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The Marr conjecture and uniqueness of wavelet transforms

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The inverse question of identifying a function from the nodes (zeroes) of its wavelet transform arises in a number of fields. These include whether the nodes of a heat or hypoelliptic equation solution determine its initial conditions, and in mathematical vision theory the Marr conjecture, on whether an image is mathematically determined by its edge information. We prove a general version of this conjecture by reducing it to the moment problem, using a basis dual to the monomial basis x^α on $\mathbb{R}^n$.

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1. Introduction

The inverse problem of determining a function f from the nodes (zeroes) of its wavelet transform has a number of applications. In partial differential equations this becomes the question of recovering the solution of a heat or hypoelliptic equation from its nodes. In mathematical vision theory it is a generalization of the problem known as the Marr conjecture, about the unique determination of a function from its multiscale edges. Here we give conditions on the wavelet and the function f for its recovery, and show that these conditions are the best of their kind.

There has been both theoretical [27, 17, 35, 7, 15, 29, 2] and empirical [24] evidence related to the Marr conjecture, regarding both its range of validity and some restrictions on it. As shown by Meyer originally [27, Ch. 8], the truth of the conjecture has limitations, and it is in general false for non-decaying f.

It is shown here that for compactly supported or exponentially decaying f, the conjecture holds in a general form; however, it is false for algebraically decaying f.

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The Marr conjecture was originally motivated by the fact that visual images are in practice often easy to reconstruct from their edges. To this extent these results are a mathematical formalization of this fact. In one dimension we apply our results to the Richter (Mexican hat) wavelet, which was the original convolving function studied by Marr [26, 25].

Our methods reduce the recovery of f to the moment problem, using the duality of two bases for functions on $\mathbb{R}^n$, the Taylor monomials $\mathbf{x}^\alpha$ and the derivatives $\delta^{(\alpha)}$ of the delta distribution at $\mathbf{0}$. The method of moments provides a natural approach to the problem, since the effects of different moments become asymptotically separated under the wavelet transform.

1.1. Background

The standard d-dimensional continuous wavelet transform of f with a smooth wavelet $\tilde{\psi}$ has the form

$$Wf(\sigma, \mathbf{x}) = \sigma^{d/2} \int_{\mathbb{R}^d} f(\mathbf{t})\, \tilde{\psi}\left(\frac{\mathbf{t}-\mathbf{x}}{\sigma}\right) d\mathbf{t} = \sigma^{d/2} f * \psi_\sigma(\mathbf{x}),$$

where for convenience we define $\psi(\mathbf{x}) = \tilde{\psi}(-\mathbf{x})$, and $\psi_\sigma(\mathbf{x}) = \sigma^{-d}\psi(\mathbf{x}/\sigma)$ is a rescaling (and normalization) of ψ by σ. We ask under what conditions a locally integrable function f is uniquely determined (up to a constant multiple) by the nodes of its wavelet transform. It is in fact possible to answer a stronger version of this question, namely whether f can be recovered from knowledge of the nodes of $Wf(\sigma, \mathbf{x})$ only at an (arbitrary) discrete sequence of scales $\{\sigma_i\}_{i\geq 0}$.

There are several versions of this question:

- In wavelet theory this is an inverse problem for the continuous wavelet transform [24, 23, 27, 17], and the dyadic transform [24, 23] (which is continuous in the space variable $\mathbf{x}$ but discrete in the scaling variable σ).
- In mathematical vision theory [25] f represents an image. Convolutions of f with rescalings of $\psi(\mathbf{x}) = G(\mathbf{x}) = (2\pi)^{-d/2}e^{-|\mathbf{x}|^2/2}$ represent Gaussian kernel smoothings (blurrings) of the image at different scales. Defining the Ricker (Mexican hat) wavelet as $M(\mathbf{x}) = \Delta G(\mathbf{x})$, and its rescaling by $\sigma > 0$ as $M_\sigma(\mathbf{x}) = \sigma^{-d}M(x/\sigma)$, it follows that the zeros of $f * M_\sigma(\mathbf{x})$ represent points of maximal change in the smoothed image, which can be interpreted as edges (generalized discontinuities) of f at scale σ. Thus the nodes of $f * M_\sigma(\mathbf{x})$ as σ increases form successively sparser "line sketches" of the image f. The

unique determination question (the Marr conjecture) asks whether these nodes (edges) form a complete representation of the image. The traditional focus on this question in mathematical vision theory has been based on the widespread use of edge perception as a model for vision.

- For hypoelliptic partial differential equations, scaled smoothing functions often arise as fundamental solutions (Green's functions). For example, the Gaussian function $u(\mathbf{x},t) = (2\pi t)^{-d/2}e^{-|\mathbf{x}|^2/2t}$ is the fundamental solution of the heat equation $u_t = \frac{1}{2}\Delta u$, and the solution to an initial value problem is obtained by convolution of the initial condition with the fundamental solution. The question is then whether the nodes of a solution uniquely determine it.

In wavelet theory this question has been studied theoretically and numerically by Meyer [27, 17] and Mallat [24, 23], and the mathematical question in vision theory has also received a good deal of attention [26, 25, 35, 7, 15, 29, 2]. Although the problem of determining nodes of parabolic equations and their properties has been studied in a number of settings [1, 21, 33], the inverse problem of determining a solution from its nodes has received less attention.

The Marr conjecture in vision theory [26, 25] is motivated by problems of edge detection and image reconstruction in biological and artificial neural systems. In this setting it is natural to restrict to functions f that are compactly supported, or more generally, satisfy some decay condition. The conjecture can be stated as

Marr Conjecture. *A locally integrable function f of sufficiently rapid decay is uniquely determined (up to a constant multiple) by the zero sets of $f * M_{\sigma_j}$ for any sequence of positive scales $\{\sigma_j\}_{j=1}^{\infty}$ tending to infinity.*

This conjecture has remained open, although special cases have been proved [35, 7]. The corresponding statement for nondecaying functions was disproved by Meyer [27], who found distinct periodic functions whose Ricker wavelet transforms have identical zero sets at all scales.

More generally, we can ask for minimal conditions on a general wavelet ψ allowing for such unique determination:

Question. *What conditions on a twice-differentiable function ψ are necessary and sufficient to imply that any function f, of sufficiently rapid decay, is uniquely determined (up to a constant multiple) by (a) the zeros in $(\sigma, \mathbf{x})$ of $Wf(\sigma, \mathbf{x})$ (b) the zero sets of $f * \psi_{\sigma_j}$, for any sequence of positive scales $\{\sigma_j\}_{j=1}^{\infty}$ tending to infinity.*

1.2. Results on unique determination

Here we answer this question by finding conditions on f and ψ that are sufficient and the best of their type for such unique determination. We require that f be integrable and of negative exponential order—meaning that f belongs to a class $\mathcal{P}'_\gamma$ of exponentially decaying functions. We require that ψ belong to a class $\mathcal{P}$ of smooth functions whose derivatives grow slower than exponentially, and satisfy the following:

Genericity Condition. *The regular zero set of any derivative of fixed order n is not contained in the zero set of any other derivative of fixed order m, for any $n, m \geq 0$.*

A regular (transverse) zero of a function ψ is a point in all of whose neighborhoods $\psi(\mathbf{x})$ takes both positive and negative values. By "derivative of fixed order m", we mean a linear combination of partial derivatives of ψ of order m, (i.e., a homogeneous linear differential operator of order m applied to ψ), modulo multiplication by a nonzero constant. As an example, the one-dimensional Gaussian wavelet $G(x)$ fails this genericity condition, in that the regular zero set of G is empty and is therefore trivially contained in the zero set of $G^{(n)}$ for any $n > 0$. However its second derivative, the Ricker wavelet $M(x)$, satisfies this condition, as we will show in Section 3.3.

Our main result can be stated as follows:

Theorem 1. *Given $\psi \in \mathcal{P}$ satisfying the above genericity condition, any function $f \in \mathcal{P}'_\gamma \cap L^1(\mathbb{R}^d)$ is uniquely determined (up to a constant multiple) by the zero sets of its wavelet transform $f * \psi_{\sigma_j}$ at any sequence of positive scales $\{\sigma_j\}_{j=1}^\infty$ tending to infinity.*

We will show that the conditions in this theorem are the best of their kind, in the following sense. First, the theorem fails if the exponential decay condition $f \in \mathcal{P}_\gamma$ is weakened to algebraic decay (see Section 7), although this leaves the conjecture open for the restricted set of functions f with decay that is between algebraic and exponential, e.g. $f(x) = e^{-|x|^{1/2}}$ Second, if the genericity condition on the regular zeroes of the wavelet ψ (see above) fails weakly, then the theorem fails to hold (see Section 3.1).

Corollary 2. *Given ψ and f as above, f is uniquely determined by the zero sets of its continuous wavelet transform $f * \psi_\sigma$ for $\sigma > 0$, and more generally its dyadic wavelet transform $f * \psi_{\sigma_j}$ with $\sigma_j = 2^j, j \in \mathbb{N}$.*

In the case of the Ricker (Gaussian derivative) wavelet $M(x) = G''(x)$, we prove the following:

Corollary 3. (Marr conjecture in one dimension)

(a) *Any $f \in \mathcal{P}'_\gamma \cap L^1(\mathbb{R})$ is uniquely determined (up to a constant multiple) by the zero sets of $f * M_{\sigma_j}$ at any sequence of positive scales $\{\sigma_j\}_{j=1}^\infty$ with a positive or infinite limit point.*

(b) *This unique determination can fail if the only limit point of $\{\sigma_j\}_{j=1}^\infty$ is zero.*

(c) *This unique determination also can fail if f is of (negative) algebraic rather than exponential order (i.e. decays algebraically rather than exponentially).*

Corollary 3 is proved using properties of the Hermite polynomials $H_n(x)$, which are defined by the relation

$$G^{(n)}(x) = (-1)^n H_n(x) G(x). \tag{1.1}$$

For dimensions $d > 1$, Theorem 1 reduces the Marr conjecture to a statement about polynomial zeros. For any multiindex of nonnegative integers $\alpha = (\alpha_1, \ldots, \alpha_d)$, we define the *Laplace-Hermite polynomial* $L_\alpha(\mathbf{x})$ in $\mathbf{x} = (x_1, \ldots, x_d)$ by

$$\Delta G^{(\alpha)}(\mathbf{x}) = (-1)^{|\alpha|} L_\alpha(\mathbf{x}) G(\mathbf{x}). \tag{1.2}$$

Above, the superscript (α) indicates a mixed partial derivative in the orders specified by α. Note that L_α is a polynomial of degree $|\alpha| + 2$, where $|\alpha| = \alpha_1 + \ldots + \alpha_d$. We thus have:

Corollary 4. (Marr conjecture in d dimensions) *If there is no pair of distinct Laplace-Hermite polynomials of degree greater than zero such that the zero set of one contains the regular zero set of the other, then any $f \in \mathcal{P}'_\gamma \cap L^1(\mathbb{R})$ is uniquely determined, up to a constant multiple, by the zero sets of $f * M_{\sigma_j}$ for any sequence of positive scales $\{\sigma_j\}_{j=1}^\infty$ tending to infinity.*

Thus in any dimension the Marr conjecture is equivalent to a condition on the zeros of Laplace-Hermite polynomials.

1.3. Results on asymptotic moment expansions

Our approach is based on moment expansions, which rely on the duality of the basis of Taylor monomials $\mathbf{x}^\alpha = x_1^{\alpha_1} \ldots x_d^{\alpha_d}$ in $\mathbb{R}^d$, with distributions $\delta^{(\alpha)}$ localized at the origin. Here $\delta^{(\alpha)}$ denotes a distributional partial derivative of the Dirac distribution δ in the orders specified by the multiindex α. The moment expansion represents a function as a series in $\delta^{(\alpha)}$, with coefficients in terms of the function's moments $\mu_\alpha = \langle f(\mathbf{x}), \mathbf{x}^\alpha \rangle$. Specifically, in a distributional sense (see Section 2.1), it is an expansion dual to a Taylor series, in the form $f \sim \sum_\alpha c_\alpha \delta^{(\alpha)}$, with $c_\alpha = (-1)^{|\alpha|}\mu_\alpha/\alpha!$. This expansion converges asymptotically in an appropriate distribution space, as we describe in Section 2.

Moment expansions have been used to study electromagnetism (in multipole expansions), gravitation, and acoustics. They have more recently also been applied to the Navier-Stokes [10, 28] and other differential equations [8, 18, 22, 11, 34].

We extend the theory of asymptotic moment expansions in two ways. First, we prove the following continuity result for convolutions of moment expansions, in terms of the rescaled variable $\mathbf{w} = \mathbf{x}/\sigma$:

Theorem 5. *If f is replaced by its asymptotic moment expansion in the convolution $f * \psi_\sigma(\sigma\mathbf{w})$, the asymptotic convergence of the resulting series, as $\sigma \to \infty$, is locally uniform in $\mathbf{w}$.*

Second, we generalize the theory of asymptotic moment expansions to distributions on $\mathbb{R}$ with only finitely many moments:

Theorem 6. *If the first N moments of f are well-defined, then f has an asymptotic moment expansion to order $N-1$. If f is replaced by this moment expansion in the convolution $f * \psi_\sigma(\sigma w)$, the asymptotic convergence of the resulting series, as $\sigma \to \infty$, is locally uniform in w.*

The phrase "asymptotic converge" is defined in the formal statements of these theorems, which appear in Sections 2.2 and 6.3 respectively.

1.4. Results on the geometry of heat equation nodes

This work leads to some new results on the nodes of solutions to the heat equation initial value problem:

$$\begin{cases} F_t = \frac{1}{2}F_{xx} & x \in (-\infty,\infty),\, t \in [0,\infty) \\ F(x,0) = f(x). \end{cases} \tag{1.3}$$

The nodes (zeros) of F form algebraic curves which we call *zero contours* of f. We show that new zero contours do not appear as t increases, strengthening and complementing previous results [1, 35, 2, 15, 21, 33]:

Theorem 7. *For any $f \in \mathcal{P}'_\gamma \cap L^1(\mathbb{R})$ and positive numbers $t_1 < t_2$, the zero contours of f intersecting the line $t = t_2$ are a subset of those that intersect $t = t_1$.*

We also obtain the following unique determination result:

Theorem 8. *Let F be a solution to (1.3) for some initial condition $f \in L^1(\mathbb{R})$. If it is known that the second integral*

$$a(x) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(z)\, dz\, dy$$

is a function of negative exponential order, then f is uniquely determined by the zeros of $F(x, t_j)$ for any sequence $\{t_j\}_{j=1}^{\infty}$ of positive real numbers with a positive or infinite limit point.

1.5. Results for discrete zero-crossings

In some applications to discrete images or signals, knowledge of exact zero sets of the wavelet transform is replaced by information only about discrete zero-crossings, i.e., pairs of adjacent lattice points between which the wavelet transform changes sign. One can ask whether such discrete zero-crossing information suffices to uniquely determine a discretized function f, which we represent as a finite sum of δ-distributions localized at integer lattice points: $f(\mathbf{x}) = \sum_{i=1}^{n} a_i \delta(\mathbf{x} - \mathbf{x}_i)$ where $a_i \in \mathbb{R}$ and $\mathbf{x}_i \in \mathbb{Z}^d$. We show that unique determination fails in this setting, even in one dimension:

Theorem 9. *There exist distributions $f(x) = \sum_{i=1}^{n} a_i \delta(x - x_i)$ and $g(x) = \sum_{i=1}^{m} b_i \delta(x - y_i)$ with $a_i, b_i \in \mathbb{R}$ and $x_i, y_i \in \mathbb{Z}$, not constant multiples of each other, such that the discrete zero-crossings (i.e. pairs of consecutive integers between which the wavelet transform changes sign) of f and g, with respect to the Ricker wavelet $M(x)$, coincide at a sequence of scales $\{\sigma_j\}_{j=1}^{\infty}$ tending to infinity.*

1.6. Outline

Section 2 derives key results on asymptotic moment expansions which are used throughout the paper. The main result (Theorem 1) and its application to the Ricker wavelet (Corollary 3(ab)) are proven in Section 3. Section 4 obtains results on the geometry of edge contours (nodal sets) in the case of the Ricker wavelet, which are used in the following sections. In Section 5 we prove that, for the one-dimensional Ricker wavelet, unique determination holds for sequences of scales with a (finite) positive limit point, but not for sequences whose only limit point is zero. Section 6 extends the theory of asymptotic moment expansions to distributions that have only finitely many moments. In Section 7 we prove Corollary 3(c), showing that the requirement of exponential decay in Theorem 1 cannot be weakened to algebraic decay. Section 8 considers the question of discrete zero-crossings and proves Theorem 9. Finally, in Section 9 we obtain the unique determination result, Theorem 8, for the heat equation.

2. Moment expansion

Moment expansions represent functions (and more generally, distributions) as series in derivatives

$$\delta^{(\alpha)}(\mathbf{x}) \equiv \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \ldots \partial x_d^{\alpha_d}} \delta(\mathbf{x}),$$

of the Dirac δ distribution. These are based on the fact that these derivatives and the monomials

$$\mathbf{x}^\alpha \equiv x_1^{\alpha_1} \ldots x_d^{\alpha_d}$$

form a biorthogonal system:

$$\langle \delta^{(\alpha)}, \mathbf{x}^\beta \rangle = \begin{cases} (-1)^{|\alpha|}\alpha! & \alpha = \beta \\ 0 & \text{otherwise}, \end{cases} \tag{2.1}$$

with

$$|\alpha| = \alpha_1 + \ldots + \alpha_d, \qquad \alpha! = \alpha_1! \ldots \alpha_d!.$$

In principle, the moment expansion of a function or distribution f is the series

$$f(\mathbf{x}) = \sum_{|\alpha| \geq 0} \frac{(-1)^{|\alpha|}}{\alpha!} \mu_\alpha \, \delta^{(\alpha)}(\mathbf{x}), \tag{2.2}$$

where μ_α is the αth moment of f:

$$\mu_\alpha = \langle f(\mathbf{x}), \mathbf{x}^\alpha \rangle.$$

We observe that, by the biorthogonality relation (2.1), the two sides of (2.2) agree when applied to any polynomial function $f(\mathbf{x})$. However, the asymptotic convergence of the sum in (2.2) requires an appropriate choice of distribution spaces.

In this section we first review the theory of asymptotic moment expansions. We then prove Theorem 5 regarding the local uniform convergence of asymptotic moment expansions applied to convolutions.

