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Projection methods for rational Riccati equations arising in stochastic optimal control

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We consider the numerical methods of large-scale rational Riccati equations, arising in stochastic optimal control. We propose a projection method or a Krylov subspace interpretation of the generalized Smith method. More importantly, we prove that some solvability conditions of the rational Riccati equation and their linearizations are inherited by the projected equation.

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1. Introduction

In this paper, we generalize the projection method to rational Riccati equations, arising in stochastic optimal control. This is the fourth paper in the series of papers on the inheritance properties of projection methods for algebraic and rational Riccati equations after [44, 45, 17].

1.1. Continuous-time rational Riccati equations

Consider the control system with state x and control u, governed by the Itô differential equation [12, 14, 19, 20]:

$$dx(t) = Ax(t)\,dt + Bu(t)\,dt + \sum_{i=1}^{N} [A_i x(t) + B_i u(t)]\,dw_i(t), \quad x(0) = x_0. \tag{1}$$

The stochastic disturbances $\{w_i(t)\}_{t\in\mathbb{R}_+}$ are independent zero mean real Wiener processes and the output y satisfies $y(t) = Cx(t) + Du(t)$. Here $A, A_i \in \mathbb{R}^{n\times n}$, $B, B_i \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{l\times n}$ and $D \in \mathbb{R}^{l\times m}$ for $i = 1, \cdots, N$. We denote the transpose by $(\cdot)^\top$, the Moore-Penrose generalized inverse by $(\cdot)^\dagger$ and the 2-norm by $\|\cdot\|$.

For the control or stabilization of (1), we may choose u to minimize

$$J(x_0, u) \equiv \mathcal{E} \int_0^\infty \left[\begin{array}{c} x \\ u \end{array} \right]^\top T \left[\begin{array}{c} x \\ u \end{array} \right] dt, \quad T \equiv \left[\begin{array}{cc} H & L \\ L^\top & R \end{array} \right] \geq 0,$$

where $H = CC^\top \in \mathbb{R}^{n \times n}$, $R \in \mathbb{R}^{m \times m}$ is positive definite, $L \in \mathbb{R}^{n \times m}$ and $\mathcal{E}$ denotes the expectation operator. This gives rise to the continuous-time rational Riccati equation (CRRE):

$$\mathcal{C}(X) \equiv A^\top X + XA + H + \Pi_1(X) - L(X)R(X)^\dagger L(X)^\top = 0, \tag{2}$$

with $L(X) \equiv L + XB + \Pi_{12}(X)$, $R(X) \equiv R + \Pi_2(X)$. The stochasticity of (1) is embodied in

$$\Pi(X) \equiv \left[\begin{array}{cc} \Pi_1(X) & \Pi_{12}(X) \\ \Pi_{12}(X)^\top & \Pi_2(X) \end{array} \right], \tag{3}$$

where $\Pi_1(X) \equiv \sum_{i=1}^N A_i^\top X A_i$, $\Pi_2(X) \equiv \sum_{i=1}^N B_i^\top X B_i$ and $\Pi_{12}(X) \equiv \sum_{i=1}^N A_i^\top X B_i$. The linear operator Π is said to be *positive* as $\Pi(X) \geq 0$ for $X \geq 0$ [36]. Note that the positivity of Π implies that of Π_1 and Π_2. The optimal control is given by

$$u = -\left[R + \Pi_2(X)\right]^\dagger \left[L + XB + \Pi_{12}(X)\right]^\top x, \tag{4}$$

with X being the unique maximal stabilizing solution to the CRRE (2). The applications of projection methods on CRREs and discrete-time rational Riccati equations (DRREs) will be discussed further in Sections 2 and 3. We refer to CRREs and DRREs collectively as RREs.

1.2. What have been done

For algebraic Riccati equations (AREs) [30] without stochastic disturbances (i.e., $\Pi(X) = 0$), they can be solved by the efficient doubling algorithms in [9, 10] (see also the references on other methods therein). Basic results in optimal control can be found in [33]; see also the surveys and reviews for Riccati equations in [1, 18, 30].

A good detailed account of RREs (in continuous-time) by Damm can be found in [12] and useful results are found in [14, 19, 20, 24]. Newton's method was applied in [12, 14] and modified Newton's methods in [21, 24], with the former considering the special case with $R > 0$ and $B_i = 0$ $(i = 1, \cdots, N)$, where the control u is deterministic. This special case, CRRE$_0$, has a constant rational term R^{-1} and has been investigated by Wonham [41, 42]. In [16], the numerical solution of (2) in large-scale (or its discrete-time cousin (14)) by Newton's method was considered, driven by an efficient algorithm for the generalized Lyapunov (or Stein) equations (GLEs or GSEs).

An algorithm of $O(n^3)$ computational complexity per iteration has been proposed for *small size* GLEs by Damm in [2, 3, 12, 13]. The modified Newton methods (MNMs) in [24, 25] have an $O(n^3)$ complexity per iteration but converge linearly. Large-scale RREs have *not* been considered previously except in [16]. For deterministic large-scale problems, please consult [5, 6, 7, 23, 27, 28, 34, 35] for CAREs, [4] for DAREs and [26, 29, 35, 37] for Lyapunov and Stein equations, all applying Galerkin or Krylov subspace methods [37, 38, 39, 40]. See also the recent surveys in [8, 38]. Alternatively, as in [16], generalized Smith methods have been utilized in [11, 31, 32].

The solvability of the CRRE$_0$ has been considered in [41]. Essentially, the assumptions limit the *influence* of the stochastic disturbances in the control system in (1). More general results on the solvability of the CRRE (2) can be found in [12, Chapter 5] and [14], under generalized stabilizability and detectability assumptions.

1.3. Main design

The initial stabilizing solution required in Newton's method is difficult to achieve [43]. The only feasible method for a general CRRE, not requiring any initial stabilizing guess, is the homotopy method in [43], driven by $n(n+1)/2$ ordinary differential equations of the homotopy flow. However, when the size of the problem n is large, this involves an overwhelming amount of computing resources. A new approach is proposed below.

Firstly, from [16], the unique maximal stabilizing solution X of the CRRE (2) (or DRRE (14)) is numerical low-rank, enabling a low-rank approximation of X. Happily, this is the basis of *all* projection method for matrix equations. Secondly, the iterative solution process in [16] suggests the Krylov subspace and Arnoldi processes in (5) and (7) (or, (15) and (16)) below for the projection method. Thirdly, after projection, the *small* projected CRRE (or DRRE) can then be solved efficiently by the homotopy method in [43]. Finally, and importantly, the solvability of the projected CRRE (or DRRE) will be inherited from that of the original CRRE (2) (or DRRE (14)), as in [44, 45] for AREs, when the quantities $\theta_1 \equiv \max_i \|r_k^{i\top} Y_k\|$ and $\theta_2 \equiv \max_{i,j} \|r_k^{i\top} Y_k r_k^j\|$ in (10) (in terms of the Arnoldi residuals r_k^i in (6) and (7) as well as the solution Y_k to the projected RRE) are small.

1.4. Projection methods for AREs

In [44, 45] for large-scale AREs, the structure-preserving doubling algorithms [11, 31] lead to the appropriate Krylov subspaces and Arnoldi processes. Stabilizability, detectability and other sufficient conditions of solvability of the ARE under consideration have been proved to pass onto the projected equations, when the Arnoldi residuals are relatively small. The results in this paper generalize those in [44, 45], with more complicated Krylov subspaces and inheritance properties, because of the more complex concept of stability relative to Π [12, 14, 19, 20, 43] and the lack of any link between the CRRE (2) or DRRE (14), and any eigenvalue problems.

1.5. Main contributions

(1) We propose an algorithm for large-scale RREs, which does not require any difficult initial stabilization (as in Newton-like methods).
(2) From [16], we propose an appropriate Krylov subspace for the projection method for RREs.
(3) We prove the solvability of the original ARE is *inherited* by the projected equation, under certain conditions.

1.6. Organization

We consider the application of projection methods to RREs, and the associated generalized Lyapunov and Stein equations, in Sections 2 and 3. We present the inheritance properties for RREs in Section 4 and some conclusions in Section 5.

