ON SINGULAR REAL ANALYTIC LEVI-FLAT FOLIATIONS[*]

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Abstract. A singular real analytic foliation $\mathcal{F}$ of real codimension one on an n-dimensional complex manifold M is Levi-flat if each of its leaves is foliated by immersed complex manifolds of dimension $n-1$. These complex manifolds are leaves of a singular real analytic foliation $\mathcal{L}$ which is tangent to $\mathcal{F}$. In this article, we classify germs of Levi-flat foliations at $(\mathbb{C}^n, 0)$ under the hypothesis that $\mathcal{L}$ is a germ of holomorphic foliation. Essentially, we prove that there are two possibilities for $\mathcal{L}$, from which the classification of $\mathcal{F}$ derives: either it has a meromorphic first integral or it is defined by a closed rational $1-$form. Our local results also allow us to classify real algebraic Levi-flat foliations on the complex projective space $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{C}}$.

Key words. Holomorphic foliation, CR-manifold, Levi-flat variety.

Mathematics Subject Classification. 37F75, 32S65, 32V40.

1. Introduction. Let $U \subset \mathbb{C}^n$ be an open subset and $H \subset U$ be a real analytic submanifold of real codimension one. For each $p \in H$, there is a unique complex vector space of dimension $n-1$ contained in the tangent space $T_p H$, which is given by $\mathcal{L}_p = T_p H \cap i T_p H$, where $i = \sqrt{-1}$. When the real analytic distribution of complex hyperplanes $p \in H \mapsto \mathcal{L}_p$ is integrable, in the sense of Frobenius, we say that H is a *Levi-flat* hypersurface. Thus, H is foliated by immersed complex manifolds of dimension $n-1$, defining the so-called *Levi foliation*. The following local normal form was proved by E. Cartan [7, Th. IV]: at each $p \in H$, there are holomorphic coordinates $(z_1, \ldots, z_n)$ in a neighborhood V of p such that

$$H \cap V = \{\text{Im}(z_n) = 0\}. \tag{1}$$

In particular, this says that the Levi foliation on $H \cap V$ is given by $z_n = c$, for $c \in \mathbb{R}$.

Consider now a (non-singular) real analytic foliation $\mathcal{F}$ of real codimension one on the open subset $U \subset \mathbb{C}^n$. We say that $\mathcal{F}$ is *Levi-flat* if its leaves are, locally, Levi-flat hypersurfaces. Thus, there exists a real analytic foliation $\mathcal{L} = \mathcal{L}(\mathcal{F})$ on U whose leaves are immersed complex manifolds of dimension $n-1$, entirely contained in the leaves of $\mathcal{F}$. Keeping the terminology of the hypersurface case, this underlying foliation is also referred to as *Levi foliation*. The above definitions make sense both in the local setting, as germs of Levi-flat hypersurfaces or foliations at $(\mathbb{C}^n, 0)$, or in a global context, as objects lying in an n-dimensional complex manifold M, locally defined in open coordinate sets.

A basic example is given by the Levi-flat foliation $\mathcal{F}_0$ defined by level sets of the form $\text{Re}(z_n) = c$, for $c \in \mathbb{R}$, in coordinates $(z_1, \ldots, z_n)$ of $\mathbb{C}^n$. Its associated Levi foliation $\mathcal{L}_0 = \mathcal{L}(\mathcal{F}_0)$ is given by $z_n = c$, for $c \in \mathbb{C}$. The germ of $\mathcal{F}_0$ at $(\mathbb{C}^n, 0)$ actually furnishes a local normal form for Levi-flat foliations under the context of

CR-conjugations [11, Th. 4.1]. This means that, given a germ of Levi-flat foliation $\mathcal{F}$ at $(\mathbb{C}^n, 0)$ with Levi foliation $\mathcal{L}$, there exists a germ of real analytic diffeomorphism $\Phi : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ that, in some neighborhood of $0 \in \mathbb{C}^n$, conjugates $\mathcal{F}$ and $\mathcal{F}_0$ together with their Levi foliations $\mathcal{L}$ and $\mathcal{L}_0$, having the additional property of being a holomorphic map when restricted to each leaf of $\mathcal{L}$ assuming values in the corresponding leaf of $\mathcal{L}_0$.

