

Stochastic metamorphosis in imaging science

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In honour of David Mumford on his 80th birthday!

In the pattern matching approach to imaging science, the process of *metamorphosis* in template matching with dynamical templates was introduced in [31]. In [17] the metamorphosis equations of [31] were recast into the Euler-Poincaré variational framework of [16] and shown to contain the equations for a perfect complex fluid [14]. This result related the data structure underlying the process of metamorphosis in image matching to the physical concept of order parameter in the theory of complex fluids [12]. In particular, it cast the concept of Lagrangian paths in imaging science as deterministically evolving curves in the space of diffeomorphisms acting on image data structure, expressed in Eulerian space. (In contrast, the landmarks in the standard LDDMM approach are Lagrangian.)

For the sake of introducing an Eulerian uncertainty quantification approach in imaging science, we extend the method of metamorphosis to apply to image matching along *stochastically* evolving time dependent curves on the space of diffeomorphisms. The approach will be guided by recent progress in developing stochastic Lie transport models for uncertainty quantification in fluid dynamics in [19, 8].

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1. Introduction

In recent work [1, 3], a new method of modelling variability of shapes has been introduced. This method uses a theory of stochastic perturbations consistent with the geometry of the diffeomorphism group corresponding to the Large Deformation Diffeomorphic Metric Mapping framework (LDDMM, see [35]). In particular, the method introduces stochastic Lie transport along stochastic curves in the diffeomorphism group of smooth invertible transformations. It models the development of variability as observed, for example, when human organs are influenced by disease processes, as analysed in computational anatomy [36]. It also provides a framework including a Hamiltonian formulation for quantifying uncertainty in the development of shape atlases in computational anatomy. Hamiltonian methods for deterministic computational anatomy were recently reviewed in [27].

The theory developed in [1, 3] treats LDDMM as a flow and uses methods based on stochastic fluid dynamics introduced in [19]. It addresses the problem of uncertainty quantification by introducing spatially correlated noise which respects the geometric structure of the data. Thus, the method provides a new way of characterising stochastic variability of shapes using spatially correlated noise in the context of the standard LDDMM framework. Numerical methods for addressing stochastic variability of shapes with landmark data structure have also been developed in [22, 21, 1, 3].

Although the examples were limited to landmark dynamics in the work [1, 3], it was clear that Lie-transport noise can be applied to any of the data structures used in the LDDMM framework, because it is compatible with the transformation theory on which LDDMM is based. The LDDMM theory was initiated by [32, 7, 9, 28, 5] building on the pattern theory of [13]. The LDDMM approach models shape comparison (registration) as dy-

namical transformations from one shape to another whose data structure is defined as a tensor valued smooth embedding. These shape transformations are expressed in terms of the action of diffeomorphic flows, regarded as time dependent curves of smooth transformations of shape spaces. This provides a unified approach to shape modelling and shape analysis, valid for a range of structures such as landmarks, curves, surfaces, images, densities and tensor-valued images. For any such data structure, the optimal shape deformations are described via the Euler-Poincaré equation of the diffeomorphism group, usually referred to as the EPDiff equation [23, 20, 36]. The work [1, 3] showed how to obtain a stochastic EPDiff equation valid for any data structure, and in particular for the finite dimensional representation of images based on landmarks. For this purpose, the work [1, 3] followed the Euler-Poincaré derivation of LDDMM of [6] based on geometric mechanics [25, 18] and the use of momentum maps to represent images and shapes. The introduction of Lie-transport noise into the EPDiff equation was implemented as cylindrical noise, obtained by pairing the deterministic momentum map the sum over eigenvectors of the spatial covariance of Stratonovich noise, each with its own Brownian motion.

The work [1, 3] was not the first to consider stochastic evolutions in LDDMM. Indeed, [33, 34] and more recently [26] had already investigated the possibility of stochastic perturbations of landmark dynamics. In the earlier works, the noise was introduced into the landmark momentum equations, as though it were an external random force acting on each landmark independently. In [26], an extra dissipative force was added to balance the energy input from the noise and to make the dynamics correspond to a certain type of heat bath used in statistical physics. In contrast, the work [1, 3] instead introduced Eulerian Stratonovich noise into the reconstruction relation used to find the deformation flows from the action of the velocity vector fields on their corresponding momenta, which are solutions of the EPDiff equation [20, 35].

As shown in [1, 3], this derivation of stochastic models is compatible with the Euler-Poincaré constrained variational principle, it preserves the momentum map structure and yields a stochastic EPDiff equation with a novel type of multiplicative noise, depending on both the gradient and the magnitude of the solution. The model in [1, 3] was based on the previous works [19, 2], where, respectively, stochastic perturbations of infinite and finite dimensional mechanical systems were considered. The Eulerian nature of this type of noise implies that the noise correlation depends on the image position and not, as for example in [33, 26], on the landmarks themselves. This property explains why the noise is compatible with any data structure while retaining the freedom in the choice of its spatial correlation.

The present work extends the Euler-Poincaré variational framework for the metamorphosis approach of [31, 17] from the deterministic setting to the stochastic setting. Section 2 reviews the derivation of the deterministic metamorphosis equations as cast by [17] into the Euler-Poincaré variational framework of [16], as well as several other developments, including the Hamilton-Pontryagin principle and two different Hamiltonian formulations of deterministic metamorphosis. Section 3 introduces metamorphosis by stochastic Lie transport and traces out its preservation and modification of the deterministic mathematical structures reviewed in Section 2. In Section 4 we summarize our results, discuss future work and mention a few open problems in mathematical analysis of stochastic partial differential equations that have been raised by the present work.

Thus, for the sake of potential applications in uncertainty quantification, this paper extends the method of metamorphosis for image registration to enable its application to image matching along stochastically time dependent curves on the space of diffeomorphisms.

2. Review of metamorphosis using deterministic Lie transport

Section summary. Before introducing stochasticity into Lie transport in the next section, this section provides notation and definitions for the general problem of metamorphoses in the deterministic case. Several formulations of the problem are given from different perspectives. These formulations are reviewed in detail, because the introduction of Lie transport stochasticity in the following sections will preserve all of the mathematical structures described in this section, although we will discuss only the last, Hamilton-Pontryagin, formulation for the stochastic case.

2.1. Notation

In the pattern matching approach to imaging science, the process of “metamorphosis” in template matching with dynamical templates was introduced in [31]. In [17] the metamorphosis equations of [31] were recast into the Euler-Poincaré variational framework of [16] and shown to contain the equations for a perfect complex fluid [14]. This result connected the data structure underlying the process of metamorphosis in image matching to the physical concept of order parameter in the theory of complex fluids. After developing the general theory in [17], various examples were reinterpreted, including point set, image and density metamorphosis. Finally, the issue was discussed of matching measures with metamorphosis, for which existence theorems for the initial and boundary value problems were provided. For more recent de-

velopments the the metamorphosis equations as well as numerical methods especially designed for metamorphosis, see [29].

Let N be manifold, which is acted upon by a Lie group G . The manifold N contains what we can refer to as “deformable objects” and G is the group of deformations, which is the group of diffeomorphisms acting on the manifold N in our applications. Several examples for the space N were developed in the Euler-Poincaré context in [17].

Definition 1. A *metamorphosis* [31] is a pair of curves $(g_t, \eta_t) \in G \times N$ parameterized by time t , with $g_0 = \text{id}$. Its **image** is the curve $n_t \in N$ defined by the action $n_t = g_t \cdot \eta_t$, where subscript t indicates explicit dependence on time, t . The quantities g_t and η_t are called, respectively, the **deformation part** of the metamorphosis, and its **template part**. When η_t is constant, the metamorphosis is said to be a **pure deformation**. In the general case, the image is a combination of a deformation and template variation.