2. Projection methods for CRREs

Inspired by the solution of continuous-time rational Riccati and generalized Lyapunov equations in [16], we apply the projection method with the generalized Krylov subspace spanned by V_k, described as the following composite Arnoldi process. With $V_0 = [C^\top,\, L^\top]$ scaled to have orthonormal columns and $A_{(\gamma)} \equiv A - \gamma I$, we construct

$$Z_k = [A_{(\gamma)}^{-\top} V_k, A_1^\top V_k, \cdots, A_N^\top V_k]. \tag{5}$$

We then scale $[V_k,\, Z_k]$ to have orthonormal columns by the QR factorization [22], i.e., $[V_k,\, Z_k] = V_{k+1} S_{k+1}$. In practice, we apply the following generalized

Arnoldi processes. From the first column block $A_{(\gamma)}^{-\top}V_k$ in (5), we have an Arnoldi relationship for $A_{(\gamma)}^{-\top}V_k$ and rearrangement leads to:

$$A^\top V_k = V_k \Phi_k^\top + v_{k+1}^0 r_k^{0\top}, \tag{6}$$

with $[V_k,\, v_{k+1}^0]$ having orthonormal columns and the Arnoldi residual r_k^0 hopefully small. Including (6) with the notation $A_0 \equiv A$, we have the Arnoldi relationship: (for $i = 0, 1, \cdots, N$)

$$A_i^\top V_k = V_k \Phi_k^{i\top} + v_{k+1}^i r_k^{i\top}, \tag{7}$$

and $[V_k,\, v_{k+1}^i]$ has orthonormal columns. Then the QR factorization

$[v_{k+1}^0, v_{k+1}^1, \cdots, v_{k+1}^N] = \widetilde{V}_{k+1}\widetilde{S}_{k+1}$ gives rise to $V_{k+1} = [V_k,\, \widetilde{V}_{k+1}]$ having orthonormal columns.

Remark 2.1. *The Arnoldi process in (5) and (7) is necessarily complicated because of the need to include the stochastic components in A_i. Since we rely on the Arnoldi relationships in (7), we may construct the Krylov subspace differently, using $A^\top$ directly in place of $A_{(\gamma)}^{-\top}$ in (5) [39]. Subsequent development uses (7), so any Krylov subspaces satisfying (7) is applicable.*

Let the projection matrix $P = [P_1,\, P_2]$ with $P_1 \equiv V_k$ and $P^\top P = I$, the Arnoldi relationships (7) then imply the useful equalities: (for all i)

$$P_1^\top A_i P_1 = \Phi_k^i, \quad P_1^\top A_i P_2 = r_k^i v_{k+1}^{i\top} P_2. \tag{8}$$

The Arnoldi residuals r_k^i $(i = 0, 1, \cdots, N)$ play important parts in our analysis. Let $X_k = V_k Y_k V_k^\top$. From (5) and (8) with $\widetilde{B}_i \equiv P_1^\top B_i$, we obtain $\widetilde{\Pi}_1(Y_k) \equiv P_1^\top \Pi_1(X_k) P_1 = \sum_{i=1}^N \Phi_k^{i\top} Y_k \Phi_k^i$, $\widetilde{\Pi}_{12}(Y_k) \equiv P_1^\top \Pi_{12}(X_k) = \sum_{i=1}^N \Phi_k^{i\top} Y_k \widetilde{B}_i$, $P_2^\top \Pi_1(X_k) P_1 = P_2^\top \sum_{i=1}^N v_{k+1}^i r_k^{i\top} Y_k \Phi_k^i$ and $P_2^\top \Pi_{12}(X_k) = P_2^\top \sum_{i=1}^N v_{k+1}^i r_k^{i\top} Y_k \widetilde{B}_i$. The projected CRRE, $P_1^\top \mathcal{C}(X_k) P_1 = 0$, has the form:

$$\mathcal{C}_{11}(Y_k) \equiv \Phi_k^{0\top} Y_k + Y_k \Phi_k^0 + H_{11} + \widetilde{\Pi}_1(Y_k) - L_1(Y_k) R_{11}(Y_k)^\dagger L_1(Y_k)^\top = 0, \tag{9}$$

where $L_1(Y_k) \equiv L_1 + Y_k \widetilde{B}_0 + \widetilde{\Pi}_{12}(Y_k)$, $R_{11}(Y_k) \equiv R + \widetilde{\Pi}_2(Y_k)$, $L_1 \equiv P_1^\top L$, $H_{11} = P_1^\top H P_1$, $\widetilde{B}_0 \equiv P_1^\top B$ and $\widetilde{\Pi}_2(Y) \equiv \sum_{i=1}^N (P_1^\top B_i)^\top Y (P_1^\top B_i)$. The projected CRRE (pCRRE) in (9) is small in size (where $\Phi_k^0 \in \mathbb{R}^{n_0 \times n_0}$ with $n^0 \ll n$), producing Y_k efficiently. We need the pCRRE (9) to satisfy the corresponding solvability condition, which will be proved in Section 4, when the Arnoldi residuals r_k^i $(i = 0, \cdots, N)$ are relatively small.

In terms of errors, apply $P^\top$ and P to the residual $R_k \equiv \mathcal{C}(X_k)$, with

$$F \equiv R(X_k)^\dagger L(X_k)^\top = R_{11}(Y_k)^\dagger L(X_k)^\top, \quad \widetilde{F} \equiv R_{11}(Y_k)^\dagger L_1(Y_k)^\top,$$

we obtain

$$\begin{aligned}
P_2^\top R_k &= P_2^\top \left[v_{k+1}^0 r_k^{0\top} Y_k P_1 + \sum_{i=1}^N v_{k+1}^i r_k^{i\top} Y_k P_1^\top (A_i - B_i F) \right] \\
P_2^\top R_k P_1 &= P_2^\top \left[v_{k+1}^0 r_k^{0\top} Y_k + \sum_{i=1}^N v_{k+1}^i r_k^{i\top} Y_k (\Phi_k^i - \widetilde{B}_i^\top \widetilde{F}) \right], \\
P_2^\top R_k P_2 &= P_2^\top \left\{ \sum_{i=1}^N v_{k+1}^i r_k^{i\top} Y_k \right. \\
&\qquad \left. \cdot \left[r_k^i v_{k+1}^{i\top} - \widetilde{B}_i [R + \widetilde{\Pi}_2(Y_k)]^\dagger \sum_{j=1}^N \widetilde{B}_j^\top Y_k r_k^j v_{k+1}^{j\top} \right] \right\} P_2.
\end{aligned}$$

Estimating $\|P^\top R_k P\|$, with $\rho_1 \equiv \|P_2^\top R_k P_1\|$ and $\rho_2 \equiv \|P_2^\top R_k P_2\|$, we have

$$\begin{aligned}
\rho_1 &\le \left[\|Y_k r_k^0\|^2 + \sum_{i=1}^N \left\| \Phi_k^i - \widetilde{B}_i^\top \widetilde{F} \right\|^2 \|Y_k r_k^i\|^2 \right]^{1/2} \le (N+1)^{1/2} \phi_1 \theta_1, \\
\rho_2 &\le \left[\sum_{i=1}^N \left\| r_k^{i\top} Y_k r_k^i \right\|^2 + \sum_{i,j=1}^N \left\| r_k^{i\top} Y_k \widetilde{B}_i [R + \widetilde{\Pi}_2(Y_k)]^\dagger \widetilde{B}_j^\top Y_k r_k^j \right\|^2 \right]^{1/2} \\
&\le N \phi_2 \theta_1^2 + N^{1/2} \theta_2, \\
\theta_1 &\equiv \max_{i \ge 0} \{\|r_k^{i\top} Y_k\|\}, \quad \theta_2 \equiv \max_{i,j \ge 1} \{\|r_k^{i\top} Y_k r_k^j\|\},
\end{aligned}$$

$$(10) \quad \phi_1 \equiv \max_{i \ge 1} \left\{ 1, \|\Phi_k^i - \widetilde{B}_i \widetilde{F}\| \right\}, \quad \phi_2 \equiv \max_{i,j \ge 1} \left\{ \|\widetilde{B}_i [R + \widetilde{\Pi}_2(Y_k)]^\dagger \widetilde{B}_j^\top\| \right\},$$

we deduce that

$$(11) \qquad \|R_k\| \le \rho_1 + O(\sqrt{\rho_1 \rho_2}) \le (N+1)^{1/2} \phi_1 \theta_1 + O(\theta_1 \theta_2).$$

Note that θ_1 and θ_2, thus ρ_1 and ρ_2, can be small even when r_k^i are large as $(Y_k)_{ij} \to 0$ as $i, j \to \infty$, a feature of projection methods [44, Section 3.1]. As in the deterministic case [44, 45], the residual R_k in (11) is bounded from above essentially by $\theta_1 \equiv \max_i \|r_k^{i\top} Y_k\|$ and $\theta_2 \equiv \max_{i,j} \|r_k^{i\top} Y_k r_k^j\|$.