In the singular case, an irreducible real analytic hypervariety $H \subset U \subset \mathbb{C}^n$ is said to be *Levi-flat* if its *regular part* — the set of points of H near which it is a real analytic manifold of real codimension one — is a Levi-flat hypersurface. We point out that, in this case, the Levi foliation of H is not always the restriction of a singular holomorphic foliation defined in a neighborhood of H — examples for this situation can be found in [3] and [12]. However, if the Levi foliation of a germ of real analytic Levi-flat hypervariety at $0 \in \mathbb{C}^n$ extends to a holomorphic foliation in a neighborhood of $0 \in \mathbb{C}^n$, a theorem by Cerveau and Lins Neto [9] asserts that the ambient foliation has a meromorphic first integral, that is, a non-constant meromorphic function that is constant along its leaves.

Likewise, we say that a singular real analytic foliation $\mathcal{F}$ is *Levi-flat* if it is Levi-flat outside its singular set $\mathrm{Sing}(\mathcal{F})$. Its *Levi foliation* $\mathcal{L} = \mathcal{L}(\mathcal{F})$ then turns out to be a singular real analytic foliation whose leaves are immersed complex manifolds of dimension $n-1$ (see Sect. 2). Note that $\mathcal{L}$ can be non-holomorphic — an example can be found in [11, Sect. 2]. In this article, our purpose is to classify Levi-flat foliations whose underlying Levi foliation is holomorphic — we assume this fact throughout the text. The same hypothesis has been considered in the article [1]. Its results, in the local case, lead to the conclusion that $\mathcal{L}$ has a Liouvillean integrating factor, which is weaker then what our classification proposes. We should also mention the paper [14], were Levi-flat foliations of class C^1 with holomorphic underlying foliation are studied.

As a first step, we examine Levi-flat foliations defined by the levels of germs of real analytic meromorphic functions with real values. We obtain the following:

THEOREM 1. *A germ of holomorphic foliation of codimension one at $(\mathbb{C}^n, 0)$, $n \geq 2$, that is tangent to the levels of a non-constant real meromorphic function with real values admits a meromorphic first integral.*

The proof of this Theorem 1 is carried out in Sect. 4. Let us describe it briefly. First, by taking two-dimensional transversal sections, we can suppose that $n = 2$. Denoting by $\mathcal{G}$ the holomorphic foliation in the theorem's statement, if its real meromorphic first integral has a fiber of codimension one accumulating to $0 \in \mathbb{C}^2$, we have a real analytic Levi-flat hypersurface tangent to $\mathcal{G}$ and the conclusion comes immediately from the aforementioned Cerveau-Lins Neto's theorem. On the other hand, if such a fiber does not exist, we can conclude that $\mathcal{G}$ is a non-dicritical foliation whose leaves accumulating to $0 \in \mathbb{C}^2$ are separatrices. A dynamical study of the (virtual) holonomy of $\mathcal{G}$ then allows us to construct a holomorphic first integral.

The main result of this article — for which Theorem 1 is part of the proof — provides a complete characterization of local Levi-flat foliations with holomorphic Levi foliations:

THEOREM 2. *Let $\mathcal{F}$ be a germ of real analytic Levi-flat foliation at $(\mathbb{C}^n, 0)$, $n \geq 2$, induced by a real analytic $1-$form ω. If the Levi foliation $\mathcal{L}$ is holomorphic, then at least one of the following two possibilities occurs:*

    (a) There exists a closed meromorphic $1-$form τ such that $\omega = h|\psi|^2 \mathrm{Re}(\tau)$, where ψ is a holomorphic equation for the polar set of τ and h is a germ of real