2.1. Asymptotic moment expansions

We begin by defining the relevant spaces of test functions and distributions. For $\gamma > 0$, let $\mathcal{P}_\gamma = \mathcal{P}_\gamma(\mathbb{R}^d)$ be the space of smooth functions ψ on $\mathbb{R}^d$ with derivatives asymptotically bounded by $e^{\gamma|\mathbf{x}|}$, so that

$$\lim_{|\mathbf{x}|\to\infty} e^{-\gamma|\mathbf{x}|}\,\psi^{(\alpha)}(\mathbf{x}) = 0,$$

for each α. The topology on $\mathcal{P}_\gamma$ is generated by the seminorms

$$||\psi||_{\gamma,\alpha} = \sup_{\mathbf{x}\in\mathbb{R}^d} \left| e^{-\gamma|\mathbf{x}|}\,\psi^{(\alpha)}(\mathbf{x}) \right|,$$

varying over multiindices α, with γ fixed. Define the space $\mathcal{P} = \mathcal{P}(\mathbb{R}^d)$ by

$$\mathcal{P} = \bigcap_{\gamma>0} \mathcal{P}_\gamma,$$

with topology generated by the seminorms $||\ ||_{\gamma,\alpha}$ as γ and α both vary. $\mathcal{P}$ is the space of smooth functions with slower-than-exponential growth. The dual spaces to $\mathcal{P}_\gamma$ and $\mathcal{P}$ are denoted by $\mathcal{P}'_\gamma$ and $\mathcal{P}'$, respectively. Informally, distributions in $\mathcal{P}'_\gamma$ decay as $e^{-\gamma|\mathbf{x}|}$ or faster, and while those in $\mathcal{P}'$ have exponential or faster decay. Clearly $\mathcal{P}'_\gamma \subset \mathcal{P}'$ for each $\gamma > 0$.

The asymptotic moment expansion of a distribution $f \in \mathcal{P}'$ is essentially the dual of a Taylor expansion, given by [9, Theorem 4.3.1]

$$f(\sigma\mathbf{x}) \sim \sum_{|\alpha|\geq 0} \frac{(-1)^{|\alpha|}}{\alpha!}\mu_\alpha\,\sigma^{-|\alpha|-d}\,\delta^{(\alpha)}(\mathbf{x}) \qquad (\sigma\to\infty),$$

where

$$\mu_\alpha = \langle f(\mathbf{x}), \mathbf{x}^\alpha\rangle$$

is the αth moment of f. This expansion holds in that for any $\psi \in \mathcal{P}$ and $N \geq 0$,

(2.3)
$$\langle f(\sigma\mathbf{x}), \psi(\mathbf{x})\rangle = \sum_{0\leq|\alpha|\leq N} \frac{\mu_\alpha}{\alpha!}\,\sigma^{-|\alpha|-d}\,\psi^{(\alpha)}(\mathbf{0}) + \mathcal{O}(\sigma^{-N-d-1}) \qquad (\sigma\to\infty),$$

where by definition $\overline{\lim}_{\sigma\to\infty}\mathcal{O}(L(\sigma))/L(\sigma)<\infty$. The above asymptotic expansion is equivalent to the following equation for all $N\geq 0$:

$$\overline{\lim_{\sigma\to\infty}}\,\sigma^{N+d}\left|\langle f(\sigma\mathbf{x}),\psi(\mathbf{x})\rangle-\sum_{0\leq|\alpha|\leq N}\frac{\mu_\alpha}{\alpha!}\,\sigma^{-|\alpha|-d}\,\psi^{(\alpha)}(\mathbf{0})\right|=0.$$

Note that for polynomial ψ of degree $\leq N$, the two sides of (2.3) coincide (without the error term) according to the biorthogonality relation (2.1). The moment expansion (2.3) for general $\psi\in\mathcal{P}$ is an asymptotic version of this biorthogonality relation.

2.2. Local uniform convergence of convolved moment expansions

Here we prove the continuity result, Theorem 5 from Section 1.3, which we state here in a more detailed form:

Theorem 5. *For all $f\in\mathcal{P}',\psi\in\mathcal{P}$, and $N\geq 0$, the σ-indexed family of functions*

$$\mathbf{w}\longmapsto\sigma^{N+d}\left((f*\psi_\sigma)(\sigma\mathbf{w})-\sum_{0\leq|\alpha|\leq N}\frac{(-1)^{|\alpha|}}{\alpha!}\mu_\alpha\,\sigma^{-|\alpha|-d}\,\psi^{(\alpha)}(\mathbf{w})\right)$$

converges locally uniformly (in $\mathbf{w}$) to the zero function of $\mathbf{w}$ as $\sigma\to\infty$.

Above, "converges locally uniformly" is shorthand for "converges uniformly on compact subsets". We remark that in the special case $\psi(x)=G(x)$ (the standard Gaussian in one dimension), letting $\sigma=t^2$, the above expression $(f*\psi_\sigma)(\sigma w)$ is just the solution of the heat equation with initial condition f, at time t, with the spatial variable x rescaled by a factor $\sqrt{t}$. In this case the expansion is interpretable precisely as an asymptotic expansion for a heat equation solution whose leading term is the Gaussian ψ, and whose higher terms are derivatives $\psi^{(\alpha)}$ of the Gaussian.

The proof of Theorem 5 is based on that of the asymptotic moment expansion in Theorem 4.3.1 of [9]. We begin with the following lemma.

Lemma 10. *Let $\rho=\rho(\mathbf{w},\mathbf{y})\in\mathcal{P}(\mathbb{R}^{2d})$, and for each fixed $\mathbf{w}\in\mathbb{R}^d$ define $\rho_{\mathbf{w}}(\mathbf{y})=\rho(\mathbf{w},\mathbf{y})$ (hence $\rho_{\mathbf{w}}\in\mathcal{P}(\mathbb{R}^d)$.) Suppose that for some integer $N\geq 0$, ρ satisfies*

$$\rho_{\mathbf{w}}^{(\alpha)}(0)\equiv\frac{\partial^{|\alpha|}\rho}{\partial\mathbf{y}^\alpha}(\mathbf{w},\mathbf{y})\bigg|_{\mathbf{y}=\mathbf{0}}=0$$

for all $\mathbf{w}$ *and each multiindex* α *with* $|\alpha| \le N$*. Then for any continuous seminorm* $|| \quad ||$ *on* $\mathcal{P}(\mathbb{R}^d)$*, the following* σ*-indexed family of functions of* $\mathbf{w}$*,*

$$\mathbf{w} \longmapsto \sigma^N ||\rho_{\mathbf{w}}(\cdot/\sigma)||,$$

converges locally uniformly (in $\mathbf{w}$*) to the zero function (of* $\mathbf{w}$*) as* $\sigma \to \infty$*.*

We use the symbol $\cdot$ to denote function or distribution arguments for the purposes the bracket operation $\langle\, , \rangle$ or seminorms. Here, the notation $\rho_{\mathbf{w}}(\cdot/\sigma)$ represents the function mapping $\mathbf{y} \in \mathbb{R}^d$ to $\rho_{\mathbf{w}}(\mathbf{y}/\sigma)$.

Proof. We prove the stronger statement that the family of functions

$$\mathbf{w} \longmapsto \overline{\lim_{\sigma\to\infty}}\, \sigma^{N+1} ||\rho_{\mathbf{w}}(\cdot/\sigma)||$$

is locally uniformly bounded in $\mathbf{w}$, where $\overline{\lim}$ denotes limit superior. Consider first the seminorm $|| \quad ||_{\gamma,\mathbf{0}}$ for fixed $\gamma > 0$, and suppose the lemma is false for this seminorm. Then there must be a compact neighborhood $K \subset \mathbb{R}^d$ and a pair of sequences $\{\mathbf{w}_j \in K\}_{j\ge 0}$, $\{\sigma_j \in \mathbb{R}\}_{j\ge 0}$, with $\sigma_j \to \infty$, such that

$$\infty = \lim_{j\to\infty} \sigma_j^{N+1} \left|\left|\rho_{\mathbf{w}_j}(\cdot/\sigma_j)\right|\right|_{\gamma,\mathbf{0}} = \lim_{j\to\infty} \sigma_j^{N+1} \sup_{\mathbf{y}\in\mathbb{R}^d} \left| e^{-\gamma|\mathbf{y}|} \rho_{\mathbf{w}_j}(\mathbf{y}/\sigma_j)\right|. \tag{2.4}$$

By passing to a subsequence if necessary, we may assume $\{\mathbf{w}_j\}$ converges to some $\mathbf{w}' \in K$.

Since

$$\lim_{|\mathbf{y}|\to\infty} \left| e^{-\gamma|\mathbf{y}|} \rho_{\mathbf{w}_j}(\mathbf{y}/\sigma_j)\right| = 0$$

for each j, the supremum on the right-hand side of (2.4) is realized at some $\mathbf{y}_j \in \mathbb{R}^d$. Hence

$$\begin{aligned} \infty &= \lim_{j\to\infty} \sigma_j^{N+1} \left|\left|\rho_{\mathbf{w}_j}(\mathbf{y}/\sigma_j)\right|\right|_{\gamma,\mathbf{0}} \\ &= \lim_{j\to\infty} \sigma_j^{N+1} \left| e^{-\gamma|\mathbf{y}_j|} \rho_{\mathbf{w}_j}(\mathbf{y}_j/\sigma_j)\right| \\ &= \lim_{j\to\infty} \left(|\mathbf{y}_j|^{N+1} e^{-\gamma|\mathbf{y}_j|(1-\sigma_j^{-1})}\right) \left(e^{-\gamma|\mathbf{y}_j/\sigma_j|} |\mathbf{y}_j/\sigma_j|^{-N-1} |\rho_{\mathbf{w}_j}(\mathbf{y}_j/\sigma_j)|\right). \end{aligned}$$

The expression $|\mathbf{y}_j|^{N+1} e^{-\gamma|\mathbf{y}_j|(1-\sigma_j^{-1})}$ is bounded in j since $\sigma_j \to \infty$, so we must have

$$\lim_{j\to\infty} e^{-\gamma|\mathbf{y}_j/\sigma_j|} |\mathbf{y}_j/\sigma_j|^{-N-1} |\rho_{\mathbf{w}_j}(\mathbf{y}_j/\sigma_j)| = \infty. \tag{2.5}$$

By passing to a subsequence if necessary, we may assume that the sequence $\{\mathbf{y}_j/\sigma_j\}_{j\geq 0}$ either approaches the origin as a limit or is bounded away from the origin. In the first case, $\lim_{j\to\infty} \mathbf{y}_j/\sigma_j = \mathbf{0}$, the quantity

$$|\mathbf{y}_j/\sigma_j|^{-N-1}\,|\rho_{\mathbf{w}'}(\mathbf{y}_j/\sigma_j)|$$

is bounded by the derivative condition on ρ, and so the quantity

$$|\mathbf{y}_j/\sigma_j|^{-N-1}\,\left|\rho_{\mathbf{w}_j}(\mathbf{y}_j/\sigma_j)\right|$$

appearing in (2.5) is bounded by continuity of the $(N+1)$st derivative of ρ. Therefore

$$e^{-\gamma|\mathbf{y}_j/\sigma_j|}|\mathbf{y}_j/\sigma_j|^{-N-1}|\rho_{\mathbf{w}_j}(\mathbf{y}_j/\sigma_j)|$$

is bounded, contradicting (2.5). In the second case, $\liminf_{j\to\infty} |\mathbf{y}_j/\sigma_j| > 0$, we note that $|\mathbf{w}_j|$ is bounded since K is compact. Therefore, the quantity

$$e^{-\gamma|\mathbf{y}_j/\sigma_j|}|\rho_{\mathbf{w}_j}(\mathbf{y}_j/\sigma_j)| = e^{-\gamma|\mathbf{y}_j/\sigma_j|}|\rho(\mathbf{w}_j, \mathbf{y}_j/\sigma_j)|$$

appearing in (2.5) is less than or equal to

$$Be^{-\gamma\sqrt{|\mathbf{w}_j|^2+|\mathbf{y}_j/\sigma_j|^2}}|\rho(\mathbf{w}_j, \mathbf{y}_j/\sigma_j)|, \tag{2.6}$$

for some $B > 0$. Quantity (2.6) is bounded in j since $\rho(\cdot,\cdot) \in \mathcal{P}(\mathbb{R}^{2d})$. Combining this with the boundedness of $|\mathbf{y}_j/\sigma_j|^{-N-1}$ (for this case) again yields a contradiction of (2.5). The lemma is therefore true for the seminorm $||\quad||_{\gamma,\mathbf{0}}$.

For the seminorm $||\quad||_{\gamma,\alpha}$ with $|\alpha| > 0$ we have

$$||\rho_{\mathbf{w}}(\cdot/\sigma)||_{\gamma,\alpha} = \sigma^{-|\alpha|}||\rho^{(\alpha)}_{\mathbf{w}}(\cdot/\sigma)||_{\gamma,\mathbf{0}},$$

whereupon we may apply the above argument to $\rho^{(\alpha)}_{\mathbf{w}}$ in place of $\rho_{\mathbf{w}}$, yielding the desired result. Since the family of seminorms $||\quad||_{\gamma,\alpha}$ generates the topology on $\mathcal{P}$, the result is true for any continuous seminorm. □

Proof of Theorem 5. Let

$$P_N(\mathbf{w}, \mathbf{y}) = \sum_{0\leq|\alpha|\leq N} \frac{(-1)^{|\alpha|}}{\alpha!}\mathbf{y}^\alpha\,\psi^{(\alpha)}(\mathbf{w})$$

be the Taylor expansion of $\psi(\mathbf{w}-\mathbf{y})$ about $\mathbf{y} = \mathbf{0}$ to order N, and define the remainder function

$$\rho_{N,\mathbf{w}}(\mathbf{y}) = \psi(\mathbf{w}-\mathbf{y}) - P_N(\mathbf{w}, \mathbf{y}).$$

Then

$$\begin{aligned}(f * \psi_\sigma)(\sigma \mathbf{w}) &= \sigma^{-d}\big\langle f(\cdot), \psi((\sigma\mathbf{w} - \cdot)/\sigma)\big\rangle \\ &= \sigma^{-d}\big\langle f(\cdot), \psi(\mathbf{w} - \cdot/\sigma)\big\rangle \\ &= \sigma^{-d}\big\langle f(\cdot), P_N(\mathbf{w}, \cdot/\sigma)\big\rangle + \sigma^{-d}\big\langle f(\cdot), \rho_{N,\mathbf{w}}(\cdot/\sigma)\big\rangle \\ &= \sum_{0\le|\alpha|\le N} \frac{(-1)^{|\alpha|}}{\alpha!}\,\mu_\alpha\,\sigma^{-|\alpha|-d}\,\psi^{(\alpha)}(\mathbf{w}) + \sigma^{-d}\big\langle f(\cdot), \rho_{N,\mathbf{w}}(\cdot/\sigma)\big\rangle.\end{aligned}$$

Rearranging, we obtain

$$\sigma^{N+d}\left((f * \psi_\sigma)(\sigma\mathbf{w}) - \sum_{0\le|\alpha|\le N} \frac{(-1)^{|\alpha|}}{\alpha!}\,\mu_\alpha\,\sigma^{-|\alpha|-d}\,\psi^{(\alpha)}(\mathbf{w})\right) = \sigma^{N}\langle f(\cdot), \rho_{N,\mathbf{w}}(\cdot/\sigma)\rangle. \tag{2.7}$$

To finish, note that the seminorm $||\rho_{N,\mathbf{w}}|| = |\langle f(\cdot), \rho_{N,\mathbf{w}}(\cdot)\rangle|$ is continuous on $\rho_{N,\mathbf{w}} \in \mathcal{P}(\mathbb{R}^d)$ for any $f \in \mathcal{P}'$, and $\rho_N(\mathbf{w}, \mathbf{y}) \equiv \rho_{N,\mathbf{w}}(\mathbf{y})$ satisfies the conditions of Lemma 10. Therefore, the family of functions

$$\mathbf{w} \longmapsto \sigma^N\langle f(\cdot), \rho_{N,\mathbf{w}}(\cdot/\sigma)\rangle$$

converges locally uniformly in $\mathbf{w}$ to the zero function as $\sigma \to \infty$, which together with (2.7) proves the theorem. □

3. Proof of unique determination

3.1. General wavelets

Here we prove our main result, Theorem 1, regarding unique determination of a function from the nodes of its wavelet transform at a discrete set of scales.

Fix $\psi \in \mathcal{P}(\mathbb{R}^d)$, and let $f \in \mathcal{P}'_\gamma \cap L^1(\mathbb{R}^d)$, for some $\gamma > 0$, be a function to be determined. We define the ψ-zeros of f at scale $\sigma > 0$ to be the zeros of $f * \psi_\sigma(\mathbf{x})$, with $\psi_\sigma(\mathbf{x}) = \sigma^{-d}\psi(\mathbf{x}/\sigma)$. We recall the genericity condition from the Introduction:

Genericity Condition. *The regular zero set of any derivative of fixed order n is not contained in the zero set of any other derivative of fixed order m, for any $n, m \ge 0$.*

Above, the *regular* (or *transverse*) zeros of ψ are those around which the function takes both positive and negative values in any open neighborhood.

A *derivative of fixed order* n (or an *order-n derivative*) of ψ is a function of the form $\sum_{|\alpha|=n} C_\alpha \psi^{(\alpha)}$ for some $n \geq 0$, where the C_α are constants not all equal to zero, defined up to multiplication by a nonzero scalar. Note that this genericity condition on non-containment of zeroes includes the case of two derivatives of the same order, i.e., where $m = n$.

Our main result can now be stated as follows:

Theorem 1. *Given $\psi \in \mathcal{P}$ satisfying the above genericity condition, any function $f \in \mathcal{P}'_\gamma \cap L^1(\mathbb{R}^d)$ (for any $\gamma > 0$) is uniquely determined (up to a constant multiple) by its ψ-zeros at any sequence of positive scales $\{\sigma_j\}_{j=1}^\infty$ tending to infinity.*

Proof. For convenience we introduce $\mathbf{w} = \mathbf{x}/\sigma$. Moment expansion (Theorem 5) gives

$$f * \psi_\sigma(\mathbf{x}) \sim \sum_{|\alpha|\geq 0} \frac{(-1)^{|\alpha|}}{\alpha!} \mu_\alpha \, \sigma^{-|\alpha|-d} \, \psi^{(\alpha)}(\mathbf{w}) \qquad (\sigma \to \infty). \tag{3.1}$$

Also for convenience, we introduce the function $Z(\sigma, \mathbf{w}) = \sigma^{n_0+d} \, (f * \psi_\sigma)(\sigma\mathbf{w})$, where n_0 is the order of the lowest-order nonzero moment of f. Z admits the moment expansion

$$Z(\sigma, \mathbf{w}) \sim \sum_{|\alpha|\geq n_0} \frac{(-1)^{|\alpha|}}{\alpha!} \mu_\alpha \, \sigma^{n_0-|\alpha|} \, \psi^{(\alpha)}(\mathbf{w}) \qquad (\sigma \to \infty). \tag{3.2}$$

By locally uniform convergence of the moment expansion (Theorem 5) in the case $N = n_0$, as $\sigma \to \infty$, $Z(\sigma, \mathbf{w})$ converges locally uniformly in $\mathbf{w}$ to

$$z(\mathbf{w}) = (-1)^{n_0} \sum_{|\alpha|=n_0} \frac{\mu_\alpha}{\alpha!} \, \psi^{(\alpha)}(\mathbf{w}). \tag{3.3}$$

The ψ-zeros at scale σ correspond to the zeros, in $\mathbf{w}$, of $Z(\sigma, \mathbf{w})$. By assumption, we are given the zero sets $E_j = \{\mathbf{w} : Z(\sigma_j, \mathbf{w}) = 0\} \subset \mathbb{R}^d$ at scale $\sigma = \sigma_j$ for each $j \geq 0$. We call the limiting set E of $\{E_j\}$ as $j \to \infty$ (i.e. the set of all limits of sequences $\{\mathbf{w}_j \in E_j\}_{j\geq 0}$) the *asymptotic zero set*.

