\ell(t,\vartheta)}{\Gamma(\alpha)} B(\gamma,\alpha) \left(\frac{b^\rho - a^\rho}{\rho} \right)^{\alpha} \right) < 1,$$

then, (20) *has a unique solution.*

Next, we shall give the definition for generalized Ulam-Hyers-Rassias (U-H-R) stability for the differential equations involving HKFD with random effects is given by

$$ {}^{\rho}D^{\alpha,\beta}\mathfrak{u}(t,\vartheta) = \mathfrak{g}(t,\vartheta,\mathfrak{u}(t,\vartheta)). \tag{25}$$

Let $\epsilon > 0$ be a positive real number and $\varphi : J \times \Omega \to \mathbb{R}^+$ be a continuous function. We consider the following inequality

$$\left| {}^{\rho}D^{\alpha,\beta}\mathfrak{v}(t,\vartheta) - \mathfrak{g}(t,\vartheta,\mathfrak{v}(t,\vartheta)) \right| \le \varphi(t,\vartheta). \tag{26}$$

Definition 4.3. *Eq.* (25) *is generalized U-H-R stable with respect to* φ *if there exists a real number* $C_{f,\varphi} > 0$ *such that for each solution* $\mathfrak{v} : \Omega \to C_{1-\gamma,\rho}$ *of the inequality* (26) *there exists a solution* $\mathfrak{u} : \Omega \to C_{1-\gamma,\rho}$ *of Eq.* (25) *with*

$$|\mathfrak{v}(t,\vartheta) - \mathfrak{u}(t,\vartheta)| \le C_{f,\varphi}\varphi(t,\vartheta), \quad t \in J, \vartheta \in \Omega.$$

Theorem 4.4. *Under the hypotheses* $(H1)$ *and* $(H5)$*, the solution of Eq.* (20) *is generalized U-H-R stable.*

Proof. Let $\mathfrak{v}$ be solution of inequality 26 and by Lemma 4.2 there exists a unique solution $\mathfrak{u}$ for the problem (20). Thus we have

$$\begin{aligned}\mathfrak{u}(t,\vartheta) = {} & \frac{\mathfrak{u}_a(\vartheta) + h(\mathfrak{u},\vartheta)}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\gamma-1} \\ & + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} s^{\rho-1}\mathfrak{g}(s,\vartheta,\mathfrak{u}(s,\vartheta))ds.\end{aligned}$$

By differentiating inequality (26) for each $t \in J,\ \vartheta \in \Omega$, we have

$$\left| \mathfrak{v}(t,\vartheta) - \frac{\mathfrak{u}_a(\vartheta)}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\gamma-1} - \frac{h(\mathfrak{v},\vartheta)}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\gamma-1} \right.$$

$$
- \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} s^{\rho-1} \mathfrak{g}(s, \vartheta, \mathfrak{v}(s,\vartheta)) ds \Bigg|
\leq \lambda_\varphi \varphi(t, \vartheta).
$$

Hence it follows

$$
\begin{aligned}
&|\mathfrak{v}(t,\vartheta) - \mathfrak{u}(t,\vartheta)| \\
&\leq \Bigg| \mathfrak{v}(t,\vartheta) - \frac{\mathfrak{u}_a(\vartheta) + h(\mathfrak{u},\vartheta)}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\gamma-1} \\
&\quad - \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} s^{\rho-1} \mathfrak{g}(s, \vartheta, \mathfrak{u}(s,\vartheta)) ds \Bigg| \\
&\leq \Bigg| \mathfrak{v}(t,\vartheta) - \frac{\mathfrak{u}_a(\vartheta)}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\gamma-1} - \frac{h(\mathfrak{v},\vartheta)}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\gamma-1} \\
&\quad - \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} s^{\rho-1} \mathfrak{g}(s, \vartheta, \mathfrak{v}(s,\vartheta)) ds \Bigg| \\
&\quad + \frac{1}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\gamma-1} |h(\mathfrak{v},\vartheta) - h(\mathfrak{u},\vartheta)| \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} s^{\rho-1} \left| \mathfrak{g}(s, \vartheta, \mathfrak{v}(s,\vartheta)) - \mathfrak{g}(s, \vartheta, \mathfrak{u}(s,\vartheta)) \right| ds \\
&\leq \lambda_\varphi \varphi(t,\vartheta) + \frac{\ell_h(t,\vartheta)}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\gamma-1} |\mathfrak{v}(t,\vartheta) - \mathfrak{u}(t,\vartheta)| \\
&\quad + \frac{\ell(t,\vartheta)}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} s^{\rho-1} |\mathfrak{v}(s,\vartheta) - \mathfrak{u}(s,\vartheta)| ds \\
&:= C_{f,\varphi} \varphi(t,\vartheta).
\end{aligned}
$$