> *analytic function with real values such that $h(0) \neq 0$; in this case, $\mathcal{L}$ is induced by τ.*
>
> *(b) There are a meromorphic function ρ and a real meromorphic function κ, with $\kappa/\bar{\kappa}$ constant along the leaves of $\mathcal{L}$, such that $\omega = \mathrm{Re}(\kappa d\rho)$; in this case, ρ is a meromorphic first integral for $\mathcal{L}$.*

We give an outline of the proof of Theorem 2, which will be detailed in Sect. 5. Our main tools are the complexification of analytic objects in $(\mathbb{C}^n, 0)$ to $(\mathbb{C}^n \times \mathbb{C}^{n*}, 0)$, where $\mathbb{C}^{n*}$ is a copy of $\mathbb{C}^n$ with the opposite complex structure defined by the complex conjugation, and the associated mirroring or $(*)$-operator, which acts on analytic objects in $(\mathbb{C}^n \times \mathbb{C}^{n*}, 0)$ (see Sect. 2). Since the Levi foliation $\mathcal{L}$ is holomorphic, it turns out that $\omega_{\mathbb{C}}$, the complexification of ω, belongs to a pencil of integrable holomorphic $1-$forms (see Sect. 3). This geometric fact lies in the core of our proof, as long as we can apply a local version of Cerveau's classification of pencil of integrable $1-$forms [8] (Proposition 3.1). An analysis exploring symmetries under the $(*)$-operator then leads to the models proposed in the assertion of our theorem. Finally, in Sect. 6, our local results are applied to the classification of algebraic Levi-flat foliations on the complex projective space $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{C}}$. This is stated in Theorem 3.

The authors are grateful to H. Reis for a comment that was essential in the development of the arguments in the proof of Theorem 1.

2. Foliations, Mirroring and complexification.

2.1. Basic notation. Consider coordinates $z = (z_1, \ldots, z_n)$ in $\mathbb{C}^n$, where $z_j = x_j + iy_j$, and the complex conjugation $\bar{z} = (\bar{z}_1, \ldots, \bar{z}_n)$, where $\bar{z}_j = x_j - iy_j$ and $i = \sqrt{-1}$. We will employ the standard multi-index notation. For instance, if $\mu = (\mu_1, \ldots, \mu_n) \in \mathbb{Z}^n_{\geq 0}$, then $z^\mu = z_1^{\mu_1} \cdots z_n^{\mu_n}$ and $\bar{z}^\mu = \bar{z}_1^{\mu_1} \cdots \bar{z}_n^{\mu_n}$. Also, if $I = \{i_1 < \ldots < i_p\} \subset \{1, \ldots, n\}$, then $|I| = p$, $dz_I = dz_{i_1} \wedge \cdots \wedge dz_{i_p}$ and $d\bar{z}_I = d\bar{z}_{i_1} \wedge \cdots \wedge d\bar{z}_{i_p}$.

We fix the following notation for rings of germs at $(\mathbb{C}^n, 0)$:
- $\mathcal{O}_n = \mathbb{C}\{z_1, \ldots, z_n\}$ is the ring of *holomorphic functions*;
- $\mathcal{A}_n = \mathbb{C}\{z_1, \ldots, z_n, \bar{z}_1, \ldots, \bar{z}_n\} = \mathbb{C}\{x_1, y_1, \ldots, x_n, y_n\}$ is the ring of *real analytic functions* (with complex values);
- $\mathcal{A}_{n\mathbb{R}} \subset \mathcal{A}_n$ is the ring of *real analytic functions with real values*.

To these rings, in the above order, we associate the fields of fractions:
- $\mathcal{M}_n$ is the field of *meromorphic functions*;
- $\mathcal{Q}_n$ is the field of *real meromorphic functions* (with complex values);
- $\mathcal{Q}_{n\mathbb{R}} \subset \mathcal{Q}_n$ is the field of *real meromorphic functions with real values*.