E contains all regular zeros of z since regular zeros persist under small locally uniform perturbations. So if $\mathbf{w}'$ is a regular zero of $z(\mathbf{w})$ we may, from knowledge of $\{E_j\}_{j\geq 0}$, choose a sequence $\{\mathbf{w}_j \in E_j\}_{j\geq 0}$ such that $\lim_{j\to\infty} \mathbf{w}_j = \mathbf{w}'$. By locally uniform convergence (Theorem 5), we may substitute $\sigma = \sigma_j$ and $\mathbf{w} = \mathbf{w}_j$ into (3.2), obtaining

$$0 = Z(\sigma_j, \mathbf{w}_j) \sim \sum_{|\alpha| \geq n_0} \frac{(-1)^{|\alpha|}}{\alpha!} \, \mu_\alpha \, \sigma_j^{n_0 - |\alpha|} \, \psi^{(\alpha)}(\mathbf{w}_j).$$

This expansion holds in the sense that for each $k \geq 0$, the partial sum of the right-hand side with $|\alpha|$ up to $n_0 + k$ vanishes up to order σ_j^{-k} as $j \to \infty$:

$$\lim_{j \to \infty} \sigma_j^k \sum_{n_0 \leq |\alpha| \leq n_0 + k} \frac{(-1)^{|\alpha|}}{\alpha!} \, \mu_\alpha \, \sigma_j^{n_0 - |\alpha|} \, \psi^{(\alpha)}(\mathbf{w}_j) = 0, \tag{3.4}$$

for all $k \geq 0$. We separate the left-hand side of (3.4) into terms involving moments of order $n_0 + k$ and those involving lower-order moments:

$$\sum_{|\alpha| = n_0 + k} \frac{(-1)^{|\alpha|}}{\alpha!} \, \mu_\alpha \psi^{(\alpha)}(\mathbf{w}') + \lim_{j \to \infty} \sum_{n_0 \leq |\alpha| < n_0 + k} \frac{(-1)^{|\alpha|}}{\alpha!} \, \mu_\alpha \, \sigma_j^{n_0 + k - |\alpha|} \, \psi^{(\alpha)}(\mathbf{w}_j) = 0. \tag{3.5}$$

The two terms of Equation (3.5) form a linear recursion relation for the moments μ_α of order $|\alpha| = n_0 + k$ in terms of lower-order moments. We now show by induction on k that Equation (3.5) recursively determines all moments of f up to a constant multiple.

As a basis step we observe that, for $k = 0$, the second term on the left-hand size of Equation (3.5) vanishes, while the first term is equal to $z(\mathbf{w}')$. Since $Z(\sigma, \mathbf{w})$ converges locally uniformly in $\mathbf{w}$ to $z(\mathbf{w})$, the asymptotic zero set E contains the regular zero set of z and is contained in the zero set of z. Furthermore, since z is an order-n_0 derivative of ψ, the genericity condition ensures that E cannot contain the regular zero set of any other fixed-order derivative of ψ (if so, this regular zero set would also be contained in the zero set of z, violating the genericity condition). Thus $z(\mathbf{w})$ is the unique fixed-order derivative of ψ whose regular zeros are contained in E. Since E is uniquely determined by the given zero sets $\{E_j\}$, it follows that n_0 and all moments μ_α with $|\alpha| = n_0$, up to a common multiple, are uniquely determined by these zero sets (using the above genericity condition for $n = m$).

Now assume for induction that for some $k \geq 1$, all moments μ_α with $|\alpha| < n_0 + k$ are known. Then the first term on the left-hand side of Equation (3.5) can be evaluated at any $\mathbf{w}' \in E$ by choosing a corresponding sequence $\{\mathbf{w}_j \in E_j\}$ with $\mathbf{w}_j \to \mathbf{w}'$ and evaluating the second term. The genericity condition ensures that the moments μ_α with $|\alpha| = n_0 + k$ are uniquely determined by the values of the first term as $\mathbf{w}'$ ranges over the regular zeros of z, since the difference between any two distinct choices for the first term of (3.5) would be an order-(n_0+k) derivative of ψ that is identically zero on the regular zero set of z, an order-n_0 derivative of ψ. Thus the moments of order n_0+k are uniquely determined by the lower-order moments together with the given zero sets $\{E_j\}$. If the lower-order moments are known only up to a common multiple, then since Equation (3.5) is linear in the moments, those of order n_0+k are determined up to this same common multiple. This completes the induction, showing that all moments of f are determined up to a constant multiple by the zero sets $\{E_j\}$.

To determine f from its moments $\{\mu_\alpha\}_\alpha$ we first determine its Fourier transform

$$\hat{f}(\boldsymbol{\omega}) = \int_{\mathbb{R}^d} f(\mathbf{x}) e^{-i\boldsymbol{\omega}\cdot\mathbf{x}}\, d\mathbf{x}.$$

We claim that $\hat{f}(\boldsymbol{\omega})$ is well-defined and analytic for all $\boldsymbol{\omega} \in \mathbb{C}^d$ with $|\operatorname{Im}\boldsymbol{\omega}| < \gamma$. This result is well-known as a version of the Payley-Wiener theorem for $f \in \mathcal{P}'_\gamma \cap L^2$; we provide the argument for $f \in \mathcal{P}'_\gamma \cap L^1$.

Fix such an $\boldsymbol{\omega}$. The Fourier transform $\hat{f}$ is well-defined at $\boldsymbol{\omega}$ since $f \in \mathcal{P}'_\gamma \cap L^1(\mathbb{R}^d)$. Furthermore, the (complex) partial derivative of $\hat{f}$ in the jth coordinate at $\boldsymbol{\omega}$ is given by

$$\frac{\partial \hat{f}}{\partial \omega_j}(\boldsymbol{\omega}) = \lim_{\substack{\epsilon\to 0\\ \epsilon\in\mathbb{C}}} \int_{\mathbb{R}^d} f(\mathbf{x}) \frac{1}{\epsilon}\left(e^{-i(\boldsymbol{\omega}\cdot\mathbf{x}+\epsilon x_j)} - e^{-i\boldsymbol{\omega}\cdot\mathbf{x}}\right)\, d\mathbf{x}. \tag{3.6}$$

Fix $\lambda \in \mathbb{R}$ satisfying $|\operatorname{Im}\boldsymbol{\omega}| < \lambda < \gamma$. For sufficiently small ϵ, the integrand in (3.6) is absolutely bounded over all $\mathbf{x} \in \mathbb{R}^d$ by

$$|\mathbf{x} f(\mathbf{x})| e^{\lambda|\mathbf{x}|},$$

which is integrable since $f \in \mathcal{P}'_\gamma \cap L^1(\mathbb{R}^d)$. By dominated convergence, the limit and integral in (3.6) can be interchanged, yielding

$$\frac{\partial \hat{f}}{\partial \omega_j}(\boldsymbol{\omega}) = \int_{\mathbb{R}^d} (-ix_j) f(\mathbf{x}) e^{-i\boldsymbol{\omega}\cdot\mathbf{x}}\, d\mathbf{x}.$$

This shows that all complex first partials of $\hat{f}(\boldsymbol{\omega})$ exist; thus $\hat{f}$ is analytic at $\boldsymbol{\omega}$.

Using dominated convergence to iteratively evaluate derivatives of $\hat{f}(\boldsymbol{\omega})$ as in (3.6), we obtain the Taylor expansion

$$\hat{f}(\boldsymbol{\omega}) = \sum_{|\alpha| \geq 0} \mu_\alpha \frac{(-i\boldsymbol{\omega})^\alpha}{\alpha!}.$$

By analytic continuation, the moments $\{\mu_\alpha\}_{|\alpha|\geq 0}$ uniquely determine $\hat{f}$ on $\mathbb{R}^d$. Since the Fourier transform is one-to-one on $L^1(\mathbb{R}^d)$, f is uniquely determined by its moments. This completes the proof. □

3.2. Counterexamples to unique determination if the conditions of Thereom 1 are weakened

Thereom 1 can be described as the strongest of its kind in two senses. First (Section 7), the theorem is false if the exponential decay required by the condition $f \in \mathcal{P}'_\gamma$ is relaxed to algebraic decay.

Second, if the above genericity condition is relaxed even mildly, the theorem becomes false for $f \in \mathcal{P}'_\gamma$. Specifically, if we consider the weakened genericity condition that the entire (not just regular) zero set of any derivative of fixed order derivative n is not contained in the zero set of any other derivative of fixed order m, there exists a $\psi \in \mathcal{P}$ and $f \in \mathcal{P}'_\gamma$ for which the conclusion of the theorem is false.

The simplest example of this involves an $\mathbb{R}^1$-wavelet based on the Gaussian, for which two finite δ-series have the same zeros. This wavelet $\psi(x)$ is based on a triplet of reals (a, b, x^*) solving the system of equations

$$-x^* e^{-(x^*)^2/2} + ax^* + b = 0 \tag{3.7a}$$

$$\left((x^*)^2 - 1\right) e^{-(x^*)^2/2} + a = 0 \tag{3.7b}$$

$$\sqrt{3}e^{-3/2} - \sqrt{3}a + b = 0, \tag{3.7c}$$

where $x^* \neq -\sqrt{3}$. A numerical solution exists with $x^* \approx 0.71$, $a \approx 0.38$, and $b \approx 0.28$. Note that x^* is necessarily a transcendental (non-algebraic) number, as we will show.

Let

$$\psi(x) = -xe^{-x^2/2} + ax + b. \tag{3.8}$$

The wavelet $\psi(x)$ has a non-regular (double) zero at $x = x^*$ and a regular zero at $x = -\sqrt{3}$. Its derivative $\psi'(x)$ has regular zeros at $x = \pm x^*$. Its second derivative $\psi''(x)$ has regular zeros at $x = 0$ and $x = \pm\sqrt{3}$ (see Figure 1). All derivatives of order $n \geq 3$ have (algebraic) zeroes given by the roots of the Hermite polynomial $H_{n+1}(x)$.

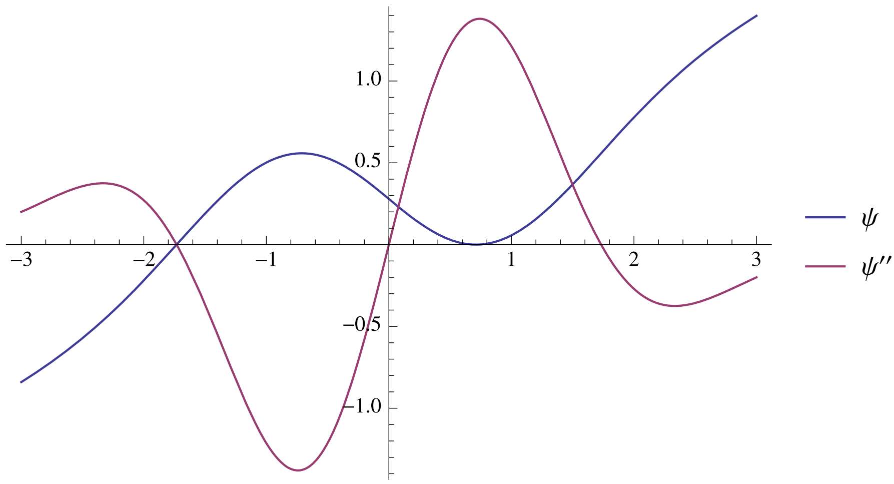

Figure 1: Graphs of $\psi(x)$, defined in Eq. (3.8), and its second derivative. Note $\psi(x)$ has a non-regular zero at $x = x^*$ and a regular zero at $x = -\sqrt{3}$, while $\psi''(x)$ has regular zeros at $x = 0$ and $x = \pm\sqrt{3}$ and no non-regular zeros.

The wavelet ψ satisfies the weakened genericity condition. Indeed, none of ψ, ψ' or ψ'' have a zero set contained in the zero set of another. Further, the zeroes of $\psi^{(n)}$ for $n \geq 3$ are algebraic, and therefore cannot contain the zero sets of ψ or ψ', which include the transcendental number x^*. Finally, the irreducibility of Hermite polynomials (see Section 3.3) implies that the zeros of $\psi^{(n)}$, $n \geq 2$, are not contained in the zeroes of any other $\psi^{(m)}$, $m \geq 2$.

Note however that ψ does *not* satisfy the original genericity condition, since its only regular zero (at $x = -\sqrt{3}$) is contained in the zero set of ψ''.

For $c > 0$ sufficiently small and $\sigma > 0$ sufficiently large, the function $\psi(w) + c\sigma^{-2}\psi''(w)$ has zeros only at $x = -\sqrt{3}$, independent of the choice of (small) c. Thus for sufficiently small $c > 0$, the initial distributions $\delta^{(0)}$ and $\delta^{(0)} + c\delta^{(2)}$ cannot be distinguished by their zeros when the scaling σ is sufficiently large. This shows that the weakened genericity condition is insufficient for the conclusion of Marr's conjecture to hold.

It remains to show that x^* is transcendental (i.e., not an algebraic number). To this end, solving (3.7b) for a we have

$$a = \left(1 - (x^*)^2\right) e^{(-x^*)^2/2}.$$

Now substituting for a in (3.7a) and solving for b:

$$b = (x^*)^3 e^{(-x^*)^2/2}.$$

Substituting in (3.7c) and rearranging,

$$\sqrt{3}e^{-3/2+(x^*)^2/2} = \sqrt{3}\left(1-(x^*)^2\right) - (x^*)^3.$$

If x^* were algebraic then the left side would be transcendental (since the exponentials of non-zero algebraics are transcendental by the Hermite-Lindemann theorem) while the right side would be algebraic, which would give a contradiction. Therefore x^* is transcendental.

3.3. Ricker wavelets and the Marr conjecture

We now specialize Theorem 1 to the Ricker (Mexican hat) wavelet $M(\mathbf{x}) = \Delta G(\mathbf{x})$, which is clearly in $\mathcal{P}$. We first consider the one-dimensional case:

Corollary 3(a). (Infinite limit case) *Any $f \in \mathcal{P}'_\gamma \cap L^1(\mathbb{R})$ is uniquely determined (up to a constant multiple) by the zero sets of $f * M_{\sigma_j}$ at any sequence of positive scales $\{\sigma_j\}_{j=1}^\infty$ tending to infinity.*

Proof. To apply Theorem 1 we need to verify the genericity condition for the one-dimensional Ricker wavelet $M(x)$. This wavelet has derivatives

$$M^{(n)}(x) = (-1)^n H_{n+2}(x)G(x),$$

where

$$H_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{n!}{k!(n-2k)!2^k} x^{n-2k} \tag{3.9}$$

is the nth (probabilists') Hermite polynomial.

We claim that two Hermite polynomials of different degree have at most the root $x = 0$ in common. This will follow from an irreducibility result of Schur [31, 30], which applies to the *physicists'* Hermite polynomials, defined by

$$\tilde{H}_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k \frac{n!}{k!(n-2k)!} (2x)^{n-2k} \tag{3.10}$$

Schur [31, 30] proved that $\tilde{H}_{2n}(x)$ and $\tilde{H}_{2n+1}(x)/x$ are irreducible over the rationals (cannot be nontrivially factored into polynomials with rational coefficients) for all $n \geq 0$. If two irreducible polynomials have a common real root x_0, then both must be scalar multiples of the minimal polynomial of x_0 (i.e., the unique monic polynomial of minimal degree that has a root at x_0; see [16, Theorem V.1.6]). Therefore, two irreducible polynomials cannot have a common root unless they are scalar multiples of each other. Inspection of Eq. (3.10) shows that no distinct polynomials from the set $\{\tilde{H}_{2n}(x)\}_{n\geq 0} \cup \{\tilde{H}_{2m+1}(x)/x\}_{m\geq 0}$ are scalar multiples of each other (note in particular that $\tilde{H}_{2n}(x)$ and $\tilde{H}_{2n+1}(x)/x$ have the same constant term but different leading coefficient.) The claim is therefore true for the physicists' Hermite polynomials. The claim for the probabilists' Hermite polynomials then follows from the relation $H_n(x) = 2^{-n/2}\tilde{H}_n(x/\sqrt{2})$.

Moreover, the relation $H_n'(x) = nH_{n-1}(x)$, together with the above irreducibility result, implies that Hermite polynomials have no multiple roots, i.e. all zeros are regular. The genericity condition on $M(x)$ follows, and the corollary is proven by application of Theorem 1. □

In higher dimensions, the genericity condition on M reduces to polynomial relations. Partial derivatives of M are described by the Laplace-Hermite polynomials $L_\alpha(\mathbf{x})$, defined in equation (1.2), which have the explicit form

$$L_\alpha(\mathbf{x}) = \sum_{i=1}^{d} \left(H_{\alpha_i+2}(x_i) \prod_{\substack{1\leq j\leq d \\ j\neq i}} H_{\alpha_j}(x_j) \right).$$

The genericity condition reduces to a statement about zeroes of these polynomials, as stated in Corollary 4 (Section 1.2). We have numerically verified the genericity condition on M in dimension $d = 2$ for $n = 0$, $0 \leq m \leq 15$.

4. Geometry of Gaussian edge contours

Having proved the Marr conjecture in one dimension (Corollary 3(a)), we ask in Sections 4–7 whether this result can be extended to other sequences of scales and to functions that decay less rapidly than those in $\mathcal{P}'_\gamma$. We therefore restrict our focus to one-dimensional Gaussian edges—that is, zeros of $f * M_\sigma$, or equivalently, of $\Delta(f * G_\sigma)$—for $f \in L^1(\mathbb{R})$. (Recall the notation $\psi_\sigma(x) = \sigma^{-1}\psi(x/\sigma)$ for any smooth function $\psi \in \mathcal{P}(\mathbb{R})$). Our results are summarized in Corollary 3(b,c) (Section 1.2).

This section gives a characterization of the geometry of one-dimensional Gaussian edges, which we will later use in proving unique determination from sequences of bounded-scale edges. Since these edges are nodes of a heat equation solution, we will represent scale using the variable $t = \sigma^2$ rather than σ.

We remark that the results of this section hold for zero contours (i.e. points where $f * G_{\sqrt{t}} = 0$) as well as edge contours ($\Delta f * G_{\sqrt{t}} = 0$), using the same arguments. The results are stated for edge contours only for convenience in applying them to further results.