Thus, Eq. (20) is generalized U-H-R stable. □

5. Random differential equation with impulsive effect

Next, we discuss the existence, uniqueness and stability of solutions of RDE with impulsive involving HKFD of the form

$$
\begin{cases}
{}^\rho D^{\alpha,\beta} \mathfrak{u}(t,\vartheta) = \mathfrak{g}(t, \vartheta, \mathfrak{u}(t,\vartheta)), \quad t \in J' := J \setminus \{t_1, \ldots, t_m\}, \\
\Delta {}^\rho I^{1-\gamma} \mathfrak{u}(t,\vartheta)|_{t=t_k} = \mathfrak{u}_{t_k}(\vartheta), \\
{}^\rho I^{1-\gamma} \mathfrak{u}(t,\vartheta)|_{t=a} = \mathfrak{u}_a(\vartheta),
\end{cases}
\tag{27}
$$

where $\mathfrak{u}_k(\vartheta) : J \times \Omega \to \mathbb{R}$ is continuous for all $k = 1, 2, \ldots, m$, and $0 = t_0 < t_1 < \ldots < t_m < t_{m+1} = b$, $\Delta^\rho I^{1-\gamma}\mathfrak{u}(t,\vartheta)|_{t=t_k} = {}^\rho I^{1-\gamma}\mathfrak{u}_{(t_k^+)}(\vartheta) - {}^\rho I^{1-\gamma}\mathfrak{u}_{(t_k^-)}(\vartheta)$, ${}^\rho I^{1-\gamma}\mathfrak{u}_{(t_k^+)}(\vartheta) = \lim_{h\to 0+}\mathfrak{u}_{(t_k+h)}(\vartheta)$ and ${}^\rho I^{1-\gamma}\mathfrak{u}_{(t_k^-)}(\vartheta) = \lim_{h\to 0-}\mathfrak{u}_{(t_k+h)}(\vartheta)$ represent the right and left limits of $\mathfrak{u}(t,\vartheta)$ at $t = t_k$.

The integral equation of the problem (27) is of the form

$$\mathfrak{u}(t,\vartheta) = \frac{\mathfrak{u}_a(\vartheta)}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \frac{\sum_{0<t_k<t}\mathfrak{u}_{t_k}(\vartheta)}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\mathfrak{g}(s,\vartheta,\mathfrak{u}(s,\vartheta))ds. \tag{28}$$

Theorem 5.1. *Assume that [H2]–[H4] are satisfied. Then, Eq. (27) has at least one solution.*

Proof. Consider the operator $T : \Omega \times PC_{1-\gamma,\rho} \to PC_{1-\gamma,\rho}$. The operator form of integral equation (28) is written as follows

$$\mathfrak{u}(t,\vartheta) = T\mathfrak{u}(t,\vartheta),$$

where

$$(T\mathfrak{u})(t,\vartheta) = \frac{\mathfrak{u}_a(\vartheta)}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \frac{\sum_{0<t_k<t}\mathfrak{u}_{t_k}(\vartheta)}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\mathfrak{g}(s,\vartheta,\mathfrak{u}(s,\vartheta))ds \tag{29}$$

Define $B_r = \left\{\mathfrak{u} \in PC_{1-\gamma,\rho} : \|\mathfrak{u}\|_{PC_{1-\gamma,\rho}} \le r\right\}$. Set

$$\omega := \frac{|\mathfrak{u}_a(\vartheta)|}{\Gamma(\gamma)} + \frac{P(\vartheta)}{\Gamma(\alpha+1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha+1-\gamma}$$

and

$$\sigma := \left(\frac{mR(\vartheta)}{\Gamma(\gamma)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \frac{B(\gamma,\alpha)Q(\vartheta)}{\Gamma(\alpha)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha}\right)$$

In order to apply Schauder fixed point theorem, we divide our proof into three steps.

Step 1: We check that $T(B_r) \subset B_r$.

$$\begin{aligned}
&\left|(T\mathfrak{u})(t,\vartheta)\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\right| \\
&\le \frac{|\mathfrak{u}_a(\vartheta)|}{\Gamma(\gamma)} + \frac{\sum_{0<t_k<t}|\mathfrak{u}_{t_k}(\vartheta)|}{\Gamma(\gamma)} \\
&\quad + \frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\int_a^t\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\left|\mathfrak{g}(s,\vartheta,\mathfrak{u}(s,\vartheta))\right| ds \\
&\le \frac{|\mathfrak{u}_a(\vartheta)|}{\Gamma(\gamma)} + \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1}\frac{\sum_{0<t_k<t} r(t,\vartheta)\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\|\mathfrak{u}\|_{PC_{1-\gamma,\rho}}}{\Gamma(\gamma)} \\
&\quad + \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\frac{1}{\Gamma(\alpha)}\int_a^t\left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\left[p(s,\vartheta) + q(s,\vartheta)\left|\mathfrak{u}(s,\vartheta)\right|\right] ds \\
&\le \frac{|\mathfrak{u}_a(\vartheta)|}{\Gamma(\gamma)} + \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1}\frac{mR(\vartheta)}{\Gamma(\gamma)}\|\mathfrak{u}\|_{PC_{1-\gamma,\rho}} \\
&\quad + \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\frac{1}{\Gamma(\alpha+1)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha} P(\vartheta) \\
&\quad + \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\frac{B(\gamma,\alpha)Q(\vartheta)}{\Gamma(\alpha)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha+\gamma-1}\|\mathfrak{u}\|_{PC_{1-\gamma,\rho}} \\
&\le \frac{|\mathfrak{u}_a(\vartheta)|}{\Gamma(\gamma)} + \frac{P(\vartheta)}{\Gamma(\alpha+1)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha+1-\gamma} \\
&\quad + \left(\frac{mR(\vartheta)}{\Gamma(\gamma)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \frac{B(\gamma,\alpha)Q(\vartheta)}{\Gamma(\alpha)}\left(\frac{b^\rho - a^\rho}{\rho}\right)^{\alpha}\right)\|\mathfrak{u}\|_{PC_{1-\gamma,\rho}}.
\end{aligned}$$

Hence

$$\|(T\mathfrak{u})\|_{PC_{1-\gamma,\rho}} \le \omega + \sigma r \le r,$$

which yields that $T(B_r) \subset B_r$. Next we prove that the operator T is completely continuous.