Note that $\phi \in \mathcal{A}_n$ is in $\mathcal{A}_{n\mathbb{R}}$ if and only if $\phi(z) = \overline{\phi(z)}$ for every sufficiently small $z \in \mathbb{C}$. In terms of the Taylor series $\phi(z) = \sum_{\mu,\nu} a_{\mu\nu} z^\mu \bar{z}^\nu$, this is equivalent to $a_{\mu\nu} = \bar{a}_{\nu\mu}$ for all μ, ν. We make the following convention: dimensions or codimensions will be real, when referring to real objects (foliations, varieties), but will be complex, when the objects in question are complex.

2.2. Levi-flat foliations. A germ of singular real analytic foliation $\mathcal{F}$ of real codimension one at $(\mathbb{C}^n, 0)$ is the object induced by a $1-$form ω with real analytic coefficients without non-trivial common factor in $\mathcal{A}_{n\mathbb{R}}$, which satisfies the Frobenius integrability condition $\omega \wedge d\omega = 0$. Writing

$$\omega = \sum_{j=1}^n A_j dx_j + B_j dy_j, \tag{2}$$

where $A_j, B_j \in \mathcal{A}_{n\mathbb{R}}$ for $j = 1, \ldots, n$, we have that the *singular set* of $\mathcal{F}$,

$$\mathrm{Sing}(\mathcal{F}) := \mathrm{Sing}(\omega) = \bigcap_{j=1}^{n} \{A_j = B_j = 0\}$$

is a real analytic set of real codimension at least two.

In complex coordinates $z = (z_1, \cdots, z_n)$ in $\mathbb{C}^n$, where $x_j = \mathrm{Re}(z_j) = (z_j + \bar{z}_j)/2$ and $y_j = \mathrm{Im}(z_j) = (z_j - \bar{z}_j)/2i$, we have

$$\omega = \sum_{j=1}^{n} \frac{A_j - iB_j}{2} dz_j + \sum_{j=1}^{n} \frac{A_j + iB_j}{2} d\bar{z}_j = \frac{\eta + \bar{\eta}}{2} = \mathrm{Re}(\eta). \tag{3}$$

The $1-$form

$$\eta = \sum_{j=1}^{n} (A_j - iB_j) dz_j \tag{4}$$

defines the (intrinsic) distribution of complex hyperplanes (outside $\mathrm{Sing}(\mathcal{F})$) associated to ω. We also set

$$\omega^{\sharp} = \mathrm{Im}(\eta) = \frac{\eta - \bar{\eta}}{2i} = \sum_{j=1}^{n} -B_j dx_j + A_j dy_j. \tag{5}$$

As explained in the introduction, the foliation $\mathcal{F}$ is *Levi-flat* if the distribution of complex hyperplanes defined outside $\mathrm{Sing}(\mathcal{F})$ by the $1-$form η in (4) is integrable, inducing the Levi foliation $\mathcal{L}$ whose leaves are immersed complex manifolds of codimension one. As a real foliation of codimension two, $\mathcal{L}$ is defined by the Pfaff system $\omega = \omega^{\sharp} = 0$, which, by (3) and (5), is equivalent to the one defined by $\eta = \bar{\eta} = 0$ and, still, to the one induced by the real analytic $2-$form $\eta \wedge \bar{\eta}$. In this paper we deal with the case where $\mathcal{L}$ is a holomorphic foliation. This can be characterized in the following way:

LEMMA 2.1. *Suppose that $\mathcal{L}$ is a holomorphic foliation, defined by an integrable holomorphic $1-$form σ at $(\mathbb{C}^n, 0)$, with coefficients without common factors in $\mathcal{O}_n$. Then there exists $\phi \in \mathcal{A}_n$ such that $\eta = \phi\sigma$.*