Given $f \in \mathcal{P}'_\gamma \cap L^1(\mathbb{R})$, define

$$F(x,t) = f * G_{\sqrt{t}}(x).$$

F is jointly analytic in both variables on the upper half-plane $H_+ = \mathbb{R} \times (0,\infty)$ (e.g. [5, Theorem 10.3.1]). Both F and $F_{xx} = \frac{\partial^2 F}{\partial x^2}$ satisfy the heat equation (1.3), and so are subject to the following maximum principle (e.g. [5, Theorem 15.3.1]):

Theorem 11. *For $t_2 > t_1 \geq 0$, let $s_1, s_2 : [t_1, t_2] \to \mathbb{R}$ be continuous functions with $s_1(t) < s_2(t)$ for all $t \in (t_1, t_2]$. Let D be the parabolic interior*

$$D = \{(x,t) : t \in (t_1, t_2], s_1(t) < x < s_2(t)\}, \tag{4.1}$$

Let $u(x,t)$ satisfy the heat equation $u_{xx} = \frac{1}{2} u_t$ on $\bar{D}$, the closure of D. Then if the maximum (or minimum) value of u over $\bar{D}$ is achieved on D, u is constant on $\bar{D}$.

An immediate consequence of the maximum principle is that F_{xx} has no isolated zeros, since such a zero would be a local extremum. Furthermore, since F_{xx} is analytic, the resolution of analytic singularities in two real dimensions [12, 13, 3] (or Puiseux series expansion, e.g. [4] or Theorem 4.2.11 of [19]) implies that for each zero $(x_0, t_0) \in H_+$ of F_{xx} there must be a neighborhood $W \subset H_+$ containing (x_0, t_0) and a collection of injective real-analytic mappings

$$\begin{aligned} (\chi_1, \tau_1) &: (-\epsilon_1, \epsilon_1) \to W, \\ &\vdots \\ (\chi_k, \tau_k) &: (-\epsilon_k, \epsilon_k) \to W, \end{aligned} \tag{4.2}$$

the images of which intersect only at $(x_0, t_0) = (\chi_1(0), \tau_1(0)) = \ldots = (\chi_k(0), \tau_k(0))$, and the union of whose images is precisely the zero set of F_{xx} in W. By analytic continuation of these mappings, the zero set of F_{xx} in H_+ can be uniquely described as a union of real-analytic curves or curve segments that have locally injective parameterizations of the form (4.2) around each point, endpoints (if they exist) only on the line $t = 0$, and whose intersections form a discrete subset of H_+. We call these curves and curve segments the *edge contours* of F. (The above statements also apply to the zero set of F rather than F_{xx}; we call the curves and curve segments comprising this set the *zero contours* of F.)

It is commonly observed computationally [35] that edge contours either form arcs from one point on the x-axis to another, or else extend from $t = 0$ to $t = \infty$. Solutions for which new edge contours are generated with increasing t have not been observed numerically or analytically. This observation has been formalized and proven in several ways [2, 15]; here we prove Theorem 7, which strengthens previous formalizations. We begin with a lemma.

Lemma 12. *If $F_{xx}(x_0, t_0) = 0$, then F_{xx} has at least one zero in any rectangle $[x_0 - \epsilon, x_0 + \epsilon] \times [t_0 - \delta, t_0)$ with $\epsilon, \delta > 0$.*

Proof. Assume, to the contrary, that there is a rectangle $[x_0-\epsilon, x_0+\epsilon]\times[t_0-\delta, t_0)$ that contains no zeros of F_{xx}. Defining $D = (x_0-\epsilon, x_0+\epsilon)\times(t_0-\delta, t_0]$, we find that F_{xx} is either maximized or minimized over $\bar{D} = [x_0 - \epsilon, x_0 + \epsilon] \times [t_0 - \delta, t_0]$ at $(x_0, t_0) \in D$. By the maximum principle (Theorem 11), $F_{xx} \equiv 0$ on $\bar{D}$, contradicting our assumption. □

As an immediate consequence, edge contours cannot have local minima in t:

Corollary 13. *If $(\chi, \tau) : (-\epsilon, \epsilon) \to H_+$ is a local parameterization of an edge contour of F, then τ has no local minimum on $(-\epsilon, \epsilon)$.*

We next define a *persistent* edge contour as an edge contour that extends to arbitrarily large values of t (or equivalently, arbitrarily large values of $\sigma = \sqrt{t}$). We immediately obtain the following two results:

Corollary 14. *A persistent edge contour can intersect the line $t = t_1$ no more than once.*

Corollary 15. *If $(\chi, \tau) : (-\epsilon, \epsilon) \to H_+$ is a local parameterization of a persistent edge contour of F, then τ has no local maximum on $(-\epsilon, \epsilon)$.*

Proof. Topologically, for each local maximum of a curve that is not a global maximum, there must also be a local minimum. Since persistent edge contours have no global maxima or local minima, they therefore cannot have local maxima. □

We now prove Theorem 7, which formalizes the observation that edge contours are not generated with increasing t. Again, we note that this result applies also to zero contours; the proof is the same but with F_{xx} and $G^{(2)}$ replaced with F and G respectively.

Theorem 7. *For any $f \in \mathcal{P}'_\gamma \cap L^1(\mathbb{R})$ and positive numbers $t_1 < t_2$, the edge contours of f intersecting the line $t = t_2$ are a subset of those that intersect $t = t_1$.*

Proof. Since edge contours are never minimized in t (Corollary 13), it only needs to be shown that there is no edge contour whose t-value decreases asymptotically to an intermediate value t', $t_1 < t' < t_2$, as its x-value diverges to positive or negative infinity.

Assume the contrary, and without loss of generality assume the x-value of the edge contour in question diverges to positive infinity as t decreases to t'. Then there is a locally analytic curve $s \mapsto (\chi(s), \tau(s))$ defined for all s greater than some s_0, with

- $F_{xx}(\chi(s), \tau(s)) = 0$, $\forall s > s_0$,
- $\tau(s)$ monotonically decreasing,
- $\lim_{s\to\infty} \chi(s) = \infty$,
- $\lim_{s\to\infty} \tau(s) = t'$.

For $x_1 > \chi(s_0)$, define $D_1 \subset H_+$ to be the closed connected region bounded by the curve $(\chi(s), \tau(s))$ and the lines $t = t_2$ and $x = x_1$ (Figure 2). Since F_{xx} is zero on the curve $(\chi(s), \tau(s))$, the maximum principle (Theorem 11) implies that $|F_{xx}(x, t)|$ achieves its maximum value over D_1 at a point on the line $x = x_1$. We denote this maximizing point (x_1, t_1^*). For any $x_2 > x_1$, $|F_{xx}|$ achieves its maximum over the domain D_2 (defined similarly to D_1 with x_1 replaced by x_2; see Figure 2) at a point on the line $x = x_2$, which we denote (x_2, t_2^*). Moreover, $|F_{xx}(x_2, t_2^*)| > |F_{xx}(x_1, t_1^*)|$ since $D_1 \subset D_2$. Iterating this argument, we obtain a sequence $\{(x_i, t_i^*)\}_{i\geq 1}$, with $x_i \to \infty$ and $|F_{xx}(x_i, t_i^*)|$ monotonically increasing. Thus

$$\lim_{i\to\infty} |F_{xx}(x_i, t_i^*)| > 0. \tag{4.3}$$

On the other hand, since $x_i \to \infty$ while t_i^* is confined to the interval $(t', t_2]$ for each i, it is easily verified that the sequence of functions $\{g_i(y)\}_{i=1}^\infty$ with

$$g_i(y) = G^{(2)}_{\sqrt{t_i^*}}(x_i - y)$$

converges as $i \to \infty$ to the zero function in the topology of $\mathcal{P}_\gamma$. Then, since $f \in \mathcal{P}'_\gamma$,

$$\lim_{i\to\infty} F_{xx}(x_i, t_i^*) = \lim_{i\to\infty} f * G^{(2)}_{\sqrt{t_i^*}}(x_i) = \lim_{i\to\infty} \langle f, g_i \rangle = 0.$$

This contradicts (4.3); hence no such edge contour exists. ∎

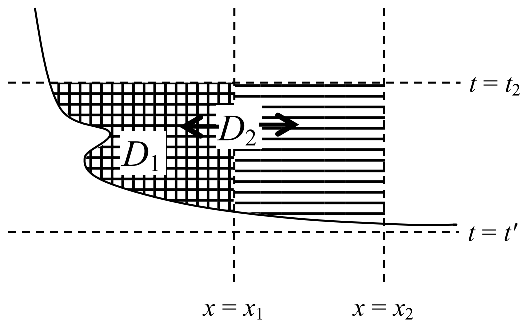

Figure 2: Regions defined in proof of Theorem 7. D_1 is bounded by the curve $(\chi(s), \tau(s))$ and the lines $t = t_2$ and $x = x_1$. D_2 is bounded by the curve $(\chi(s), \tau(s))$ and the lines $t = t_2$ and $x = x_2$, with $x_2 > x_1$.

5. Reconstruction from other edge sequences

The above proof of unique determination from Gaussian edges (Marr's conjecture, Corollary 3(a)) uses only the asymptotics of the edges of f for large scales σ. This is unexpected, since one would anticipate more information would arise from small-scale rather than large-scale edges. We show here that in the one-dimensional Gaussian case, a sequence of bounded-scale edges (i.e. with σ remaining bounded) is also sufficient to uniquely determine any $f \in \mathcal{P}'_\gamma \cap L^1(\mathbb{R})$, as long as the sequence of scales has a positive limit point.

5.1. Sequences of scales with a positive limit point

With f and F as above, let $\{t_j\}_{j\geq 1}$ be a sequence of positive real numbers with a limit point $t' > 0$, for which the solutions (in x) to $F_{xx}(x, t_j) = 0$ are given.

The asymptotic edge (Section 3.1) of the Ricker wavelet transform of f is given by the zeros of $H_{n_0+2}\left(x/\sqrt{t}\right)$, where n_0 is the order of the first nonzero moment of f. Since H_{n_0+2} has $n_0 + 2$ distinct regular real roots, there are exactly $n_0 + 2$ persistent edge contours. Theorem 7 implies that the persistent edge contours intersect the lines $t = t_j$ for all j, as well as the limiting line $t = t'$. Further, by Lemma 12, the persistent edge contours cross the lines $t = t_j$ rather than achieving local minima at the intersection points. Thus by analytic continuation, the persistent edge contours are uniquely determined by the given solutions to $F_{xx}(x, t_j) = 0$. We now recall that the edge contours of f are the zeros of $f * M_\sigma$, with $M_\sigma(x) = \sigma^{-1}M(x/\sigma)$ and $\sigma = \sqrt{t}$. The infinite-limit ($\sigma_j \to \infty$) case of Corollary 3(a), proven above, guarantees that the persistent edge contours uniquely determine f. We have thus proved:

Corollary 3(a). (General case) *Any $f \in \mathcal{P}'_\gamma \cap L^1(\mathbb{R})$ is uniquely determined (up to a constant multiple) by the zero sets of $f * M_{\sigma_j}$ at any set of scales $\{\sigma_j\}_{j=1}^\infty$ with a positive or infinite limit point.*

5.2. Sequences of scales converging only to zero

Perhaps surprisingly, unique determination is not guaranteed in the case that the scales $\{\sigma_j\}$ have zero as their only limit point, as we now show:

Corollary 3(b). *For any $\gamma > 0$, there exist two functions $f_1, f_2 \in \mathcal{P}'_\gamma \cap L^1(\mathbb{R})$, which are not constant multiples of each other, and a sequence of scales $\{\sigma_j\}_{j=1}^\infty$ converging to zero, such that $f_1 * M_{\sigma_j}$ and $f_2 * M_{\sigma_j}$ have identical zero sets for every $j \geq 1$.*

Proof. Fix $\gamma > 0$. We will prove this statement by constructing a compactly supported function $h \in \mathcal{P}'_\gamma \cap L^1(\mathbb{R})$ and defining $f_1(x) = G(x) \equiv (2\pi)^{-1/2}e^{-x^2/2}$ and $f_2(x) = G(x) + h(x)$. The function h will be defined by its second derivative $\Delta h \in \mathcal{P}'$, which we represent as an infinite sum

$$\Delta h = \sum_{n=1}^{\infty} c_n J_{\alpha_n,\beta_n}.$$

Above, for any real numbers $0 < |\beta| < \alpha < 1$, the distribution $J_{\alpha,\beta} \in \mathcal{P}'_\gamma$ is defined as a combination δ-distributions localized at $x = \pm(1+\beta) \pm \alpha$:

$$\begin{aligned} J_{\alpha,\beta}(x) = \delta(x+1+\alpha+\beta) - \delta(x+1-\alpha+\beta) \\ - \delta(x-1+\alpha-\beta) + \delta(x-1-\alpha-\beta). \end{aligned}$$

We will choose c_n, α_n, and β_n inductively, so that the edge contours of $G+h$ oscillate about those of G as $\sigma \to 0$.

We begin by setting $c_1 = 1$ and choosing $0 < -\beta_1 < \alpha_1 < 1$ arbitrarily. We define h_1 by

$$\Delta h_1(x) = c_1 J_{\alpha_1,\beta_1}(x),$$

together with the requirement that h_1 be compactly supported. (We assume all h_n are compactly supported, and so can be defined by their second derivatives.) The function h_1 is illustrated in Figure 3.

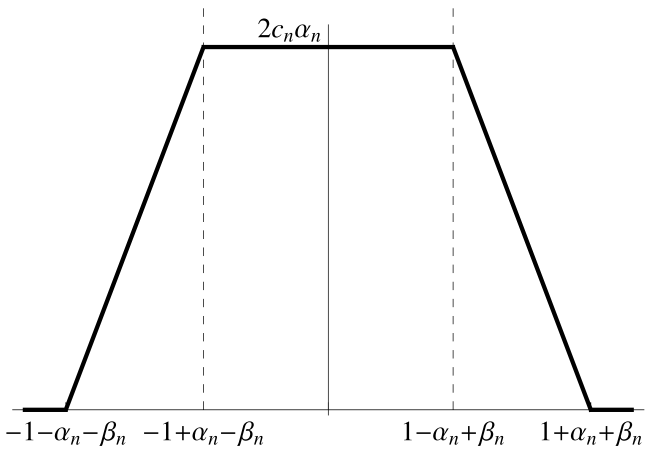

Figure 3: The graph of $h_n - h_{n-1}$, which satsifies $\Delta(h_n - h_{n-1}) = c_n J_{\alpha_n,\beta_n}$. Note that $\int_{-\infty}^{\infty} |h_n(x) - h_{n-1}(x)|\, dx = 4c_n\alpha_n(1+\beta_n)$. For $n = 1$ this is the graph of h_1. h is constructed as an infinite sum of functions of this form.

There are two edge contours of G (i.e. zero curves of $\Delta(G * G_\sigma) = G^{(2)}_{\sqrt{\sigma^2+1}}$), described by $x = \pm\sqrt{\sigma^2+1}$. Since J_{α_1,β_1} is nonnegative/nonpositive wherever $G^{(2)}$ is, the addition of h_1 to G creates no new edge contours (any such created edge contours would have to manifest themselves at arbitrarily small scales by Theorem 7), and perturbs the edge contours of G symmetrically about the σ-axis (by the symmetry of J_{α_1,β_1}).

Furthermore, since $\beta_1 < 0$, the positive point masses of J_{α_1,β_1} are closer to ± 1 than the negative ones. Thus there is a sufficiently small $\sigma_1 > 0$, for which

$$\Delta(h_1 * G_{\sigma_1})\left(\pm\sqrt{\sigma_1^2+1}\right) > 0.$$

Now suppose inductively that for some $n \geq 1$ we have a compactly supported function $h_n \in L^1(\mathbb{R})$ such that Δh_n is zero in neighborhoods of ± 1, and a strictly decreasing sequence of positive scales $\{\sigma_1, \ldots \sigma_n\}$ such that

$$\begin{cases} \Delta(h_n * G_{\sigma_k})\left(\pm\sqrt{\sigma_k^2+1}\right) > 0 & \text{for } k \text{ odd} \\ \Delta(h_n * G_{\sigma_k})\left(\pm\sqrt{\sigma_k^2+1}\right) < 0 & \text{for } k \text{ even}, \qquad 1 \leq k \leq n. \end{cases} \tag{5.1}$$

As an induction step, we will choose real numbers c_{n+1}, α_{n+1}, β_{n+1}, and σ_{n+1}, with $c_{n+1} > 0$, $0 < (-1)^{n+1}\beta_{n+1} < \alpha_{n+1} < 1$, $|\beta_{n+1}| < \alpha_{n+1}$, and $0 < \sigma_{n+1} < \sigma_n$, such that if $h_{n+1} \in L^1(\mathbb{R})$ is defined by

$$\Delta h_{n+1}(x) = \Delta h_n(x) + c_{n+1} J_{\alpha_{n+1},\beta_{n+1}},$$

then

$$\begin{cases} \Delta(h_{n+1} * G_{\sigma_k}) \left(\pm\sqrt{\sigma_k^2+1}\right) > 0 & \text{for } k \text{ odd} \\ \Delta(h_{n+1} * G_{\sigma_k}) \left(\pm\sqrt{\sigma_k^2+1}\right) < 0 & \text{for } k \text{ even}, \qquad 1 \le k \le n+1. \end{cases} \tag{5.2}$$

First, note that for any fixed σ (in particular for $\sigma = \sigma_k$ with $1 \le k \le n$), the quantity

$$\left| J_{\alpha_{n+1},\beta_{n+1}} * G_\sigma(x) \right|$$

is uniformly bounded over all choices of α_{n+1} and β_{n+1} and all x. Thus for c_{n+1} sufficiently small, the desired relationships (5.2) hold for $k \le n$ no matter the values of α_{n+1} and β_{n+1}. We choose c_{n+1} so that this property is satisfied and also $c_{n+1} < c_n/2$.

Second, since Δh_n is zero in neighborhoods of ± 1, there exist arbitrarily small σ_{n+1}, α_{n+1} and β_{n+1} such that

$$\Delta(h_n * G_{\sigma_{n+1}}) \left(\pm\sqrt{\sigma_{n+1}^2+1}\right)$$

is arbitrarily small in magnitude relative to

$$J_{\alpha_{n+1},\beta_{n+1}} * G_{\sigma_{n+1}} \left(\pm\sqrt{\sigma_{n+1}^2+1}\right),$$

and therefore the sign of

$$\begin{aligned} &\Delta(h_{n+1} * G_{\sigma_{n+1}}) \left(\pm\sqrt{\sigma_{n+1}^2+1}\right) \\ &= \Delta(h_n * G_{\sigma_{n+1}}) \left(\pm\sqrt{\sigma_{n+1}^2+1}\right) + c_{n+1} J_{\alpha_{n+1},\beta_{n+1}} * G_{\sigma_{n+1}} \left(\pm\sqrt{\sigma_{n+1}^2+1}\right) \end{aligned}$$

coincides with that of

$$J_{\alpha_{n+1},\beta_{n+1}} * G_{\sigma_{n+1}} \left(\pm\sqrt{\sigma_{n+1}^2+1}\right).$$

Finally, since α_{n+1} and β_{n+1} were chosen to satisfy $0 < (-1)^{n+1}\beta_{n+1} < \alpha_{n+1}$, it follows that

$$(-1)^n J_{\alpha_{n+1},\beta_{n+1}} * G_{\sigma_{n+1}} \left(\pm\sqrt{\sigma_{n+1}^2 + 1} \right) > 0,$$

and hence

$$(-1)^n \, \Delta(h_{n+1} * G_{\sigma_{n+1}}) \left(\pm\sqrt{\sigma_{n+1}^2 + 1} \right) > 0,$$

as desired. This completes the inductive construction of h_{n+1} and σ_{n+1}.