2.1. Generalized Lyapunov equations

Consider the linearized version of the CRRE (2) or the GLE:

$$\mathcal{L}_{\Pi_1}(X) \equiv A^\top X + XA + \Pi_1(X) + H = 0, \tag{12}$$

where $\mathcal{L}_{\Pi_1}$ is stable (i.e., $\mathcal{L}_0(\cdot) \equiv A^\top(\cdot) + (\cdot)A$ is stable with respect to Π_1 [12]) and $H \geq 0$.

The projection method, with the same Krylov subspace as in (5) and (7) for CRREs, yields the projected GLE:

$$\widetilde{\mathcal{L}}_{\Pi_1}(Y_k) \equiv P_1^\top \mathcal{L}_{\Pi_1}(P_1 Y_k P_1^\top) P_1 = \Phi_k^{0\top} Y_k + Y_k \Phi_k^0 + \widetilde{\Pi}_1(Y_k) + H_{11} = 0. \tag{13}$$

2.2. Truncation

Counter-intuitively, including more vectors in V_k during the Arnoldi process in (5) and (7) may actually harm the accuracy or even the viability of the projection method. For the "quality" of V_k and the condition of Y_k, it is important to truncate nearly dependent basis vectors during the Arnoldi process. For a detailed discussion consult [44, Section 3.1].

3. Projection methods for DRREs

For DRREs and GSEs, we shall share notations with CRREs and GLEs without confusion.

From [15, 19, 25], consider the following discrete-time stochastic control system for the state x and output y:

$$x(t+1) = Ax(t) + Bu(t) + \sum_{i=1}^{N} [A_i x(t) + B_i u(t)]\, w_i(t), \quad y(t) = Cx(t) + Dv(t).$$

In stochastic optimal control, we minimize

$$J_d(x_0, u) \equiv \mathcal{E} \sum_{t=0}^{\infty} \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}^\top T \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}.$$

For the optimal control $u = -\widetilde{F}_X(X)x$, we require the maximal stabilizing solution X of the discrete-time rational Riccati equation (DRRE):

$$\mathcal{D}(X) \equiv -X + H + \Pi_1(X) + A^\top XA - L(X)R(X)^\dagger L(X)^\top = 0, \tag{14}$$

with $L(X) \equiv L + A^\top XB + \Pi_{12}(X))$, $R(X) \equiv R + B^\top XB + \Pi_2(X)$, and $\widetilde{F}_Y(Z) \equiv R(X)^\dagger L(X)^\top$. The stochastic disturbances associated with A_i and B_i manifest themselves in Π in (3).

The solution to (14) has been investigated theoretically and numerically in [19, 25] and in more general forms in [15]. Inspired by the solution of discrete-time algebraic Riccati and generalized Stein equations in [16], we apply the projection method with the generalized Krylov subspace spanned by V_k, described as the following composite Arnoldi process. From $V_0 = [C^\top,\, L^\top]$ (scaled to have orthonormal columns), we construct

$$Z_k = [A^\top V_k, A_1^\top V_k, \cdots, A_N^\top V_k]. \tag{15}$$

We may scale $[V_k,\, Z_k]$ to have orthonormal columns by the QR factorization, i.e., $[V_k,\, Z_k] = V_{k+1}S_{k+1}$. In practice, we apply the following generalized Arnoldi processes. From the first column block $A^\top V_k$, it is easy to see that we have a similar Arnoldi relationships as (7): (for $i = 0, 1, \cdots, N$)

$$A_i^\top V_k \quad = \quad V_k \Phi_k^{i\top} + v_{k+1}^i r_k^{i\top}, \tag{16}$$

with $A_0 \equiv A$, $[V_k,\, v_{k+1}^i]$ having orthonormal columns. Then from the QR factorization $[v_{k+1}^0, \cdots, v_{k+1}^N] = \widetilde{V}_{k+1}\widetilde{S}_{k+1}$, we built $V_{k+1} = [V_k,\, \widetilde{V}_{k+1}]$ with orthonormal columns.

Let the projection matrix $P = [P_1,\, P_2]$ with $P_1 \equiv V_k$ and $P^\top P = I$, and $X_k = V_k Y_k V_k^\top$. The projected DRRE is $P_1^\top \mathcal{D}(X_k) P_1 = 0$, of the form:

$$\mathcal{D}_{11}(Y_k) \equiv -Y_k + H_{11} + \widetilde{\Pi}_1(Y_k) + \Phi_k^{0\top} Y_k \Phi_k^0 - L_1(Y_k) R_{11}(Y_k)^\dagger L_1(Y_k)^\top = 0,$$

with $L_1(Y_k) \equiv L_1 + \Phi_k^{0\top} Y_k \widetilde{B}_0 + \widetilde{\Pi}_{12}(Y_k)$, $\widetilde{R}_{11}(Y_k) \equiv R + \widetilde{B}_0^\top Y_k \widetilde{B}_0 + \widetilde{\Pi}_1(Y_k)$ and $R_{11}(Y_k) \equiv R + \widetilde{B}_0^\top Y_k \widetilde{B}_0 + \widetilde{\Pi}_2(Y_k)$. For errors, with

$$F \equiv R(X_k)^\dagger L_1(X_k)^\top = R_{11}(Y_k)^\dagger L(X_k)^\top, \quad \widetilde{F} \equiv R_{11}(Y_k)^\dagger L_1(Y_k)^\top,$$

$A_0 \equiv A$ and $B_0 \equiv B$, we obtain

$$P_2^\top R_k = P_2^\top \sum_{i=0}^{N} v_{k+1}^i r_k^{i\top} Y_k P_1^\top (A_i - B_i F),$$

$$P_2^\top R_k P_1 = P_2^\top \sum_{i=0}^{N} v_{k+1}^i r_k^{i\top} Y_k (\Phi_k^i - \widetilde{B}_i \widetilde{F}),$$

$$P_2^\top R_k P_2 = P_2^\top \left\{ v_{k+1}^0 r_k^{0\top} Y_k \left[r_k^0 v_{k+1}^{0\top} - \widetilde{B}_0 \widetilde{R}_{11}(Y_k)^\dagger \sum_{i=1}^N \widetilde{B}_i^\top Y_k r_k^i v_{k+1}^i \right] \right.$$
$$\left. + \sum_{i=1}^N v_{k+1}^i r_k^{i\top} Y_k \left[r_k^i v_{k+1}^{i\top} - \widetilde{B}_i R_{11}(Y_k)^\dagger \sum_{j=1}^N \widetilde{B}_j^\top Y_k r_k^j v_{k+1}^{i\top} \right] \right\} P_2.$$

With θ_1 and θ_2 as defined in (10), and $\phi_1 \equiv \max_{i\geq 0}\left\{\|\Phi_k^i - \widetilde{B}_i\widetilde{F}\|\right\}$ and $\phi_2 \equiv \max_{i,j\geq 0}\left\{\|\widetilde{B}_i[R + \widetilde{B}_0^\top Y_k \widetilde{B}_0 + \widetilde{\Pi}_2(Y_k)]^\dagger \widetilde{B}_j^\top\|\right\}$, we obtain

$$\begin{aligned}
\rho_1 &\equiv \|P_2^\top R_k P_1\| \leq \left[\sum_{i=0}^N \left\| \Phi_k^i - \widetilde{B}_i \widetilde{F} \right\|^2 \|Y_k r_k^i\|^2 \right]^{1/2} \leq (N+1)^{1/2} \phi_1 \theta_1, \\
\rho_2 &\equiv \|P_2^\top R_k P_2\| \leq \left[\sum_{i=1}^N \left\| r_k^{0\top} Y_k \widetilde{B}_0 \widetilde{R}_{11}(Y_k)^\dagger \widetilde{B}_i^\top Y_k r_k^i \right\|^2 \right. \\
&\quad \left. + \sum_{i=0}^N \left\| r_k^{i\top} Y_k r_k^i \right\|^2 + \sum_{i,j=1}^N \left\| r_k^{i\top} Y_k \widetilde{B}_i R_{11}(Y_k)^\dagger \widetilde{B}_j^\top Y_k r_k^j \right\|^2 \right]^{1/2} \\
&\leq (N+N^2)^{1/2} \phi_2 \theta_1^2 + (N+1)^{1/2} \theta_2,
\end{aligned}$$

treating the second-ordered term in ρ_2 as a perturbation, we deduce that

$$\|R_k\| \leq \rho_1 + O(\sqrt{\rho_1 \rho_2}) \leq (N+1)^{1/2} \phi_1 \theta_1 + O(\theta_1 \theta_2). \tag{17}$$

Again from (17), $\theta_1 \equiv \max_i \|r_k^{i\top} Y_k\|$ and $\theta_2 \equiv \max_{i,j} \|r_k^{i\top} Y_k r_k^j\|$ can be small even when r_k^i are large as $(Y_k)_{ij} \to 0$ as $i, j \to \infty$.