Proof. Write $\eta = \sum_{j=1}^{n} \varepsilon_j dz_j$, where $\varepsilon_j \in \mathcal{A}_n$, and $\sigma = \sum_{j=1}^{n} \alpha_j dz_j$, where $\alpha_j \in \mathcal{O}_n$. In a small neighborhood of $0 \in \mathbb{C}^n$, for z outside $\mathrm{Sing}(\eta) \cup \mathrm{Sing}(\sigma)$, the equations $\eta(z) = 0$ and $\sigma(z) = 0$ define same hyperplane, so that there exists $\phi(z) \in \mathbb{C}^*$ such that $\eta(z) = \phi(z)\sigma(z)$. For each j such that $\alpha_j(z) \neq 0$, we have that $\phi(z) = \varepsilon_j(z)/\alpha_j(z)$, so that ϕ extends to a neighborhood of $0 \in \mathbb{C}^n$ as a real meromorphic function that is real analytic outside $\mathrm{Sing}(\sigma)$, a complex analytic set of codimension two. This is possible, if and only if, α_j divides ε_j in $\mathcal{A}_n$, whenever $\alpha_j \neq 0$. Consequently, ϕ is real analytic in a neighborhood of $0 \in \mathbb{C}^n$. $\square$

2.3. Mirroring. Let $\mathbb{C}^{n*} \simeq \mathbb{C}^n$ be the space obtained by endowing $\mathbb{C}^n$ with the opposite complex structure, induced by the complex conjugation, where we consider complex coordinates $w = (w_1, \ldots, w_n) = \bar{z}$, with $w_j = \bar{z}_j = x_j - iy_j$. The conjugation map $z = x + iy \mapsto x - iy = w$ defines a biholomorphism between $\mathbb{C}^n$ and $\mathbb{C}^{n*}$ that we call *mirroring*. We define the $(*)$-*operator*, taking sets, functions or differential forms in $\mathbb{C}^n$ to their *mirrors* in $\mathbb{C}^{n*}$, in the following way:

- If $\gamma \subset \mathbb{C}^n$ then $\gamma^* = \{z; \bar{z} \in \gamma\} \subset \mathbb{C}^{n*}$.
- If $\phi : \gamma \subset \mathbb{C}^n \to \mathbb{C}$ is a function then

$$\phi^* : \quad \gamma^* \subset \mathbb{C}^{n*} \quad \to \quad \mathbb{C}$$
$$w \quad \mapsto \quad \overline{\phi(\,\bar{w})}.$$

- For a differential $p-$form $\varpi = \sum_{I,J} \phi_{I,J}(z) dz_I \wedge d\bar{z}_J$, with $|I| + |J| = p$, we set

$$\varpi^* = \sum_{I,J} \phi^*_{I,J}(w) dw_I \wedge d\bar{w}_J.$$

Note that if $\phi \in \mathcal{A}_n$ has Taylor series $\phi(z) = \sum_{\mu,\nu} a_{\mu\nu} z^\mu \bar{z}^\nu$, then

$$\phi^*(w) = \overline{\sum_{\mu,\nu} a_{\mu\nu} \bar{w}^\mu w^\nu} = \sum_{\mu,\nu} \bar{a}_{\mu\nu} w^\mu \bar{w}^\nu. \tag{6}$$

Therefore, mirroring preserves the class of analyticity, real or complex. As a consequence, $\gamma \subset \mathbb{C}^n$ is a real or complex analytic subvariety if and only if the same holds for $\gamma^* \subset \mathbb{C}^{n*}$. Of our particular interest is the mirroring of foliations. If $\mathcal{G}$ is a holomorphic foliation of codimension p at $(\mathbb{C}^n, 0)$, defined by a holomorphic $p-$form ϖ — that is integrable and locally decomposable outside the singular set — then $\mathcal{G}^*$ is the foliation defined by ϖ^*. It is straightforward to check that the leaves of $\mathcal{F}^*$ are obtained by the mirroring of those of $\mathcal{F}$.