Having defined the partial sums h_n inductively, we now define h to be their limit in the L^1 topology. This limit exists because $h_n - h_{n-1}$ has L^1-norm $4\alpha_n c_n(1 + \beta_n)$ (see Figure 3), and therefore the L^1-norm of h_n is bounded for each n by $4\sum_{m=1}^{\infty} c_m\alpha_m(1+\beta_m)$. This sum converges since the c_n are bounded by a geometric sequence ($c_{n+1} < c_n/2$), and $|\beta_n| < \alpha_n < 1$ for each n. The latter inequality also implies that h is compactly supported; therefore, $h \in \mathcal{P}'_\gamma \cap L^1(\mathbb{R})$.

In the above limit, the relationships (5.1) are preserved with $\leq$ in place of $<$:

$$\begin{cases} \Delta(h * G_{\sigma_n}) \left(\pm\sqrt{\sigma_n^2 + 1} \right) \geq 0 & \text{for } n \text{ odd} \\ \Delta(h * G_{\sigma_n}) \left(\pm\sqrt{\sigma_n^2 + 1} \right) \leq 0 & \text{for } n \text{ even.} \end{cases}$$

Thus, letting $x_n = \sqrt{\sigma_n^2 + 1}$, we have shown that the sign of $\Delta((G + h) * G_{\sigma_n})(x_n) - \Delta(G * G_{\sigma_n})(x_n)$ alternates with n. Since $\sigma_n \to 0$, this implies that the edge contours of G and $G + h$ cross infinitely often as $\sigma \to 0$. The two edge contours of both G and $G + h$ are symmetric about the σ-axis, the intersections of edge contours on each side of the σ-axis occur at the same σ-values. Thus the edges of G and $G + h$ agree on an infinite sequence of scales tending to zero. □

6. Distributions with finitely many moments

We have shown that any one-dimensional function with exponential decay is uniquely determined by a sequence of scaled Gaussian edges, thus giving a sufficient condition for the Marr conjecture in one dimension. One can ask whether this result could be extended to functions that decay less rapidly—for example, functions with algebraic decay. Addressing this question requires a formal notion of distributions with only finitely many moments. To that end, this section introduces the space $\mathcal{M}_N$ of smooth test functions of asymptotic order $|x|^N$ or less, and its dual $\mathcal{M}'_N$, whose elements are distributions with moments through order N. We first define these spaces, then consider derivatives and antiderivatives of distributions in $\mathcal{M}'_N$, and finally we prove the existence and continuity of asymptotic moment expansions for such distributions.

We consider only one-dimensional distributions, but the definitions and results presented here can readily be generalized to arbitrary dimensions.

6.1. Definitions

For any nonnegative integer N, let $\mathcal{M}_N$ denote the space of smooth test functions ψ on $\mathbb{R}$ such that, for each integer $n \geq 0$, the seminorm

$$||\psi||_{N,n} = \sup_{x\in\mathbb{R}} (1+|x|)^{-(N-n)}|\psi^{(n)}(x)| \tag{6.1}$$

is finite. (These seminorms were first introduced by Hörmander [14] and are often used to define symbol classes of pseudodifferential operators; e.g. [32].)

The topology on $\mathcal{M}_N$ is generated by the family of seminorms $||\ ||_{N,n}$ for $n \geq 0$. Functions in $\mathcal{M}_N$ behave asymptotically as $|x|^N$ or less, and their nth derivatives behave asymptotically as $|x|^{N-n}$ or less. In particular, $x^m \in \mathcal{M}_N$ for each integer $0 \leq m \leq N$. We also observe from (6.1) that for $M \leq N$, $||\psi||_{N,n} \leq ||\psi||_{M,n}$ for each $\psi \in \mathcal{M}_M$ and $n \geq 0$, from which it follows that $\mathcal{M}_M \subset \mathcal{M}_N$.

We denote the dual space of distributions on $\mathcal{M}_N$ by $\mathcal{M}'_N$. Distributions in $\mathcal{M}'_N$ have moments through order N, where the nth moment of $f \in \mathcal{M}'_N$ is defined as

$$\mu_n(f) = \langle f, x^n\rangle, \qquad 0 \leq n \leq N.$$

For $M \leq N$, we have $\mathcal{M}'_N \subset \mathcal{M}'_M$ since $\mathcal{M}_M \subset \mathcal{M}_N$. We also note that for all N, $\mathcal{M}_N \subset \mathcal{P}$ and hence $\mathcal{P}' \subset \mathcal{M}'_N$.

6.2. Derivatives and antiderivatives

We observe from (6.1) that for each $n \geq 0$, $0 \leq m \leq N$,

$$||\psi^{(m)}||_{N-m,n} = ||\psi||_{N,n+m}, \tag{6.2}$$

and therefore, $\psi^{(m)} \in \mathcal{M}_{N-m}$ whenever $\psi \in \mathcal{M}_N$. This relation also shows that the derivative is a continuous linear operator from $\mathcal{M}_N$ to $\mathcal{M}_{N-1}$.

The derivative of a distribution $f \in \mathcal{M}'_N$ is defined, according to integration by parts, as the element of $f' \in \mathcal{M}'_{N+1}$ that satisfies

$$\langle f', \psi \rangle = -\langle f, \psi' \rangle \tag{6.3}$$

for all $\psi \in \mathcal{M}_{N+1}$. By extension, the mth derivative of $f \in \mathcal{M}'_N$, denoted $f^{(m)}$, is an element of $\mathcal{M}'_{N+m}$, for each integer $m \geq 0$.

We can also define the antiderivative of a distribution $f \in \mathcal{M}'_N$, provided that f has vanishing zeroth moment. This definition requires the following lemma regarding antiderivatives of test functions:

Lemma 16. *If ψ is a smooth function and $\psi' \in \mathcal{M}_{N-1}$, $N \geq 1$, then $\psi \in \mathcal{M}_N$.*

Proof. Since $||\psi||_{N,n} = ||\psi'||_{N-1,n-1}$ for all $n \geq 1$, we need only verify that $||\psi||_{N,0}$ is finite. To show this, we note that $\psi' \in \mathcal{M}_{N-1}$ implies that $||\psi'||_{(N-1,0)}$ is finite, and thus there exists some constant $C > 0$ such that $|\psi'(x)| \leq C(1 + |x|)^{N-1}$ for all $x \in \mathbb{R}$. In particular, we have

$$-C(1+x)^{N-1} \leq \psi'(x) \leq C(1+x)^{N-1} \qquad \text{for } x > 0 \tag{6.4a}$$

$$-C(1-x)^{N-1} \leq \psi'(x) \leq C(1-x)^{N-1} \qquad \text{for } x < 0. \tag{6.4b}$$

Upon integrating (6.4a) from $x = 0$ to $x = \infty$, and (6.4b) from $x = -\infty$ to $x = 0$ (and recalling that $N \geq 1$), it follows that there exists some K such that $|\psi(x)| \leq K(1 + |x|)^N$. Thus $||\psi||_{N,0}$ is finite, completing the proof. □

Using the above lemma, we show that any $f \in \mathcal{M}'_N$ with $\mu_0(f) = 0$ has an antiderivative in $\mathcal{M}'_{N-1}$.

Corollary 17. *If $f \in \mathcal{M}'_N$, $N \geq 1$, and $\mu_0(f) = 0$, then there exists a unique $g \in \mathcal{M}'_{N-1}$ with $g' = f$.*

Proof. For $\psi \in \mathcal{M}_{N-1}$, define $\langle g, \psi \rangle = -\langle f, \phi \rangle$, where ϕ is any antiderivative of ψ. It is clear from Eq. (6.3) that any antiderivative of f must have this form. The quantity $\langle f, \phi \rangle$ is well-defined since $\phi \in \mathcal{M}_N$ by Lemma 16. To show uniqueness, we note that the value of $\langle g, \psi \rangle$ does not depend on the choice of ϕ since

$$\langle f, \phi + C \rangle = \langle f, \phi \rangle + C\mu_0(f) = \langle f, \phi \rangle.$$

To show that g is a continuous functional on $\mathcal{M}_{N-1}$, consider a sequence $\{\psi_i \in \mathcal{M}_{N-1}\}_{i \geq 1}$ converging to the zero function in the topology of $\mathcal{M}_{N-1}$. We define a corresponding sequence $\{\phi_i \in \mathcal{M}_N\}_{i \geq 1}$ by

$$\phi_i(x) = \int_{-\infty}^{x} \psi_i(y)\, dy \qquad \text{for each } i.$$

We claim that $\{\phi_i\}_{i\geq 1}$ converges as $i \to \infty$ to the zero function in the topology of $\mathcal{M}_N$. Indeed, for $n \geq 1$ we have from (6.2) that

$$\lim_{i\to\infty} ||\phi_i||_{N,n} = \lim_{i\to\infty} ||\psi_i||_{N-1,n-1} = 0.$$

It therefore only remains to show that $\lim_{i\to\infty} ||\phi_i||_{N,0} = 0$. This can be shown by observing that, since $\lim_{i\to\infty} ||\psi_i||_{N-1,0} = 0$, there is a sequence of positive numbers $\{C_i\}_{i\geq 1}$, $C_i \to 0$, with $\psi_i(x)$ bounded in absolute value by $C_i(1+|x|)^{N-1}$. Integrating separately over the domains $(-\infty, 0]$ and $[0,\infty)$ as in the proof of Lemma 16, it follows that there is a sequence of positive numbers $\{K_i\}_{i\geq 1}$, $K_i \to 0$, such that $\phi_i(x)$ is bounded in absolute value by $K_i(1+|x|)^N$. This proves that $\lim_{i\to\infty} ||\phi_i||_{N,0} = 0$ and thereby verifies the claim that $\{\phi_i\}_{i\geq 1}$ converges to the zero function in the topology of $\mathcal{M}_N$.

The continuity of g as a functional on $\mathcal{M}_{N-1}$ now follows from its definition and the continuity of f:

$$\lim_{i\to\infty} \langle g, \psi_i\rangle = -\lim_{i\to\infty} \langle f, \phi_i\rangle = 0.$$

We conclude that $g \in \mathcal{M}'_{N-1}$ as required. □

Iterating Corollary 17, a distribution in $\mathcal{M}'_N$ whose first m moments vanish has a unique mth antiderivative in $\mathcal{M}'_{N-m}$:

Corollary 18. *Consider $f \in \mathcal{M}'_N$, $N \geq 1$ such that $\mu_0(f) = \ldots = \mu_{m-1}(f) = 0$, for some positive integer $m \leq N$. Then there exists a unique $g \in \mathcal{M}'_{N-m}$ with $g^{(m)} = f$.*

Proof. We proceed by induction on m, with Corollary 17 serving as a base ($m = 1$) case. Suppose the claim holds in the case $m = m^*$ for some positive integer $m^* \leq N-1$; we will prove it for $m = m^*+1$. Consider $f \in \mathcal{M}'_N$ with $\mu_0(f) = \ldots = \mu_{m^*}(f) = 0$. By the inductive hypothesis, there exists a unique $h \in \mathcal{M}'_{N-m^*}$ with $h^{(m^*)} = f$. Iteratively applying (6.3) m^* times, we have that

$$\begin{aligned} 0 &= \mu_{m^*}(f) \\ &= \left\langle h^{(m^*)}, x^{m^*} \right\rangle \\ &= (-1)^{m^*}(m^*)!\, \langle h, 1\rangle \\ &= (-1)^{m^*}(m^*)!\, \mu_0(h). \end{aligned}$$

Thus $\mu_0(h) = 0$, which allows us to apply Corollary 17 to h. We obtain that there exists a unique $g \in \mathcal{M}'_{N-m^*-1}$ with $g' = h$. Taking m^* derivatives of both sides yields $g^{(m^*+1)} = f$, completing the induction step. □

6.3. Asymptotic moment expansion

Here introduce the asymptotic moment expansion for distributions with finitely many moments. A distribution $f \in \mathcal{M}'_N$ has an asymptotic moment expansion to order $N-1$ in the moments of f, convolutions of which converge locally uniformly, as we show in the following analogue of Theorem 5:

Theorem 6. *For all integers $0 \leq M \leq N-1$ and $f \in \mathcal{M}'_N, \psi \in \mathcal{M}_N$, the σ-indexed family of functions*

$$w \longmapsto \sigma^{M+1} \left(f * \psi_\sigma(\sigma w) - \sum_{n=0}^{M} \frac{(-1)^n}{n!} \, \mu_n \, \sigma^{-n-1} \, \psi^{(n)}(w) \right),$$

converges locally uniformly (in w) to the zero function (of w) as $\sigma \to \infty$.

Once the following analogue of Lemma 10 is proved, the proof of Theorem 6 follows exactly the proof of Theorem 5 (in the case $d = 1$, which is the only case we consider here).

Lemma 19. *Let $\rho(w, y)$ be a smooth function and, for fixed $w \in \mathbb{R}$, define ρ_w by $\rho_w(y) = \rho(w, y)$. Suppose that*

(a) $\rho_w \in \mathcal{M}_N$ for some fixed $N \geq 1$,
(b) For each $n \geq 0$, $||\rho_w||_{N,n}$ is locally uniformly bounded in w, and
(c) There is some integer M, $0 \leq M \leq N-1$, such that for each $w \in \mathbb{R}$ and $0 \leq m \leq M$,

$$\frac{d^m}{dy^m} \rho_w(0) = 0.$$

Then for any continuous seminorm $||\quad||$ on $\mathcal{M}_N$, the σ-indexed family of functions

$$w \longmapsto \sigma^M ||\rho_w(\cdot/\sigma)||$$

converges locally uniformly (in w) to the zero function (of w) as $\sigma \to \infty$.

(We recall that the symbol $\cdot$ represents function or distribution arguments with regard to the bracket and seminorm operations.)

Proof. Suppose the conclusion is false for the seminorm $||\quad||_{N,n}$. Then there is a compact neighborhood $K \subset \mathbb{R}$ and a pair of sequences $\{w_j \in K\}_{j \geq 0}$, $\{\sigma_j \in \mathbb{R}\}_{j \geq 0}$, with $\sigma_j \to \infty$, such that

$$0 < \lim_{j\to\infty} \sigma_j^M \left|\left|\rho_{w_j}(\cdot/\sigma_j)\right|\right|_{N,n} = \lim_{j\to\infty} \sigma_j^{M-n} \sup_{y\in\mathbb{R}} (1+|y|)^{-(N-n)} \, |\rho_{w_j}^{(n)}(y/\sigma_j)|.$$

The equality above uses

$$\frac{d^n}{dy^n}\big(\rho_{w_j}(y/\sigma_j)\big) = \sigma_j^{-n} \rho_{w_j}^{(n)}(y/\sigma_j).$$

By passing to a subsequence if necessary, we may assume $\{w_j\}$ converges to a $w' \in K$. Since for fixed w_j, values of y can be chosen to make the quantity

$$(1+|y|)^{-(N-n)} \, |\rho_{w_j}^{(n)}(y/\sigma_j)|$$

arbitrary close to its supremum over $y \in \mathbb{R}$, there is a sequence $\{y_j\}_{j\geq 0}$ such that

$$\lim_{j\to\infty} \sigma_j^{M-n} (1+|y_j|)^{-(N-n)} \, |\rho_{w_j}^{(n)}(y_j/\sigma_j)| > 0. \tag{6.5}$$

Passing to further subsequences if necessary, we may assume that $\{y_j/\sigma_j\}$ either converges to 0 or is bounded away from 0 in absolute value as $j \to \infty$.

Case 1: $\lim_{j\to\infty} y_j/\sigma_j = 0$ *and* $n \leq M$*.* In this case we rewrite (6.5) as

$$\lim_{j\to\infty} \left(\frac{|y_j|^{M-n}}{(1+|y_j|)^{N-n}}\right) \left(|y_j/\sigma_j|^{-(M-n)} \, |\rho_{w_j}^{(n)}(y_j/\sigma_j)|\right) > 0. \tag{6.6}$$

Above, the first parenthesized quantity $\frac{|y_j|^{M-n}}{(1+|y_j|)^{N-n}}$ is bounded above by 1 for all j since $0 \leq M-n < N-n$. For the second parenthesized quantity, we have that

$$\lim_{j\to\infty} |y_j/\sigma_j|^{-(M-n)} \, |\rho_{w'}^{(n)}(y_j/\sigma_j)| = 0,$$

by condition (c) of the statement of the lemma, so

$$\lim_{j\to\infty} |y_j/\sigma_j|^{-(M-n)} \, |\rho_{w_j}^{(n)}(y_j/\sigma_j)| = 0,$$

by the smoothness of ρ in both arguments. Thus

$$\lim_{j\to\infty} \left(\frac{|y_j|^{M-n}}{(1+|y_j|)^{N-n}}\right) \left(|y_j/\sigma_j|^{-(M-n)} \, |\rho_{w_j}^{(n)}(y_j/\sigma_j)|\right) = 0,$$

contradicting (6.6).

Case 2: $\lim_{j\to\infty} y_j/\sigma_j = 0$ *and* $n > M$*.* In this case we rewrite (6.5) as

$$\lim_{j\to\infty} \frac{(\sigma_j^{-1} + |y_j/\sigma_j|)^{n-M}}{(1+|y_j|)^{N-M}} \, |\rho_{w_j}^{(n)}(y_j/\sigma_j)| > 0. \tag{6.7}$$

The quantity

$$\frac{(\sigma_j^{-1} + |y_j/\sigma_j|)^{n-M}}{(1+|y_j|)^{N-M}}$$

converges to 0 as $j \to \infty$ since σ_j^{-1} and y_j/σ_j both converge to 0, and $n-M$ and $N-M$ are both positive in this case. On the other hand,

$$\lim_{j\to\infty} |\rho_{w_j}^{(n)}(y_j/\sigma_j)| = |\rho_{w'}^{(n)}(0)|,$$

by the smoothness of ρ. Thus

$$\lim_{j\to\infty} \frac{(\sigma_j^{-1} + |y_j/\sigma_j|)^{n-M}}{(1+|y_j|)^{N-M}} |\rho_{w_j}^{(n)}(y_j/\sigma_j)| = 0,$$

contradicting (6.7).

