Annals of Mathematical Sciences and Applications
Volume 3, Number 1, 7–29, 2018

Accuracy of Ptolemy's *Almagest* in predicting solar eclipses

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Predicting eclipses, especially solar eclipses, was one of the important challenges in ancient and medieval astronomy. Using a statistical approach, David Mumford tested the accuracy of the Chinese algorithm for predicting solar eclipses as formulated in the *Shoushihli* [1]. I carried out a similar analysis of the Indian *Tantrasaṅgraha* [3] using his appoach. In this paper, I report on the accuracy of Ptolemy's algorithm and compare it with the *Shoushihli* and the *Tantrasaṅgraha*.

AMS 2000 subject classifications: Primary 85-03; secondary 01A20, 01A25, 01A32.
Keywords and phrases: *Almagest*, Ptolemy, Ancient eclipse prediction, Medieval Chinese Astronomy, *Shoushihli*, Medieval Indian Astronomy, *Tantrasaṅgraha*.

1. Introduction

One evening at a mathematics conference in 2012, David Mumford mentioned to me that it would be interesting to compare the accuracy of algorithms used in ancient and medieval astronomy across cultures. In particular, he was interested in comparing the Chinese, the Indian and the Greek traditions. Some time later, he implemented the Chinese method of predicting solar eclipses as formulated in the *Shoushihli* (1280 C.E.) and compared its predictions with the predictions computed using modern theory [1]. I adapted his framework and computer code to carry out a similar analysis of *Nīlakaṇṭha*'s *Tantrasaṅgraha* (c.1500 C.E.) which is mathematically the most refined version of medieval Indian astronomy [3]. This paper is a report on a similar analysis of Ptolemy's *Almagest* [4].

Solar eclipses are one of the most spectacular heavenly phenomena. Explanations of eclipses by ancient cultures range from their interpretation as omens to the geometric models of the Greeks. Ptolemy's *Almagest* (c.140 C.E.) represents the final version of ancient Greek astronomy which held sway over Islamic and European astronomy until the advent of Copernicus and Kepler.

Geometric astronomy emerged in India in the early first millennium C.E. Its geometric models are based on epicycles. It is not ptolemaic, but is closely related to pre-ptolemaic Greek astronomy. Details of this contact and its sources remain unknown. The basic framework of Indian astronomy was single epicycle models for the Sun and the Moon, and two epicycle models for the planets. A unique feature of Indian tradition is its use of a variable epicycle radius. Also unique is the Indian use of iteration to solve for interdependent variables. Except for the introduction of lunar evection by *Mañjula* in the 10th century, this framework remained unchanged. Menalaus' spherical trigonometry was apparently unknown in India. Indian astronomers derived the necessary formulas for dealing with spherical geometry from a set of basic planar right triangles aligned with the three coordinate systems (ecliptical, equatorial and horizontal). Its most mathematically accurate version is *Nīlakaṇṭha*'s *Tantrasaṅgraha*[1].

Early Chinese astronomy relied on polynomial interpolations which were later replaced by explicit algebraic formulas. Geometric models are conspicuously absent from Chinese astronomy. Simple algebraic models gradually grew in their complexity over time as more and more phenomena were taken into account[2]. Since predictions of celestial phenomena were solely the emperor's prerogative, the astronomical texts were compiled only by royal astronomers. Failed predictions had the potential for undermining the authority of the emperor and hence frequent revisions were made over time. There were some 200 systems proposed through history and about a quarter of them were officially adopted[3]. Chinese astronomers were aware of the nonuniform lunar motion as early as the first century B.C.E. and astronomical tables for the lunar inequality were formulated in the 3rd century C.E. Nonuniform motion of the Sun was first noticed in the middle of the 6h century. The graph of first forward differences of the solar equation of center based on these early observations consists of two segments spanning the equinoxes. Each segment is an oscillating saw-tooth wave function. If we ignore the small oscillations, the graph becomes a step-wise constant function. A century later, *Yixing* revised these values obtaining approximately correct trend line for the solar equation of center. He also tabulated values of lunar parallax when the Moon is at its highest point in the sky. A ma-

[1]K. Ramasubramanian and M. S. Sriram, *Tantrasaṅgraha of Nīlakaṇṭha Somayājī*, Hindustan Book Agency, New Delhi, 2011

[2]See for example Kiyoshi Yabuuti, "Astronomical Tables in China, from the Han to the T'ang Dynasties', in *Chūgoku chūsei Kagaku gijutsushi no kenkyū* (Studies in the history of medieval Chinese science and technology), 1963, pp. 445–492.

[3]Nathan Sivin, *Granting the Seasons*, Springer, 2009.

jor improvement in computation of the parallax was made by *Xuang* in the ninth century who provided a method for calculating the parallax for all positions of the Moon. Lunar evection was not taken into account in China. The culmination of the Chinese approach is the *Shoushihli* (13th century). After the *Shoushihli*, elements of Islamic astronomy and later, western astronomy were introduced into Chinese astronomy. The algebraic formulas in the *Shoushihli* are specialized for Beijing.

The problem in assessing accuracy of ancient algorithms is that the actual eclipses are too few to properly analyze prediction errors. Mumford's solution is to statistically generate synthetic solar eclipses to assess the accuracy of predictions. He randomly chooses longitudes of mean conjunctions, lunar perigee and a lunar node (ascending or descending) within 20 degrees of the conjunction. The clock is set to start at local noon on the vernal equinox in a given year, 140 C.E. for the *Almagest* and 1280 C.E. for the *Shoushihli* and 1500 C.E. for the *Tantrasaṅgraha*. The locations are Alexandria for the *Almagest*, Beijing for the *Shoushihli* and Kochi in Kerala, India for the *Tantrasaṅgraha*.

The next step after choosing the time and the longitude of a mean conjunction is to determine the time and the longitude of the true conjunction as it would be seen from the Earth's center. Since the actual Sun and the Moon move with variable speed, it is necessary to correct their mean longitudes to obtain their true longitudes. This correction is called the "equation of center". Unless the Moon happens to be directly overhead, the apparent longitude and the latitude of the Moon observed from a point on the surface of the Earth are different from their geocentrically observed values due to lunar parallax. The same is true for the Sun, but the effect is very small. Consequently, the next step is to determine the time and the longitude of the apparent conjunction for a given location on the surface of the Earth. The final step is the computation of the angular distance between centers of the apparent Sun and the apparent Moon. A solar eclipse is locally observed if this distance is less than the sum of the angular radii of the Sun and the Moon. In ancient times, the most difficult step was the computation of parallax. From the modern point of view, it is the lunar motion which is the most difficult (see [1]).

Ptolemy illustrates each algorithm by a numerical example and then proceeds to construct numerical tables. All a calendar maker would need are these tables and he would not have to know the algorithms that created them. Of course this introduces an additional source of error, namely the interpolation error. This could be significant especially when interpolating functions of two or more variables. Ptolemy's algorithms relevant to prediction of solar eclipses are contained in the first six chapters of the *Al-*

magest. I have put these together in the form of modern formulas in Sections 3 and 4. The formulas are mostly from Pedersen's book [2]. The numbers quoted from the *Almagest* are in sexagecimal notation. Ptolemy's algorithm for predicting solar eclipses is described in Section 5. Mumford's statistical framework is described in Section 6. The three traditions are compared in Section 7. The accuracy of eclipse prediction depends on the algorithm as well as the values of the parameters. In the case of the Greek and Indian traditions, I have evaluated their accuracy before and after correcting the values of the critical parameters. The Chinese algebraic formulas without explicit geometric parameters are not amenable to similar analysis.