The $(*)$-operator can be defined in a broader context, acting in objects in $\mathbb{C}^n \times \mathbb{C}^{n*} \simeq \mathbb{C}^{2n}$. We have:

- If $\Gamma \subset \mathbb{C}^n \times \mathbb{C}^{n*}$, then $\Gamma^* = \{(z, w); (\bar{w}, \bar{z}) \in \Gamma\}$.
- $\Phi : \Gamma \subset \mathbb{C}^n \times \mathbb{C}^{n*} \to \mathbb{C}$ is a function, then

$$\Phi^* : \quad \Gamma^* \quad \to \quad \mathbb{C}$$
$$(z, w) \quad \mapsto \quad \overline{\Phi(\bar{w}, \bar{z})}\,.$$

- For a differential $p-$form $\varpi = \sum_{I,I',J,J'} \phi_{I,I',J,J'}(z, w) dz_I \wedge d\bar{z}_{I'} \wedge dw_J \wedge d\bar{w}_{J'}$, where $|I| + |I'| + |J| + |J'| = p$, we define

$$\varpi^* = \sum_{I,I',J,J'} \phi^*_{I,I',J,J'}(z, w) dw_I \wedge d\bar{w}_{I'} \wedge dz_J \wedge d\bar{z}_{J'}.$$

Note in particular that, if $\Gamma = \gamma \times \{w\}$, with $\gamma \subset \mathbb{C}^n$ and $w \in \mathbb{C}^{n*}$, then $\Gamma^* = \{\bar{w}\} \times \gamma^*$. For a germ of holomorphic function Φ at $(\mathbb{C}^n \times \mathbb{C}^{n*}, 0)$ with Taylor series $\Phi(z, w) = \sum_{\mu,\nu} a_{\mu\nu} z^\mu w^\nu$, we have

$$\Phi^*(z, w) = \overline{\sum_{\mu,\nu} a_{\mu\nu} \bar{w}^\mu \bar{z}^\nu} = \sum_{\mu,\nu} \bar{a}_{\mu\nu} z^\nu w^\mu. \tag{7}$$

2.4. Complexification. We work in $\mathbb{C}^n \times \mathbb{C}^{n*} \simeq \mathbb{C}^{2n}$ with coordinates (z, w). We assume henceforth that $\mathbb{C}^{2n}$ is endowed with this decomposition, in such a way that any function in $\mathcal{O}_{2n}$ or $\mathcal{M}_{2n}$ is intrinsically a function in the coordinates (z, w). Let $\phi \in \mathcal{A}_n$ be a germ of real analytic function having Taylor series

$$\phi(z, \bar{z}) = \sum_{\mu,\nu} a_{\mu\nu} z^\mu \bar{z}^\nu. \tag{8}$$

The *complexification* of ϕ is the germ of holomorphic function $\phi_{\mathbb{C}} \in \mathcal{O}_{2n}$ whose Taylor series is

$$\phi_{\mathbb{C}}(z, w) = \sum_{\mu,\nu} a_{\mu\nu} z^{\mu} w^{\nu}. \tag{9}$$

The complexification of a germ of real analytic p-form $\varpi = \sum_{I,J} \phi_{I,J} dz_I \wedge d\bar{z}_J$ at $(\mathbb{C}^n, 0)$ is the germ of holomorphic p-form at $(\mathbb{C}^{2n}, 0)$ with expression

$$\varpi_{\mathbb{C}} = \sum_{I,J} (\phi_{I,J})_{\mathbb{C}} \, dz_I \wedge dw_J. \tag{10}$$

This is evidently a two-way process: the *decomplexification* of a germ of holomorphic function $F \in \mathcal{O}_{2n}$ is the unique $\phi \in \mathcal{A}_n$ such that $\phi_{\mathbb{C}} = F$. The *decomplexification* of a holomorphic $p-$form $(\mathbb{C}^{2n}, 0)$ is done in an obvious way. The ideas of complexification and decomplexification can be canonically extended to meromorphic functions and differential forms.

If $\phi \in \mathcal{A}_n$ is as in (8), we have

$$\phi^*(w, \bar{w}) = \sum_{\mu,\nu} \bar{a}_{\mu\nu} w^{\mu} \bar{w}^{\nu},$$

which gives, by the correspondence $w = \bar{z}$,

$$(\phi^*)_{\mathbb{C}}(z, w) = \sum_{\mu,\nu} \bar{a}_{\mu\nu} z^{\nu} w^{\mu}. \tag{11}$$

Comparing this with (7) and (9), we get the commutative property $(\phi^*)_{\mathbb{C}} = (\phi_{\mathbb{C}})^*$, allowing us to denote both expressions by $\phi_{\mathbb{C}}^*$. Similar remarks apply to successive mirror and complexification of a real analytic $1-$form ϖ, making the notation $\varpi_{\mathbb{C}}^*$ unambiguous.