3.1. Generalized Stein equations

Consider the linearized version of the DRRE (14) or the GSE:

$$\mathcal{S}_{\Pi_1}(X) \equiv -X + A^\top X A + \Pi_1(X) + H = 0,$$

where $\mathcal{S}_{\Pi_1}$ is stable (i.e., $\mathcal{S}_0(\cdot) \equiv -(\cdot) + A^\top(\cdot)A$ is stable with respect to Π_1 [12]) and $H \geq 0$.

The projection method, with the same Krylov subspace as in (5) and (7) for CRREs, leads to the projected GSE:

$$\widetilde{\mathcal{S}}_{\Pi_1}(Y_k) \equiv P_1^\top \mathcal{S}_{\Pi_1}(P_1 Y_k P_1^\top) P_1 = -Y_k + \Phi_k^{0\top} Y_k \Phi_k^0 + \widetilde{\Pi}_1(Y_k) + H_{11} = 0.$$

4. Inheritance properties for solvability

There are many sufficient conditions for the solvability of the CRREs. For example, a unique stabilizing solution X exists for the RRE if the underlying system is stabilizable relatively to Π, Condition (A6) on T, as well as some null space condition holds for the generalized inverse. Particularly useful to our discussion, we required $X > 0$, which is guaranteed by [43, Lemma 3.11]. See also Corollaries 3.7 and 3.10, and Lemma 3.8 in [43] for more related results. Solvability of RREs is nontrivial, associated with different sufficient conditions. From [12, Lemma 1.8.4] and [43, Section 3.1.1], the following assumptions are required, in various combinations, for solvability:

(A1) (Stabilizability) (A, B) is c-stabilizable relative to Π; i.e., there exist $F \in \mathbb{R}^{m\times n}$, $X > 0$ and

$$\widehat{\Pi} \equiv \left[\begin{array}{c} I \\ -F \end{array} \right]^\top \Pi \left[\begin{array}{c} I \\ -F \end{array} \right]^\top,$$

such that $(A - BF)^\top X + X(A - BF) + \widehat{\Pi}(X) < 0$;

(A2) (Detectability) $(H - LR^\dagger L^\top,\ A - BR^\dagger L^\top)$ is c-detectable relative to Π_1; i.e., there exist $X > 0$ such that

$$(A - BR^\dagger L^\top)^\top X + X(A - BR^\dagger L^\top) + \Pi_1(X) < H - LR^\dagger L^\top;$$

(A3) There exists $\widehat{X}$ where $\mathrm{Null}[R + \Pi_2(\widehat{X})] \subseteq \mathrm{Null}(B)$ and $\mathcal{C}(\widehat{X})$ is (semi-)positive definite.

(A4) Let $R(X) \equiv R + \Pi_2(X)$, $L(X) \equiv L + XB + \Pi_{12}(X)$ for CRREs, or $R(X) \equiv R + B^\top XB + \Pi_2(X)$, $L(X) \equiv L + A^\top XB + \Pi_{12}(X)$ for DRREs, we require

$$\mathrm{Null}(R(X)) \subseteq \mathrm{Null}(S(X)); \tag{18}$$

(A5) $\mathrm{Null}(R) \subseteq \mathrm{Null}(L)$, $\mathrm{Null}(R) \subseteq \mathrm{Null}(B)$ or $\mathrm{Null}(R+\Pi_2(X)) \subseteq \mathrm{Null}(B)$;

(A6) $H > LR^\dagger L^\top$; and

(A7) $T \equiv \left[\begin{array}{cc} R & L^\top \\ L & H \end{array} \right] \geq 0.$

We shall prove the inheritance properties for the sufficient conditions (A1)–(A7) associated with the solvability of RREs. Previously in other papers on projection methods for AREs, as discussed in [44, 45], the solvability of the projected AREs has been *assumed*. Only results for CRREs will be shown and the analogous results for DRREs can be deduced similarly.

4.1. Stabilizability and detectability

By [12, Lemma 1.7.2], the stabilizability of $[A, (A_i),\, B, (B_i)]$ is equivalent to the following:

> if $X \in \mathbb{C}^{n\times n}$ is an eigenvector of $\mathcal{L}_A + \Pi_1$ corresponding to an eigenvalue $\lambda \notin \mathcal{C}_-$, then $B^* X,\, B_i^* X \neq 0$.

Detectability and stabilizability are adjoint properties, with detectability of $[A, (A_i),\, C]$ being equivalent to the stabilizability of $[A^\top, (A_i^\top),\, C^\top, (0)]$.

4.1.1. Inheritance of stabilizability. Expanding using the Kronecker product, stabilizability is thus equivalent to

$$\mathcal{M}(s) \equiv [\mathcal{A} - sI,\, \mathcal{B}] \;\text{ f.r.,} \tag{19}$$

for $s \notin \mathcal{C}_-$, with "f.r." abbreviating "full-rank" and

$$\mathcal{A} \equiv I \otimes A + A \otimes I + \sum_{i=1}^{N} A_i \otimes A_i, \quad \mathcal{B} \equiv I \otimes [B,\, B_1, \cdots, B_N].$$

More rigorously in terms of the minimum singular value, let

$$\tau[A, (A_i);\, B, (B_i)] \equiv \min_{s\notin\mathcal{C}_-} \sigma_{\min}\mathcal{M}(s). \tag{20}$$

The stabilizability of $[A, (A_i); B, (B_i)]$ is equivalent to $\tau[A, (A_i); B, (B_i)] > 0$.

First with $A_\otimes \equiv I \otimes (P^\top AP) + (P^\top AP) \otimes I$, consider $x^1 = y^1 \otimes z^1 = [y_1^{1\top},\, y_2^{1\top}]^\top \otimes [z_1^{1\top},\, z_2^{1\top}]^\top$, we have

$$\begin{aligned}
&\tau[A, (A_i);\, B, (B_i)] = \min_{s\notin\mathcal{C}_-} \sigma_{\min}\, [\mathcal{A} - sI,\, \mathcal{B}] \\
= \;& \min_{s\notin\mathcal{C}_-} \min_{\|x^1\|=1} \left\| x^{1\top} (P^\top \otimes P^\top)\, [\mathcal{A} - sI,\, \mathcal{B}] \begin{bmatrix} P\otimes P & 0 \\ 0 & P \otimes I \end{bmatrix} \right\| \\
= \;& \min_{s\notin\mathcal{C}_-} \min_{\|x^1\|=1} \left\| \left\{ x^{1\top} \left[A_\otimes + \sum_{i=1}^{N} (P^\top A_i P) \otimes (P^\top A_i P) - sI \right], \right.\right. \\
& \qquad \left.\left. \Big(I \otimes [P^\top B,\, P^\top B_i, \cdots, P^\top B_N] \Big) \right\} \right\| \\
= \;& \min_{s\notin\mathcal{C}_-} \min_{\|x^1\|=1} \left\| \left\{ y^{1\top} \otimes (z^{1\top} P^\top AP) + (y^{1\top} P^\top AP) \otimes z^{1\top} \right. \right.
\end{aligned}$$

$$+\sum_{i=1}^{N}(y^{1\top}P^{\top}A_iP)\otimes(z^{1\top}P^{\top}A_iP)-sx^{1\top},$$
$$y^{1\top}\otimes(z^{1\top}[P^{\top}B,\,P^{\top}B_i,\cdots,P^{\top}B_N])\Big\}\Big\|.$$