2. Notation

a_m, a_v: mean and true anomaly respectively
c: double elongation
e: eccentricity of the lunar deferent
h: hour angle
k: equation of center
R: radius of the deferent
r: radius of the epicycle
Y: length of a tropical year
α, δ: equatorial coordinates, right ascension and declination
λ, β: ecliptical coordinates, longitude and latitude
$\dot{\lambda}$: instantaneous velocity
Δ: distance of a celestial body from the Earth's center
ϵ: obliquity of the ecliptic
γ: angle between the ecliptic and a local vertical circle
ζ: angular zenith distance
θ: local rising time of an arc measured from the vernal equinox
ι: inclination of the lunar orbit
ν: angular radius of a celestial body as seen from the Earth
Π: parallax in altitude
Π_λ, Π_β: parallax in longitude and in latitude
Π_H: horizontal parallax
ρ: radius of the Sun, the Moon or the Earth
σ: $\sigma = +1$ if the nearest node is an ascending node, -1 otherwise
φ: terrestrial latitude
Ω: distance from the nearest lunar node
ω: mean angular velocity
ω_a, ω_t: anomalistic and tropical mean angular lunar velocity

The following subscripts have specific meaning
S: the Sun
M: the Moon
Z: zenith
P: the Sun or the Moon or a point on the ecliptic
ah: aphelion
ph: perihelion
ag: apogee
pg: perigee
asc: ascendant
mh: midheaven
ac: apparent conjunction (observed from the surface of the Earth)
tc: true conjunction (geocentrically observed)
mc: mean conjunction
ap: apparent (observed locally from the surface of the Earth)

3. Motion of the Sun and the Moon

3.1. Solar equation of center

The mean solar longitude $\lambda_{m,S}$ at time t is given by the equation

$$\lambda_{m,S}(t) = \lambda_{m,S}(t_0) + \omega_S(t - t_0)$$

where ω_S is Sun's mean tropical velocity along the ecliptic. The length of Ptolemy's tropical year, Y, is 365;14,48 days ([2], p. 131)[4]. Therefore, $\omega_S = \frac{360}{Y}$ degrees/day. We will take t_0 as the time of mean conjunction so that $\lambda_{m,S}(t_0) =$ the longitude of the mean conjunction, λ_{mc}.

The mean solar anomaly a_m is defined as $\lambda_{m,S} - \lambda_{ah}$ where λ_{ah} is the longitude of the aphelion. Ptolemy derives the values of λ_{ah} and the ratio of the radii of the epicycle and the deferent, $\frac{r_S}{R_S}$, from the observed lengths of seasons. He confirms Hipparchus' value $\lambda_{ah} = 65;30$ and believed it to remain constant over time. In fact aphelion moves forward at the rate of 0.0171° per year[5]. The correct value of λ_{ah} for 140 C.E. is 71°. The error in aphelion would produce an ever greater error in the solar equation of center which would have a larger and larger impact on the accuracy of eclipse prediction as time went on.

[4]The reference is to Pedersen's Survey [2]. For each formula or constant, Pedersen cites the relevant section of the *Almagest*.

[5]*al Battani* (9th century) is credited to be the first astronomer to have discovered the motion of aphelion and assigned the value 82;17 to λ_{ah} while *al Zarqali* (11th century) assigned the value 77;50 to it.

Ptolemy obtains $\frac{r_S}{R_S} = \frac{1}{24}$. He sets the radius of the Sun's deferent R_S equal to 60 parts so that the radius of the Sun's epicycle $r_S = 2;30$ parts.

The solar equation of center is given by

$$k_S = -\arcsin\left(\frac{\frac{r_S}{R_S}\sin a_m}{\sqrt{(\frac{r_S}{R_S}\sin a_m)^2 + (\frac{r_S}{R_S}\cos a_m + 1)^2}}\right) \quad ([2], 5.27)$$

and the true longitude of the Sun is given by $\lambda_S = \lambda_{m,S} + k_S$.

3.2. Lunar equation of center

The longitude of the mean Moon is given by $\lambda_{m,M}(t) = \lambda_{m,M}(t_0) + \omega_t(t - t_0)$ where ω_t is the mean tropical velocity of the moon $= 13;10,34,58,33,30,30$ degrees/day. The longitude of the lunar perigee is given by $\lambda_{pg}(t) = \lambda_{pg}(t_0) + \omega_{pg}(t - t_0)$ where the velocity of the lunar perigee is given by $\omega_{\mathrm{pg}} = \omega_t - \omega_a$ with the mean anomalistic velocity of the moon $\omega_a = 13;3,53,56,17,51,59$ degrees/day. The longitude of the apogee $\lambda_{ag}(t) = \lambda_{pg}(t) - 180$. The Moon's tropical, draconitic, (i.e. node to node), anomalistic (from apogee to apogee) and synodic (from conjunction to conjunction) velocities are given in [2] on page 164.

The mean lunar anomaly is given by $a_m = \lambda_{m,M} - \lambda_{\mathrm{ag}}$. Ptolemy's first lunar model is similar to his solar model. He introduces evection in the second model which depends on the elongation, that is the longitudinal difference between the Sun and the Moon. He then introduces a correction to the mean anomaly in the third model. Near a conjunction, the elongation is nearly zero and hence the two corrections to the first model are small. Example 12 in Appendix A of Toomer's *Almagest* [4] indicates that only the first lunar model was used for eclipse calculations. In this paper, I have used the third model. Ptolemy deduced the distance of the Moon from the Earth from its observed angular distance from zenith. From this, it is possible to express the radii of the epicycle, deferent and the eccentricity of the lunar orbit in terms of the radius of the Earth. ([2], p. 207):

Radius of Moon's epicycle $r_M = 5;10$ Earth radii

Radius of Moon's deferent $R_M = 48;52$ Earth radii

Lunar orbit's eccentricity $e = 10;8$ Earth radii

The formulas for calculating the true longitude λ_M of the Moon are:

$$c = 2(\lambda_{m,M} - \lambda_{m,S})$$

$$p = \sqrt{(1 - (\frac{e}{R_M}\sin c)^2)} + \frac{e}{R_M}\cos c \quad ([2], 6.46)$$

$$q = -\arcsin\left(\frac{\frac{e}{R_M}\sin c}{\sqrt{(\frac{e}{R_M}\sin c)^2+(p+\frac{e}{R_M}\cos c)^2}}\right) \quad ([2],\ 6.48)$$

$$\text{True anomaly: } a_v = a_m + q \quad ([2],\ 6.45)$$

$$\Delta = \sqrt{(\tfrac{r_M}{R_M}\sin a_v)^2 + (p + \tfrac{r_M}{R_M}\cos a_v)^2)} \quad ([2],\ 6.51)$$

$$\text{Equation of center: } k_M = -\arcsin(\frac{\frac{r_M}{R_M}\sin a_v}{\Delta}) \quad ([2],\ 6.50)$$

$$\lambda_M = \lambda_{m,M} + k_M$$

The distance of the Moon from the Earth's center is $R_M\Delta$ Earth radii.