We say that a function F in $\mathcal{O}_{2n}$ is $(*)$-*symmetric* or *mirror symmetric* if $F^* = F$. Observe that, for some $\phi \in \mathcal{A}_n$, the equality $\phi_{\mathbb{C}} = \phi_{\mathbb{C}}^*$ holds if and only if $\phi \in \mathcal{A}_{n\mathbb{R}}$. Indeed, comparing (9) and (11), we find that the mirror invariance is equivalent to $a_{\mu\nu} = \bar{a}_{\nu\mu}$ for all indices μ, ν, giving the conclusion. More generally, $(*)$-symmetric functions in $\mathcal{O}_{2n}$ are precisely those that decomplexify as real analytic functions with real values. The following fact is straightforward: if $\varphi \in \mathcal{O}_1$ is a function whose Taylor series has real coefficients and $F \in \mathcal{O}_{2n}$, then $(\varphi \circ F)^* = \varphi \circ F^*$. In particular, $\varphi \circ F$ is $(*)$-symmetric if F is. This fact can be applied, for instance, to the function $\exp(F)$ and, when F is a unity such that $\mathrm{Re}(F(0)) > 0$, to $\sqrt{F}$ (taking the principal branch of the square root). The notion of $(*)$-symmetry can be extended in an obvious way to meromorphic functions at $(\mathbb{C}^{2n}, 0)$.

PROPOSITION 2.2. *We have the following facts:*
(a) A $(*)$-symmetric germ of meromorphic function in $\mathcal{M}_{2n}$ can be written as a quotient of $(*)$-symmetric holomorphic functions in $\mathcal{O}_{2n}$.
(b) A real meromorphic function $f \in \mathcal{Q}_n$ is in $\mathcal{Q}_{n\mathbb{R}}$ if and only if $f = \bar{f}$.

Proof. Let F be the meromorphic function as in item (a), written in the form $F = G/H$, where $G, H \in \mathcal{O}_{2n}$ are without common factors. Since $F^* = F$, we have $G^*/H^* = G/H$. Comparing zeroes and poles, we find a unity $U \in \mathcal{O}_{2n}$ satisfying

$UU^* = 1$ such that $G^* = UG$ and $H^* = UH$. We have $|U(0)| = 1$ and, in fact, we can suppose that $U(0) = 1$. Actually, taking $\alpha \in \mathbb{C}^*$ such that $\alpha^2 = U(0)$, we have

$$(\alpha G)^* = \frac{U}{\alpha^2}(\alpha G) \qquad \text{and} \qquad (\alpha H)^* = \frac{U}{\alpha^2}(\alpha H).$$

Thus, it suffices to replace G by αG and H by αH in the rational decomposition of F. Now, by putting $\tilde{G} = \sqrt{U}\, G$, we have

$$\tilde{G}^* = (\sqrt{U})^* G^* = \sqrt{U^*}\, UG = \sqrt{U}\, G = \tilde{G}.$$

In a similar way, the function $\tilde{H} = \sqrt{U}\, H$ is such that $\tilde{H}^* = \tilde{H}$. Therefore we can write $F = \tilde{G}/\tilde{H}$, where $\tilde{G}, \tilde{H} \in \mathcal{O}_{2n}$ are $(*)$-symmetric.