With $y^1=[y_1^{1\top},\,0^{\top}]^{\top}$, $y^1=[y_1^{1\top},\,0^{\top}]^{\top}$, $x_1^1=y_1^1\otimes z_1^1$ and (8), we have

$$\begin{aligned}
&\tau[A,(A_i);\,B,(B_i)]\le\min_{s\notin\mathcal{C}_-}\min_{\|x_1^1\|=1}\\
&\left\|\left\{\begin{bmatrix} y_1^1\\ 0\end{bmatrix}^{\top}\otimes(z_1^{1\top}[\Phi_k^0,\,r_k^0v_{k+1}^{0\top}P_2])+(y_1^{1\top}[\Phi_k^0,\,r_k^0v_{k+1}^{0\top}P_2])\otimes\begin{bmatrix} z_1^1\\ 0\end{bmatrix}^{\top}\right.\right.\\
&+\sum_{i=1}^{N}(y_1^{1\top}[\Phi_k^i,\,r_k^iv_{k+1}^{i\top}P_2])\otimes(z_1^{1\top}[\Phi_k^i,\,r_k^iv_{k+1}^{i\top}P_2])\\
&\left.\left.-s\begin{bmatrix} y_1^1\\ 0\end{bmatrix}^{\top}\otimes\begin{bmatrix} z_1^1\\ 0\end{bmatrix}^{\top},\ \begin{bmatrix} y_1^1\\ 0\end{bmatrix}^{\top}\otimes(z_1^{1\top}[\widetilde{B}_0,\,\widetilde{B}_1,\cdots,\widetilde{B}_N])\right\}\right\|
\end{aligned}\tag{21}$$

$$\begin{aligned}
&\le\min_{s\notin\mathcal{C}_-}\min_{\|x_1^1\|=1}\left\|\left\{y_1^{1\top}\otimes(z_1^{1\top}\Phi_k^0)+(y_1^{1\top}\Phi_k^0)\otimes z_1^{1\top}+\sum_{i=1}^{N}(y_1^{1\top}\Phi_k^i)\otimes(z_1^{1\top}\Phi_k^i)\right.\right.\\
&\qquad\left.\left.-sy_1^{1\top}\otimes z_1^{1\top},\ y_1^{1\top}\otimes(z_1^{1\top}[\widetilde{B}_0,\,\widetilde{B}_1,\cdots,\widetilde{B}_N])\right\}\right\|+\psi_k^1\\
&\qquad=\tau[\Phi_k^0,(\Phi_k^i);\,\widetilde{B}_0,(\widetilde{B}_i)]+\psi_k^1,
\end{aligned}\tag{22}$$

where $\|y^1\otimes z^1\|=\|y^1\|\cdot\|z^1\|$, $\varphi^1\equiv\max_i\{\|\Phi_k^{i\top}y_1^1\|,\|\Phi_k^{i\top}z_1^1\|\}$,
$r^1\equiv\max_{i\ge0}\{\|r_k^{i\top}y_1^1\|,\|r_k^{i\top}z_1^1\|\}$ and for $j=1$:

$$\begin{aligned}
\psi_k^j\ &\equiv\ \left\|\begin{bmatrix} y_1^j\\ 0\end{bmatrix}^{\top}\otimes\begin{bmatrix} 0\\ P_2^{\top}v_{k+1}^0r_k^{0\top}z_1^j\end{bmatrix}^{\top}+\begin{bmatrix} 0\\ P_2^{\top}v_{k+1}^0r_k^{0\top}y_1^j\end{bmatrix}^{\top}\otimes\begin{bmatrix} z_1^j\\ 0\end{bmatrix}^{\top}\right.\\
&\quad+\sum_{i=1}^{N}\left\{\begin{bmatrix} 0\\ P_2^{\top}v_{k+1}^ir_k^{i\top}y_1^j\end{bmatrix}^{\top}\otimes\begin{bmatrix} 0\\ P_2^{\top}v_{k+1}^ir_k^{i\top}z_1^j\end{bmatrix}^{\top}+\begin{bmatrix} \Phi_k^{i\top}y_1^j\\ 0\end{bmatrix}^{\top}\right.\\
&\quad\left.\left.\otimes\begin{bmatrix} 0\\ P_2^{\top}v_{k+1}^ir_k^{i\top}z_1^j\end{bmatrix}^{\top}+\begin{bmatrix} 0\\ P_2^{\top}v_{k+1}^ir_k^{i\top}y_1^j\end{bmatrix}^{\top}\otimes\begin{bmatrix} \Phi_k^{i\top}z_1^j\\ 0\end{bmatrix}^{\top}\right\}\right\|\\
&\le\ \left[\|r_k^{0\top}y_1^j\|^2+\|r_k^{0\top}z_1^j\|^2+\sum_{i=1}^{N}\Big(\|\Phi_k^{i\top}y_1^j\|^2\|r_k^{i\top}z_1^j\|^2\right.\\
&\qquad\left.+\|\Phi_k^{i\top}z_1^j\|^2\|r_k^{i\top}y_1^j\|^2+\|r_k^{i\top}y_1^j\|^2\|r_k^{i\top}z_1^j\|^2\Big)\right]^{1/2}
\end{aligned}\tag{23}$$

$$\leq \quad \sqrt{2(N\varphi^1+1)\,(r^1)^2+N(r^1)^4},$$

with y_1^1 and z_1^1 optimize the first term in (4.1.1). For a general $x=\sum_j \alpha_j y^j \otimes z^j$ with $\{y^j\otimes z^j\}$ being orthonormal, $y^j=[y_1^{j\top}, y_2^{j\top}]^\top$, $z^j=[z_1^{j\top}, z_2^{j\top}]^\top$ and $\|x\|^2=\sum_j \alpha_j^2=1$, the proof follows similarly, with additional summations with respect to j and applications of the triangular inequality in (4.1.1) and (23). Let $\varphi^j \equiv \max_i\{\|\Phi_k^{i\top} y_1^j\|, \|\Phi_k^{i\top} z_1^j\|\}$, $\varphi \equiv \max_j\{\varphi^j\}$, $r^j \equiv \max_{i\geq 0}\{\|r_k^{i\top} y_1^j\|, \|r_k^{i\top} z_1^j\|\}$, $\tilde{r} \equiv \max_j\{r^j\}$ and from (23):

$$\psi_k \equiv \sqrt{\sum_j (\psi_k^j)^2} \leq \sqrt{2(N\varphi+1)\tilde{r}^2+N\tilde{r}^4}, \tag{24}$$

the result in (22) for the general case has the form

$$\tau[A,(A_i);\, B,(B_i)] \leq \tau[\Phi_k^0,(\Phi_k^i);\, \widetilde{B}_0,(\widetilde{B}_i)] + \psi_k. \tag{25}$$

Thus (24) implies that $\psi_k=[2(N\varphi+1)]^{1/2}\tilde{r}+O(\tilde{r}^3)$ which will be small if $\tilde{r}$ is, or $\|r_k^{i\top} y_1^j\|$ and $\|r_k^{i\top} z_1^j\|$ are. From (25), the inheritance of stabilizability, in the terms of τ in (20), holds when $\tau[A,(A_i);\, B,(B_i)] > \psi_k$, bounding the original system from unstabilizability by a distance of at least ψ_k.

4.1.2. Difficulties with detectability. The inheritance of the adjoint property of detectability *cannot* be deduced similarly. Note that detectability is substituted by (A6) in some theorems on solvability of RREs.

Expanding using the Kronecker product, detectability is equivalent to

$$\mathcal{N}(s) \equiv \begin{bmatrix} \mathcal{A}-sI \\ I\otimes C \end{bmatrix} \ \text{f.r.}, \tag{26}$$

for $s\notin \mathcal{C}_-$, with $\mathcal{A}\equiv I\otimes A + A\otimes I + \sum_{i=1}^N A_i\otimes A_i$ as defined in (19), without any C_i.

In terms of the minimum singular value, ignoring the argument for degenerate C_i in τ, let

$$\tau[A^\top,(A_i^\top);\, C^\top] \equiv \min_{s\notin\mathcal{C}_-} \sigma_{\min}\mathcal{N}(s).$$

The detectability of $[A,(A_i),C]$ is equivalent to $\tau[A^\top,(A_i^\top);\, C^\top]>0$.

First consider $x^1=y^1\otimes z^1=[y_1^{1\top},\, y_2^{1\top}]^\top \otimes [z_1^{1\top},\, z_2^{1\top}]^\top$, we have

$$\tau[A^\top,(A_i^\top);\, C^\top] \equiv \min_{s\notin\mathcal{C}_-} \sigma_{\min} \begin{bmatrix} \mathcal{A}-sI \\ I\otimes C \end{bmatrix}$$

$$= \min_{s\notin\mathcal{C}_-}\min_{\|x^1\|=1}\left\|\begin{bmatrix} P^\top\otimes P^\top & 0 \\ 0 & P^\top\otimes I\end{bmatrix}\begin{bmatrix}\mathcal{A}-sI \\ I\otimes C\end{bmatrix}(P\otimes P)x^1\right\|$$
$$= \min_{s\notin\mathcal{C}_-}\min_{\|x^1\|=1}\left\|\begin{bmatrix} A_\otimes+\sum_{i=1}^N(P^\top A_iP)\otimes(P^\top A_iP)-sI \\ I\otimes(CP)\end{bmatrix}x^1\right\|.$$

Let $A_0\equiv A$, $\widetilde{C}\equiv CP=[C_1,0]$ and for $i=0,1,\cdots,N$:

$$P^\top A_iP=\begin{bmatrix}\Phi_k^i & r_k^iv_{k+1}^{i\top}P_2 \\ A_{21}^i & A_{22}^i\end{bmatrix},\ \widetilde{A}_i\equiv\begin{bmatrix}\Phi_k^i & 0 \\ A_{21}^i & A_{22}^i\end{bmatrix},\ \widehat{A}_i\equiv\begin{bmatrix}\Phi_k^i & 0 \\ 0 & 0\end{bmatrix}.$$