Following [31, 17], we will use either letters η or n to denote elements of N , the former being associated to the template part of a metamorphosis, and the latter to its image.

The variational problem we shall study optimizes over metamorphoses (g_t, η_t) by minimizing, for some Lagrangian $L : TG \times TN \rightarrow \mathbb{R}$, the action integral

$$(2.1) \quad S = \int_0^1 L(g_t, \dot{g}_t, \eta_t, \dot{\eta}_t) dt,$$

with fixed endpoint conditions for the initial and final images n_0 and n_1 (with $n_t = g_t \eta_t$) and $g_0 = \text{id}_G$. That is, the images n_t are constrained at the end-points, with the initial condition $g_0 = \text{id}$.

Let \mathfrak{g} denote the Lie algebra of the Lie group G . We will consider Lagrangians defined on $TG \times TN$, that satisfy the following invariance conditions: there exists a function ℓ defined on $\mathfrak{g} \times TN$ such that

$$(2.2) \quad L(g, U_g, \eta, \xi_\eta) = \ell(U_g g^{-1}, g\eta, g\xi_\eta).$$

In other words, L is invariant under the right action of G on $G \times N$ defined by $(g, \eta)h = (gh, h^{-1}\eta)$.

For a metamorphosis (g_t, η_t) , we therefore have a reduced Lagrangian, upon defining $u_t = \dot{g}_t g_t^{-1}$, $n_t = g_t \eta_t$ and $\nu_t = g_t \dot{\eta}_t$, given by

$$(2.3) \quad L(g_t, \dot{g}_t, \eta_t, \dot{\eta}_t) = \ell(u_t, n_t, \nu_t).$$

The Lie derivative with respect to a vector field X will be denoted \mathcal{L}_X . The Lie algebra of G is identified with the set of right invariant vector fields $U_g = ug$, $u \in T_{\text{id}}G = \mathfrak{g}$, $g \in G$, and we will use the notation $\mathcal{L}_u = \mathcal{L}_U$.

The Lie bracket $[u, v]$ on the Lie algebra of smooth vector fields \mathfrak{g} is defined by

$$(2.4) \quad \mathcal{L}_{[u,v]} = -(\mathcal{L}_u \mathcal{L}_v - \mathcal{L}_v \mathcal{L}_u)$$

and the associated adjoint operator is $\text{ad}_u v = [u, v]$. Letting $I_g(h) = ghg^{-1}$ and $Ad_v g = \mathcal{L}_v I_g(\text{id})$, we also have $\text{ad}_u v = \mathcal{L}_u(Ad_v)(\text{id})$. When G is a group of diffeomorphisms, this yields $\text{ad}_u v = duv - dvu$.

The pairing between a linear form μ and a vector field u will be denoted $\langle \mu, u \rangle$. Duality with respect to this pairing will be denoted with an asterisk. For example, N^* is the dual space of the manifold N with respect to this pairing.

When G acts on a manifold \tilde{N} , the diamond operator (\diamond) is defined on $\tilde{N}^* \times \tilde{N}$ and takes values in the dual Lie algebra \mathfrak{g}^* . That is, $\diamond : \tilde{N}^* \times \tilde{N} \rightarrow \mathfrak{g}^*$. For $\tilde{n}^* \in \tilde{N}^*$ and $\tilde{n} \in \tilde{N}$ the diamond operation is defined by

$$(2.5) \quad \langle \tilde{n}^* \diamond \tilde{n}, u \rangle_{\mathfrak{g}} := -\langle \tilde{n}^*, u\tilde{n} \rangle_{T\tilde{N}},$$

where the action of a vector field $u \in \mathfrak{g}$ on $\tilde{n} \in \tilde{N}$ is denoted by simple concatenation, $u\tilde{n} \in T\tilde{N}$. For example, the Lie algebra action of the vector field $u \in \mathfrak{g}$ on $\tilde{n} \in \tilde{N}$ is denoted $u\tilde{n} = \mathcal{L}_u \tilde{n} \in T\tilde{N}$. Subscripts on the pairings in the definition (2.5) indicate, as follows, $\langle \cdot, \cdot \rangle_{\mathfrak{g}} : \mathfrak{g}^* \times \mathfrak{g} \rightarrow \mathbb{R}$ and $\langle \cdot, \cdot \rangle_{T\tilde{N}} : T^*\tilde{N} \times T\tilde{N} \rightarrow \mathbb{R}$. In what follows, for brevity of notation we will often suppress these subscripts t , except where we wish to emphasise the presence or absence of explicit time dependence. Suppressing these subscripts when explicit time dependence is understood should cause no confusion.

2.2. Euler-Poincaré theorem for the deterministic case

Theorem 2 (Euler-Poincaré theorem). *With the preceding notation, the following four statements are equivalent for a metamorphosis Lagrangian L that is invariant under the right action of G on $G \times N$ defined by $(g, \eta)h = (gh, h^{-1}\eta)$, with fixed endpoint conditions for the initial and final images n_0 and n_1 :*

i *Hamilton's variational principle*

$$(2.6) \quad \delta S = \delta \int_0^1 L(g_t, \dot{g}_t, \eta_t, \dot{\eta}_t) dt = 0 \quad \text{for}$$

$$L(g_t, \dot{g}_t, \eta_t, \dot{\eta}_t) = \ell(\dot{g}_t g_t^{-1}, g_t \eta_t, g_t \dot{\eta}_t)$$

holds, for variations δg_t of g_t and $\delta \eta_t$ of η_t vanishing at the endpoints.

- ii** g_t and η_t satisfy the Euler–Lagrange equations for L on $TG \times TN$.
- iii** *The constrained variational principle*

$$(2.7) \quad \delta S = \delta \int_0^1 \ell(u_t, n_t, \nu_t) dt = 0$$

holds for Lagrangian ℓ defined on $\mathfrak{g} \times TN$ using variations of $u_t = \dot{g}_t g_t^{-1}$, $n_t = g_t \eta_t$ and $\nu_t = g_t \dot{\eta}_t$ of the form

$$(2.8) \quad \delta u = \dot{\xi}_t - ad_{u_t} \xi_t, \quad \delta n = \varpi_t + \xi_t n_t, \quad \text{and} \quad \delta \nu = \dot{\varpi}_t + \xi_t \nu_t - u_t \varpi_t,$$

where $\xi_t = \delta g_t g_t^{-1}$, $\varpi_t = g_t \delta \eta_t$ and these variations vanish at the endpoints.

iv *The Euler–Poincaré equations hold on $\mathfrak{g} \times TN$*

$$(2.9) \quad \frac{\partial}{\partial t} \frac{\delta \ell}{\delta u} + ad_{u_t}^* \frac{\delta \ell}{\delta u} + \frac{\delta \ell}{\delta n} \diamond n_t + \frac{\delta \ell}{\delta \nu} \diamond \nu_t = 0,$$

$$\frac{\partial}{\partial t} \frac{\delta \ell}{\delta \nu} + u_t \star \frac{\delta \ell}{\delta \nu} - \frac{\delta \ell}{\delta n} = 0,$$

with auxiliary equation

$$(2.10) \quad \dot{n}_t = \nu_t + u_t n_t,$$

obtained from the definitions $u_t = \dot{g}_t g_t^{-1}$ and $n_t = g_t \eta_t$, with $\nu_t = g_t \dot{\eta}_t$, provided the endpoint condition holds, that

$$(2.11) \quad \frac{\delta \ell}{\delta u}(1) + \frac{\delta \ell}{\delta \nu}(1) \diamond n_1 = 0,$$

at time $t = 1$.