Step 2: The operator T is continuous.

Let $\mathfrak{u}_n$ be a sequence such that $\mathfrak{u}_n \to \mathfrak{u}$ in B_r. Then for each $t \in J$,

$$\left|T\mathfrak{u}_n(t,\vartheta) - \mathfrak{P}\mathfrak{u}(t,\vartheta))\left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma}\right|$$

$$\leq \frac{1}{\Gamma(\gamma)} \sum_{0<t_k<t} |\psi_k(\mathfrak{u}_{k_n}(t_k)) - \psi_k(\mathfrak{u}(t_k))|$$
$$+ \frac{1}{\Gamma(\alpha)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{1-\gamma} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}$$
$$\times |\mathfrak{g}(s,\vartheta,\mathfrak{u}_n(s,\vartheta)) - \mathfrak{g}(s,\vartheta,\mathfrak{u}(s,\vartheta))|\, ds.$$

Since $\mathfrak{g}$ is continuous, then we have

$$\|T\mathfrak{u}_n - T\mathfrak{u}\|_{PC_{1-\gamma,\rho}} \to 0 \quad \text{as} \quad n \to \infty.$$

This proves the continuity of T.

Step 3: $T(B_r)$ is relatively compact.

Thus $T(B_r)$ is uniformly bounded. And for any $t_l,\ t_m \in J,\ t_l > t_m$ then, we have

$$\left|\left(\frac{t_l^\rho - a^\rho}{\rho}\right)^{1-\gamma} (\mathfrak{P}\mathfrak{u})(t_l,\vartheta) - \left(\frac{t_m^\rho - a^\rho}{\rho}\right)^{1-\gamma} (\mathfrak{P}\mathfrak{u})(t_m,\vartheta))\right|$$
$$= \left|\frac{\sum_{0<t_k<t_l} \mathfrak{u}_{t_k}(\vartheta)}{\Gamma(\gamma)} + \frac{\left(\frac{t_l^\rho - a^\rho}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha)} \int_0^{t_l} \left(\frac{t_l^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\mathfrak{g}(s,\vartheta,\mathfrak{u}(s,\vartheta))ds\right.$$
$$\left. - \frac{\sum_{0<t_k<t_m} \mathfrak{u}_{t_k}(\vartheta)}{\Gamma(\gamma)} - \frac{\left(\frac{t_m^\rho - a^\rho}{\rho}\right)^{1-\gamma}}{\Gamma(\alpha)} \int_0^{t_m} \left(\frac{t_m^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\mathfrak{g}(s,\vartheta,\mathfrak{u}(s,\vartheta))ds\right|,$$

tending to zero as $t_l \to t_m$. That is, T is equicontinuous. Hence, $T(B_r)$ is relatively compact. As a outcome of Steps 1–3 together with Arzelä-Ascoli theorem, we can conclude that that T is completely continuous. Finally by Schauder fixed point theorem the proof is complete. □

Theorem 5.2. *Assume that [H1] and [H3] are satisfied. If*

$$\left(\frac{m\ell^*(t,\vartheta)}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \frac{\ell(t,\vartheta)}{\Gamma(\alpha)} \left(\frac{t^\rho - a^\rho}{\rho}\right)^{\alpha} B(\gamma,\alpha)\right) < 1, \tag{30}$$

then, the Eq. (27) *has a unique solution.*

Now, we shall give the definition for U-H-R stability of random implicit differential equation with impulsive effect involving HKFD of the form

$$\begin{cases} {}^\rho D^{\alpha,\beta}\mathfrak{u}(t,\vartheta) = \mathfrak{g}(t,\vartheta,\mathfrak{u}(t,\vartheta)), \\ \Delta^\rho I^{1-\gamma}\mathfrak{u}(t,\vartheta)|_{t=t_k} = \mathfrak{u}_{t_k}(\vartheta). \end{cases} \tag{31}$$

Let $\epsilon > 0$ be a positive real number and $\varphi : J \times \Omega \to \mathbb{R}^+$ be a continuous function. We consider the following inequalities

$$\begin{cases} \left|{}^\rho D^{\alpha,\beta}\mathfrak{u}(t,\vartheta) - \mathfrak{g}(t,\vartheta,\mathfrak{v}(t,\vartheta))\right| \le \varphi(t,\vartheta), \\ \left|\Delta^\rho I^{1-\gamma}\mathfrak{v}(t,\vartheta)|_{t=t_k} - \mathfrak{v}_{t_k}(\vartheta)\right| \le \varphi(t,\vartheta). \end{cases} \tag{32}$$

Definition 5.3. *Eq. (31) is generalized U-H-R stable with respect to φ if there exists a real number $C_{f,\varphi} > 0$ such that for each solution $\mathfrak{u} : \Omega \to PC_{1-\gamma}$ of the inequality (32) there exists a solution $\mathfrak{v} : \Omega \to PC_{1-\gamma}$ of Eq. (31) with*

$$|\mathfrak{v}(t,\vartheta) - \mathfrak{u}(t,\vartheta)| \le C_{f,\varphi}\varphi(t,\vartheta), \quad t \in J, \vartheta \in \Omega.$$