4. Parallax

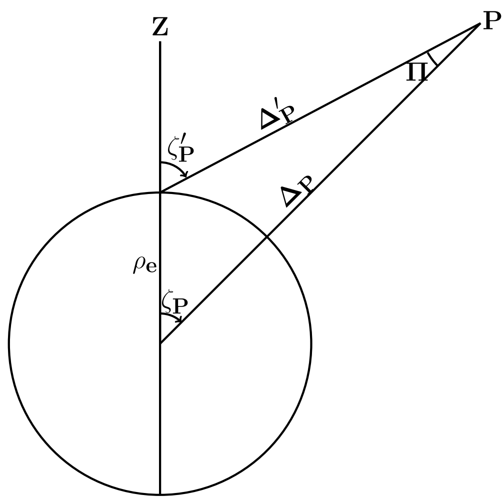

Figure 1: Parallax.

The parallax Π of a celestial body P when viewed from a point on the surface of the Earth is illustrated in Figure 1. The circle represents the surface of the Earth. Let ζ_P be the angular distance of P from the local zenith Z if observed from the center of the Earth and let ζ'_P be its angular distance from Z observed from the surface of the Earth. Then, $\Pi = \zeta'_P - \zeta_P$.

$$\sin\Pi \approx \tan\Pi = \frac{\rho_e \sin\zeta_P}{\Delta_P - \rho_e\cos\zeta_P} \quad ([2],\ 7.9)$$

where

ρ_e the radius of the Earth.
Δ_P distance of P from the Earth's center.

Ptolemy assumes that the Sun's distance from the Earth's center is constant and equal to 1210 Earth radii ([2], p. 211). His formula for the distance of the Moon from the Earth's center is given in the section on the lunar equation of center.

The maximum value of the parallax is called the horizontal parallax. It occurs when P is on the observer's horizon since $\sin \Pi = \rho_e \sin \zeta'_P / \Delta_P$. $\Pi_H = \arcsin \frac{\rho_e}{\Delta_P} \approx \frac{\rho_e}{\Delta_P}$.

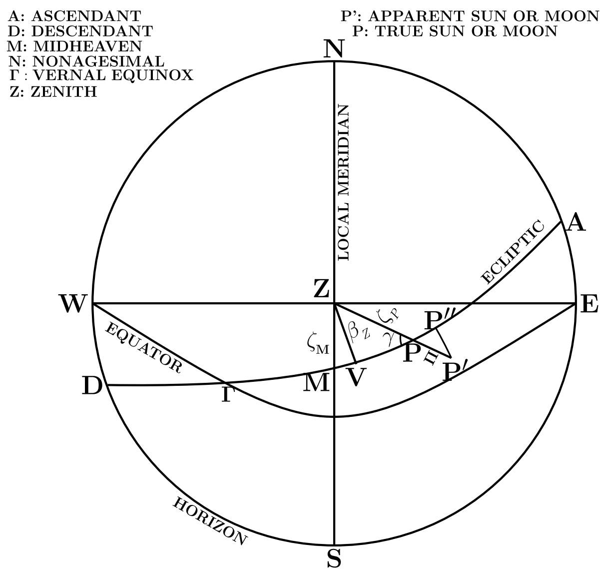

Figure 2: Salient points on the ecliptic.

For predicting the time and magnitude of solar eclipses, it is necessary to calculate the components of Π along the ecliptical coordinates, namely, the parallax in longitude Π_λ and the parallax in latitude Π_β. (Π is called the parallax in altitude.) In Figure 2, P is geocentrically observed to be on the ecliptic. P' is the position of P as seen by an observer on the surface of the Earth. The arc $P'P''$ is the parallax in latitude and the arc PP'' is the parallax in longitude.

Let γ denote the angle between the ecliptic and the vertical circle through P. Then, treating the triangle $PP'P''$ as a plane triangle,

Parallax in latitude $\Pi_\beta = \Pi \sin \gamma$
Parallax in longitude $\Pi_\lambda = \Pi \cos \gamma$

These are the formulas given by Ptolemy in the *Almagest*. For the purpose of calculating its parallax, Ptolemy assumes that the Moon is on the ecliptic. The main problem in calculating the parallax is how to determine the zenith distance and the angle γ.

4.1. Zenith distance ζ and angle γ

This is essentially a question of coordinate transformation and spherical trigonometry. There are three spherical coordinate systems relevant here. One is the ecliptical coordinates defined by ecliptical longitude λ and latitude β. Another is the equatorial coordinates defined by right ascension α and declination δ. Finally, the horizontal coordinates at a given location are defined by azimuth τ and altitude a. The zenith distance is the complement of the altitude. Instead of azimuth, hour angle is specified to fix the position of a point on ecliptic with respect to the local meridian. The hour angle is defined as the right ascension of the zenith minus the right ascension of the given point on the ecliptic. The angle γ is easily computed using spherical trigonometry.

Ptolemy's method is somewhat circuitous involving the ascendant and the midheaven (Figure 2). The ascendant is the point where the ecliptic intersects the horizon in the east. The midheaven is the point where it intersects the local meridian. Before we list formulas related to these, we note the following two basic transformations:

$$\sin\delta = \sin\lambda \sin\epsilon \quad ([2],\ 4.2) \text{ and } \sin\alpha = \tan\delta \cot\epsilon \quad ([2],\ 4.3)$$

where ϵ is the obliquity of the ecliptic. In the *Almagest*, $\epsilon = 23;51,20°$. A simpler formula relating α and λ is $\tan\lambda = \frac{\tan\alpha}{\cos\epsilon}$.

Let P denote either the Sun or the Moon with longitude λ_P and hour angle h. Note that because of the way we have set the clock, the right ascension of the zenith $\alpha_Z = (Y+1)\lambda_{m,S}$ modulo 360 where Y is the length of the tropical year in days. The hour angle of P equals $\alpha_Z - \alpha_P$ degrees or $(\alpha_Z - \alpha_P)/15$ hours..

Ptolemy calculates the zenith distance ζ_P of P by first determining the longitudes λ_{asc} and λ_{mh} of the ascendant and the midheaven respectively by means of their rising times. The rising time, also called the oblique ascension, $\theta(\lambda,\varphi)$ of a point on ecliptic at longitude λ_P at a given location with terrestrial latitude φ is the local rising time of the arc of the ecliptic from the vernal equinox to the point P. It is given in degrees by the formula

$$\theta(\lambda_P,\varphi) = \arcsin\left(\frac{\sin\lambda_P \cos\epsilon}{\cos\delta_P}\right) - \arcsin\left(\frac{\sin\lambda_P \sin\epsilon \tan\varphi}{\cos\delta_P}\right) \quad ([2],\ 4.22)$$

The first term on the right side is the right ascension α_P of P. The last term is called the ascensional difference.