Corollary 3 (Coadjoint motion). *Equations (2.9) and the auxiliary equation (2.10) for n_t together imply the following coadjoint motion equation,*

$$(2.12) \quad \frac{\partial}{\partial t} \left(\frac{\delta \ell}{\delta u} + \frac{\delta \ell}{\delta \nu} \diamond n \right) + ad_{u_t}^* \left(\frac{\delta \ell}{\delta u} + \frac{\delta \ell}{\delta \nu} \diamond n \right) = 0.$$

The equivalence of statements **i** and **ii** in Theorem 2 is classical, and no other proof will be offered here. The proofs of the other equivalences in Euler-Poincaré Theorem 2 and its Corollary 3 for deterministic metamorphosis are laid out in the sections below.

2.3. Deterministic Euler-Lagrange equations

We compute the Euler-Lagrange equations associated with the minimization of the symmetry reduced action

$$\mathcal{S} = \int_0^1 \ell(u_t, n_t, \nu_t) dt$$

with fixed boundary conditions n_0 and n_1 . We therefore consider variations δu and $\omega = \delta n$. The variation $\delta \nu$ can be obtained from $n = g\eta$ and $\nu = g\dot{\eta}$ yielding $\dot{n} = \nu + un$ and $\dot{\omega} = \delta \nu + u\omega + (\delta u)n$. Here and in the following of this paper, we assume that computations are performed in a local chart on TN with respect to which we take partial derivatives.

We therefore have

$$\delta \mathcal{S} = \delta \int_0^1 \left(\left\langle \frac{\delta \ell}{\delta u}, \delta u_t \right\rangle + \left\langle \frac{\delta \ell}{\delta n}, \omega_t \right\rangle + \left\langle \frac{\delta \ell}{\delta \nu}, \dot{\omega}_t - u_t \omega_t - (\delta u_t) n_t \right\rangle \right) dt = 0.$$

The δu term yields the equation

$$\frac{\delta \ell}{\delta u} + \frac{\delta \ell}{\delta \nu} \diamond n_t = 0.$$

where, in a slight abuse of notation, $\delta \ell / \delta \nu \in T(TN)^*$ has been considered as a linear form on TN by $\langle \delta \ell / \delta \nu, z \rangle := \langle \delta \ell / \delta \nu, (0, z) \rangle$. From the terms involving ω , we find, after an integration by parts

$$(2.13) \quad \frac{\partial}{\partial t} \frac{\delta \ell}{\delta \nu} + u_t \star \frac{\delta \ell}{\delta \nu} - \frac{\delta \ell}{\delta n} = 0.$$

Here, we have introduced notation for the star (\star) operation,

$$(2.14) \quad \left\langle \frac{\delta \ell}{\delta \nu}, u\omega \right\rangle = \left\langle u \star \frac{\delta \ell}{\delta \nu}, \omega \right\rangle = \left\langle \mathcal{L}_u^T \frac{\delta \ell}{\delta \nu}, \omega \right\rangle.$$

That is, for $u\omega = \mathcal{L}_u \omega$, the star (\star) operation denotes the dual of the Lie derivative, $u \star \nu^* = \mathcal{L}_u^T \nu^*$.

We therefore obtain the system of equations

$$(2.15) \quad \begin{aligned} \frac{\delta \ell}{\delta u} + \frac{\delta \ell}{\delta \nu} \diamond n_t &= 0, \\ \frac{\partial}{\partial t} \frac{\delta \ell}{\delta \nu} + u_t \star \frac{\delta \ell}{\delta \nu} &= \frac{\delta \ell}{\delta n}, \\ \dot{n}_t &= \nu_t + u_t n_t. \end{aligned}$$

Note that the sum $\frac{\delta \ell}{\delta u} + \frac{\delta \ell}{\delta \nu} \diamond n$ is the momentum map arising from Noether's theorem for the considered invariance of the Lagrangian. The special form of the boundary conditions (fixed n_0 and n_1) ensures that this momentum map vanishes.

2.4. Deterministic Euler-Poincaré reduction

As explained in [17], a system equivalent to that in (2.15) can be obtained via Euler-Poincaré reduction [16]. In this setting, we make the variation in the group element and in the template instead of the velocity and the image. We denote $\xi_t = \delta g_t g_t^{-1}$ and $\varpi_t = g_t \delta \eta_t$. From these definitions, we obtain the expressions of the variations, δu , δn and $\delta \nu$.

We first have $\delta u_t = \dot{\xi}_t + [\xi_t, u_t]$, which arises from the standard Euler-Poincaré reduction theorem, as provided in [16, 25]. We also have $\delta n_t = \delta(g_t \eta_t) = \varpi_t + \xi_t n_t$. From $\nu_t = g_t \dot{\eta}_t$, we get $\delta \nu_t = g_t \delta \dot{\eta}_t + \xi_t \nu_t$ and from $\varpi_t = g_t \delta \eta_t$ we also have $\dot{\varpi}_t = u_t \varpi_t + g_t \dot{\eta}_t$. This yields $\delta \nu_t = \dot{\varpi}_t + \xi_t \nu_t - u_t \varpi_t$.

We also compute the boundary conditions for ξ and ϖ . At $t = 0$, we have $g_0 = \text{id}$ and $n_0 = g_0 \eta_0 = \text{cst}$ which implies $\xi_0 = 0$ and $\varpi_0 = 0$. At $t = 1$, the relation $g_1 \eta_1 = \text{cst}$ yields $\xi_1 n_1 + \omega_1 = 0$.

Now, the first variation of is

$$\int_0^1 \left(\left\langle \frac{\delta \ell}{\delta u}, \dot{\xi}_t - \text{ad}_{u_t} \xi_t \right\rangle + \left\langle \frac{\delta \ell}{\delta n_t}, \varpi_t + \xi_t n_t \right\rangle + \left\langle \frac{\delta \ell}{\delta \nu}, \dot{\varpi}_t + \xi_t \nu_t - u_t \varpi_t \right\rangle \right) dt = 0.$$

In the integration by parts to eliminate $\dot{\xi}_t$ and $\dot{\varpi}_t$, the boundary term is $\langle (\delta \ell / \delta u)_1, \xi_1 \rangle + \langle (\delta \ell / \delta \nu)_1, \omega_1 \rangle$. Using the boundary condition, the last term can be re-written

$$-\langle (\delta \ell / \delta \nu)_1, \xi_1 n_1 \rangle = \langle (\delta \ell / \delta \nu)_1 \diamond n_1, \xi_1 \rangle.$$

We therefore obtain the endpoint equation

$$\frac{\delta \ell}{\delta u}(1) + \frac{\delta \ell}{\delta \nu}(1) \diamond n_1 = 0.$$

The evolution equation for $\delta\ell/\delta u$ is

$$\frac{\partial}{\partial t} \frac{\delta\ell}{\delta u} + \text{ad}_{u_t}^* \frac{\delta\ell}{\delta u} + \frac{\delta\ell}{\delta n} \diamond n_t + \frac{\delta\ell}{\delta\nu} \diamond \nu_t = 0,$$

and $\delta\ell/\delta\nu$ evolves by

$$\frac{\partial}{\partial t} \frac{\delta\ell}{\delta\nu} + u_t \star \frac{\delta\ell}{\delta\nu} - \frac{\delta\ell}{\delta n} = 0.$$