Theorem 5.4. *The assumptions [H1], [H4], [H5] and (30) hold. Then, Eq. (27) is generalized U-H-R stable.*

Proof. Let $\mathfrak{v}$ be solution of inequality (32) and by Theorem 4.2 there $\mathfrak{u}$ is unique solution of the problem

$$\begin{aligned} {}^\rho D^{\alpha,\beta}\mathfrak{u}(t,\vartheta) &= \mathfrak{g}(t,\vartheta,\mathfrak{u}(t,\vartheta)), \\ \Delta^\rho I^{1-\gamma}\mathfrak{u}(t,\vartheta)|_{t=t_k} &= \mathfrak{u}_{t_k}(\vartheta), \\ {}^\rho I^{1-\gamma}\mathfrak{u}(t,\vartheta)|_{t=a} &= \mathfrak{u}_a(\vartheta). \end{aligned}$$

Then, we have

$$\begin{aligned} \mathfrak{u}(t,\vartheta) =& \frac{\mathfrak{u}_a(\vartheta)}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \frac{\sum_{0<t_k<t}\mathfrak{u}_{t_k}(\vartheta)}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} \\ &+ \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\mathfrak{g}(s,\vartheta,\mathfrak{u}(s,\vartheta))ds. \end{aligned}$$

By differentiating inequality (32), for each $t \in (t_k, t_{k+1}]$, we have

$$\begin{aligned} &\Bigg|\mathfrak{v}(t,\vartheta) - \frac{\mathfrak{u}_a(\vartheta)}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} - \frac{\sum_{0<t_k<t}\mathfrak{v}_{t_k}(\vartheta)}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} \\ &\quad - \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\mathfrak{g}(s,\vartheta,\mathfrak{u}(s,\vartheta))ds\Bigg| \\ &\le \left|\frac{\sum_{0<t_k<t} g_k}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1} + \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\varphi(s,\vartheta)ds\right| \\ &\le \frac{m}{\Gamma(\gamma)}\left(\frac{t^\rho - a^\rho}{\rho}\right)^{\gamma-1}\varphi(t,\vartheta) + \lambda_\varphi\varphi(t,\vartheta) \end{aligned}$$

$$\leq \left(\frac{m}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\gamma-1} + \lambda_\varphi \right) \varphi(t, \vartheta).$$

Hence for each $t \in (t_k, t_{k+1}]$, it follows

$$\begin{aligned}
&|\mathfrak{v}(t,\vartheta) - \mathfrak{u}(t,\vartheta)| \\
&\leq \Bigg| \mathfrak{v}(t,\vartheta) - \frac{\mathfrak{u}_a(\vartheta)}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\gamma-1} - \frac{\sum_{0<t_k<t} \mathfrak{u}_{t_k}(\vartheta)}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\gamma-1} \\
&\quad - \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} s^{\rho-1} \mathfrak{g}(s,\vartheta,\mathfrak{u}(s,\vartheta)) ds \Bigg| \\
&\leq \Bigg| \mathfrak{v}(t,\vartheta) - \frac{\mathfrak{u}_0(\vartheta)}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\gamma-1} - \frac{\sum_{0<t_k<t} \mathfrak{v}_k(\vartheta)}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\gamma-1} \\
&\quad - \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} s^{\rho-1} \mathfrak{g}(s,\vartheta,\mathfrak{v}(s,\vartheta)) ds \Bigg| \\
&\quad + \frac{\sum_{0<t_k<t} |\mathfrak{v}_k(\vartheta) - \mathfrak{u}_k(\vartheta)|}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\gamma-1} \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} s^{\rho-1} \left| \mathfrak{g}(s,\vartheta,\mathfrak{v}(s,\vartheta)) - \mathfrak{g}(s,\vartheta,\mathfrak{u}(s,\vartheta)) \right| ds \\
&\leq \left(\frac{m}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\gamma-1} + \lambda_\varphi \right) \varphi(t,\vartheta) \\
&\quad + \frac{m\ell^*(t,\vartheta)}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\gamma-1} |\mathfrak{v}(t,\vartheta) - \mathfrak{u}(t,\vartheta)| \\
&\quad + \frac{\ell(t,\vartheta)}{\Gamma(\alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{\alpha-1} s^{\rho-1} |\mathfrak{v}(s,\vartheta) - \mathfrak{u}(s,\vartheta)| ds
\end{aligned}$$

By Lemma 2.7, there exists a constant $K > 0$ independent of $\lambda_\varphi \varphi(t, \vartheta)$ such that

$$|\mathfrak{v}(t,\vartheta) - \mathfrak{u}(t,\vartheta)| \leq K \left(\frac{m}{\Gamma(\gamma)} \left(\frac{t^\rho - a^\rho}{\rho} \right)^{\gamma-1} + \lambda_\varphi \right) \varphi(t,\vartheta) := C_{f,\varphi} \varphi(t,\vartheta).$$