To determine λ_{asc}, solve for λ_{asc} the following equation:

$$\theta(\lambda_{asc}, \varphi) = 15\bar{h} + \theta(\lambda_P, \varphi) \quad ([2], 4.29)$$

where $\bar{h}$ = hours after the rise of P = half day length + h. The day length = $(\theta(\lambda_P + 180, \varphi) - \theta(\lambda_P, \varphi))/15$ hours ([2], 4.26).

To determine λ_{mh}, solve for λ_{mh}:

$$\theta(\lambda_{mh}, 0) = 15\bar{h} + \theta(\lambda_P, 0) \quad ([2], 4.30)$$

Zenith distance of Midheaven: $\zeta_{mh} = \varphi - \delta_{mh}$ ([2], 4.36)

$$\cos\zeta_P = \frac{\cos\zeta_{mh}\sin(\lambda_{asc} - \lambda_P)}{\sin(\lambda_{asc} - \lambda_{mh})} \quad ([2], 4.37)$$

$$\cos\gamma_P = -\cot\zeta_P\cot(\lambda_{asc} - \lambda_P) \quad ([2], 438a)$$

The last formula becomes numerically unstable as either $\zeta_P \to 0$ or $|\lambda_{asc} - \lambda_P| \to 0$. The following reformulation removes this singularity.

$$\sin\zeta_P\cos\gamma_P = -\frac{\cos\zeta_{mh}\cos(\lambda_{asc} - \lambda_P)}{\sin(\lambda_{asc} - \lambda_{mh})}$$

Assuming $\sin\Pi \approx \Pi$, the components of the parallax are:

$$\Pi_\lambda \approx \frac{\rho_e\sin\zeta_P\cos\gamma_P}{\Delta_P - \rho_e\cos\zeta_P}$$

$$\Pi_\beta \approx \sqrt{\Pi^2 - \Pi_\lambda^2}$$

$$\Pi_\beta > 0 \text{ if } \varphi < \delta_{mh}, \quad \Pi_\beta < 0 \text{ if } \varphi > \delta_{mh}$$

The problem in implementing these formulas is that Ptolemy does not have an explicit formula for the inverse of function θ. He solves the equations for λ_{asc} and λ_{mh} by interpolating his table for θ. The simpler formulas described in the next section circumvent the use of rising times and interpolation.

4.2. Calculating ζ_P and γ without using rising times

ζ_P and $\cos\gamma_P$ may be calculated using the standard formulas of coordinate transformation.

$\cos\zeta_P = \sin\varphi\sin\delta_P + \cos\varphi\cos\delta_P\cos h.$

This formula was known to *Āryabhaṭa* in India in the 5th century C.E. In view of the close connection between medieval Indian astronomy and the pre-ptolemaic astronomy in Greece, it is strange that this formula is absent from the *Almagest*. $\cos\gamma_P$ may be calculated from the equatorial coordinates of the zenith as follows.

Declination of the zenith $\delta_Z = \varphi$.

Right ascension of zenith $\alpha_Z = \alpha_P + h$.

$\sin\beta_Z = \cos\epsilon\sin\varphi - \sin\epsilon\cos\varphi\sin\alpha_Z$. (This formula is given in the *Tantrasaṅgraha*.)

$\sin\lambda_Z = \frac{\cos\varphi\cos\epsilon\sin\alpha_Z + \sin\varphi\sin\epsilon}{\cos\beta_Z}$.

$\cos\lambda_Z = \frac{\cos\varphi\cos\alpha_Z}{\cos\beta_Z}$.

The longitudes of the midheaven and the ascendant may also be easily calculated.

$\alpha_{mh} = \alpha_Z$ and $\lambda_{mh} = \arctan(\frac{\tan\alpha_{mh}}{\cos\epsilon})$.

$\lambda_{asc} = \lambda_Z + 90$.

The formula for $\cos\gamma$ now follows by considering the right spherical triangle PVZ in Figure 2. The point V on the ecliptic is called the nonagesimal where the ecliptic has the maximum altitude. Use of ecliptical coordinates λ_Z, β_Z of the zenith which are basic quantities in the Indian tradition is absent from the *Almagest*, although β_Z is implicit in Ptolemy's formula for the angle between the ecliptic and the horizon.

5. Ptolemy's algorithm for predicting solar eclipses

The following algorithm is described in the *Almagest* (VI 9, H528-533) and Example[6] 12, Appendix A in [4]. Start with the longitude λ_{mc} of the mean conjunction at time t_{mc} and the longitudinal distance Ω_{mc} of the mean conjunction from the nearest lunar node. $\Omega_{mc} = \lambda_{mc} - \lambda_{node}$. The small motion of the nodes is ignored as the following steps are carried out.

5.1. Determine the true geocentric conjunction

Let k_S and k_M be the equations of center for the Sun and the Moon at the time of mean conjunction. Let λ_{tc} be the longitude of the true conjunction occurring at time t_{tc}. Ptolemy assumes that the instantaneous lunar velocity $\dot\lambda_M$ and instantaneous solar velocity $\dot\lambda_S$ are constant and their ratio is equal to $\frac{1}{13}$ during the time interval between t_{mc} and t_{tc}. Then

$$\Delta\lambda = \lambda_{tc} - \lambda_{mc} = \frac{13}{12}(k_S - k_M)$$

[6]This example is due to Theon of Alexandria (c.4th century C.E.)

$$t_{tc} = t_{mc} + \Delta\lambda/\dot{\lambda}_M$$

He approximates $\dot{\lambda}_M$ by the formula ([2], 7.51)

$$\dot{\lambda}_M = \omega_t + \frac{\partial k_M}{\partial a_v}\omega_a$$

$\frac{\partial k_M}{\partial a_v}$ is the partial derivative of the lunar equation of center k_M with respect to the true anomaly a_v at mean conjunction, that is, when the elongation is zero.

The distance of the true conjunction from the nearest node $\Omega_{tc} = \Omega_{mc} + k_M + \Delta\lambda$.

5.2. Determine the locally observed apparent conjunction

First Approximation:

- Calculate the longitudinal component $\bar{\Pi}_\lambda^{(1)}$ of the net parallax (that is, the parallax of the Moon minus the parallax of the Sun) at the time of true conjunction. The longitudes of the apparent Moon and the apparent Sun now differ by $\bar{\Pi}_\lambda^{(1)}$.
- Time of apparent conjunction $t_{ac} \approx t_{tc} + \frac{\bar{\Pi}_\lambda^{(1)}}{\dot{\lambda}_M}$.