Consequently, we obtain the following system of equations,

$$(2.16) \quad \begin{aligned} \frac{\partial}{\partial t} \frac{\delta\ell}{\delta u} + \text{ad}_{u_t}^* \frac{\delta\ell}{\delta u} + \frac{\delta\ell}{\delta n} \diamond n_t + \frac{\delta\ell}{\delta\nu} \diamond \nu_t &= 0, \\ \frac{\partial}{\partial t} \frac{\delta\ell}{\delta\nu} + u_t \star \frac{\delta\ell}{\delta\nu} - \frac{\delta\ell}{\delta n} &= 0, \end{aligned}$$

as well as the auxiliary equation

$$(2.17) \quad \frac{\partial}{\partial t} n_t = \nu_t + u_t n_t,$$

obtained from the definitions $u_t = \dot{g}_t g_t^{-1}$ and $n_t = g_t \eta_t$. Moreover, the endpoint condition holds, that

$$(2.18) \quad \frac{\delta\ell}{\delta u}(1) + \frac{\delta\ell}{\delta\nu}(1) \diamond n_1 = 0,$$

at time $t = 1$. □

As discussed in [17], the system (2.9) is equivalent to (2.15), since they characterize the same critical points. Direct evidence of this fact may be obtained from the proof of Corollary 3, that

$$(2.19) \quad \frac{\partial}{\partial t} \left(\frac{\delta\ell}{\delta u} + \frac{\delta\ell}{\delta\nu} \diamond n \right) + \text{ad}_{u_t}^* \left(\frac{\delta\ell}{\delta u} + \frac{\delta\ell}{\delta\nu} \diamond n \right) = 0.$$

Proof of Corollary 3. A solution of (2.9) satisfies the coadjoint motion equation,

$$\begin{aligned} & \frac{\partial}{\partial t} \left(\frac{\delta\ell}{\delta u_t} + \frac{\delta\ell}{\delta\nu} \diamond n_t \right) \\ &= \frac{\partial}{\partial t} \frac{\delta\ell}{\delta u} + \left(\frac{\partial}{\partial t} \frac{\delta\ell}{\delta\nu} \right) \diamond n_t + \frac{\delta\ell}{\delta\nu} \diamond \dot{n}_t \end{aligned}$$

$$\begin{aligned}
 &= \frac{\partial}{\partial t} \frac{\delta \ell}{\delta u} + \left(\frac{\delta \ell}{\delta n} - u_t \star \frac{\delta \ell}{\delta \nu} \right) \diamond n_t + \frac{\delta \ell}{\delta \nu} \diamond (\nu_t + u_t n_t) \\
 &= \frac{\partial}{\partial t} \frac{\delta \ell}{\delta u} + \frac{\delta \ell}{\delta n} \diamond n_t + \frac{\delta \ell}{\delta \nu} \diamond \nu_t - \left(u_t \star \frac{\delta \ell}{\delta \nu} \right) \diamond n_t + \frac{\delta \ell}{\delta \nu} \diamond (u_t n_t) \\
 &= -\text{ad}_{u_t}^* \frac{\delta \ell}{\delta u} - \text{ad}_{u_t}^* \left(\frac{\delta \ell}{\delta \nu} \diamond n_t \right).
 \end{aligned}$$

In the last equation, we have used the fact that, for any $\alpha \in \mathfrak{g}$,

$$\begin{aligned}
 \left\langle \frac{\delta \ell}{\delta \nu} \diamond (un) - \left(u \star \frac{\delta \ell}{\delta \nu} \right) \diamond n, \alpha \right\rangle &= \left\langle \frac{\delta \ell}{\delta \nu}, \alpha(un) - u(\alpha n) \right\rangle \\
 &= -\left\langle \frac{\delta \ell}{\delta \nu}, [u, \alpha]n \right\rangle \\
 &= -\left\langle \frac{\delta \ell}{\delta \nu} \diamond n, [u, \alpha] \right\rangle \\
 &= -\left\langle \text{ad}_{u_t}^* \left(\frac{\delta \ell}{\delta \nu} \diamond n_t \right), \alpha \right\rangle. \quad \square
 \end{aligned}$$

Remark 4. *Corollary 3 combined with the relation $(\delta \ell / \delta u)_1 + (\delta \ell / \delta \nu)_1 \diamond u_1 = 0$ implies the first equation in (2.15). Namely, the zero level set of the momentum map is preserved by coadjoint motion.*

2.5. Deterministic Hamiltonian formulation

The Euler-Poincaré formulation of metamorphosis in (2.9) and (2.10) in Theorem 2 allows passage to its Hamiltonian formulation via the following **Legendre transformation** of the reduced Lagrangian ℓ in the velocities u and ν , in the Eulerian fluid description,

$$(2.20) \quad \mu = \frac{\delta \ell}{\delta u}, \quad \sigma = \frac{\delta \ell}{\delta \nu}, \quad h(\mu, \sigma, n) = \langle \mu, u \rangle + \langle \sigma, \nu \rangle - \ell(u, \nu, n).$$

Accordingly, one computes the variational derivatives of h as

$$(2.21) \quad \frac{\delta h}{\delta \mu} = u, \quad \frac{\delta h}{\delta \sigma} = \nu, \quad \frac{\delta h}{\delta n} = -\frac{\delta \ell}{\delta n}.$$

Consequently, the Euler-Poincaré equations (2.9) and the auxiliary kinematic equation (2.10) for metamorphosis imply the following equations, for the Legendre-transformed variables, (μ, σ, n) ,

$$\begin{aligned}
 (2.22) \quad &\partial_t \mu + \text{ad}_{\delta h / \delta \mu}^* \mu + \sigma \diamond \frac{\delta h}{\delta \sigma} - \frac{\delta h}{\delta n} \diamond n = 0, \\
 &\partial_t \sigma + \mathcal{L}_{\delta h / \delta \mu}^T \sigma + \frac{\delta h}{\delta n} = 0,
 \end{aligned}$$

as well as the auxiliary equation

$$(2.23) \quad \partial_t n = \mathcal{L}_{\delta h / \delta \mu} n + \frac{\delta h}{\delta \sigma}.$$

These equations are **Hamiltonian**. That is, they may be expressed in the form

$$(2.24) \quad \frac{\partial \mathbf{z}}{\partial t} = \{\mathbf{z}, h\} = \mathbf{b} \cdot \frac{\delta h}{\delta \mathbf{z}},$$

where $\mathbf{z} \in (\mu, \sigma, n)$ and the Hamiltonian matrix \mathbf{b} defines the Poisson bracket

$$(2.25) \quad \{f, h\} = \int d^n x \frac{\delta f}{\delta \mathbf{z}} \cdot \mathbf{b} \cdot \frac{\delta h}{\delta \mathbf{z}},$$

which is bilinear, skew symmetric and satisfies the **Jacobi identity**,

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

Assembling the metamorphosis equations (2.22) - (2.23) into the Hamiltonian form (2.24) gives,

$$(2.26) \quad \begin{bmatrix} \partial_t \mu \\ \partial_t \sigma \\ \partial_t n \end{bmatrix} = - \begin{bmatrix} \text{ad}_{\square}^* \mu & \sigma \diamond \square & -\square \diamond n \\ \mathcal{L}_{\square}^T \sigma & 0 & 1 \\ -\mathcal{L}_{\square} n & -1 & 0 \end{bmatrix} \begin{bmatrix} \delta h / \delta \mu (= u) \\ \delta h / \delta \sigma (= \nu) \\ \delta h / \delta n \end{bmatrix}$$

In this expression, the operators act to the right on all terms in a product by the chain rule.

Remarks about the Hamiltonian matrix. The Hamiltonian matrix in equation (2.26) was discovered some time ago in the context of complex fluids in [15]. There, it was proven to be a valid Hamiltonian matrix by associating its Poisson bracket as defined in equation (2.25) with the dual space of a certain Lie algebra of semidirect-product type which has a canonical two-cocycle on it. The mathematical discussion of Lie algebras with two-cocycles is given in [15]. See also [14, 10, 11, 12] for further discussions of semidirect-product Poisson brackets with cocycles for complex fluids.

Being dual to the semidirect-product Lie algebra $\mathfrak{g} \circledast T^*N$, our Hamiltonian matrix in equation (2.26) is in fact a **Lie-Poisson Hamiltonian matrix**. See, e.g., [25] and references therein for more discussions of such Hamiltonian matrices. For our present purposes, its rediscovery in the context of metamorphosis links the physical and mathematical interpretations of the variables in the theory of imaging science with earlier work in complex

fluid dynamics and with the gauge theory approach to condensed matter, see, e.g., [24].