Thus, Eq. (27) is generalized U-H-R stable. □

6. Continuous dependence

In this section, first we study the continuous dependence of solution of FRDEs involving HFD by applying generalized Gronwall inequality an important tool. Consider the Eq. (4). To present dependence of solution on the order, let us consider the solutions of two equations with the neighbouring orders. Before studying the continuous dependence the Cauchy-type problem (4), we will discuss some results for the RDEs involving Katugampola fractional derivative

$$\begin{cases} {}^{\rho}D^{\alpha}\mathfrak{u}(t,\vartheta) = \mathfrak{g}(t,\mathfrak{u}(t,\vartheta)), t \in J \\ {}^{\rho}I^{1-\alpha}\mathfrak{u}(a,\vartheta) = \mathfrak{u}(\vartheta), \end{cases} \tag{33}$$

and equivalent to the integral equation which is of the form

$$\mathfrak{u}(t,\vartheta) = \frac{\mathfrak{u}(\vartheta)}{\Gamma(\alpha)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\alpha-1} + \frac{1}{\Gamma(\alpha)}\int_a^t \left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1} s^{\rho-1}\mathfrak{g}(s,\vartheta,\mathfrak{u}(s,\vartheta))ds. \tag{34}$$

First we present the continuous dependence of the solution of the Cauchy-type problem involving Katugampola fractional differential equation

Theorem 6.1. *Let $\alpha > 0$, $\nu > 0$ such that $0 < \alpha - \nu < \alpha \leq 1$. Let $\mathfrak{u}$ is continuous function satisfying Lipschitz condition [H2] in R. Foe $a \leq t < h < b$, assume that $\mathfrak{u}$ is the solution of Eq. (4) and $\overline{\mathfrak{u}}$ is the solution of equation*

$$\begin{aligned} {}^{\rho}D^{\alpha-\nu}\overline{\mathfrak{u}}(t,\vartheta) &= \mathfrak{g}(t,\vartheta,\overline{\mathfrak{u}}(t,\vartheta)), \\ {}^{\rho}I^{1-(\alpha-\nu)}\overline{\mathfrak{u}}(t,\vartheta)|_{t=a} &= \overline{\mathfrak{u}}(\vartheta). \end{aligned} \tag{35}$$

Then, for $a < t \leq h$, the estimate of the following

$$|\overline{\mathfrak{u}}(t,\vartheta) - \mathfrak{u}(t,\vartheta)| \leq K_1(t) + \int_a^t \left[\sum_{k=1}^{\infty}\left(\frac{\ell(t,\vartheta)\Gamma(\alpha-\nu)}{\Gamma(\alpha)}\right)^k \frac{\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{k(\alpha-\nu)-1} s^{\rho-1}}{\Gamma(k(\alpha-\nu))} K_1(s)\right] ds$$

holds, where

$$K_1(t) = \left|\frac{\overline{\mathfrak{u}}_a}{\Gamma(\alpha-\nu)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\alpha-\nu-1} - \frac{\mathfrak{u}_a}{\Gamma(\alpha)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\alpha-1}\right|$$

$$+\|\mathfrak{g}\|\left|\frac{\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}-\frac{\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha-\nu}}{(\alpha-\nu)\Gamma(\alpha)}\right|$$

$$+\|\mathfrak{g}\|\left|\frac{\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha-\nu}}{(\alpha-\nu)\Gamma(\alpha)}-\frac{\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha}}{\Gamma(\alpha+1)}\right|$$

Proof. Solutions of the problems (33) and (35) are given by

(36)

$$\mathfrak{u}(t,\vartheta)=\frac{\mathfrak{u}(\vartheta)}{\Gamma(\alpha)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\mathfrak{g}(s,\vartheta,\mathfrak{u}(s,\vartheta))ds$$

and

$$\text{(37)}\qquad \overline{\mathfrak{u}}(t,\vartheta)=\frac{\overline{\mathfrak{u}}(\vartheta)}{\Gamma(\alpha-\nu)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha-\nu-1}+\frac{1}{\Gamma(\alpha-\nu)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-\nu-1}s^{\rho-1}\mathfrak{g}(s,\vartheta,\overline{\mathfrak{u}}(s,\vartheta))ds$$

respectively, it follows that

$$\begin{aligned}
&|\overline{\mathfrak{u}}(t,\vartheta)-\mathfrak{u}(t,\vartheta)|\\
&=\left|\frac{\overline{\mathfrak{u}}(\vartheta)}{\Gamma(\alpha-\nu)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha-\nu-1}-\frac{\mathfrak{u}_a}{\Gamma(\alpha)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha-1}\right.\\
&+\frac{1}{\Gamma(\alpha-\nu)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-\nu-1}s^{\rho-1}\mathfrak{g}(s,\vartheta,\overline{\mathfrak{u}}(s,\vartheta))ds\\
&\left.-\frac{1}{\Gamma(\alpha)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\mathfrak{g}(s,\vartheta,\mathfrak{u}(s,\vartheta))ds\right|\\
&\le\left|\frac{\overline{\mathfrak{u}}(t,\vartheta)}{\Gamma(\alpha-\nu)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha-\nu-1}-\frac{\mathfrak{u}_a}{\Gamma(\alpha)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha-1}\right|\\
&+\left|\int_a^t\left(\frac{\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-\nu-1}}{\Gamma(\alpha-\nu)}-\frac{\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-\nu-1}}{\Gamma(\alpha)}\right)s^{\rho-1}\mathfrak{g}(s,\vartheta,\overline{\mathfrak{u}}(s,\vartheta))ds\right|\\
&+\left|\int_a^t\frac{1}{\Gamma(\alpha)}\left(\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-\nu-1}-\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}\right)s^{\rho-1}\mathfrak{g}(s,\vartheta,\mathfrak{u}(s,\vartheta))ds\right|
\end{aligned}$$