With x^1 optimizing the first term in (27), we then have

$$\tau[A^\top,(A_i^\top);C^\top]\le\min_{s\notin\mathcal{C}_-}\min_{\|x^1\|=1}$$
$$\left\|\begin{bmatrix} I\otimes\widetilde{A}_0+\widetilde{A}_0\otimes I+\sum_{i=1}^N\widetilde{A}_i\otimes\widetilde{A}_i-sI \\ I\otimes[C_1,0]\end{bmatrix}x^1\right\|+\zeta^1, \tag{27}$$

where $\breve{y}_2^i\equiv v_{k+1}^{i\top}P_2y_2^1$, $\breve{z}_2^i\equiv v_{k+1}^{i\top}P_2z_2^1$ and

$$\zeta^1 \equiv \|r_k^i\breve{y}_2^1\|+\|r_k^i\breve{y}_2^1\|+\sum_{i=1}^N\|r_k^i\breve{y}_2^1\|\|r_k^i\breve{z}_2^1\|.$$

Assume that

$$\tau[A^\top,(A_i^\top);C^\top]>\zeta^1, \tag{28}$$

the first term in (27) is positive, indicating that the system associated with A_{22}^0 is stable. In particular, for some $x^1=y^1\otimes[0,z_2^{1\top}]^\top$, we have

$$\min_{s\notin\mathcal{C}_-}\min_{\|x^1\|=1}\left\|\begin{bmatrix} I\otimes\widetilde{A}_0+\widetilde{A}_0\otimes I+\sum_{i=1}^N\widetilde{A}_i\otimes\widetilde{A}_i-sI \\ I\otimes[C_1,0]\end{bmatrix}x^1\right\|>0. \tag{29}$$

Minimum singular value. Expanding (29), we have

$$\left\|\begin{bmatrix} I\otimes\widetilde{A}_0+\widetilde{A}_0\otimes I+\sum_{i=1}^N\widetilde{A}_i\otimes\widetilde{A}_i-sI \\ I\otimes[C_1,0]\end{bmatrix}x^1\right\|$$
$$= \left\|\begin{bmatrix}\mathcal{A}(s) \\ I\otimes[C_1,0]\end{bmatrix}\left[\begin{array}{c} y_1^1\otimes z_1^1 \\ \hline y_2^1\otimes z_1^1 \\ \hline y_1^1\otimes z_2^1 \\ y_2^1\otimes z_2^1\end{array}\right]\right\|,$$

(after appropriate reordering of rows) with $I_1 \equiv I_d$, $I_2 \equiv I_{n-d}$, $I_{11} \equiv I_{d^2}$, $I_{12} \equiv I_{d(n-d)}$, $I_{22} \equiv I_{(n-d)^2}$ and

$$\begin{bmatrix} \mathcal{A}(s) \\ I_n \otimes [C_1, 0] \end{bmatrix} \equiv \left[\begin{array}{ccc} \mathcal{A}_{11}(s) & 0 & 0 \\ \mathcal{A}_{21} & \mathcal{A}_{22}(s) & 0 \\ \mathcal{A}_{31} & \mathcal{A}_{32} & \mathcal{A}_{33}(s) \\ \hline I_1 \otimes C_1 & 0 & 0 \\ 0 & I_2 \otimes C_1 & 0 \end{array}\right],$$

$$\mathcal{A}_{11}(s) \equiv I_1 \otimes \Phi_k^0 + \Phi_k^0 \otimes I_1 + \sum_i \Phi_k^i \otimes \Phi_k^i - sI_{11},$$

$$\mathcal{A}_{21} \equiv A_{21}^0 \otimes I_1 + \sum_i A_{21}^i \otimes \Phi_k^i,$$

$$\mathcal{A}_{22}(s) \equiv I_2 \otimes \Phi_k^0 + A_{22}^0 \otimes I_1 + \sum_i A_{22}^i \otimes \Phi_k^i - sI_{12},$$

$$\mathcal{A}_{31} \equiv \begin{bmatrix} I_1 \otimes A_{21}^0 + \sum_i \Phi_k^i \otimes A_{21}^i \\ \sum_i A_{21}^i \otimes A_{21}^i \end{bmatrix},$$

$$\mathcal{A}_{32} \equiv \begin{bmatrix} 0 \\ I_2 \otimes A_{21}^0 + \sum_i A_{22}^i \otimes A_{21}^i \end{bmatrix},$$

$$\mathcal{A}_{33}(s) \equiv \left[\begin{array}{c|c} I_1 \otimes A_{22}^0 + \Phi_k^0 \otimes I_2 + \sum_i \Phi_k^i \otimes A_{22}^i - sI_{12} & 0 \\ A_{21}^0 \otimes I_2 + \sum_i A_{21} \otimes A_{22}^i & I_2 \otimes A_{22}^0 + A_{22}^0 \otimes I_2 + \sum_i A_{22}^i \otimes A_{22}^i - sI_{22} \end{array}\right].$$

From the zeroes in $\mathcal{C}$ and the fact that (26) implying the full-rank of

$[\mathcal{A}(s)^\top, \mathcal{C}^\top]^\top$, we have the nonsingularity of $\mathcal{A}_{33}(s)$, which also implies the same for $\mathcal{A}_{22}(s)$. These imply the full rank of

$$\left[\begin{array}{cc} \mathcal{A}_{11}(s) & 0 \\ \mathcal{A}_{21} & \mathcal{A}_{22}(s) \\ \hline I_1 \otimes C_1 & 0 \\ 0 & I_2 \otimes C_1 \end{array}\right] \quad \text{or} \quad \left[\begin{array}{c} \mathcal{A}_{11}(s) \\ \hline I_1 \otimes C_1 \\ -(I_2 \otimes C_1)\mathcal{A}_{22}(s)^{-1}\mathcal{A}_{21} \end{array}\right].$$

Either full-rank matrix does not lead to the full-rank of $[\mathcal{A}_{11}(s)^\top, I \otimes C_1^\top]^\top$ we require, unless $\mathcal{A}_{21} = 0$ which is untrue in general.

Generalized Lyapunov equations. From (29), under the assumption of (28), the system $[\widetilde{A}^{0\top}, (\widetilde{A}_i^\top); \widetilde{C}^\top]$ is detectable with respect to

$$\widetilde{\Pi}_1(X) \equiv \begin{bmatrix} \widetilde{\Pi}_{11}^{(1)}(X) & \widetilde{\Pi}_{12}^{(1)}(X) \\ \widetilde{\Pi}_{21}^{(1)}(X) & \widetilde{\Pi}_{22}^{(1)}(X) \end{bmatrix} = \sum_{i=1}^{N} \widetilde{A}_i^\top X \widetilde{A}_i.$$

Expanding with $X = (X_{ij})$ $(X_{21} = X_{12}^\top)$, it is easy to see that

$$\widetilde{\Pi}_{11}^{(1)}(X) = \widehat{\Pi}_1(X_{11}) + \sum_{i=1}^{N} (\Phi_k^{i\top} X_{12} A_{21}^i + A_{21}^{i\top} X_{12}^\top \Phi_k^i + A_{21}^{i\top} X_{22} A_{21}^i),$$

$$\widehat{\Pi}_1(X_{11}) \equiv \sum_{i=1}^{N} \Phi_k^{i\top} X_{11} \Phi_k^i, \quad \widetilde{\Pi}_{12}^{(1)}(X) = \sum_{i=1}^{N} (\Phi_k^{i\top} X_{12} A_{22}^i + A_{21}^{i\top} X_{22} A_{22}^i),$$

$$\widetilde{\Pi}_{21}^{(1)}(X) = [\widetilde{\Pi}_{12}^{(1)}(X)]^\top, \quad \widetilde{\Pi}_{22}^{(1)}(X) = \sum_{i=1}^{N} A_{22}^{i\top} X_{22} A_{22}.$$

From [43, Theorem 3.1], there exists G such that $\widetilde{A_0} - G\widetilde{C}$ is stable with respect to $\widetilde{\Pi}_1$, or there exists a positive definite solution X such that

$$\widetilde{A}_0^\top X + X\widetilde{A}_0 + \widetilde{\Pi}_1(X) + Q = 0, \quad Q > 0.$$

To prove that $[\Phi_k^{0\top}, (\Phi_k^{i\top}); C_1^\top]$ is detectable with respect to $\widehat{\Pi}_1$, we need to find G such that $\Phi_k^0 - G_1 C_1$ is stable with respect to $\widehat{\Pi}$, or there exists a positive definite solution X_{11} such that

$$\Phi_k^{0\top} X_{11} + X_{11} \Phi_k^i + \widehat{\Pi}_1(X_{11}) + Q_{11} = 0, \quad Q_{11} > 0.$$

Because $\widetilde{\Pi}$ is quadratic in $\widetilde{A}_i$, inheritance of detectability can only be proved to be inherited if $A_{21}^i = 0$ $(i = 0, 1, \cdots, N)$.