2.6. Deterministic Hamilton-Pontryagin formulation

An alternative formulation to either the Euler-Lagrange equations, or the Euler-Poincaré approach is obtained in the Hamilton-Pontryagin principle. In this approach, the diffeomorphic paths appear explicitly.

Theorem 5 (Hamilton-Pontryagin principle). *The Euler-Poincaré equations in Corollary 3 for coadjoint motion given by*

$$(2.27) \quad \begin{aligned} \frac{\partial}{\partial t} \left(\frac{\delta \ell}{\delta u} + \frac{\delta \ell}{\delta \nu} \diamond n \right) + \text{ad}_u^* \left(\frac{\delta \ell}{\delta u} + \frac{\delta \ell}{\delta \nu} \diamond n \right) &= 0, \\ \frac{\partial}{\partial t} \frac{\delta \ell}{\delta \nu} + u \star \frac{\delta \ell}{\delta \nu} - \frac{\delta \ell}{\delta n} &= 0, \end{aligned}$$

as well as the auxiliary equation

$$(2.28) \quad \dot{n} = \nu + un,$$

on the space $\mathfrak{g}^* \times T^*N \times TN$ are equivalent to the following implicit variational principle,

$$(2.29) \quad \delta S(u, n, \dot{n}, \nu, g, \dot{g}) = 0,$$

for a constrained action

$$(2.30) \quad \begin{aligned} S(u, n, \dot{n}, \nu, g, \dot{g}) \\ = \int_0^1 \left[\ell(u, n, \nu) + \langle M, (\dot{g}g^{-1} - u) \rangle + \langle \sigma, (\dot{n} - \nu - un) \rangle \right] dt. \end{aligned}$$

Proof. The variations of S in formula (2.30) are given by

$$(2.31) \quad \begin{aligned} 0 = \delta S = \int_0^1 \left\langle \frac{\delta \ell}{\delta u} - M + \sigma \diamond n, \delta u \right\rangle - \left\langle \dot{\sigma} + u \star \sigma - \frac{\delta \ell}{\delta n}, \delta n \right\rangle \\ + \left\langle \frac{\delta \ell}{\delta \nu} - \sigma, \delta \nu \right\rangle + \left\langle M, \delta(g^{-1}\dot{g}) \right\rangle dt. \end{aligned}$$

After a side calculation, one finds $\delta(\dot{g}g^{-1}) = \dot{\xi} - \text{ad}_u \xi$, with $\xi = \delta g g^{-1}$ for the last term in (2.30). Then, integrating by parts yields the familiar relation

$$\int_0^1 \left\langle M, \delta(\dot{g}g^{-1}) \right\rangle dt = \int_0^1 \left\langle M, (\dot{\xi} - \text{ad}_u \xi) \right\rangle dt$$

$$= \int_0^1 \left\langle -\dot{M} - \text{ad}_u^* M, \eta \right\rangle dt + \left\langle M, \xi \right\rangle \Big|_0^1,$$

where $\xi = \delta g g^{-1}$ vanishes at the endpoints in time.

Thus, stationarity $\delta S = 0$ of the Hamilton–Pontryagin variational principle with constrained action integral (2.30) yields the following set of equations:

$$(2.32) \quad M = \frac{\delta \ell}{\delta u} + \sigma \diamond n, \quad \sigma = \frac{\delta \ell}{\delta \nu}, \quad \frac{\partial \sigma}{\partial t} + u \star \sigma - \frac{\delta \ell}{\delta n} = 0, \quad \frac{\partial M}{\partial t} + \text{ad}_u^* M = 0,$$

as well as the constraint equations

$$(2.33) \quad \dot{g} g^{-1} - u = 0 \quad \text{and} \quad \dot{n} - \nu - un = 0.$$

This finishes the proof of the Hamilton–Pontryagin principle in Theorem 5. \square

Proposition 6 (Untangling the Lie–Poisson structure (2.26)).

By the change of variables

$$(2.34) \quad h(\mu, \sigma, n) = H(M, \sigma, n),$$

the Lie–Poisson structure (2.26) transforms into

$$(2.35) \quad \begin{bmatrix} \partial_t M \\ \partial_t \sigma \\ \partial_t n \end{bmatrix} = - \begin{bmatrix} \text{ad}_{\square}^* M & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \delta H / \delta M (= u) \\ \delta H / \delta \sigma + \mathcal{L}_{\delta H / \delta M} n (= \nu + \mathcal{L}_u n) \\ \delta H / \delta n + \mathcal{L}_{\delta H / \delta M}^T \sigma (= \delta H / \delta n + \mathcal{L}_u^T \sigma) \end{bmatrix},$$

and thereby recovers equations (2.27) and (2.28) in Hamiltonian form.

Proof. The proof follows from the expanding out the change of variables formula for variational derivatives,

$$\delta h(\mu, \sigma, n) = \delta H(M, \sigma, n),$$

where $(M, \sigma, n) := (\mu + \sigma \diamond n, \sigma, n)$. Namely, one substitutes the corresponding terms,

$$\begin{aligned} \delta h(\mu, \sigma, n) &= \left\langle \frac{\delta h}{\delta \mu}, \delta \mu \right\rangle + \left\langle \frac{\delta h}{\delta \sigma}, \delta \sigma \right\rangle + \left\langle \frac{\delta h}{\delta n}, \delta n \right\rangle, \\ \delta H(M, \sigma, n) &= \left\langle \frac{\delta H}{\delta M}, \delta \mu \right\rangle + \left\langle \frac{\delta H}{\delta \sigma} + \mathcal{L}_{\frac{\delta H}{\delta M}} n, \delta \sigma \right\rangle - \left\langle \frac{\delta H}{\delta n} + \mathcal{L}_{\frac{\delta H}{\delta M}}^T \sigma, \delta n \right\rangle, \end{aligned}$$

into the transformed Hamiltonian structure. \square

Remark 7. *The Lie-Poisson Hamiltonian structure (2.35) is the variable transformation (2.34) of the corresponding structure (2.26). The corresponding Lie-Poisson bracket is defined on the dual Lie algebra of the vector fields over the domain, \mathcal{D} ; namely, $\mathfrak{g} = \mathfrak{X}(\mathcal{D})$ with canonical 2-cocycle $\mathfrak{X}(\mathcal{D})^* \times T^*\mathcal{F}(\mathcal{D}, N)$, where $\mathcal{F}(\mathcal{D}, N)$ denotes smooth functions from the domain, \mathcal{D} , to the data structure manifold, N . For more details about how the untangling of Lie-Poisson structures is applied in geometric mechanics in the theory of complex fluids and for further citations in this literature to earlier work, see [11, 12].*

3. Metamorphosis by stochastic Lie transport

Section summary. As we have seen in the previous review section, the metamorphosis of images applies the Lie group of diffeomorphisms to deform a template image that is undergoing its own internal dynamics as it deforms. As we have discussed, this type of deformation allows considerable freedom for image matching and has an analogy with complex fluids, in which the template properties are regarded as order parameters (coset spaces of broken symmetries) for the complex fluids.

In this section, we consider stochastic perturbations corresponding to uncertainty due to random errors in the reconstruction of the deformation map from its vector field. The paper [4] shows that one may compound the uncertainty in the deformation map, treated here, by also introducing uncertainty in the reconstruction of the template position from its velocity field. The paper [4] also applies this more general geometric theory to several classical examples, including landmarks, images, closed curves, as well as discussing its use for functional data analysis.