$$+\left|\int_a^t \frac{\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-\nu-1}}{\Gamma(\alpha)} s^{\rho-1}\left(\mathfrak{g}(s,\overline{\mathfrak{u}}(s))-\mathfrak{g}(s,\mathfrak{u}(s))\right)ds\right|$$
$$\leq \left|\frac{\overline{\mathfrak{u}}(\vartheta)}{\Gamma(\alpha-\nu)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha-\nu-1}-\frac{\mathfrak{u}(\vartheta)}{\Gamma(\alpha)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha-1}\right|$$
$$+\|\mathfrak{g}\|\left|\frac{\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}-\frac{\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha-\nu}}{(\alpha-\nu)\Gamma(\alpha)}\right|+\|\mathfrak{g}\|\left|\frac{\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha-\nu}}{(\alpha-\nu)\Gamma(\alpha)}-\frac{\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha}}{\Gamma(\alpha+1)}\right|$$
$$+\frac{\ell(t,\vartheta)}{\Gamma(\alpha)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-\nu-1} s^{\rho-1}\left|\overline{\mathfrak{u}}(s)-\mathfrak{u}(s)\right|ds.$$

Then we have by Grownwall Lemma 2.5,

$$|\overline{\mathfrak{u}}(t,\vartheta)-\mathfrak{u}(t,\vartheta)|$$
$$\leq K_1(t)+\int_a^t\left[\sum_{k=1}^{\infty}\left(\frac{\ell(t,\vartheta)\Gamma(\alpha-\nu)}{\Gamma(\alpha)}\right)^k\frac{\left(\frac{t^\rho-s^\rho}{\rho}\right)^{k(\alpha-\nu)-1}}{\Gamma(k(\alpha-\nu))}s^{\rho-1}K_1(s)\right]ds.$$

Hence, the proof of theorem is complete. □

Next, we study the continuous dependence of the solution on the order of the Cauchy-type problem (4) involving HKFD equation using the Grownwall Lemma, for that we consider the initial condition that given in (4), and the solutions of two initial value problems with a neighbouring orders and a neighbouring initial values.

Theorem 6.2. *Let $\alpha > 0$, $\nu > 0$ such that $0 < \alpha-\nu < \alpha \leq 1$. Let $\mathfrak{u}$ is continuous function satisfying Lipschitz condition [H1] in R. For $a \leq t < h < b$, assume that $\mathfrak{u}$ is the solution of Eq. (4) and $\overline{\mathfrak{u}}$ is the solution of equation*

$$\begin{aligned}&{}^\rho D^{\alpha-\nu,\beta}\overline{\mathfrak{u}}(t,\vartheta)=\mathfrak{g}(t,\vartheta,\overline{\mathfrak{u}}(t,\vartheta)),\\&{}^\rho I^{1-\gamma-\nu(\beta-1)}\overline{\mathfrak{u}}(t,\vartheta)|_{t=a}=\overline{\mathfrak{u}}(\vartheta).\end{aligned}\tag{38}$$

Then, for $a < t \leq h$, the estimate of the following

$$
\begin{aligned}
&|\overline{\mathfrak{u}}(t,\vartheta)-\mathfrak{u}(t,\vartheta)| \\
&\leq K_2(t)+\int_a^t \left[\sum_{k=1}^{\infty}\left(\frac{\ell(t,\vartheta)\Gamma(\alpha-\nu)}{\Gamma(\alpha)}\right)^k \frac{\left(\frac{t^\rho-s^\rho}{\rho}\right)^{k(\alpha-\nu)-1}}{\Gamma(k(\alpha-\nu))} s^{\rho-1}K_1(s)\right] ds,
\end{aligned}
$$

where

$$
\begin{aligned}
K_2(t)=&\left|\frac{\overline{\mathfrak{u}}_a}{\Gamma(\gamma+\nu(\beta-1))}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma+\nu(\beta-1)}-\frac{\mathfrak{u}_a}{\Gamma(\gamma)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}\right| \\
&+\|\mathfrak{g}\|\left|\frac{\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}-\frac{\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha-\nu}}{(\alpha-\nu)\Gamma(\alpha)}\right| \\
&+\|\mathfrak{g}\|\left|\frac{\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha-\nu}}{(\alpha-\nu)\Gamma(\alpha)}-\frac{\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha}}{\Gamma(\alpha+1)}\right|.
\end{aligned}
$$