Case 3: $|y_j/\sigma_j| > B$ *for some* $B > 0$ *and all* $j \geq 0$*.* In this case, we rewrite (6.5) as

$$\lim_{j\to\infty} \left(\frac{\sigma_j^{M-n}(1+|y_j/\sigma_j|)^{N-n}}{(1+|y_j|)^{N-n}} \right) \left((1+|y_j/\sigma_j|)^{-(N-n)} |\rho_{w_j}^{(n)}(y_j/\sigma_j)| \right) > 0. \tag{6.8}$$

The second parenthesized quantity in (6.8),

$$(1+|y_j/\sigma_j|)^{-(N-n)} |\rho_{w_j}^{(n)}(y_j/\sigma_j)|,$$

is positive, less than or equal to $||\rho_{w_j}||_{N,n}$ by this norm's definition, and therefore bounded in j since $||\rho_w||_{N,n}$ is locally uniformly bounded in w. As for the first parenthesized quantity, since $|y_j/\sigma_j| > B$ implies $\lim_{j\to\infty} |y_j| = \infty$,

$$\begin{aligned}
\lim_{j\to\infty} \frac{\sigma_j^{M-n}(1+|y_j/\sigma_j|)^{N-n}}{(1+|y_j|)^{N-n}} &= \lim_{j\to\infty} \frac{\sigma_j^{M-n}(1+|y_j/\sigma_j|)^{N-n}}{|y_j|^{N-n}} \\
&= \lim_{j\to\infty} \sigma_j^{M-N} (|y_j/\sigma_j|^{-1} + 1)^{N-n} \\
&= 0.
\end{aligned}$$

The last equality follows from the facts that $|y_j/\sigma_j|^{-1} < B^{-1}$ for all j, and that $\sigma_j^{M-N} \to 0$ since $M < N$. We conclude

$$\lim_{j\to\infty} \left(\frac{\sigma_j^{M-n}(1+|y_j/\sigma_j|)^{N-n}}{(1+|y_j|)^{N-n}} \right) \left((1+|y_j/\sigma_j|)^{-(N-n)} |\rho_{w_j}^{(n)}(y_j/\sigma_j)| \right) = 0,$$

contradicting (6.8).

We have shown that the σ-indexed family of functions

$$w \longmapsto \sigma^M ||\rho_w(\cdot/\sigma)||_{N,n}$$

converges locally uniformly (in w) to the zero function of w as $\sigma \to \infty$ for each $n \geq 0$. Since the family of seminorms $||\ ||_{N,n}$ generates the topology on $\mathcal{M}_N$, the result is true for any continuous seminorm. □

For the Gaussian wavelet $\psi = G$ we have:

Corollary 20. *For all $f \in \mathcal{M}'_N$, $N \geq 0$, and all M, $0 \leq M \leq N-1$, the family of functions*

$$w \mapsto \sigma^{M+1} \left(f * G_\sigma(\sigma w) - \sum_{n=0}^{M} \frac{\mu_n}{n!} \sigma^{-n-1} H_n(w) G(w) \right)$$

converges locally uniformly to the zero function of w as $\sigma \to \infty$.

7. Necessity of strong decay

Corollary 3(a) states that a real-valued function with exponential decay is uniquely determined by its Gaussian edges at any sequence of scales not converging to zero. On the other hand, Meyer's counterexample [27] shows that such unique determination fails for non-decaying functions. This raises the question of what requirements on a function f guarantee it to be uniquely determined by a sequence of its Gaussian edges at a sequence of scales. One might conjecture that unique determination can be extended to all functions vanishing at infinity.

Here we prove Corollary 3(c), showing that the above conjecture is false. We will prove this by constructing two functions, f and g, that have a fixed, arbitrarily large, number of finite moments, and whose Gaussian edges coincide on an infinite sequence of scales tending to infinity. Thus the unique determination result, Corollary 3(a), does not, in general, extend to functions with algebraic decay, leaving open only classes of functions with decay rates between exponential and algebraic, e.g. classes decaying as the log-normal function $\ell(x) = \frac{1}{x}e^{-(\ln|x|)^2}$ or faster.

Let N be a positive even integer such that $H_N(\pm 1) > 0$. (We will show in Lemma 23 below that infinitely many such N exist. From inspection of the first six Hermite polynomials, we see that the smallest such N is $N = 6$.) Let $h \in L^1(\mathbb{R})$ be a positive symmetric function satisfying the following conditions:

(i) For all $\phi \in \mathcal{M}_{N-1}$,

$$\int_{\mathbb{R}} |\phi(x)|h(x)\, dx < \infty. \tag{7.1}$$

(Thus h can be regarded as an element of $\mathcal{M}'_{N-1}$; see Section 6.1 for definitions.)

(ii) h has divergent Nth moment: $\displaystyle\int_{\mathbb{R}} x^N h(x)\, dx = \infty$.

(iii) h has second moment < 2: $\displaystyle\int_{\mathbb{R}} x^2 h(x)\, dx < 2$.

(We note that the third condition can always be arranged by multiplying h by an appropriate constant.)

Starting with any such h we will construct a pair of distributions $f, g \in \mathcal{M}'_{N-3}$, which have finite moments up to order $N-3$ but divergent $(N-2)$nd moment. We will show that f and g have exactly two persistent edge contours each, which are symmetric in the coordinate $w = x/\sigma$. We will further show that there is a sequence of pairs $\{(w_i, \sigma_i)\}_{i\geq 1}$, with $w_i, \sigma_i > 0$ and σ_i increasing to infinity, such that

$$\begin{aligned} &\Delta\big(f * G_{\sigma_i}\big)\,(\sigma_i w_i) \geq 0 \geq \Delta\big(g * G_{\sigma_i}\big)\,(\sigma_i w_i) \quad \text{for } i \text{ odd} \\ &\Delta\big(f * G_{\sigma_i}\big)\,(\sigma_i w_i) \leq 0 \leq \Delta\big(g * G_{\sigma_i}\big)\,(\sigma_i w_i) \quad \text{for } i \text{ even}. \end{aligned} \tag{7.2}$$

These statements together imply that edge contours of f and g intersect on a sequence of scales tending to infinity. Finally, to obtain a violation of Marr's conjecture, we replace the distributions $f, g \in \mathcal{M}'_{N-3}$ by the integrable functions $f * G$ and $g * G$, whose edge contours are the same as those of f and g, but shifted by one unit in σ.

The argument consists of two parts. The first constructs f, g, and $\{(w_i, \sigma_i)\}_{i\geq 1}$, demonstrates the existence of two persistent, symmetric edge contours, and verifies (7.2). The second part shows that f and g have no other persistent edge contours. Condition (iii) above will only be invoked in the second part.

7.1. Part 1: iterative construction of f and g

We construct f, g, and $\{(w_i, \sigma_i)\}_{i\geq 1}$ inductively, similarly to the argument of Section 5. At each step k of the induction we will construct a pair of distributions $f_{k+1}, g_{k+1} \in \mathcal{M}'_{N-3}$ and pairs (w_{2k}, σ_{2k}) and $(w_{2k+1}, \sigma_{2k+1})$ such that (7.2) holds for $1 \leq i \leq 2k+1$, with f_{k+1} and g_{k+1} in place of f and g. After the induction, the final distributions f and g will be obtained as the weak-* limits of f_k and g_k, respectively, as $k \to \infty$.

Base step. As a base step, we construct distributions $f_1, g_1 \in \mathcal{M}'_{N-3}$ and a pair (w_1, σ_1) such that (7.2) holds for $i = 1$ with f_1 and g_1 in place of f and g. For arbitrary real numbers c_1 and d_1 with $d_1 > c_1 > 0$, let $C_1, D_1 \subset \mathbb{R}$ be the intervals $[-c_1, c_1]$ and $[-d_1, d_1]$, respectively. We define f_1 and g_1 by specifying their second derivatives $\Delta f_1, \Delta g_1 \in \mathcal{M}'_{N-1}$:

$$\begin{aligned} \Delta f_1 &= \delta^{(2)} + \chi_{C_1} h - \sum_{\substack{m \text{ even} \\ 0\leq m\leq N-2}} a_{1,m}\delta^{(m)} \\ \Delta g_1 &= \delta^{(2)} + \chi_{D_1} h - \sum_{\substack{m \text{ even} \\ 0\leq m\leq N-2}} b_{1,m}\delta^{(m)}. \end{aligned} \tag{7.3}$$

Here, χ_U denotes the characteristic function of $U \subset \mathbb{R}$, which has value 1 on U and zero elsewhere. The coefficients $a_{1,m}$ and $b_{1,m}$ in (7.3), for m even and $0 \leq m \leq N-2$, are set as

$$\begin{aligned} a_{1,m} &= \int_{C_1} \frac{x^m}{m!} h(x)\, dx \\ b_{1,m} &= \int_{D_1} \frac{x^m}{m!} h(x)\, dx. \end{aligned}$$

This guarantees that $\mu_2(\Delta f_1) = \mu_2(\Delta g_1) = 2$, and $\mu_n(\Delta f_1) = \mu_n(\Delta g_1) = 0$ for $0 \leq n \leq N-1, n \neq 2$. Thus, by construction, the moments of Δf_1 and Δg_1 coincide with those of $\delta^{(2)}$ to order $N-1$. (Note that the odd moments of Δf_1 and Δg_1 vanish due to the symmetry of h.) Since, in particular, the zeroth and first moments of Δf_1 and Δg_1 are zero, Corollary 18 guarantees that f_1 and g_1 are uniquely defined from (7.3) as elements of $\mathcal{M}'_{N-3}$. Moreover, Δf_1 and Δg_1 are compactly supported and therefore in $\mathcal{P}'$. Since $\mathcal{P}$ is closed under derivatives, $\mathcal{P}'$ is closed under antiderivatives; thus f_1 and g_1 are in $\mathcal{P}'$ as well.

Expanding $\Delta f_1 * G_\sigma$ and $\Delta g_1 * G_\sigma$ as in (3.1) (with Δf replacing f) and invoking (1.1), we can describe the edges of f_1 and g_1 in w and σ as the respective zeros of

$$\begin{aligned}\Delta\big(f_1 * G_\sigma\big)(\sigma w) &= \sigma^{-3} H_2(w) G(w) + \frac{\mu_N\big(\Delta f_1\big)}{N!}\sigma^{-N-1} H_N(w) G(w)\\ &\quad + \mathcal{O}(\sigma^{-N-2})\\ \Delta\big(g_1 * G_\sigma\big)(\sigma w) &= \sigma^{-3} H_2(w) G(w) + \frac{\mu_N\big(\Delta g_1\big)}{N!}\sigma^{-N-1} H_N(w) G(w)\\ &\quad + \mathcal{O}(\sigma^{-N-2}) \qquad (\sigma \to \infty).\end{aligned} \tag{7.4}$$

For $|w|$ close to 1, both coefficients of σ^{-N-1} in (7.4) are positive. This follows since h is positive—hence so are $\mu_N(\Delta f_1)$ and $\mu_N(\Delta g_1)$—and $H_N(\pm 1)$ is positive by assumption. Furthermore, since $C_1 \subset D_1$, we have $\mu_N(\Delta f_1) < \mu_N(\Delta g_1)$ and so the larger of the two coefficients of σ^{-N-1} is that associated to g_1. We conclude from this analysis of the coefficients in (7.4) that for any fixed w sufficiently close to ± 1,

$$\Delta\big(f_1 * G_\sigma\big)(\sigma w) < \Delta\big(g_1 * G_\sigma\big)(\sigma w) \tag{7.5}$$

for all sufficiently large σ.

We also observe from (7.4) that f_1 and g_1 each have (at least) two persistent edge contours, corresponding to the roots $w = \pm 1$ of $H_2(w) = w^2 - 1$. By Corollary 14, each of these edge contours can be parameterized with w as a function of σ; moreover, by the symmetry of f_1 and g_1, these parameterizations can be written in the form $w = \pm e_{f_1}(\sigma)$ and $w = \pm e_{g_1}(\sigma)$. Since the coefficients of σ^{-N-1} in (7.4) are both positive for $|w| \approx 1$ as previously stated, and the coefficients of σ^{-3} have the sign of $H_2(w) = w^2 - 1$, $e_{f_1}(\sigma)$ and $e_{g_1}(\sigma)$ both approach 1 from below as $\sigma \to \infty$ (see Figure 4). Therefore, for any w_1 less than but sufficiently close to 1, the line $w = w_1$ intersects both edge contours described by $w = e_{f_1}(\sigma)$ and $w = e_{g_1}(\sigma)$. Combining this observation with (7.5) implies that for w_1 less than but sufficiently close to 1, there is a range of σ values satisfying

$$\Delta\big(f_1 * G_\sigma\big)(\sigma w_1) < 0 < \Delta\big(g_1 * G_\sigma\big)(\sigma w_1). \tag{7.6}$$

(See Figure 4.) Moreover, the upper bound of σ values satisfying (7.6) increases without bound as w_1 increases to 1. Fix w_1 and $\sigma = \sigma_1$ such that (7.6) is satisfied. We have thus constructed $f_1, g_1 \in \mathcal{P}' \subset \mathcal{M}'_{N-3}$ and the pair (w_1, σ_1), as required for the base step.

Inductive hypothesis. Let k be an arbitrary natural number, and suppose we have distributions $f_k, g_k \in \mathcal{P}' \subset \mathcal{M}'_{N-3}$ defined by

$$\Delta f_k = \delta^{(2)} + \chi_{C_k} h - \sum_{\substack{m \text{ even} \\ 0 \le m \le N-2}} a_{k,m} \delta^{(m)}$$

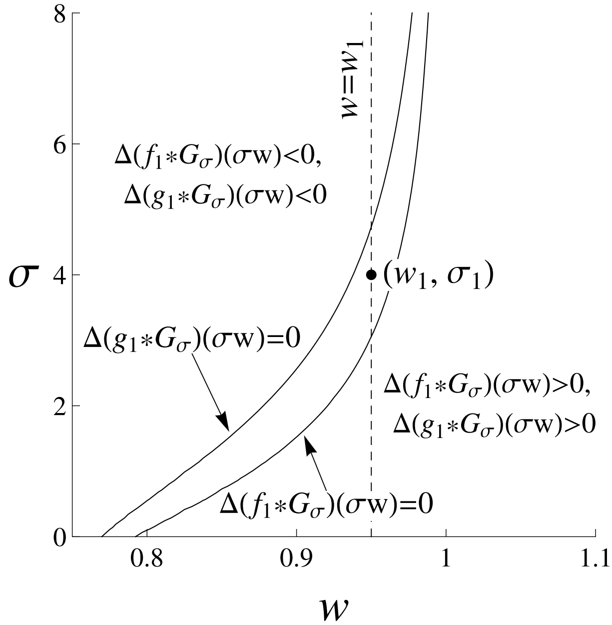

Figure 4: The edge contours of example distributions f_1 and g_1 are shown together with a choice of w_1 and σ_1.

$$\Delta g_k = \delta^{(2)} + \chi_{D_k} h - \sum_{\substack{m \text{ even} \\ 0 \le m \le N-2}} b_{k,m} \delta^{(m)}.$$

Here $C_k, D_k \subset \mathbb{R}$ are compact and symmetric about the origin, and

$$\begin{aligned} a_{k,m} &= \int_{C_k} \frac{x^m}{m!} h(x)\, dx \\ b_{k,m} &= \int_{D_k} \frac{x^m}{m!} h(x)\, dx, \qquad 0 \le m \le N-2, \end{aligned}$$

so that as in the base case, the moments of Δf_k and Δg_k agree with those of $\delta^{(2)}$ to order $N-1$. Suppose further that there are pairs $(w_1, \sigma_1), \ldots,$ $(w_{2k-1}, \sigma_{2k-1})$, with σ_i increasing in i, satisfying

(7.7)
$$\begin{aligned} &\Delta\big(f_k * G_{\sigma_i}\big)(\sigma_i w_i) < 0 < \Delta\big(g_k * G_{\sigma_i}\big)(\sigma_i w_i) \quad \text{for } i \text{ odd} \\ &\Delta\big(f_k * G_{\sigma_i}\big)(\sigma_i w_i) > 0 > \Delta\big(g_k * G_{\sigma_i}\big)(\sigma_i w_i) \quad \text{for } i \text{ even}, \quad 1 \le i \le 2k-1. \end{aligned}$$

Induction step. The induction step consists of two halves. In the first, we construct the distribution f_{k+1} and the pair (w_{2k}, σ_{2k}) from f_k and g_k. In the second, we construct g_{k+1} and the pair $(w_{2k+1}, \sigma_{2k+1})$, from f_{k+1} and g_k.

For the first half, we will construct $f_{k+1} \in \mathcal{P}'$ of the same form as above,

$$\Delta f_{k+1} = \delta^{(2)} + \chi_{C_{k+1}} h - \sum_{\substack{m \text{ even} \\ 0 \le m \le N-2}} a_{k+1,m} \delta^{(m)},$$

with

$$a_{k+1,m} = \int_{C_{k+1}} \frac{x^m}{m!} h(x)\, dx, \tag{7.8}$$

such that

(a) the relationships (7.7) are preserved,

$$\begin{aligned} &\Delta\big(f_{k+1} * G_{\sigma_i}\big)(\sigma_i w_i) < 0 < \Delta\big(g_k * G_{\sigma_i}\big)(\sigma_i w_i) \quad \text{for } i \text{ odd} \\ &\Delta\big(f_{k+1} * G_{\sigma_i}\big)(\sigma_i w_i) > 0 > \Delta\big(g_k * G_{\sigma_i}\big)(\sigma_i w_i) \quad \text{for } i \text{ even} \\ &\qquad\qquad 1 \le i \le 2k-1, \end{aligned} \tag{7.9}$$

(b) $\mu_N(\Delta f_{k+1}) > \mu_N(\Delta g_k) + 1$.