4.2. Null space and other conditions

4.2.1. (A4). The inheritance the null space requirement (A4) or (18) by the projected quantities

$$\widetilde{R}(Y_k) \equiv R(X_k), \quad \widetilde{L}(Y_k) \equiv P_1^\top L(P_1 Y_k P_1^\top).$$

is obvious, provided that (18) holds from the structure of R, B_i, L and B, independent of X. It is the result of the above definitions and the fact that, ignoring X or X_k, we have

$$\mathrm{Null}(\widetilde{R}(X_k)) = \mathrm{Null}(R(X_k)) \subseteq \mathrm{Null}(L(X_k)) \subseteq \mathrm{Null}(\widetilde{L}(X_k)).$$

4.2.2. (A3). For (A3), the null space conditions follows as that for (A4). We need $\widehat{X} = P_1 Y_k P_1^\top$ then the (semi-)definiteness of $\mathcal{C}(\widehat{X})$ is inherited by $C_{11}(Y_k)$. Condition (A3) provides an upper bound for the monotonely increasing sequence of approximate solution X_k in the proof of solvability using Newton's iteration. This technical assumption is unavoidable but is rarely checked. For the definiteness result, any numerically low-ranked solution $X = P_1 Y_k P_1^\top + O(\epsilon)$ of $\mathcal{C}(X) > 0$ enables the choice $\widehat{X} = P_1 Y_k P_1^\top$ provided that the $O(\epsilon)$ term is small enough to preserve the strict inequality after projection. For the semi-definite result, $\widehat{X} = 0$ implies $\mathcal{C}(\widehat{X}) = H \geq 0$ or the semi-definite requirement. The inheritance holds because $\mathcal{C}(\widehat{Y}) = H_{11} \geq 0$ for $\widehat{Y} = 0$.

Because of the difficulty in the definiteness case (requiring a numerically low-ranked solution $X = P_1 Y_k P_1^\top + O(\epsilon)$ of $\mathcal{C}(X) > 0$), some may prefer to abandon the existence results, for instance, in [43, Corollary 3.7].

4.2.3. (A5)–(A7). For (A5), it is clear, for $\widetilde{R} = \widetilde{R}(X)$ independent of X, that

$$\begin{aligned}
&\mathrm{Null}(\widetilde{R}) = \mathrm{Null}(R) \subseteq \mathrm{Null}(L) \subseteq \mathrm{Null}(P_1^\top L) = \mathrm{Null}(L_1),\\
&\mathrm{Null}(\widetilde{R}) = \mathrm{Null}(R) \subseteq \mathrm{Null}(B) \subseteq \mathrm{Null}(P_1^\top B) = \mathrm{Null}(\widetilde{B}_0),\\
&\mathrm{Null}(\widetilde{R} + \widetilde{\Pi}_2(Y_k)) = \mathrm{Null}(R + \Pi_2(X_k)) \subseteq \mathrm{Null}(B) \subseteq \mathrm{Null}(\widetilde{B}_0).
\end{aligned}$$

For (A6), we have

$$\lambda_{\min}(H - LR^\dagger L^\top) \leq \lambda_{\min}\left\{P_1^\top (H - LR^\dagger L^\top) P_1\right\} = \lambda_{\min}(H_{11} - L_1 \widetilde{R}^\dagger L_1^\top).$$

For (A7), we have

$$\begin{bmatrix} I & \\ & P^\top \end{bmatrix} T \begin{bmatrix} I & \\ & P \end{bmatrix} = \left[\begin{array}{cc|c} R & L_1^\top & L_2^\top \\ L_1 & H_{11} & 0 \\ \hline L_2 & 0 & 0 \end{array}\right] \geq 0,$$

implying the definiteness of the analogous quantity after projection:

$$\widetilde{T} \equiv \begin{bmatrix} \widetilde{R} & L_1^\top \\ L_1 & H_{11} \end{bmatrix} = \begin{bmatrix} R & L_1^\top \\ L_1 & H_{11} \end{bmatrix} \geq 0.$$

Remark 4.1. *Inheritance properties for DRREs can be considered using similar approaches and techniques for CRREs. The definitions of stability for GLEs and GSEs are similar to the stabilizability of CRREs and DRREs, respectively. The inheritance of stability for projection methods for GLEs and GSEs is a special case of the inheritance of stabilizability.*

4.3. Inheritance for GLEs and GSEs

For the GLE (12), we want to show that stability of $\mathcal{L}_{\Pi_1}$ passes onto $\widetilde{\mathcal{L}}_{\Pi_1}$ of the projected GLE (13). The stability of $\mathcal{L}_{\Pi_1}$ means that any eigenvalue λ is stable (with negative real part) and the corresponding eigenvector $X \neq 0$ satisfies

$$A^\top X + XA + \Pi_1(X) = \lambda X.$$

Denote $X_{ij} \equiv P_i^\top X P_j$ $(i, j = 1, 2)$. Apply $P^\top$ and P from the left and the right, with the help of the Arnoldi relationships (7) or the results in (8), the (1,1)-subblock is

$$\begin{aligned}
&\widetilde{\mathcal{L}}_{11}(X_{11}) + \widetilde{R}_k = \lambda X_{11}, \\
&\widetilde{R}_k \equiv X_{12} P_2^\top v_{k+1}^0 r_k^{0\top} + r_k^0 v_{k+1}^{0\top} P_2 X_{12}^\top \\
&+ \sum_{i=1}^{N} \left(\Phi_k^{i\top} X_{12} P_2^\top v_{k+1}^i r_k^{i\top} + r_k^i v_{k+1}^{i\top} P_2 X_{12}^\top \Phi_k^i + r_k^i v_{k+1}^{i\top} P_2 X_{22} P_2^\top v_{k+1}^i r_k^{i\top} \right),
\end{aligned}$$

considered only when $X_{11} \neq 0$. Note that there are many more eigenvalues (counting multiplicity) of $\mathcal{L}_{\Pi_1}$ than those of $\widetilde{\mathcal{L}}_{\Pi_1}$ and it is impossible for all $X_{11} = 0$. The residual $\widetilde{R}_k$ for the eigenvalue problem is hopefully small of $O(\tilde{r})$ and the stability of $\mathcal{L}_{\Pi_1}$ will pass onto $\widetilde{\mathcal{L}}_{\Pi_1}$ if the distance of the spectrum of $\widetilde{\mathcal{L}}_{\Pi_1}$ from the imaginary axis (stability radius) is greater than $O(\tilde{r})$, excluding the original GLE being very ill-conditioned.

A more precise statement on the relationship between the spectra of A and Φ_k^0 can be found in [44, Theorem 2.1] and a similar result for the spectra $\mathcal{L}_{\Pi_1}$ and $\widetilde{\mathcal{L}}_{\Pi_1}$ may be obtained using perturbation techniques. However, the above qualitative argument is adequate for our purpose. Inheritance of solvability can also be investigated using the perturbation approach in [44, Section 2.2.6].

5. Conclusions

For RREs of moderate sizes, only a few solution methods are available. Apart from the homotopy method, Newton-type methods are difficult to initialize.

Large-scale problems are difficult, if at all possible, to solve. The first paper of its kind, we propose a projection method based on a generalized Krylov subspace for large-scale RREs and the associated linear equations. The small projected equations will be efficient to solve, possibly using the homotopy method. We have also extended the inheritance properties of some solvability conditions of projection methods, or the solvability of the original equation is inherited by the projected one, when the Arnoldi residuals are relatively small. We have only presented some theoretical results in this paper and illustrative numerical examples for the projection method will follow in later publications. A comprehensive comparison of different Krylov subspaces is also left for the future.