Second Approximation (Epiparallax):

- Calculate $\bar{\Pi}_\lambda^{(2)}$ at time t_{ac}. The longitudes of the Sun and the Moon set equal to λ_{tc}. The distance of the Moon from the Earth, Δ_M is calculated using the value of the mean anomaly at time t_{ac} and elongation set equal to zero.
- $\Delta\bar{\Pi}_\lambda = \bar{\Pi}_\lambda^{(2)} - \bar{\Pi}_\lambda^{(1)}$.
- The longitudinal component of the total net parallax $\bar{\Pi}_\lambda = \bar{\Pi}_\lambda^{(1)} + \Delta\bar{\Pi}_\lambda + \frac{(\Delta\bar{\Pi}_\lambda)^2}{\bar{\Pi}_\lambda^{(1)}}$.
- Time of apparent conjunction: $t_{ac} = t_{tc} + \frac{13\bar{\Pi}_\lambda}{12\dot{\lambda}_M}$.
- Longitude of the apparent conjunction: $\lambda_{ac} = \lambda_{tc} + \frac{13}{12}\bar{\Pi}_\lambda$.
- With longitude λ_{ac} of the Moon at time t_{ac}, calculate the net latitudinal parallax $\bar{\Pi}_\beta$. (Elongation = 0.) The corresponding longitudinal distance is $12\bar{\Pi}_\beta$. (Ptolemy's approximation: The inclination of the Moon's orbit $\iota = 5$ degrees. $\frac{1}{\sin\iota} \approx 12$.) Let $\sigma = 1$ if the nearest node is the ascending node; $\sigma = -1$ if the nearest node is the descending

node. Then, the distance Ω_{ap} of the apparent conjunction from the parallax corrected nearest node is given by

$$\Omega_{ap} = \Omega_{tc} + \frac{13}{12}\bar{\Pi}_\lambda + 12\sigma\bar{\Pi}_\beta.$$

5.3. Magnitude of the eclipse

Let ν_M and ν_S be the angular radii of the Moon and the Sun respectively as seen from the Earth. The angular distance between the apparent Sun and the apparent Moon $\beta_{ap} = \sigma\iota\sin\Omega_{ap}$. If $|\beta_{ap}| < \nu_S + \nu_M$, a solar eclipse is locally observed with

$$\text{magnitude of the eclipse } = 1 - \frac{|\beta_{ap}|}{\nu_S + \nu_M}.$$

Since the distance of the Sun from the Earth is assumed to be constant, ν_S is constant and assumed to be equal to the minimum value of ν_M at syzygies. Ptolemy determines the maximum and minimum values of the angular radius of the moon at syzygies from pairs of lunar eclipses occurring near the perigee and the apogee. These are 0;17,40° and 0;15,40° respectively. These values are inconsistent with Ptolemy's theory of lunar motion. Since $\nu_M = \arctan\frac{\rho_M}{\Delta_M}$ where ρ_M is the Moon's radius and Δ_M is its distance from the Earth, ν_M approximately varies inversely as Δ_M. From the equations of lunar motion, it may be readily seen that ρ_M varies between $53;50$ and $64;10$ Earth radii at syzygies ([2], page 207). Therefore according to Ptolemy's lunar theory, the maximum of ν_M should deviate from its average by 8.8%, but Ptolemy's observed values deviate from their average only by 6.0%. In the statistical trials reported below, the values of ν_M according to the modern theory deviated from their average at most by 6.4% which is close to Ptolemy's observed value.

Instead of determining the magnitude as defined above, Ptolemy expresses the extent to which the Sun is occluded in digits. The Sun is assumed to be 12 digits wide. His tables for eclipses occurring at syzygies follow from the following formulas.

At the apogee, $\nu_M = \nu_S$. Therefore, the Sun is fully occluded ($d = 12$) when $\Omega_{ap} = 0$. If the Sun and the Moon just touch each other, $d = 0$ and $|\Omega_{ap}| = (\nu_M + \nu_S)/\tan 5° \approx 6°$. Interpolating linearly, we get

$$d_a(\Omega_{ap}) = 12 - 2|\Omega_{ap}| \quad \text{digits}$$

At the perigee, $\nu_M > \nu_S$. The Sun becomes fully occluded as soon as $|\Omega_{ap}| = (\nu_M - \nu_S)/\tan 5^\circ \approx 0.4^\circ$. Therefore, Ptolemy sets $d = 12$ when $|\Omega_{ap}| = 0.4$. $d = 0$ when $|\Omega_{ap}| = (\nu_M + \nu_S)(/\tan 5^\circ) \approx 6.4^\circ$. The interpolation formula at the perigee is

$$d_p(\Omega_{ap}) = 12.8 - 2|\Omega_{ap}| \quad \text{digits}$$

Note that $d_p = 12.8$ when $\Omega_{ap} = 0$.

Occlusion when the lunar position is between the syzygies with nodal distance Ω_{ap} and the distance Δ from the Earth is obtained by linear interpolation:

$$\begin{aligned} d(\Omega_{ap}, \Delta) &= d_a(\Omega_{ap}) + [d_p(\Omega_{ap}) - d_a(\Omega_{ap})]\,\frac{\Delta_a - \Delta}{2r_M} \\ &= 16.968 - 0.07742\Delta - 2|\Omega_{ap}| \end{aligned}$$

In the above, the distance from the Earth at apogee, $\Delta_a = 64;10$ and the radius of the epicycle, $r_M = 5;10$.

To convert the digits as defined by Ptolemy to the magnitude of the eclipse as defined above, note that when $\Omega_{ap} = 0$, $d(0, \Delta) = 16.968 - 0.07742\Delta$ and the magnitude equals 1. Therefore,

$$\text{magnitude of the eclipse} = \frac{d(\Omega_{ap}, \Delta)}{16.968 - 0.07742\Delta} = 1 - \frac{|\Omega_{ap}|}{8.484 - 0.03871\Delta}$$

and

$$\frac{|\beta_{ap}|}{\nu_S + \nu_M} = \frac{|\Omega_{ap}|}{8.484 - 0.03871\Delta}$$

6. Mumford's statistical framework

A sample of hypothetical solar eclipses is created by randomly choosing longitudes of the mean conjunction, perigee and a lunar node. The node is randomly chosen to be ascending or descending. The longitude of the node is chosen to be within 20 degrees of the conjunction. A solar eclipse is predicted if $|\beta_{ap}| < \nu_S + \nu_M$ and it occurs during daylight. A prediction is considered *strong* if its magnitude is greater than 0.1 and the time of the nearest approach is at least one hour after sunrise or one hour before sunset. A prediction is *weak* if it is not strong. A strong prediction by an ancient algorithm is considered to be *correct* if it is predicted at least weakly by modern algorithms; it is a *false positive* otherwise. A strong modern prediction is called *false negative* if it is not even weakly predicted by the ancient algorithm.

In order to assess accuracy of an algorithm, instead of comparing magnitudes, *normalized latitudes* $\frac{\beta_{ap}}{\nu_S+\nu_M}$ are compared and the standard deviation of the error is computed. The hour angle and the equation of center of the Moon and the Sun are compared as well and the standard deviation of the errors is computed.

7. Results

I have used the same format as in [1] for summarizing the results below. In addition to presenting an analysis of Ptolemy's algorithm, I have also included analyses of the *Tantrasaṅgraha* [3] and the *Shoushihli* [1] for comparison across different traditions. In each case, I ran 10,000 trials. Table 1 shows the rate of false positives and false negatives for each of the algorithms. The *Almagest* is the most accurate with an error rate of 3% while the *Tantrasaṅgraha* is the least accurate with an error rate of 11.8%. The *Shoushihli* is not much better with an error rate of 9.9%.