We do this by setting $C_{k+1} = C_k \cup C'_{k+1}$ where $C'_{k+1} = [-c_{k+1}, -c'_{k+1}] \cup [c'_{k+1}, c_{k+1}]$ for some appropriately chosen positive real numbers c_{k+1} and c'_{k+1} with $c_{k+1} > c'_{k+1} > c_k$, to be determined later. Note

$$\begin{aligned} &\big((\Delta f_{k+1} - \Delta f_k) * G_\sigma\big)(\sigma w) \\ &= \left(\left(\chi_{C'_{k+1}} h - \sum_{\substack{m \text{ even} \\ 0 \le m \le N-2}} a'_{k+1,m} \delta^{(m)} \right) * G_\sigma \right)(\sigma w) \\ &= \int_{C'_{k+1}} h(y) G_\sigma(\sigma w - y)\, dy - \sum_{\substack{m \text{ even} \\ 0 \le m \le N-2}} a'_{k+1,m} G_\sigma^{(m)}(\sigma w), \end{aligned} \tag{7.10}$$

where

$$a'_{k+1,m} = \int_{C'_{k+1}} \frac{x^m}{m!} h(x)\, dx = 2 \int_{c'_{k+1}}^{c_{k+1}} \frac{x^m}{m!} h(x)\, dx.$$

By (7.1), the integrals $\int_{\mathbb{R}} \frac{x^m}{m!} h(x)\, dx$ converge for all nonegative integers $m \le N-1$. It follows that the coefficients $a'_{k+1,m}$ can be made arbitrarily small uniformly over all choices of c_{k+1}, by choosing a sufficiently large value of c'_{k+1}. The decay properties of G_σ and integrability of h imply that for any fixed σ and w, the first term of (7.10)—and hence the full quantity (7.10)—can also be made arbitrarily small uniformly over c_{k+1}, by a sufficiently large choice of c'_{k+1}. Since this holds in particular for $w = w_i$ and $\sigma = \sigma_i$, $1 \le i \le 2k-1$, we can choose c'_{k+1} such that condition (7.9) holds regardless of the value later chosen for c_{k+1}, validating condition (a). We fix such a c'_{k+1}. Then since h is positive and has divergent Nth moment, a sufficiently large choice of c_{k+1} will guarantee $\mu_N(\chi_{C_{k+1}} h) > \mu_N(\chi_{D_k} h) + 1$, and hence $\mu_N(\Delta f_{k+1}) > \mu_N(\Delta g_k) + 1$, validating condition (b).

We now construct the pair (w_{2k}, σ_{2k}). By our choice of the coefficients $a_{k+1,m}$ in (7.8), the moments of Δf_{k+1} coincide with those of $\delta^{(2)}$ through order $N-1$ (as do the moments of Δg_k according to our inductive assumption). Furthermore, condition (b) implies that $\mu_N(\Delta f_{k+1}) > \mu_N(\Delta g_k)$. These observations enable us, using an argument similar to that used in the base case above, to choose $w_{2k} > 0$ and $\sigma_{2k} > \sigma_{2k-1} + 1$ satisfying

$$\Delta(f_{k+1} * G_{\sigma_{2k}})\,(\sigma_{2k} w_{2k}) > 0 > \Delta(g_k * G_{\sigma_{2k}})\,(\sigma_{2k} w_{2k})\,.$$

We observe that since Δf_{k+1} is compactly supported, it is in $\mathcal{P}'$ and hence also in $\mathcal{M}'_{N-1}$. This finishes the first half of the induction step.

For the second half of the induction step we construct, in similar fashion, a distribution $g_{k+1} \in \mathcal{P}' \subset \mathcal{M}'_{N-3}$ satisfying

$$\Delta g_{k+1} = \delta^{(2)} + \chi_{D_{k+1}} h - \sum_{\substack{m \text{ even} \\ 0 \le m \le N-2}} b_{k+1,m} \delta^{(m)},$$

with

$$b_{k+1,m} = \int_{D_{k+1}} \frac{x^m}{m!} h(x)\, dx,$$

where $D_{k+1} \subset \mathbb{R}$ is compact and symmetric about the origin, such that

(a) the relationships (7.7) are preserved now for i up to $2k$ rather than $2k-1$,

$$\begin{aligned} &\Delta\big(f_{k+1} * G_{\sigma_i}\big)(\sigma_i w_i) < 0 < \Delta\big(g_{k+1} * G_{\sigma_i}\big)(\sigma_i w_i) \quad \text{for } i \text{ odd} \\ &\Delta\big(f_{k+1} * G_{\sigma_i}\big)(\sigma_i w_i) > 0 > \Delta\big(g_{k+1} * G_{\sigma_i}\big)(\sigma_i w_i) \quad \text{for } i \text{ even}, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 1 \le i \le 2k, \end{aligned}$$

(b) $\mu_N(\Delta g_{k+1}) > \mu_N(\Delta f_{k+1}) + 1$.

After fixing g_{k+1} we choose w_{2k+1} and $\sigma_{2k+1} > \sigma_{2k} + 1$ such that

$$\Delta(f_{k+1} * G_{\sigma_{2k+1}})\,(\sigma_{2k+1}w_{2k+1}) < 0 < \Delta(g_{k+1} * G_{\sigma_{2k+1}})\,(\sigma_{2k+1}w_{2k+1})\,.$$

To summarize, we have constructed distributions $f_{k+1}, g_{k+1} \in \mathcal{P}' \subset \mathcal{M}'_{N-3}$ and pairs (w_{2k}, σ_{2k}) and $(w_{2k+1}, \sigma_{2k+1})$ such that (7.7) holds with k replaced by $k+1$. This completes the induction step.

Conclusion of induction. By induction, we have constructed two sequences of distributions $\{f_k\}_{k\geq 1}$, $\{g_k\}_{k\geq 1}$ and pairs $\{(w_i, \sigma_i)\}_{i\geq 1}$ such that (7.7) holds for all $k \geq 1$.

Limit construction of f and g. We claim that the sequence $\left\{\Delta f_k\right\}_{k\geq 1}$ converges in the weak-* topology on $\mathcal{M}'_{N-1}$ to the distribution

$$\Delta f = \delta^{(2)} + \chi_C h - \sum_{\substack{m \text{ even} \\ 0\leq m\leq N-2}} a_m \delta^{(m)}, \tag{7.11}$$

where

$$a_m = \int_C \frac{x^m}{m!} h(x)\, dx, \qquad C = \bigcup_{k=1}^{\infty} C_k, \tag{7.12}$$

and similarly for $\left\{\Delta g_k\right\}_{k\geq 1}$, with $D = \bigcup_k D_k$ in place of C. To verify this claim, consider an arbitrary test function $\phi \in \mathcal{M}_{N-1}$. For each $k \geq 0$ we have

$$\langle \Delta f_k, \phi\rangle = \phi^{(2)}(0) + \int_{\mathbb{R}} \chi_{C_k}(x) h(x)\phi(x)\, dx - \sum_{\substack{m \text{ even} \\ 0\leq m\leq N-2}} a_{k,m}\phi^{(m)}(0). \tag{7.13}$$

The integrand $\chi_{C_k}(x)h(x)\phi(x)$ of the middle term of (7.13) is bounded in absolute value by the function $h(x)|\phi(x)|$—which is integrable by (7.1)—and converges pointwise to $\chi_C(x)h(x)\phi(x)$. It follows from the dominated convergence theorem that the middle term of (7.13) converges to the finite quantity

$$\int_{\mathbb{R}} \chi_C(x) h(x)\phi(x)\, dx,$$

as desired. To verify convergence of the third term of (7.13), it suffices to show that for each even m, $0 \leq m < N$, the sequence $\{a_{k,m}\}_{k\geq 0}$ converges to a_m as given by (7.12). Since each $a_{k,m}$ is a constant multiple of the integral of $\chi_{C_k}(x)\, x^m h(x)$ and $x^m \in \mathcal{M}_{N-1}$ for $0 \leq m < N$, convergence of each sequence $\{a_{k,m}\}_{k\geq 0}$ follows with the same argument used to prove convergence of the middle term of (7.13). We conclude that $\{\Delta f_k\}$ converges as claimed, and a similar argument establishes the convergence of $\{\Delta g_k\}$.

We have thus constructed Δf and Δg as elements of $\mathcal{M}'_{N-1}$. Since $G_\sigma \in \mathcal{M}_{N-1}$ for each $\sigma > 0$, the relationships (7.7) are preserved under the weak-* limits $\Delta f_k \to \Delta f$, $\Delta g_k \to \Delta g$, with $\leq$ in place of $<$ as in (7.2). We define $f, g \in \mathcal{M}'_{N-3}$ as the second antiderivatives of Δf and Δg respectively. (This construction is allowed by Corollary 18 since the zeroth and first moments of Δf and Δg are zero. It can also be shown that f and g are the respective limits of the sequences $\{f_k\}$ and $\{g_k\}$ in the weak-* topology on $\mathcal{M}'_{N-3}$, but we will not use this fact.)

Properties of f and g. We know the following about $f, g \in \mathcal{M}'_{N-3}$: They are symmetric about the origin since C and D are. It can be seen from (7.11) that the moments of Δf coincide with those of $\delta^{(2)}$ through order $N-1$, and thus f and g each have a pair of persistent edge contours, also symmetric about the origin, approaching $w = \pm 1$. Since (7.2) is satisfied, these edge contours of intersect on an infinite sequence of scales. Finally, since we required $\sigma_{i+1} > \sigma_i + 1$ for all $i \geq 1$, the sequence $\{\sigma_i\}_{i\geq 1}$ is increasing and diverges to positive infinity. This completes the first part of the proof of Corollary 3(c): the existence of f and g with edge contours that intersect at a sequence of scales tending to infinity.

It remains to be shown that there are no other edge contours of f and g that might allow these distributions to be distinguished from their edges at the scales $\{\sigma_i\}_{i\geq 1}$. This will be shown in the second part. For the second part we will need that, since Condition (b) on f_{k+1} and g_{k+1} holds for each k,

$$\mu_N(\Delta f_{k+1}) > \mu_N(\Delta g_k) + 1 \quad \text{and} \quad \mu_N(\Delta g_{k+1}) > \mu_N(\Delta f_{k+1}) + 1,$$

and it follows that

$$\int_C x^N h(x)\, dx = \infty \qquad \text{and} \qquad \int_D x^N h(x)\, dx = \infty. \tag{7.14}$$

7.2. Part 2: non-existence of divergent edge contours

For the second (final) part of the argument, we show that the persistent edge contours approaching $w = \pm 1$ are the only persistent edge contours of f and g. We state this as a theorem:

Theorem 21. *Let N be an even number greater than or equal to 4, and consider a positive symmetric function $\tilde{h} \in L^1(\mathbb{R})$ satisfying the following conditions:*

(i) For all $\phi \in \mathcal{M}_{N-1}$, $\int_{\mathbb{R}} |\phi(x)|\tilde{h}(x)\, dx < \infty$. *(Thus* $\tilde{h}$ *can be regarded as an element of* $\mathcal{M}'_{N-1}$.*)*

(ii) $\tilde{h}$ *has infinite Nth moment:* $\displaystyle\int_{\mathbb{R}} x^N \tilde{h}(x)\, dx = \infty$.

(iii) $\tilde{h}$ *has second moment* < 2*:* $\displaystyle\int_{\mathbb{R}} x^2 \tilde{h}(x)\, dx < 2$.

Define the distribution $u \in \mathcal{M}'_{N-3}$ *by its second derivative:*

$$\Delta u = \delta^{(2)} + \tilde{h} - \sum_{\substack{m \text{ even} \\ 0 \le m \le N-2}} c_m \delta^{(m)}, \qquad c_m = \int_{\mathbb{R}} \frac{x^m}{m!} \tilde{h}(x)\, dx. \tag{7.15}$$

Then u *has exactly two persistent edge contours, which approach* $w = \pm 1$ *as* $\sigma \to \infty$.

This theorem can be applied to the distributions f and g (in place of u) by setting $\tilde{h} = \chi_C h$ and $\tilde{h} = \chi_D h$ respectively. This will complete the argument that the edge contours of f and g intersect on a sequence of scales tending to infinity.

The proof of Theorem 21 requires the following lemma:

Lemma 22. *Let* $\tilde{h} \in L^1(\mathbb{R})$ *and* c_m *be as in the statement of Theorem 21. For* $x, \sigma \in \mathbb{R}$, $x > 0$, *define*

$$Q(x,\sigma) = -c_0 G_\sigma(x) + \tilde{h} * G_\sigma(x).$$

Then for each $\sigma > 0$, $Q(x,\sigma)$ *has exactly two zeros in* x, *is negative between these zeros, and positive outside of them. Moreover, these zeros satisfy* $x/\sigma \to \pm 1$ *as* $\sigma \to \infty$.

Proof. First we show that $Q(0,\sigma) < 0$ for each $\sigma > 0$. To see this we expand

$$\begin{aligned} Q(0,\sigma) &= \frac{1}{\sqrt{2\pi}\sigma} \left(-c_0 + \int_{\mathbb{R}} \tilde{h}(y) e^{-y^2/(2\sigma^2)}\, dy \right) \\ &= \frac{1}{\sqrt{2\pi}\sigma} \int_{\mathbb{R}} \tilde{h}(y) \left(-1 + e^{-y^2/(2\sigma^2)} \right)\, dy. \end{aligned}$$

The integrand is negative for each $y \neq 0$ and $\sigma > 0$; therefore, $Q(0,\sigma) < 0$.

We now consider the absolute ratio, $R(x,\sigma)$, of the two terms in $Q(x,\sigma)$. This ratio can be written

$$R(x,\sigma) = \left(\tilde{h} * G_\sigma(x) \right) \Big/ \left(c_0 G_\sigma(x) \right) = \frac{1}{c_0} \int_{\mathbb{R}} \tilde{h}(y) e^{(2xy - y^2)/(2\sigma^2)}\, dy.$$

We note that $Q(x,\sigma)$ has the same sign as $R(x,\sigma)-1$. Our above observation that $Q(0,\sigma)<0$ implies that $R(0,\sigma)<1$. Using the symmetry of $\tilde{h}$, we can rewrite the ratio $R(x,\sigma)$ as

$$
\begin{aligned}
R(x,\sigma) &= \frac{1}{c_0}\int_0^\infty \tilde{h}(y)\left(e^{(2xy-y^2)/(2\sigma^2)}+e^{(-2xy-y^2)/(2\sigma^2)}\right)\,dy \\
&= \frac{2}{c_0}\int_0^\infty \tilde{h}(y)\,e^{-y^2/(2\sigma^2)}\cosh\left(\frac{xy}{\sigma^2}\right)\,dy.
\end{aligned}
\tag{7.16}
$$

We note that $\cosh(xy/\sigma^2)$ grows monotonically without bound in $|x|$ for each fixed $y>0$ and $\sigma>0$. Therefore, $R(x,\sigma)$ grows monotonically without bound in $|x|$ for fixed $\sigma>0$, surpassing 1 at exactly one value of $|x|$. This proves that $Q(x,\sigma)$ has exactly two zeros in x for each $\sigma>0$, and is negative between them and zero outside of them.

For the convergence claim, we note that the zeroth moment $\mu_0\big(\tilde{h}-c_0\delta^{(0)}\big)$ vanishes by the definition of c_0, while the first moment $\mu_1\big(\tilde{h}-c_0\delta^{(0)}\big)$ vanishes since $\tilde{h}$ is symmetric. Moment expansion (Corollary 20 with $M=2$), applied to the distribution $\tilde{h}-c_0\delta^{(0)}$, therefore implies that the quantity

$$
\sigma^3\left(\left(\tilde{h}-c_0\delta^{(0)}\right)*G_\sigma\right)(\sigma w)-\frac{\mu_2\left(\tilde{h}-c_0\delta^{(0)}\right)}{2!}H_2(w)G(w)
$$

converges to zero locally uniformly in w as $\sigma\to\infty$. The first term above is equal to $\sigma^3 Q(\sigma w,\sigma)$. It follows that, as $\sigma\to\infty$, the zeros of $Q(x,\sigma)$ satisfy $x/\sigma\to\pm1$, corresponding to the zeros $w=\pm1$ of $H_2(w)$. □

Proof of Theorem 21. We begin by applying the moment expansion (Corollary 20 with $M=2$) to the distribution $\Delta u\in\mathcal{M}'_{N-1}$. (Recall $N\geq4$ and thus $M<N-1$ for $M=2$.) Since the moments of Δu coincide with those of $\delta^{(2)}$ through order $N-1$, the quantity

$$
\sigma^3\Delta\big(u*G_\sigma\big)(\sigma w)-H_2(w)G(w)
$$

converges to zero locally uniformly in w, as $\sigma\to\infty$. Thus any persistent edge contours of u must either approach the roots $w=\pm1$ of $H_2(w)$, or diverge in w as $\sigma\to\infty$. We now show that the second case cannot occur.

Assume, to the contrary, that a persistent edge contour $Z\subset H_+$ of u diverges to (without loss of generality) $+\infty$ in w as $\sigma\to\infty$. Define a mapping $\sigma=s(x)$ so that for each x greater than or equal to some $x_0>0$, $(x,s(x))\in Z$. (There is some freedom in this construction, since a line $x=x'$ may intersect Z multiple times.) By Corollary 15, local parameterizations of

Z have no local maxima, so $s(x)$ can be chosen to be monotone increasing in x. However, $s(x)$ is not necessarily continuous—it may jump between branches of the set-valued function $S(x) = \{\sigma : (x, \sigma) \in Z\}$.

For all $x \geq x_0$, $(x, s(x))$ lies on an edge contour of u, so convolving (7.15) with $G_{s(x)}$ and applying (1.1) yields

$$
\begin{aligned}
(7.17) \qquad 0 &= (\Delta u) * G_{s(x)}(x) \\
&= s(x)^{-2} H_2\left(\frac{x}{s(x)}\right) G_{s(x)}(x) + \big(\tilde{h} * G_{s(x)}\big)(x) \\
&\qquad - \sum_{\substack{m \text{ even} \\ 0 \leq m \leq N-2}} c_m s(x)^{-m} H_m\left(\frac{x}{s(x)}\right) G_{s(x)}(x).
\end{aligned}
$$

(Here expressions $u * G_{s(x)}(x)$ are calculated by first evaluating the convolution $u * G_\sigma(x)$ and then substituting $\sigma = s(x)$. The argument of s is thus not considered part of the argument of $G_{s(x)}(\cdot)$ in the convolution.)

Since Z diverges to infinity in $w = x/s(x)$, we have

$$\lim_{x \to \infty} \frac{x}{s(x)} = \infty.$$

We consider two cases, depending on the asymptotic behavior of $x/s(x)^2$.

Case 1: $\liminf_{x \to \infty} x/s(x)^2 = 0$. In this case we rewrite the right-hand side of (7.17) as a sum of two expressions (separately enclosed in parentheses):

$$
\begin{aligned}
(7.18) \quad & \left(\left(\tilde{h} * G_{s(x)}\right)(x) - c_0 G_{s(x)}(x)\right) \\
+ & \left((1 - c_2) s(x)^{-2} H_2\left(\frac{x}{s(x)}\right) - \sum_{\substack{m \text{ even} \\ 4 \leq m \leq N-2}} c_m s(x)^{-m} H_m\left(\frac{x}{s(x)}\right)\right) G_{s(x)}(x).
\end{aligned}
$$

We will show that there is an x for which both of these expressions are positive, contradicting (7.17).

The first expression of (7.18) can be written as as $Q\big(x, s(x)\big)$, with $Q(x, \sigma)$ defined as in Lemma 22. Lemma 22 asserts that $Q(x, \sigma)$ has two zero curves, approaching $x/\sigma = \pm 1$, and is negative between these curves and positive outside of them. Since $\lim_{x \to \infty} x/s(x) = \infty$, the point $(x, s(x))$ lies outside of the zero curves of $Q(x, \sigma)$ for all sufficiently large x. Therefore, the first expression of (7.18) is positive for sufficiently large x.