3.1. Notation and approach for the stochastic case

To derive the stochastic partial differential equations (SPDEs) for uncertainty quantification in the metamorphosis approach to imaging science, we combine the recent developments for uncertainty quantification in fluid dynamics in [19] with the Hamilton-Pontryagin principle for metamorphosis discussed in the previous section. The idea is to replace the deterministic evolutionary constraints in equation (2.33) by the following stochastic processes,

$$(3.1) \quad \begin{aligned} dg g^{-1} - \left(u_t(x) dt + \sum_i \xi_i(x) \circ dW_t^i \right) &= 0 \quad \text{and} \\ dn_t - \nu_t dt - \left(u_t dt + \sum_i \xi_i \circ dW_t^i \right) n &= 0, \end{aligned}$$

where \mathbf{d} is brief notation for the stochastic evolution operator, which strictly speaking is an integral operator. The first of these stochastic processes may be written equivalently as a stochastic version of the Lagrange-to-Euler map by using the notation g_t^* for pullback by the stochastic diffeomorphism g_t ,

$$(3.2) \quad \mathbf{d}g_t - g_t^* \left(u_t(X) dt + \sum_i \xi_i(X) \circ dW_t^i \right) = 0.$$

In this form, one sees that g_t is a stochastic process with time dependent drift term given by the pullback operation, $u_t(x) dt = g_t^* u_t(X) dt = u_t(g_t X) dt$, in which subscript t on g_t and $u_t(X)$ indicates that both u_t and g_t depend explicitly on time, t . Thus, the dynamical drift velocity $u_t(x)$ depends on time explicitly and also through the Lagrange-to-Euler map $x = g_t X$ governed by (3.2) with initial value $g_0 = Id$. The Lagrange-to-Euler map in (3.2) also contains a Stratonovich stochastic term, comprising a finite sum over time independent spatial functions ξ_i , $i = 1, 2, \dots, N$, each composed in a Stratonovich sense (denoted by the symbol \circ) with its own Brownian motion in time, dW_t^i . This type of Stratonovich stochasticity, called “cylindrical noise”, was introduced in [30]. In the cylindrical noise term, the $\xi_i(x)$, $i = 1, 2, \dots, N$, are interpreted as describing the spatial correlations of the noise in fixed Eulerian space, e.g., as eigenvectors of the correlation tensor, or covariance, for a process which is assumed to be statistically stationary.

3.2. Stochastic Hamilton-Pontryagin approach

Theorem 8 (Stochastic Hamilton–Pontryagin principle). *Stochastic metamorphosis is governed by coadjoint motion represented as SPDE given by*

$$(3.3) \quad \begin{aligned} \mathbf{d} \left(\frac{\delta \ell}{\delta u} + \frac{\delta \ell}{\delta \nu} \diamond n \right) + a d_{(u_t(x) dt + \sum_i \xi_i(x) \circ dW_t^i)}^* \left(\frac{\delta \ell}{\delta u} + \frac{\delta \ell}{\delta \nu} \diamond n \right) &= 0, \\ \mathbf{d} \frac{\delta \ell}{\delta \nu} + \left(u_t(x) dt + \sum_i \xi_i(x) \circ dW_t^i \right) \star \frac{\delta \ell}{\delta \nu} - \frac{\delta \ell}{\delta n} &= 0, \end{aligned}$$

as well as the auxiliary equation

$$(3.4) \quad \mathbf{d}n = \nu + \left(u_t(x) dt + \sum_i \xi_i(x) \circ dW_t^i \right) n,$$

on the space $\mathfrak{g}^* \times T^*N \times TN$ are equivalent to the following implicit variational principle,

$$(3.5) \quad \delta S(u, n, \mathbf{d}n, \nu, g, \mathbf{d}g) = 0,$$

for a stochastically constrained action

$$(3.6) \quad \begin{aligned} S(u, n, \mathbf{d}n, \nu, g, \mathbf{d}g) &= \int_0^1 \left[\ell(u_t, n_t, \nu_t) dt + \left\langle M, \mathbf{d}g_t g_t^{-1} - u_t(x) dt - \sum_i \xi_i(x) \circ dW_t^i \right\rangle \right. \\ &\quad \left. + \left\langle \sigma, (\mathbf{d}n_t - \nu_t dt - (u_t(x) dt + \sum_i \xi_i(x) \circ dW_t^i) n_t) \right\rangle \right]. \end{aligned}$$

Remark 9 (Stratonovich versus Itô representations). *In dealing with the stochastic variational principle, we will work in the Stratonovich representation, because it admits ordinary variational calculus. However, later, when we consider expected values for the solutions, we will transform to the equivalent Itô representation. In transforming to the Itô representation, we will discover that the effective diffusion from the Itô contraction term is by no means a Laplacian. Instead, the Itô contraction term turns out to produce a double Lie derivative with respect to the sum of vector fields $\xi_i(x)$.*

After this remark, we return to the proof of Theorem 8 for the Stochastic Hamilton–Pontryagin principle.

Proof. The variations of S in formula (3.6) are given by

$$(3.7) \quad \begin{aligned} 0 = \delta S &= \int_0^1 \left\langle \frac{\delta \ell}{\delta u} - M + \sigma \diamond n, \delta u \right\rangle dt \\ &\quad - \left\langle \mathbf{d}\sigma + (u_t(x) dt + \sum_i \xi_i(x) \circ dW_t^i) \star \sigma - \frac{\delta \ell}{\delta n} dt, \delta n \right\rangle \\ &\quad + \left\langle \frac{\delta \ell}{\delta \nu} - \sigma, \delta \nu \right\rangle dt + \left\langle M, \delta(\mathbf{d}g g^{-1}) \right\rangle dt. \end{aligned}$$

In a side calculation, one finds

$$\delta(\mathbf{d}g g^{-1}) = \mathbf{d}\xi - \text{ad}_{(u_t(x) dt + \sum_i \xi_i(x) \circ dW_t^i)} \xi, \quad \text{with } \xi = \delta g g^{-1},$$

for substitution into the last term. Then, integrating by parts yields the relation

$$\int_0^1 \left\langle M, \delta(\mathbf{d}g g^{-1}) \right\rangle dt$$

$$\begin{aligned}
 &= \int_0^1 \left\langle M, \mathbf{d}\xi - \text{ad}_{(u_t(x) dt + \sum_i \xi_i(x) \circ dW_t^i)} \xi \right\rangle dt \\
 &= \int_0^1 \left\langle -\mathbf{d}M - \text{ad}_{(u_t(x) dt + \sum_i \xi_i(x) \circ dW_t^i)}^* M, \eta \right\rangle dt + \left\langle M, \xi \right\rangle \Big|_0^1,
 \end{aligned}$$

where $\xi = \delta g g^{-1}$ vanishes at the endpoints in time.