Table 1: **Eclipse Predictions**

	Almagest	*Tantrasaṅgraha*	*Shoushihli*
Strong Prediction	973	1065	1099
False Positive	24 (2.5%)	116 (10.9%)	101 (9.2%)
False Negative	5 (0.5%)	10 (0.9%)	8 (0.7%)

The rate of false positives for the *Shoushihli* reported in [1] is 8.2% which is lower than the 9.2% found here, but it is within the statistical margin of error. The 95% confidence interval is $9.2 \pm 1.7\%$. The sample size in [1] is twice the sample size used in this paper.

In the case of the *Tantrasaṅgraha*, the rate of false positives reported in [3] is 3.9% which is much lower than the rate of 10.9% reported here. The explanation is the following. Comparison of the *Shoushihli* and the *Tantrasaṅgraha* in [3] is made using the same place and the same year for both. As explained below, relative sizes of the error in the aphelion in the two cases depend on the choice of the year. Therefore a neutral choice was to use the theoretically correct value of aphelion in both cases, thus making the comparison independent of the choice of the year. The results in [3] correspond to the results shown in Table 3 below. The rate of 3.9% for false positives in [3] is close to the corresponding value of 4.3% in Table 3.

False predictions in all three cases are mostly false positive. The standard deviation of errors in normalized latitude, hour angle and equations of center is tabulated in Table 2. All three have large standard deviation of error in

Table 2: **Standard Deviation of errors**

	Std. dev. of error		
	Almagest	*Tantrasaṅgraha*	*Shoushihli*
Normalized Latitude	0.1131	0.2022	0.1873
Hour angle (minutes)	50 min.	73 min.	29 min.
Solar equation of center	0.3625°	0.4465°	0.3620°
Lunar equation of center	0.1827°	0.1538°	0.3330°

the solar equation of center. The *Shoushihli* has the smallest standard deviation of error in the hour angle but almost twice as large a standard deviation of error in the lunar equation of center. Values of the three algorithms for normalized latitude, hour angle and equations of center vs the corresponding modern values are plotted in Figures 3 and 4. Black circles in the plot of normalized latitude represent correct predictions. Red crosses represent false negatives and the green stars represent false positives. (In the print version of the figure, false negatives are represented by dark grey crosses and false positives are represented by light grey squares.) The two black diagonals represent predictions differing from modern predictions by $\pm 0.25\%$.

In the case of the *Almagest* and the *Tantrasaṅgraha*, it is possible to analyze the effect of the error in the parameter values on the prediction error. The motion of the Sun is the simplest and close to modern theory. The solar equation of center[7] is $\sin k_S \approx (r_S/R_S)\sin(\lambda_{m,S} - \lambda_{ah}) \bmod (r_S/R_S)^2$. In all three traditions, the aphelion was believed to be at a permanently fixed position on the ecliptic. An error in the longitude λ_{ah} of the aphelion amounts to a phase difference between the ancient and the modern values of anomalies giving the plot in Figure 4 an oval shape. Ptolemy deduced the values $\lambda_{ah} = 65.5°$ and the ratio $\frac{r_S}{R_S} = \frac{1}{24}$ from the observed lengths of seasons. λ_{ah} according to the modern theory was about 71° in 140 C.E. In India, the value of λ_{ah} was fixed at 78° at the time of *Āryabhaṭa* (5th century C.E.). The correct value at that time was between 77° and 78°. Its correct value in 1500 C.E. was 97.3°. Although there was no explicit concept of perihelion and aphelion in the Chinese tradition, Chinese astronomers since the 6th century always assumed that the aphelion coincides with the summer solstice. The correct value of λ_{ah} in 600 C.E. was 79°, but at the time of the *Shoushihli*, the correct value was 90.6°, very close to the traditional Chinese value of 90°. Consequently, the plot for the *Shoushihli* in Figure 4 is not

[7] In Indian astronomy, r_S is not held constant, but assumed to depend on the solar anomaly. Consequently, $\sin k_S = (r_S/R_S)\sin(\lambda_{m,S} - \lambda_{ah})$ exactly in the *Tantrasaṅgraha*.

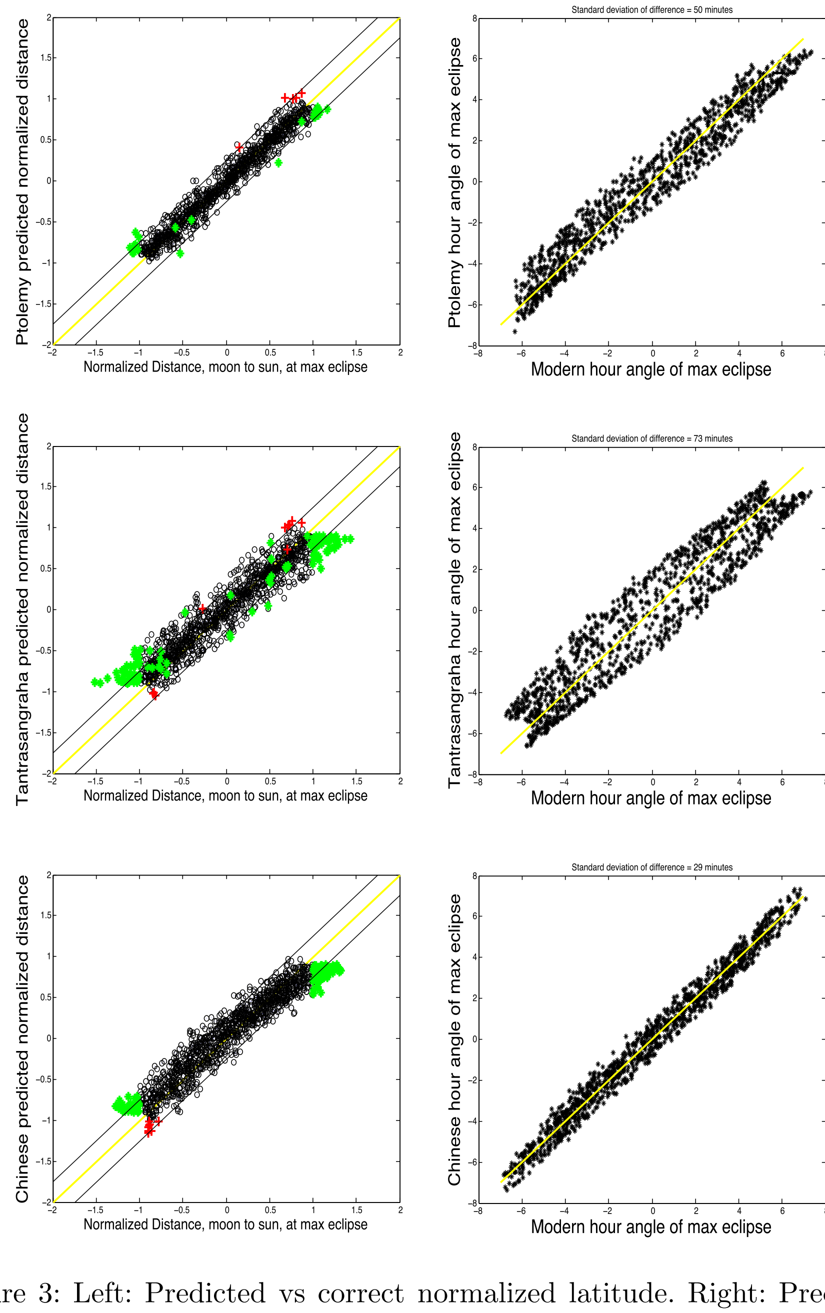

Figure 3: Left: Predicted vs correct normalized latitude. Right: Predicted vs correct hour angle. Top: *Almagest*, Middle: *Tantrasaṅgraha*, Bottom: *Shoushihli*.