The sign of the second expression of (7.18) is that of

$$(1-c_2)s(x)^{-2}H_2\left(\frac{x}{s(x)}\right)-\sum_{\substack{m \text{ even}\\ 4\le m\le N-2}} c_m s(x)^{-m}H_m\left(\frac{x}{s(x)}\right). \tag{7.19}$$

Since $\lim_{x\to\infty} x/s(x)=\infty$, each of the Hermite polynomials $H_m\big(x/s(x)\big)$ in (7.19) becomes dominated as $x\to\infty$ by its highest-order term, $\big(x/s(x)\big)^m$. Thus, for sufficiently large x, the sign of (7.19) coincides with the sign of

$$\begin{aligned}(1-c_2)s(x)^{-2}\left(\frac{x}{s(x)}\right)^2 &- \sum_{\substack{m \text{ even}\\ 4\le m\le N-2}} c_m s(x)^{-m}\left(\frac{x}{s(x)}\right)^m\\ =(1-c_2)\left(\frac{x}{\big(s(x)\big)^2}\right)^2 &- \sum_{\substack{m \text{ even}\\ 4\le m\le N-2}} c_m \left(\frac{x}{\big(s(x)\big)^2}\right)^m.\end{aligned} \tag{7.20}$$

This expression is a polynomial in the variable $x/\big(s(x)\big)^2$. Since we have assumed (for Case 1) that $\liminf_{x\to\infty} x/\big(s(x)\big)^2=0$, there exist arbitrarily large x for which the sign of (7.20) coincides with the sign of its lowest-order term's coefficient $1-c_2$. By Condition (iii) on $\tilde{h}$,

$$1-c_2=1-\int_{\mathbb{R}}\frac{x^2}{2!}\tilde{h}(x)\,dx>0.$$

Thus there exist arbitrarily large x for which (7.20)—and hence also the second expression of (7.18)—is positive. Since the first expression of (7.18) is positive for sufficiently large x, there are values of x for which both expressions in (7.18) are positive, contradicting (7.17). Thus Case 1 is impossible.

Case 2: $\liminf\limits_{x\to\infty} x/s(x)^2>0$. In this case we multiply both sides of (7.17) by $\sqrt{2\pi}s(x)$ and rewrite as

(7.21)

$$\left(s(x)^{-2}H_2\left(\frac{x}{s(x)}\right)-\sum_{\substack{m \text{ even}\\ 0\le m< N}} c_m s(x)^{-m}H_m\left(\frac{x}{s(x)}\right)\right)\exp\left(-\frac{x}{2}\frac{x}{s(x)^2}\right) + \int_{\mathbb{R}}\exp\left(-\frac{(x-y)^2}{2s(x)^2}\right)\tilde{h}(y)\,dy=0.$$

Since (in Case 2) $x/s(x)^2$ is bounded below for sufficiently large x, the quantity $\exp\left(-\frac{x}{2}\frac{x}{s(x)^2}\right)$ is bounded above by an exponentially decreasing function of x. Further, since $s(x)$ is monotone increasing, $s(x)^{-1}$ is bounded, and so the polynomial

$$s(x)^{-2}H_2\left(\frac{x}{s(x)}\right) - \sum_{\substack{m \text{ even} \\ 0\le m\le N-2}} c_m s(x)^{-m} H_m\left(\frac{x}{s(x)}\right)$$

has at most polynomial growth in x. Combining these bounds, it follows that the first term of (7.21) is absolutely bounded above for all sufficiently large x by a function $Ke^{-\gamma x}$, with $K, \gamma > 0$. The two terms of (7.21) sum to zero, so the second term also satisfies this bound, giving

$$Ke^{-\gamma x} > \int_{\mathbb{R}} \exp\left(-\frac{(x-y)^2}{2s(x)^2}\right) \tilde{h}(y)\, dy, \tag{7.22}$$

for sufficiently large x. Also for x sufficiently large,

$$\exp\left(-\frac{(x-y)^2}{2s(x)^2}\right) > \frac{1}{2}\chi_{[-1,1]}(x-y),$$

with $\chi_{[-1,1]}$ an indicator function as above. Combining with (7.22) yields

$$Ke^{-\gamma x} > \frac{1}{2}\int_{x-1}^{x+1} \tilde{h}(y)\, dy,$$

again for sufficiently large x. Multiplying by x^N and integrating from a sufficiently large x_0 to infinity,

$$K\int_{x_0}^{\infty} x^N e^{-\gamma x}\, dx > \frac{1}{2}\int_{x_0}^{\infty} x^N \int_{x-1}^{x+1} \tilde{h}(y)\, dy\, dx.$$

The left-hand side is finite, and thus the right-hand side is finite as well. Interchanging order of integration on the right-hand side and noting that the integrand is nonnegative,

$$\begin{aligned}\frac{1}{2}\int_{x_0}^{\infty} x^N \int_{x-1}^{x+1} \tilde{h}(y)\, dy\, dx &\ge \frac{1}{2}\int_{x_0+1}^{\infty} \left(\int_{y-1}^{y+1} x^N\, dx\right) \tilde{h}(y)\, dy \\ &\ge \frac{1}{2}\int_{x_0+1}^{\infty} 2(y-1)^N \tilde{h}(y)\, dy\end{aligned}$$

$$= \int_{x_0+1}^{\infty} \left(\frac{y-1}{y}\right)^N y^N \tilde{h}(y)\, dy$$
$$\geq \left(\frac{x_0}{x_0+1}\right)^N \int_{x_0+1}^{\infty} y^N \tilde{h}(y)\, dy.$$

Thus

$$\int_{x_0+1}^{\infty} y^N \tilde{h}(y)\, dy < \infty.$$

Since $\tilde{h}$ is symmetric, it follows that

$$\int_{\mathbb{R}} y^N \tilde{h}(y)\, dy < \infty.$$

But this contradicts Condition (ii) that the Nth moment of $\tilde{h}$ diverges. Thus Case 2 is also impossible. We conclude that there are no edge contours of u that diverge in w, and the only persistent edge contours of u are those for which $w \to \pm 1$ as $\sigma \to \infty$. □

To summarize, Theorem 21 shows that the only persistent edge contours of f and g are those that approach $w = \pm 1$. We showed in the first part (Section 7.1) that these edge contours intersect on a sequence of scales tending to infinity. Though f and g are distributions (rather than functions) we can take the convolutions $f * G$ and $g * G$ as initial functions to obtain a violation of Marr's conjecture. This completes the proof of Corollary 3(c).

We conclude this section by proving that there are infinitely many even N such that $H_N(\pm 1) > 0$. Recall that these conditions on N were required in the proof of Corollary 3(c). The lemma below shows that infinitely many such N can be chosen, and consequently, there exist counterexamples to unique determination having an arbitrarily large number of finite moments.

Lemma 23. *There exist infinitely many positive even integers N such that $H_N(\pm 1) > 0$.*

Proof. Consider the sequence $\{a_n\}_{n=3}^{\infty}$ with $a_n = H_n(1)$. The irreducibility of Hermite polynomials (see Section 3.3) implies that $a_n \neq 0$ for $n \geq 3$.

We will prove that, if there is a sign change from a_{n-1} to a_n for $n \geq 4$, the next sign change occurs either from a_{n+1} to a_{n+2} or a_{n+2} to a_{n+3}. This means that, in the sequence $\{a_n\}_{n=3}^{\infty}$, a sign change occurs every two or three elements.

The proof is based on the recurrence relation $H_{n+1}(x) = xH_n(x) - nH_{n-1}(x)$. Substituting $x = 1$ gives

$$a_{n+1} = a_n - na_{n-1}. \tag{7.23}$$

Suppose for some $n \geq 4$ that $a_{n-1} < 0$ and $a_n > 0$. Then Eq. (7.23) guarantees that $a_{n+1} > 0$. Similarly, if $a_{n-1} > 0$ and $a_n < 0$ then $a_{n+1} < 0$. Overall, if there is a sign change from a_{n-1} to a_n, then there is no sign change from a_n to a_{n+1}.

Now suppose for some $n \geq 4$ that a_{n-1}, a_n, and a_{n+1} are all positive. Eq. (7.23) implies that $a_{n+1} < a_n$, from which it follows that $a_{n+2} = a_{n+1} - (n+1)a_n < 0$. Therefore, there cannot be more than three consecutive positive elements of $\{a_n\}_{n=3}^{\infty}$. A similar argument shows there cannot be more than three consecutive negative elements either.

Overall, we have shown that a sign change occurs every two or three elements in the sequence $\{a_n\}_{n=3}^{\infty}$. It follows that if $a_n > 0$, then either $a_{n+2} > 0$ or $a_{n+4} > 0$. Since $a_6 = H_6(1) > 0$, there are infinitely many positive even integers n such that $a_n > 0$. Finally, since $H_N(-1) = H_N(1) = a_N$ for N even, there are infinitely many even N such that $H_N(\pm 1) > 0$. □

8. Discrete zero-crossings

One version of the conjecture investigated here arose in mathematical vision theory with regard to whether the multiscale edges of an image determine the image. It may be natural to ask for the purpose of applications whether our results extend to digital images and signals, which are discrete rather than continuous. In this context, the given data consist of *zero-crossings*—pairs of adjacent lattice points at which the scaled (discrete) wavelet transform of a function changes sign. At issue is whether such zero-crossings contain sufficient information about the function to allow for its unique determination.

To address this question within the framework of our paper, we formalize it in the following way: Consider a distribution f that is a sum of δ-distributions located at a finite set of integer points: $f(x) = \sum_{i=1}^{n} a_i \delta(x - x_i)$ where $a_i \in \mathbb{R}$ and $x_i \in \mathbb{Z}$. We define a (discrete) zero-crossing at scale σ, with respect to the Ricker wavelet $M(x)$, to be a pair of consecutive integers x and $x+1$ for which $f * M_\sigma(x)$ and $f * M_\sigma(x+1)$ have opposite sign. We ask whether f is uniquely determined by its zero-crossings at a sequence of scales $\{\sigma_j\}_{j=1}^{\infty}$ tending to infinity.

We answer this question in the negative, as stated in the following theorem. This suggests that, in contrast to the continuous case, the information contained in the (discrete) zero-crossings of $f * M_{\sigma_j}$ may be too coarse to allow for the unique determination of f.

Theorem 9. *There exist distributions $f(x) = \sum_{i=1}^{n} a_i \delta(x - x_i)$ and $g(x) = \sum_{i=1}^{m} b_i \delta(x - y_i)$ with $a_i, b_i \in \mathbb{R}$ and $x_i, y_i \in \mathbb{Z}$, that are not constant multiples* *of each other, such that the (discrete) zero-crossings of f and g with respect to the Ricker wavelet $M(x)$ coincide at a sequence of scales $\{\sigma_j\}_{j=1}^{\infty}$ tending to infinity.*

Proof. Fix $a > 0$. We let $f(x) = \delta(x)$ and

$$g(x) = \delta(x) + \frac{a}{2}\delta(x-1) + \frac{a}{2}\delta(x+1).$$

Convolving these distributions with the scaled Ricker wavelet yields

$$\begin{aligned} f * M_\sigma(x) &= M_\sigma(x) \\ g * M_\sigma(x) &= M_\sigma(x) + \frac{a}{2}\left(M_\sigma(x-1) + M_\sigma(x+1)\right) \\ &= H_2(x/\sigma)G_\sigma(x) \\ &\quad + \frac{a}{2}\left(H_2\left(\frac{x-1}{\sigma}\right)G_\sigma(x-1) + H_2\left(\frac{x+1}{\sigma}\right)G_\sigma(x+1)\right). \end{aligned}$$

Simplifying $g * M_\sigma(x)$ using the identities

$$\begin{aligned} G_\sigma(x+1) + G_\sigma(x-1) &= 2e^{-1/(2\sigma^2)}\,G_\sigma(x)\cosh\left(x/\sigma^2\right) \\ G_\sigma(x+1) - G_\sigma(x-1) &= -2e^{-1/(2\sigma^2)}\,G_\sigma(x)\sinh\left(x/\sigma^2\right), \end{aligned}$$

we write

$$g * M_\sigma(x) = \sigma^{-2}G_\sigma(x)\ Z(x,\sigma),$$

with

$$Z(x,\sigma) = x^2 - \sigma^2 + ae^{-1/(2\sigma^2)}\left((x^2-\sigma^2+1)\cosh\left(\frac{x}{\sigma^2}\right) - 2x\sinh\left(\frac{x}{\sigma^2}\right)\right).$$

Since $\sigma^{-2}G_\sigma(x)$ is always positive, the zero-crossings of $g*M_\sigma(x)$ and $Z(x,\sigma)$ coincide for each $\sigma > 0$.

We claim that $Z(x,\sigma)$, and hence $g * M_\sigma(x)$, has no more than two zero-crossings for sufficiently large σ. For this, it suffices to show that for all sufficiently large σ, $Z(x,\sigma)$ is convex in x. Moreover, since $x^2-\sigma^2$ is convex and $ae^{-1/(2\sigma^2)}$ is a positive constant in x, it further suffices to show the convexity of $(x^2-\sigma^2+1)\cosh\left(\frac{x}{\sigma^2}\right) - 2x\sinh\left(\frac{x}{\sigma^2}\right)$. We compute:

$$\begin{aligned} &\frac{d^2}{dx^2}\left((x^2-\sigma^2+1)\cosh(x/\sigma^2) - 2x\sinh(x/\sigma^2)\right) \\ &\qquad = \frac{1}{\sigma^4}\left((x^2+2\sigma^4-5\sigma^2+1)\cosh\left(\frac{x}{\sigma^2}\right) + (4\sigma^2-2)x\sinh\left(\frac{x}{\sigma^2}\right)\right). \end{aligned}$$

For $\sigma > \frac{1}{2}\sqrt{5+\sqrt{17}} \approx 1.51$, the polynomials $2\sigma^4 - 5\sigma^2 + 1$ and $4\sigma^2 - 2$ are both positive. Since $\cosh(x/\sigma^2)$ is positive and $x\sinh(x/\sigma^2)$ is nonnegative for all values of x and σ, it follows that $Z(x,\sigma)$ is convex for such σ. This verifies the claim that $Z(x,\sigma)$, and hence $g * M_\sigma(x)$, has no more than two zero-crossings for sufficiently large σ.

We now compare the scaled zero-crossings of g to those of f. The zero-crossings of $f * M_\sigma(x)$ are $x = \pm\sigma$. Suppose that the given sequence of scales $\{\sigma_j\}_{j=1}^\infty$ is restricted to the set $\mathbb{N} + \frac{3}{2} = \{\frac{5}{2}, \frac{7}{2}, \frac{9}{2}, \ldots\}$. Scales σ in this set have the properties that (i) $g*M_\sigma(x)$ has no more than two zero-crossings, and (ii) the (discrete) zero-crossings of $f * M_\sigma(x)$ occur at the pairs $(-\sigma - \frac{1}{2}, -\sigma + \frac{1}{2})$ and $(\sigma - \frac{1}{2}, \sigma + \frac{1}{2})$. We compute the values of $Z(x,\sigma)$ at the second pair:

$$Z\left(\sigma + \tfrac{1}{2}, \sigma\right) = \left(\sigma + \frac{1}{4}\right)\left(1 + ae^{-1/(2\sigma^2)}\cosh\left(\frac{\sigma + \frac{1}{2}}{\sigma^2}\right)\right) + ae^{-1/(2\sigma^2)}\left(\cosh\left(\frac{\sigma + \frac{1}{2}}{\sigma^2}\right) - 2(\sigma + \tfrac{1}{2})\sinh\left(\frac{\sigma + \frac{1}{2}}{\sigma^2}\right)\right).$$

Observing that $\lim_{\sigma\to\infty}\cosh(\sigma^{-1}) = \lim_{\sigma\to\infty}\sigma\sinh(\sigma^{-1}) = 1$, we obtain

$$Z\left(\sigma + \tfrac{1}{2}, \sigma\right) = (1+a)\sigma + \mathcal{O}(1) \qquad (\sigma \to \infty).$$

A similar computation shows that

$$Z\left(\sigma - \tfrac{1}{2}, \sigma\right) = -(1+a)\sigma + \mathcal{O}(1) \qquad (\sigma \to \infty).$$

Therefore, for all sufficiently large $\sigma \in \mathbb{N} + \frac{3}{2}$, $g * M_\sigma(x)$ also has a zero-crossing at the pair $(\sigma - \frac{1}{2}, \sigma + \frac{1}{2})$. By symmetry, $g*M_\sigma(x)$ has a zero-crossing at $(-\sigma - \frac{1}{2}, -\sigma + \frac{1}{2})$ as well. Since $g * M_\sigma(x)$ cannot have any other zero-crossings for $\sigma \in \mathbb{N} + \frac{3}{2}$, we conclude that there is a sequence of scales $\{\sigma_j\}_{j=1}^\infty$, tending to infinity, at which the (discrete) zero-crossings of $f * M_{\sigma_j}(x)$ and $g * M_{\sigma_j}(x)$ coincide. □

9. Uniqueness of heat equation solutions

Our results also yield a uniqueness condition for solutions to the heat equation (1.3). If it is known that $F(x,t)$ solves (1.3) for some initial condition $f \in \mathcal{P}'_\gamma \cap L^1(\mathbb{R}^d)$, then by Corollary 3(a), both f and F are uniquely determined (up to a multiplicative constant) by the zeros of $F_{xx}(x, t_j)$ for any sequence $\{t_j\}_{j=1}^\infty$ of positive reals with a positive or infinite limit point.

As stated in Theorem 8 (Section 1.4), a similar result holds for the zeros of F rather than F_{xx} provided it is known that that the second integral

$$a(x) = \int_{-\infty}^{x} \int_{-\infty}^{y} f(z)\, dz\, dy \tag{9.1}$$

is in $\mathcal{P}'_\gamma \cap L^1(\mathbb{R}^d)$. (In particular this requires $\mu_0(f) = \mu_1(f) = 0$.) Letting $A(x,t)$ be the heat equation solution with initial condition $A(x,0) = a(x)$, Theorem 8 follows from applying Corollary 3(a) to the zeros of $A_{xx} = F$.

The condition $a \in \mathcal{P}'_\gamma \cap L^1(\mathbb{R}^d)$ above cannot be dispensed with. To see this, let $f_1(x)$ and $f_2(x)$ be distinct anti-symmetric functions that are positive for $x > 0$ and negative for $x < 0$. The respective solutions of (1.3) with initial conditions given by such f_1 and f_2 have the same zero set, consisting only of the line $x = 0$. In this case, f_1 and f_2 have positive first moment, so their respective second integrals a_1 and a_2, defined as in (9.1), are not in $\mathcal{P}'_\gamma \cap L^1(\mathbb{R}^d)$.

Theorem 8 appears to be a new type of uniqueness theorem for the heat equation. In particular, it requires a type of global agreement between two functions in order to imply their identity. In contrast, most heat equation uniqueness theorems [20, 6] are based on local agreement to infinite order.

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