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References

[1] H. Abou-Kandil, G. Freiling, V. Ionescu and G. Jank, *Matrix Riccati Equations in Control and Systems Theory.* Birkhäuser Verlag, Basel, 2003. MR1997753

[2] P. Benner and T. Breiten, Low rank methods for a class of generalized Lyapunov equations and related issues. *Numer. Math.* **124** (2013), 441–470. MR3066036

[3] P. Benner and T. Damm, Lyapunov equations, energy functionals, and model order reduction of bilinear and stochastic systems. *SIAM J. Contr. Optim.* **49** (2011), 686–711. MR2801215

[4] P. Benner and H. Fassbender, On the numerical solution of large-scale sparse discrete-time Riccati equations. *Adv. Comput. Maths.* **35** (2011), 119–147. MR2827083

[5] P. Benner and J. Saak, A semi-discretized heat transfer model for optimal cooling of steel profiles. In *Dimension Reduction of Large-Scale Systems*, ed. P. Benner, V. Mehrmann and D. C. Sorensen, Lecture Notes in Computational Science and Engineering, Springer-Verlag, Berlin/Heidelberg, Germany; **45** (2005), 353–356. MR2516498

[6] P. Benner and J. Saak, *A Newton-Galerkin-ADI Method for Large-Scale Algebraic Riccati Equations.* Applied Linear Algebra 2010, GAMM Workshop Applied and Numerical Linear Algebra, Novi Sad, May 27, 2010 (http://ala2010.pmf.uns.ac.rs/presentations/4g1220pb.pdf).

[7] P. Benner and J. Saak, *A Newton-Galerkin-ADI Method for Large-Scale Algebraic Riccati Equations.* Applied Linear Algebra 2010, GAMM Workshop Applied and Numerical Linear Algebra, Novi Sad, May 27, 2010; available at ala2010.pmf.uns.ac.rs/presentations/4g1220pb.pdf.

[8] P. Benner and J. Saak, Numerical solution of large and sparse continuous time algebraic matrix Riccati and Lyapunov equations: a state of the art survey. GAMM-Reports, **6**(1) (2013), 32–52. MR3095913

[9] E. K.-W. Chu, H.-Y. Fan and W.-W. Lin, A structure-preserving doubling algorithm for continuous-time algebraic Riccati equations. *Linear Alg. Appl.* **396** (2005), 55–80. MR2112199

[10] E. K.-W. Chu, H.-Y. Fan, W.-W. Lin and C.-S. Wang, A structure-preserving doubling algorithm for periodic discrete-time algebraic Riccati equations. *Int. J. Control* **77** (2004), 767–788. MR2072208

[11] E. K.-W. Chu and P. C.-Y. Weng, Large-scale discrete-time algebraic Riccati equations — doubling algorithm and error analysis. *J. Comp. Appl. Maths.* **277** (2015), 115–126. MR3272169

[12] T. Damm, *Rational Matrix Equations in Stochastic Control.* Lecture Notes in Control and Information Sciences, Vol. **297**, Springer Verlag, Berlin, 2004. MR2038790

[13] T. Damm, Direct methods and ADI-preconditioned Krylov subspace methods for generalized Lyapunov equations. *Numer. Linear Algebra Applic.* **15** (2008), 853–871. MR2464173

[14] T. Damm and D. Hinrichsen, Newton's method for a rational matrix equation occurring in stochastic control. *Linear Algebra Applic.* **332–334** (2001), 81–109. MR1839427

[15] V. Dragan, T. Morozan and A.-M. Stoica, *Mathematical Methods in Robust Control of Discrete-Time Linear Stochastic Systems.* Springer, New York, 2010. MR2573211

[16] H.-Y. Fan, P. C.-Y. Weng and E. K.-W. Chu, Smith method for generalized Lyapunov, Stein and rational Riccati equations in stochastic control. *Numer. Algor.* **71** (2016), 245–272. MR3452930

[17] H.-Y. Fan, L. Zhang and E. K.-W. Chu, Projection methods for large-scale Riccati differential equations. *Tech. Report*, NCTS, National Taiwan University, 2017.

[18] G. Freiling, A survey of nonsymmetric Riccati equations. *Linear Algebra Applic.* **351–352** (2002), 243–270. MR1917481

[19] G. Freiling and A. Hochhaus, Properties of the solutions of rational matrix difference equations. *Comput. Math. Applic.* **45** (2003), 1137–1154. MR2000584

[20] G. Freiling and A. Hochhaus, On a class of rational matrix differential equations arising in stochastic control. *Linear Algebra Applic.* **379** (2004), 43–68. MR2039296

[21] C.-H. Guo, Iterative solution of a matrix Riccati equation arising in stochastic control. *Oper. Theory Adv. Appl.* **130** (2001), 209–221. MR1902010

[22] G. H. Golub and C. F. Van Loan, *Matrix Computations*, 3rd ed. Johns Hopkins University Press, Baltimore, MD, 1996. MR1417720

[23] M. Heyouni and K. Jbilou, An extended block Arnoldi algorithm for large-scale solutions of the continuous-time algebraic Riccati equations. *Elect. Trans. Numer. Anal.* **33** (2009), 53–62. MR2590127

[24] I. G. Ivanov, Iterations for solving a rational Riccati equations arising in stochastic control. *Comput. Math. Applic.* **53** (2007), 977–988. MR2333341

[25] I. G. Ivanov, Properties of Stein (Lyapunov) iterations for solving a general Riccati equation. *Nonlinear Anal.* **67** (2007), 1155–1166. MR2325369

[26] I. M. Jaimoukha and E. M. Kasenally, Krylov subspace methods for solving large Lyapunov equations. *SIAM J. Numer. Anal.* **31** (1994), 227–251. MR1259974

[27] K. Jbilou, Block Krylov subspace methods for large continuous-time algebraic Riccati equations. *Numer. Algorithms* **34** (2003), 339–353. MR2043907

[28] K. Jbilou, An Arnoldi based algorithm for large algebraic Riccati equations. *Appl. Math. Lett.* **19** (2006), 437–444. MR2213145

[29] K. Jbilou, Low rank approximate solutions to large Sylvester matrix equations. *Appl. Math. Comput.* **177** (2006), 365–376. MR2234524

[30] P. Lancaster and L. Rodman, *Algebraic Riccati Equations*. Clarendon Press, Oxford, 1995. MR1367089

[31] T. Li, E. K.-W. Chu, W.-W. Lin and P. C.-Y. Weng, Solving large-scale continuous-time algebraic Riccati equations by doubling. *J. Comput. Appl. Math.* **237** (2013), 373–383. MR2966913

[32] T. Li, P. C.-Y. Weng, E. K.-W. Chu and W.-W. Lin, Large-scale Stein and Lyapunov equations, Smith method, and applications. *Numer. Algor.* **63** (2013), 727–752. MR3077971

[33] V. L. Mehrmann, *The Autonomous Linear Quadratic Control Problem*. Lecture Notes in Control and Information Sciences, Vol. **163**, Springer Verlag, Berlin, 1991. MR1133203

[34] J. Saak, *Efficient Numerical Solution of Large Scale Algebraic Matrix Equations in PDE Control and Model Order Reduction*. Dr. rer. nat. Dissertation, Chemnitz University of Technology, Germany, 2009.

[35] J. Saak, H. Mena and P. Benner, *Matrix Equation Sparse Solvers (MESS): A MATLAB Toolbox for the Solution of Sparse Large-Scale Matrix Equations*. Chemnitz University of Technology, Germany, 2010.

[36] H. Schneider, Positive operators and an inertia theorem. *Numer. Math.* **7** (1965), 11–17. MR0173678

[37] V. Simoncini, A new iterative method for solving large-scale Lyapunov matrix equations. *SIAM J. Sci. Comput.* **29** (2007), 1268–1288. MR2318706

[38] V. Simoncini, Computational methods for linear matrix equations. *Tech. Report*, January 2014; available from http://www.dm.unibo.it/~simoncin/public_matrixeq_rev.pdf.

[39] V. Simoncini, Analysis of the rational Krylov subspace projection method for large-scale algebraic Riccati equations. *SIAM J. Matrix Anal. Applic.* **37** (2016), 1655–1674 MR3570279

[40] V. Simoncini, D. B. Szyld and M. Monsalve, On two numerical methods for the solution of large-scale algebraic Riccati equations. *IMA J. Numer. Anal.* **34** (2014), 904–920. MR3232439

[41] W. M. Wonham, On a matrix Riccati equation of stochastic control. *SIAM J. Control* **6** (1968), 681–797. MR0239161

[42] W. M. Wonham, Erratum: On a matrix Riccati equation of stochastic control. *SIAM J. Control* **7** (1969), 365. MR0249167

[43] L. Zhang, H.-Y. Fan, E. K.-W. Chu and Y. Wei, Homotopy for rational Riccati equations arising in stochastic control. *SIAM J. Sci. Comp.* **37** (2015), B103–B125. MR3313834

[44] L. Zhang, H.-Y. Fan and E. K.-W. Chu, Inheritance properties of projection methods for continuous-time algebraic Riccati equations. *Tech. Report*, NCTS, National Taiwan University, 2017.

[45] L. Zhang, H.-Y. Fan and E. K.-W. Chu, Inheritance properties of projection methods for discrete-time algebraic Riccati equations. *Tech. Report*, NCTS, National Taiwan University, 2017.

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