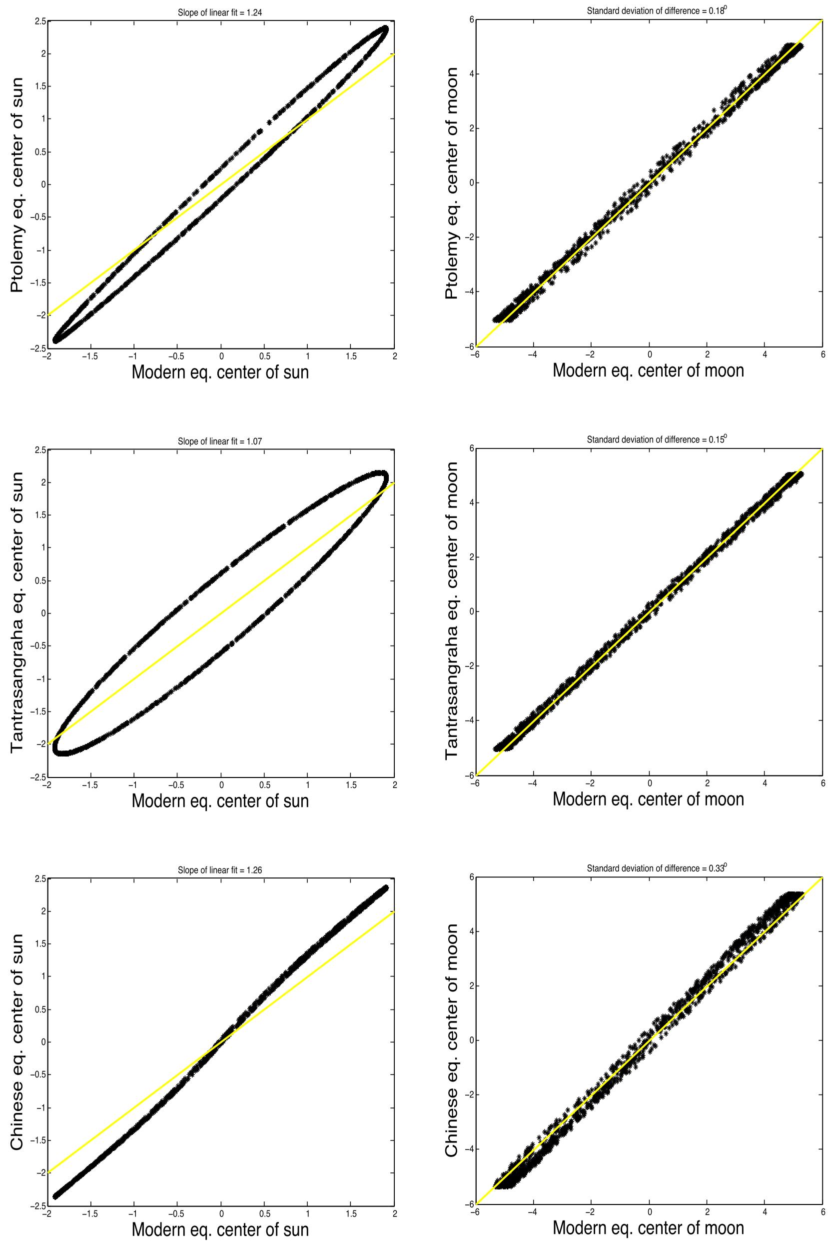

Figure 4: Left: Predicted vs correct solar equation of center. Right: Predicted vs correct lunar equation of center. Top: *Almagest*, Middle: *Tantrasaṅgraha*, Bottom: *Shoushihli*.

oval, but almost linear. The small deviation from the linear fit is due to the fact that the *Shoushihli* approximates the solar equation in each season by a cubic polynomial.

Results of running the statistical trials after correcting Ptolemy's and *Nīlakaṇṭha*'s values of λ_{ah} are shown in Tables 3 and 4, and plotted in Figure 5. The standard deviation of error in the solar equation of center drops from 0.3625° to 0.3329° in the case of the *Almagest* and from 0.4465° to 0.1664° in the case of the *Tantrasaṅgraha*. The performance of the *Almagest* has improved, but that of the *Tantrasaṅgraha* has improved more because the error in the *Tantrasaṅgraha*'s λ_{ah} is larger. The *Tantrasaṅgraha*'s rate of prediction error drops from 11.8% to 4.3% and the standard deviation of error in the hour angle drops from 73 minutes to mere 17 minutes compared to Ptolemy's 42 minutes.

Table 3: **Eclipse Predictions (correct λ_{ah})**

	Almagest	*Tantrasaṅgraha*
Strong Prediction	974	1079
False Positive	15 (1.5%)	46 (4.3%)
False Negative	0 (0%)	0 (0%)

Table 4: **Standard Deviation of errors (correct λ_{ah})**

	Std. dev. of error	
	Almagest	*Tantrasaṅgraha*
Normalized Latitude	0.0882	0.0963
Hour angle (minutes)	42 min.	17 min.
Solar equation of center	0.3329°	0.1664°
Lunar equation of center	0.1829°	0.1558°

An error in the ratio r_S/R_S produces a tilt in the plot with respect to the 45° diagonal. The linear fit to its plot in Figure 5 has a slope 1.25 in the case of the *Almagest* and 1.12 in the case of the *Tantrasaṅgraha*. The reason for the systematic error is that the value of the ratio $\frac{r_S}{R_S}$ is too large. If we put $\frac{r_0}{R_S} = 2e$ where $e =$ the eccentricity of the Earth's orbit, the equation of center coincides with the Keplerian model up to first order. The theoretical value of $2e$ is $\frac{1}{29.9239} \approx \frac{1}{30}$. So the correct value of $\frac{r_S}{R_S}$ is approximately $\frac{1}{30}$. Ptolemy's value is $\frac{1}{24}$ while *Nīlakaṇṭha*'s value is $\frac{3}{80}$. In the case of the *Shoushihli*, there is no explicit parameter $\frac{r_S}{R_S}$, but note that $\frac{1}{24}/\frac{1}{30} = 1.25$ and $\frac{3}{80}/\frac{1}{30} = 1.125$ which are the values of the slopes of the linear fit of plots of Ptolemy and *Nīlakaṇṭha*'s solar equation in Figure 5. The plot for the

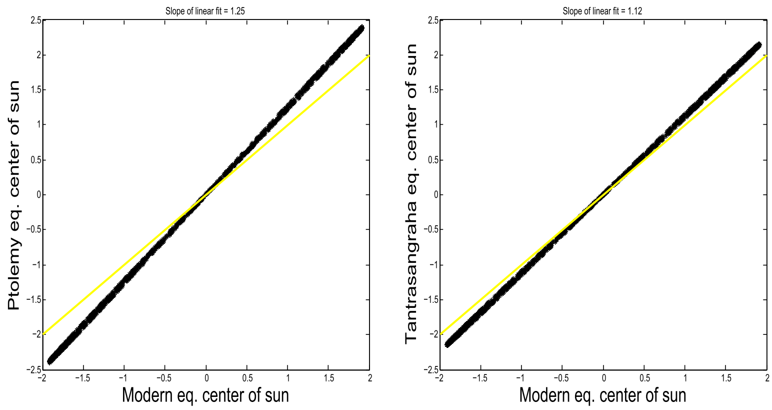

Figure 5: Solar equation of center with correct aphelion. Left: *Almagest*, Right: *Tantrasaṅgraha*.