Thus, stationarity $\delta S = 0$ of the Hamilton–Pontryagin variational principle with stochastically constrained action integral (3.7) yields the following set of SPDEs:

$$\begin{aligned}
 &\mathbf{d}M + \text{ad}_{(u_t(x) dt + \sum_i \xi_i(x) \circ dW_t^i)}^* M = 0, \\
 (3.8) \quad &\mathbf{d}\sigma + \left(u_t(x) dt + \sum_i \xi_i(x) \circ dW_t^i \right) \star \sigma - \frac{\delta \ell}{\delta n} dt = 0,
 \end{aligned}$$

for the quantities

$$(3.9) \quad M = \frac{\delta \ell}{\delta u} + \sigma \diamond n, \quad \text{and} \quad \sigma = \frac{\delta \ell}{\delta \nu},$$

as well as the stochastic constraint equations

$$\begin{aligned}
 &\mathbf{d}g g^{-1} - \left(u_t(x) dt + \sum_i \xi_i(x) \circ dW_t^i \right) = 0, \\
 (3.10) \quad &\mathbf{d}n - \nu - \left(u_t(x) dt + \sum_i \xi_i(x) \circ dW_t^i \right) n = 0.
 \end{aligned}$$

This finishes the proof of the Hamilton–Pontryagin principle for stochastic metamorphosis formulated in Theorem 8. □

3.3. Stochastic Hamiltonian formulation

By Corollary 3, the stochastic equations (3.8) through (3.10) above imply the corresponding stochastic versions of in (2.9) and (2.10) in Theorem 2. Namely,

$$\begin{aligned}
 &\mathbf{d} \frac{\delta \ell}{\delta u} + \text{ad}_{\tilde{u}}^* \frac{\delta \ell}{\delta u} + \frac{\delta \ell}{\delta n} \diamond n dt + \frac{\delta \ell}{\delta \nu} \diamond \nu dt = 0, \\
 (3.11) \quad &\mathbf{d} \frac{\delta \ell}{\delta \nu} + \mathcal{L}_{\tilde{u}}^T \frac{\delta \ell}{\delta \nu} - \frac{\delta \ell}{\delta n} dt = 0, \\
 &\mathbf{d}n = \mathcal{L}_{\tilde{u}} n_t + \nu dt
 \end{aligned}$$

with Stratonovich stochastic transport velocity \tilde{u} given by

$$(3.12) \quad \tilde{u} := u_t(x) dt + \sum_i \xi_i(x) \circ dW_t^i.$$

At this point the Hamiltonian structure of the deterministic metamorphosis equations reveals how we can write the stochastic metamorphosis equations in Hamiltonian form. Namely, we deform the deterministic Hamiltonian by adding the stochastic part to it as being linear in the momentum μ and paired with the Stratonovich noise perturbation,

$$(3.13) \quad \tilde{h} := h(\mu, \sigma, n) dt + \left\langle \mu, \sum_i \xi_i(x) \circ dW_t^i \right\rangle.$$

We then Legendre transform to the Hamiltonian side.

Assembling the metamorphosis equations (3.11) into the Hamiltonian form (2.24) gives,

$$(3.14) \quad \begin{aligned} \begin{bmatrix} d\mu \\ d\sigma \\ dn \end{bmatrix} &= - \begin{bmatrix} \text{ad}_{\square}^* \mu & \sigma \diamond \square & -\square \diamond n \\ \mathcal{L}_{\square}^T \sigma & 0 & 1 \\ -\mathcal{L}_{\square} n & -1 & 0 \end{bmatrix} \\ &\times \begin{bmatrix} \delta \tilde{h} / \delta \mu (= \tilde{u} := u_t(x) dt + \sum_i \xi_i(x) \circ dW_t^i) \\ \delta \tilde{h} / \delta \sigma = (\delta h / \delta \sigma) dt (= \nu dt) \\ \delta \tilde{h} / \delta n = (\delta h / \delta n) dt \end{bmatrix} \\ &= \begin{bmatrix} -\text{ad}_{\tilde{u}}^* (\mu + \sigma \diamond n) - \sigma \diamond \nu dt + (\delta h / \delta n) \diamond n dt \\ -\mathcal{L}_{\tilde{u}}^T \sigma - (\delta h / \delta n) dt \\ \mathcal{L}_{\tilde{u}} n + \nu dt \end{bmatrix}. \end{aligned}$$

As before, the operators in the Hamiltonian matrix (3.14) act to the right on all terms in a product by the chain rule. We see that the Hamiltonian formulation of the stochastic metamorphosis equations has summoned the same Lie-Poisson Hamiltonian structure as in the deterministic case in (2.26).

By the change of variables corresponding to (2.34) for this stochastic case,

$$(3.15) \quad \tilde{h}(\mu, \sigma, n) = \tilde{H}(M, \sigma, n) = H(M, \sigma, n) dt + \left\langle M, \sum_i \xi_i(x) \circ dW_t^i \right\rangle,$$

one finds that the Lie-Poisson structure in (3.14) transforms into

$$(3.16) \quad \begin{bmatrix} dM \\ d\sigma \\ dn \end{bmatrix} = - \begin{bmatrix} \text{ad}_{\square}^* M & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} \delta \tilde{H} / \delta M (= \tilde{u}) \\ \delta \tilde{H} / \delta \sigma + \mathcal{L}_{\delta \tilde{H} / \delta M} n (= \nu dt + \mathcal{L}_{\tilde{u}} n) \\ \delta \tilde{H} / \delta n + \mathcal{L}_{\delta \tilde{H} / \delta M}^T \sigma (= (\delta H / \delta n) dt + \mathcal{L}_{\tilde{u}}^T \sigma) \end{bmatrix},$$

and thereby recovers equations (3.8) – (3.10) in Hamiltonian form, in which σ and n are canonically conjugate variables.

Remark 10. *The advantage of writing equations (3.16) in terms of the total momentum 1-form density $M := \mathbf{M} \cdot d\mathbf{x} \otimes d^3x$ may be seen by recalling that $\text{ad}_{\tilde{u}}^* M = \mathcal{L}_{\tilde{u}} M$ for 1-form densities. Consequently, the first equation in (3.16) implies $(\mathbf{d} + \mathcal{L}_{\tilde{u}})M = 0$, which in turn implies that*

$$(3.17) \quad \mathbf{d}(g^* M) = g^* \left((\mathbf{d} + \mathcal{L}_{\tilde{u}})M \right) = 0.$$

This means the total momentum 1-form density $M := \mathbf{M} \cdot d\mathbf{x} \otimes d^3x$ is preserved by the stochastic flow given by the pullback of the stochastic diffeomorphism g_t in (3.2), which is the flow of the stochastic vector field \tilde{u} . That is, the stochastic Lagrange-to-Euler flow g_t , which is the solution of $\mathbf{d}g g^{-1} = \tilde{u}$ in (3.1),

$$(3.18) \quad \mathbf{d}g_t - g_t^* \tilde{u} = \mathbf{d}g_t - g_t^* \left(u_t(X) dt + \sum_i \xi_i(X) \circ dW_t^i \right) = 0,$$

preserves the quantity M along its flow. Hence, we say that the total momentum 1-form density $M := \mathbf{M} \cdot d\mathbf{x} \otimes d^3x$ is stochastically advected.

3.4. Itô representation

In preparation for writing the Itô representation, we first substitute $\text{ad}_u^* \mu = \mathcal{L}_u \mu$ to show in the more familiar Lie derivative notation the action of the vector field u on its dual momentum, the 1-form density $\mu = \delta\ell/\delta u$. The equivalent Itô representations of equations (3.11) are then given by

$$(3.19) \quad \begin{aligned} & \mathbf{d} \frac{\delta\ell}{\delta u} + \mathcal{L}_u \frac{\delta\ell}{\delta u} dt + \sum_i \mathcal{L}_{\xi_i(x)} \frac{\delta\ell}{\delta u} dW_t^i - \frac{1}{2} \sum_i \mathcal{L}_{\xi_i(x)} \left(\mathcal{L}_{\xi_i(x)} \frac{\delta\ell}{\delta u} \right) dt + \frac{\delta\ell}{\delta n} \diamond n dt \\ & \quad + \frac{\delta\ell}{\delta\nu} \diamond \nu dt = 0, \\ & \mathbf{d} \frac{\delta\ell}{\delta\nu} + \mathcal{L}_u^T \frac{\delta\ell}{\delta\nu} dt + \sum_i \mathcal{L}_{\xi_i(x)}^T \frac{\delta\ell}{\delta\nu} dW_t^i - \frac{1}{2} \sum_i \mathcal{L}_{\xi_i(x)}^T \left(\mathcal{L}_{\xi_i(x)}^T \frac{\delta\ell}{\delta\nu} \right) dt - \frac{\delta\ell}{\delta n} dt = 0, \\ & \mathbf{d}n = \mathcal{L}_u n_t dt + \sum_i \mathcal{L}_{\xi_i(x)} n_t dW_t^i - \frac{1}{2} \sum_i \mathcal{L}_{\xi_i(x)} \left(\mathcal{L}_{\xi_i(x)} n_t \right) dt + \nu dt, \end{aligned}$$