Proof. Solutions of the problems (20) and (38) are given by

(39)

$$
\mathfrak{u}(t,\vartheta)=\frac{\mathfrak{u}(\vartheta)}{\Gamma(\alpha)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha-1}+\frac{1}{\Gamma(\alpha)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\mathfrak{g}(s,\vartheta,\mathfrak{u}(s,\vartheta))ds
$$

and

$$
\begin{aligned}
(40)\qquad \overline{\mathfrak{u}}(t,\vartheta)=&\frac{\overline{\mathfrak{u}}_a}{\Gamma(\gamma+\nu(\beta-1))}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma+\nu(\beta-1)} \\
&+\frac{1}{\Gamma(\alpha-\nu)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-\nu-1}s^{\rho-1}\mathfrak{g}(s,\vartheta,\overline{\mathfrak{u}}(s,\vartheta))ds
\end{aligned}
$$

it follows that

$$
\begin{aligned}
&|\overline{\mathfrak{u}}(t,\vartheta)-\mathfrak{u}(t,\vartheta)| \\
=&\left|\frac{\overline{\mathfrak{u}}(\vartheta)}{\Gamma(\gamma+\nu(\beta-1))}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma+\nu(\beta-1)-1}-\frac{\mathfrak{u}(\vartheta)}{\Gamma(\alpha)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha-1}\right. \\
&+\frac{1}{\Gamma(\alpha-\nu)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-\nu-1}s^{\rho-1}\mathfrak{g}(s,\vartheta,\overline{\mathfrak{u}}(s,\vartheta))ds
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{\Gamma(\alpha)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\mathfrak{g}(s,\vartheta,\mathfrak{u}(s,\vartheta))ds\Bigg| \\
\le&\left|\frac{\overline{\mathfrak{u}}(\vartheta)}{\Gamma(\gamma+\nu(\beta-1))}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma+\nu(\beta-1)-1}-\frac{\mathfrak{u}(\vartheta)}{\Gamma(\alpha)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha-1}\right| \\
&+\left|\int_a^t\left(\frac{\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-\nu-1}}{\Gamma(\alpha-\nu)}-\frac{\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-\nu-1}}{\Gamma(\alpha)}\right)s^{\rho-1}\mathfrak{g}(s,\vartheta,\overline{\mathfrak{u}}(s,\vartheta))ds\right| \\
&+\left|\int_a^t\frac{1}{\Gamma(\alpha)}\left(\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-\nu-1}-\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}\right)s^{\rho-1}\mathfrak{g}(s,\vartheta,\mathfrak{u}(s,\vartheta))ds\right| \\
&+\left|\int_a^t\frac{\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-\nu-1}}{\Gamma(\alpha)}s^{\rho-1}\left(\mathfrak{g}(s,\vartheta,\overline{\mathfrak{u}}(s,\vartheta))-\mathfrak{g}(s,\vartheta,\mathfrak{u}(s,\vartheta))\right)ds\right| \\
\le&\left|\frac{\overline{\mathfrak{u}}(\vartheta)}{\Gamma(\gamma+\nu(\beta-1))}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma+\nu(\beta-1)-1}-\frac{\mathfrak{u}(\vartheta)}{\Gamma(\alpha)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha-1}\right| \\
&+\|\mathfrak{g}\|\left|\frac{\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha-\nu}}{\Gamma(\alpha-\nu+1)}-\frac{\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha-\nu}}{(\alpha-\nu)\Gamma(\alpha)}\right| \\
&+\|\mathfrak{g}\|\left|\frac{\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha-\nu}}{(\alpha-\nu)\Gamma(\alpha)}-\frac{\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha}}{\Gamma(\alpha+1)}\right| \\
&+\frac{\ell(t,\vartheta)}{\Gamma(\alpha)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-\nu-1}s^{\rho-1}\left|\overline{\mathfrak{u}}(s,\vartheta)-\mathfrak{u}(s,\vartheta)\right|ds.
\end{aligned}
$$

Then, we have by Grownwall Lemma 2.5,

$$
\begin{aligned}
&|\overline{\mathfrak{u}}(t,\vartheta)-\mathfrak{u}(t,\vartheta)| \\
&\quad\le K_2(t)+\int_a^t\left[\sum_{k=1}^{\infty}\left(\frac{\ell(t,\vartheta)\Gamma(\alpha-\nu)}{\Gamma(\alpha)}\right)^k\frac{\left(\frac{t^\rho-s^\rho}{\rho}\right)^{k(\alpha-\nu)-1}}{\Gamma(k(\alpha-\nu))}s^{\rho-1}K_1(s)\right]ds.
\end{aligned}
$$

Hence, the proof of the theorem is complete. □

In the next theorem, we shall make a small change of the initial condition that given in (4), as follows

$$I^{1-\gamma}\overline{\mathfrak{u}}(a,\vartheta)=\overline{\mathfrak{u}}(\vartheta)+\epsilon, \tag{41}$$

where ϵ is arbitrary constant.

We state and prove the result as follows:

Theorem 6.3. *Let $\gamma=\alpha+\beta-\alpha\beta$, where $0<\alpha<1$ and $0\leq\beta\leq1$. Let $\mathfrak{g}:J\times\mathbb{R}\to\mathbb{R}$ be a function such that $\mathfrak{g}(\cdot,\mathfrak{u}(\cdot))\in C_{1-\gamma;\rho}(J,\mathbb{R})$ for any $\mathfrak{u}\in C_{1-\gamma;\rho}(J,\mathbb{R})$, and satisfies the condition [H1]. For $a\leq t<h<b$, assume that $\mathfrak{u}$ is the solution of Eq. (20) and $\overline{\mathfrak{u}}$ is the solution of equation*

$$\begin{cases}{}^{\rho}D^{\alpha,\beta}\overline{\mathfrak{u}}(t,\vartheta)=\mathfrak{g}(t,\vartheta,\overline{\mathfrak{u}}(t,\vartheta)),t\in J,\\ {}^{\rho}I^{1-\gamma}\overline{\mathfrak{u}}(t,\vartheta)|_{t=a}=\overline{\mathfrak{u}}(\vartheta)+\epsilon.\end{cases} \tag{42}$$

Then,

$$|\mathfrak{u}(t,\vartheta)-\overline{\mathfrak{u}}(t)|\leq\frac{\epsilon}{\Gamma(\gamma)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\gamma-1}E_{\alpha,\gamma}\left(\ell\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\alpha}\right)$$

holds, where $E_{\alpha,\gamma}=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+\gamma)}$ is Mittag-Leffler function.