Table 5: **Eclipse Predictions (corrected solar equation)**

	Almagest	*Tantrasaṅgraha*	*Shoushihli*
# Prediction	983	1,059	1,112
False Positive	16 (1.6%)	42 (4.0%)	136 (12.2%)
False Negative	4 (0.4%)	0 (0%)	40 (3.6%)

Table 6: **Standard Deviation of errors (corrected solar equation)**

	Std. dev. of error		
	Almagest	*Tantrasaṅgraha*	*Shoushihli*
Magnitude	0.0905	0.1082	0.2331
Hour angle	39 min.	29 min.	64 min.
Solar eqn of cntr	0.0091°	0.0145°	0.0407°
Lunar eqn of cntr	0.1862°	0.1578°	0.3389°

Shoushihli in Figure 5 has a slope 1.26. Therefore, the *Shoushihli*'s equation of center may be corrected by dividing it by 1.26. The results of the 10,000 random trials with corrected equation of center are shown in Tables 5 and 6.

The standard deviation of error in the solar equation of center drops further from 0.3329° to 0.0091° in the case of the *Almagest* and from 0.1664° to 0.0145° in the case of the *Tantrasaṅgraha*. The prediction error in the case of the *Almagest* goes up slightly while there is a slight improvement in the hour angle. In the case of the *Tantrasaṅgraha*, it is exactly the opposite. In the case of the *Shoushihli*, the effect is more dramatic. The standard deviation of error in the solar equation of center does drop from 0.3620° to 0.0407° but the prediction error goes up from 9% to 15.8%. The standard deviation of error in the hour angle goes up from 29 minutes to 64 minutes. This is quite surprising. Standard deviation of error in the distance of the true conjunction (which does not involve parallax) from the node after correcting for $\frac{r_S}{R_S}$ drops from 0.3497 to 0.0315 for the *Almagest* and from 0.1694 to 0.0223 for the *Tantrasaṅgraha*, but in the case of the *Shoushihli* it drops from 0.3868 to a still high 0.3256. Perhaps this is related to the large error in the *Shoushihli*'s lunar equation of center.

The plot of the lunar equation of center in Figure 6 shows no systematic error as in the case of the solar equation, but there two other important sources which introduce systematic error. One is the inclination ι of the lunar orbit. An error in ι introduces a proportionate amount of error in the lunar latitude. Ptolemy's value $\iota = 5°$ is a little too small compared to the theoretical value $\iota = 5.145°$. Indian astronomers always used a grossly inaccurate value $\iota = 4.5°$.

Since $\sin \Pi = \frac{\rho_e}{d} \sin \zeta'$, the horizontal parallax $\Pi_H = \frac{\rho_e}{d}$ is the other source of systematic error. Because the solar parallax is very small, we focus only on the horizontal parallax of the Moon. The lunar distance from the Earth's center varies and so the error in the lunar horizontal parallax is intimately connected with the theory of lunar motion. A linear fit of the plot of Ptolemy's horizontal parallax vs the theoretical has the equation

$$\Pi_H(\textit{Almagest}) = 1.3804 \Pi_H(\text{Modern}) - 0.0061$$

Ptolemy estimates the distance of the Moon from an observation he made of the zenith distance of the Moon in Alexandria. This is enough to deduce that Ptolemy's mean distance of the Moon equals 59 Earth radii. This is the only parameter available for adjustment of the horizontal parallax without changing the rest of the theory of the Moon. The theoretically correct value is 60.27. The linear fit of the plot of the horizontal parallax in the case the *Tantrasaṅgraha* has the equation

$$\Pi_H(\textit{Tantrasaṅgraha}) = 1.9841 \Pi_H(\text{Modern}) - 0.0177$$

Again, the mean lunar distance is the only parameter we can adjust. *Nīlakaṇṭha* assumes that the mean lunar distance is 65.46 Earth radii.

Results of running the statistical trials after the following modifications are shown in Tables 7 and 8: ι was changed to its correct value $= 5.145°$. The lunar distance calculated using Ptolemy's lunar theory was multiplied by

$\frac{60.27}{59}$ and the lunar distance computed according to the *Tantrasaṅgraha* was multiplied by $\frac{60.27}{65.46}$. The algebraic model of the *Shoushihli* is not amenable to such an adjustment[8].

Again, the impact of the latest corrections is insignificant in the case of the *Almagest*, but they significantly improve the accuracy of the *Tantrasaṅgraha*'s eclipse prediction.

Table 7: **Eclipse Predictions (corrected parameters)**

	Almagest	*Tantrasaṅgraha*
Strong Prediction	982	942
False Positive	12 (1.2%)	12 (1.3%)
False Negative	2 (0.2%)	0 (0%)

Table 8: **Standard Deviation of errors (corrected parameters)**

	Std. dev. of error	
	Almagest	*Tantrasaṅgraha*
Normalized Latitude	0.0892	0.0777
Hour angle (minutes)	40 min.	26 min.
Solar equation of center	0.0092°	0.0146°
Lunar equation of center	0.1863°	0.1565°

8. Conclusion

The *Almagest* is the most accurate among the three algorithms considered in this paper with the rate of error in eclipse prediction of 3%. The *Shoushihli* is a distant second with an error rate of 9.9%. The least accurate is the *Tantrasaṅgraha* with an error rate of 11.8%. Almost all of the false predictions are false positive in all three traditions. Accuracy of eclipse prediction by the *Tantrasaṅgraha* suffers most from its continued use of the longitude of aphelion which was fixed a millenium earlier and its inaccurate value of the inclination of the lunar orbit. Tests of the *Tantrasaṅgraha*'s algorithm show that its accuracy after correcting crucial parameters is comparable to that of the *Almagest*. With its reliance on empirical algebraic formulas, the Chinese formulation is inherently limited to the time and place for which it was designed. With no underlying parametric model, it is unclear if and how its accuracy could be improved.

[8] The *Shoushihli*'s value for ι is 5.91°.

Acknowledgement

I thank David Mumford for introducing me to this subject and making his computer code for eclipse prediction by modern and the *Shoushihli*'s algorithms available. The code was essential for carrying out the comparisons reported here. I also thank Lovy Singhal of the Tata Institute of Fundamental Research for his help with figures in this paper.

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Received March 6, 2017