with stochastic transport velocity \widehat{u} given in Itô form by

$$(3.20) \quad \widehat{u} := u_t(x) dt + \sum_i \xi_i(x) dW_t^i,$$

plus the Itô contraction drift terms. Likewise, the stochastic advection of the total momentum 1-form density $M = \mu + \sigma \diamond n = \mathbf{M} \cdot d\mathbf{x} \otimes d^3x$, expressed in Stratonovich form as $(\mathbf{d} + \mathcal{L}_{\widehat{u}})M = 0$, is expressed in Itô form as

$$(3.21) \quad \mathbf{d}M + \mathcal{L}_u M dt + \sum_i \mathcal{L}_{\xi_i(x)} M dW_t^i - \frac{1}{2} \sum_i \mathcal{L}_{\xi_i(x)} \left(\mathcal{L}_{\xi_i(x)} M \right) dt = 0,$$

in which the last sum is the Itô contraction term.

In Itô form, the expectation of the noise terms vanish. The noise interacts multiplicatively with both the solution \mathbf{M} and the gradient of the solution $\nabla \mathbf{M}$, through the Lie derivative, as

$$(3.22) \quad \begin{aligned} & \sum_i \mathcal{L}_{\xi_i(x)} M dW_t^i \\ &= \sum_i \left\{ \left[(\xi_i(\mathbf{x}) \cdot \nabla) \mathbf{M} + (\nabla \xi_i(\mathbf{x}))^T \cdot \mathbf{M} + \mathbf{M} \operatorname{div} \xi_i(\mathbf{x}) \right] dW_t^i \right\} \cdot d\mathbf{x} \otimes d^3x. \end{aligned}$$

Likewise, the Itô contraction drift terms are not Laplacians, as would have been the case for additive noise with constant amplitude. Instead, in (3.21) they are sums over double Lie derivatives with respect to the vector fields $\xi_i(x)$ associated with the spatial correlations of the stochastic perturbation. This double Lie derivative combination was called the *Lie Laplacian* in [19]. As an operator, it has many properties of potential interest in the mathematical analysis of these equations. See [8] for an example of its use in developing analytical estimates, with an application to three-dimensional stochastic incompressible fluid dynamics, for which local-in-time existence, uniqueness and a regularity condition for well-posedness of the equations are proven using these analytical estimates.

4. Conclusions

Summary. The preservation of the Hamiltonian structure achieved in (3.14) for the present formulation of the stochastic metamorphosis equations provides the interpretation of the stochastic part of the flow. The Hamiltonian flow of the momentum $\langle \mu, \sum_i \xi_i(x) \circ dW_t^i \rangle$ produces stochastic translation

in Eulerian space with velocity $\sum_i \xi_i(x) \circ dW_t^i$. Thus, adding the stochastic part, linear in the momentum, to the metamorphosis Hamiltonian $h(\mu, \nu, n)$ adds a stochastic transport to the deterministic flow. This is consistent with our intention of modelling stochastic metamorphosis as motion generated by a temporally stochastic flow on the diffeomorphisms, with spatial correlations given by the prescribed, time-independent correlation eigenvectors determined from data assimilation.

As we have seen, metamorphosis is a combination of diffeomorphic action and template variation. In the present paper only the diffeomorphic dynamics has been made stochastic, so far. In fact, the template dynamics may easily be made stochastic, if a proper rationale can be made for the choice in modifying the stochastic part of the Hamiltonian to include effects of template noise. The example Hamiltonian we consider for this modification is

$$(4.1) \quad \tilde{k} := h(\mu, \sigma, n) dt + \left\langle \mu + \sigma \diamond n, \sum_i \xi_i(x) \circ dW_t^i \right\rangle,$$

cf. equations (3.13) and (3.15), noting the difference between \tilde{k} in (4.1) and \tilde{h} in (3.15).

As in (3.14), we again use the same Lie-Poisson Hamiltonian structure (2.26) as in the deterministic case, to find the stochastic metamorphosis equations with template noise,

$$(4.2) \quad \begin{aligned} \begin{bmatrix} d\mu \\ d\sigma \\ dn \end{bmatrix} &= - \begin{bmatrix} \text{ad}_{\square}^* \mu & \sigma \diamond \square & -\square \diamond n \\ \mathcal{L}_{\square}^T \sigma & 0 & 1 \\ -\mathcal{L}_{\square} n & -1 & 0 \end{bmatrix} \\ &\times \begin{bmatrix} \delta \tilde{k} / \delta \mu (= \tilde{u} := u_t(x) dt + \sum_i \xi_i(x) \circ dW_t^i) \\ \delta \tilde{k} / \delta \sigma = \nu dt - \mathcal{L}_{(\sum_i \xi_i(x) \circ dW_t^i)} n \\ \delta \tilde{k} / \delta n = (\delta h / \delta n) dt - \mathcal{L}_{(\sum_i \xi_i(x) \circ dW_t^i)}^T \sigma \end{bmatrix} \\ &= - \begin{bmatrix} \text{ad}_{\tilde{u}}^* \mu + \text{ad}_{(\sum_i \xi_i(x) \circ dW_t^i)}^* (\sigma \diamond n) + \sigma \diamond \nu dt - (\delta h / \delta n) \diamond n dt \\ \mathcal{L}_{u dt}^T \sigma + (\delta h / \delta n) dt \\ -\mathcal{L}_{u dt} n - \nu dt \end{bmatrix}. \end{aligned}$$

Consequently,

$$(4.3) \quad \mathbf{d}(\sigma \diamond n) = (d\sigma) \diamond n + \sigma \diamond (dn) = -[\mathcal{L}_{u dt}(\sigma \diamond n) - \sigma \diamond \nu dt + (\delta h / \delta n) \diamond n dt],$$

and we find from (4.2) that

$$(4.4) \quad \mathbf{d}(\mu + \sigma \diamond n) = -\mathcal{L}_{\tilde{u}}(\mu + \sigma \diamond n), \quad \mathbf{d}\sigma = -\left(\mathcal{L}_u^T \sigma + \frac{\delta h}{\delta n}\right) dt, \quad \mathbf{d}n = (\mathcal{L}_u n + \nu) dt.$$

Upon comparing the equations in (4.2) with those in (3.14) and equations in (4.4) with those in (3.16), we see that in this case adding stochasticity to the template variables via the stochastic Hamiltonian in (4.1) has cancelled out some of the effects of the pure stochastic diffeomorphic transport, since $\tilde{u} := u_t(x) dt + \sum_i \xi_i(x) \circ dW_t^i$ in the σ and n equations in (3.14), has been replaced by $u_t(x) dt$ in (4.2). Thus, the interaction between the two types of stochasticity for the diffeomorphic transport and the template evolution in some cases can be quite significant, and is easily calculable in the Hamiltonian formulation.

See [4] for further discussions of this more general geometric theory of stochastic metamorphosis, with stochasticity in both the diffeomorphic and template evolutions, and applications to several classical examples, including landmarks, images, closed curves, as well as a discussion its use for functional data analysis.

The present paper has not provided any results about the well-posedness of solutions for this new class of stochastic partial differential equations (SPDEs), and in particular nothing has been said about the existence of their solutions, even locally in time. We are not aware of any analytical results in the literature about this class of SPDEs, except to remark that the double Lie derivative in their Itô forms suggests that some analytical estimates may be available by following the methods of [8]. Thus, the present paper leaves open the question of the well-posedness of this class of SPDEs for future mathematical work.

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