Proof. In accordance with Theorem 3.3 we have $\mathfrak{u}(t,\vartheta)=\lim_{n\to\infty}\mathfrak{u}_m(t,\vartheta)$ with $\mathfrak{u}_0(t,\vartheta)$ and $\mathfrak{u}_m(t,\vartheta)$ are as defined in equations (8) and (9). Clearly, we can write $\overline{\mathfrak{u}}(t,\vartheta)=\lim_{n\to\infty}\overline{\mathfrak{u}}_m(t,\vartheta)$, and

$$\overline{\mathfrak{u}}_0(t,\vartheta)=\frac{\overline{\mathfrak{u}}(\vartheta)+\epsilon}{\Gamma(\gamma)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\gamma-1}, \tag{43}$$

$$\overline{\mathfrak{u}}_m(t,\vartheta)=\overline{\mathfrak{u}}_0(t,\vartheta)+\frac{1}{\Gamma(\alpha)}\int_a^t\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}s^{\rho-1}\mathfrak{g}(s,\vartheta,\overline{\mathfrak{u}}_{m-1}(s,\vartheta))ds. \tag{44}$$

It follows from (8) and (43) that

$$\begin{aligned}|\mathfrak{u}_0(t,\vartheta)-\overline{\mathfrak{u}}_0(t,\vartheta)|&=\left|\frac{\mathfrak{u}(\vartheta)}{\Gamma(\gamma)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\gamma-1}-\frac{\overline{\mathfrak{u}}(\vartheta)+\epsilon}{\Gamma(\gamma)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\gamma-1}\right|\\&\leq\frac{\epsilon}{\Gamma(\gamma)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\gamma-1}.\end{aligned} \tag{45}$$

Now, by using equations (9) and (44) and applying the Lipschitz condition [H1], we get

$$\begin{aligned}
&|\mathfrak{u}_1(t,\vartheta)-\overline{\mathfrak{u}}_1(t,\vartheta)|\\
&=\Bigg|\frac{\epsilon}{\Gamma(\gamma)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}\\
&\quad+\frac{1}{\Gamma(\alpha)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\left(\mathfrak{g}(s,\vartheta,\mathfrak{u}_0(s,\vartheta))-\mathfrak{g}(s,\vartheta,\overline{\mathfrak{u}}_0(s,\vartheta))\right)ds\Bigg|\\
&\le\frac{\epsilon}{\Gamma(\gamma)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}+\frac{\ell(t,\vartheta)}{\Gamma(\alpha)}\int_a^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}|\mathfrak{u}_0(s,\vartheta)-\overline{\mathfrak{u}}_0(s,\vartheta)|\,ds\\
&\le\epsilon\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}\left[\frac{1}{\Gamma(\gamma)}+\frac{\ell(t,\vartheta)}{\Gamma(\alpha+\gamma)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha}\right].
\end{aligned}$$

Then, we have

$$|\mathfrak{u}_1(t,\vartheta)-\overline{\mathfrak{u}}_1(t,\vartheta)|\le\epsilon\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}\sum_{i=0}^{1}\left[\frac{\ell^i(t,\vartheta)}{\Gamma(\alpha i+\gamma)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha i}\right]. \tag{46}$$

Similarly,

$$|\mathfrak{u}_2(t,\vartheta)-\overline{\mathfrak{u}}_2(t,\vartheta)|\le\frac{\epsilon}{\Gamma(\gamma)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}\sum_{i=0}^{2}\left[\frac{\ell^i(t,\vartheta)}{\Gamma(\alpha i+\gamma)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha i}\right]. \tag{47}$$

By using the mathematical induction method, we conclude that

$$|\mathfrak{u}_m(t,\vartheta)-\overline{\mathfrak{u}}_m(t,\vartheta)|\le\frac{\epsilon}{\Gamma(\gamma)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}\sum_{i=0}^{m}\left[\frac{\ell^i(t,\vartheta)}{\Gamma(\alpha i+\gamma)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha i}\right]. \tag{48}$$

Taking limit as $m\to\infty$, we have

$$\begin{aligned}
|\mathfrak{u}(t,\vartheta)-\overline{\mathfrak{u}}(t,\vartheta)|&\le\frac{\epsilon}{\Gamma(\gamma)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}\sum_{i=0}^{\infty}\left[\frac{\ell^i(t,\vartheta)}{\Gamma(\alpha i+\gamma)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha i}\right]\\
&=\frac{\epsilon}{\Gamma(\gamma)}\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\gamma-1}E_{\alpha,\gamma}\left(\ell(t,\vartheta)\left(\frac{t^\rho-a^\rho}{\rho}\right)^{\alpha}\right),
\end{aligned} \tag{49}$$

which completes the proof. □

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