Item (b) follows straight by applying (a) to the complexification $F = f_{\mathbb{c}}$. □

We state the following result for future reference:

PROPOSITION 2.3. *Let $\phi \in \mathcal{A}_n$. Then $\phi_{\mathbb{c}}$ and $\phi_{\mathbb{C}}^*$ have a common non-trivial factor in $\mathcal{O}_{2n}$ if and only if ϕ has a non trivial factor that lies in $\mathcal{A}_{n\mathbb{R}}$.*

Proof. On the one hand, if $f \in \mathcal{A}_{n\mathbb{R}}$ is a factor of ϕ, then $f_{\mathbb{c}} = f_{\mathbb{c}}^*$ is a factor of both $\phi_{\mathbb{c}}$ and $\phi_{\mathbb{c}}^*$. On the other hand, suppose that $G \in \mathcal{O}_{2n}$ is a non-unity that divides both $\phi_{\mathbb{c}}$ and $\phi_{\mathbb{C}}^*$. Then the decomplexification $g \in \mathcal{A}_n$ of G divides both ϕ and $\bar{\phi}$ in $\mathcal{A}_n$. That is, there are functions $\alpha, \beta \in \mathcal{A}_n$ such that $\phi = g\alpha$ and $\bar{\phi} = g\beta$. If g has an irreducible factor, say g_1, not lying in $\mathcal{A}_{n\mathbb{R}}$, then, writing $g = g_1 h$ for some $h \in \mathcal{A}_n$, we have $\phi = g_1 h\alpha$ and $\bar{\phi} = g_1 h\beta$. Thus

$$\phi = g_1 h\alpha = \bar{g}_1 \bar{h} \bar{\beta}.$$

This shows that $\bar{g}_1$ also divides ϕ. Hence $g_1 \bar{g}_1 = |g_1|^2$ is a factor of ϕ in $\mathcal{A}_{n\mathbb{R}}$. □

3. Pencils of integrable $1-$forms. Let η_1 and η_2 be germs of integrable holomorphic $1-$forms at $(\mathbb{C}^n, 0)$, $n \geq 3$. Suppose that they are independent — meaning that $\eta_1 \wedge \eta_2 \neq 0$ — and that their singular sets do not have a common component of codimension one. We say that η_1 and η_2 define a *pencil of integrable $1-$forms* or, shortly, an *integrable pencil* if, for every $a, b \in \mathbb{C}$, with at least one of them non-zero, the $1-$form $\eta_{(a,b)} = a\eta_1 + b\eta_2$ is integrable. Thus, $\eta_{(a,b)}$ defines a holomorphic foliation of codimension one $\mathcal{F}_t$, where $t = (a : b) \in \mathbb{P}^1$. It may turn out that $\mathrm{codim}_{\mathbb{C}} \mathrm{Sing}(\eta_{(a,b)}) = 1$, but, since $\mathrm{Sing}(\eta_1)$ and $\mathrm{Sing}(\eta_2)$ do not have a common component of codimension one, this will happen for only finitely many values of $t = (a : b)$ (see [17, Lem. 2]). For these values, an equation for $\mathcal{F}_t$ is obtained by cancelling in $\eta_{(a,b)}$ the common factors of its coefficients. The integrability condition reads

$$\begin{aligned}
0 = \eta_{(a,b)} \wedge d\eta_{(a,b)} &= (a\eta_1 + b\eta_2) \wedge d(a\eta_1 + b\eta_2) \\
&= ab\,(\eta_1 \wedge d\eta_2 + \eta_2 \wedge d\eta_1)\,.
\end{aligned}$$

Thus η_1 and η_2 define an integrable pencil if and only if

$$\eta_1 \wedge d\eta_2 + \eta_2 \wedge d\eta_1 = 0. \tag{12}$$

This equation will be referred to as *pencil condition*. Observe that, knowing that η_1 and η_2 are integrable, the pencil condition is satisfied once we find a pair of values $a, b \in \mathbb{C}^*$ for which $\eta_{(a,b)}$ is integrable. We denote the integrable pencil engendered by η_1 and η_2 by $\mathcal{P} = \mathcal{P}(\eta_1, \eta_2)$. Note that if $h \in \mathcal{O}_n$ is a unity, then $h\eta_1$ and $h\eta_2$

also define an integrable pencil, denoted by $h\mathcal{P} = \mathcal{P}(h\eta_1, h\eta_2)$. Clearly, the foliations associated to both pencils are the same.

Given an integrable pencil $\mathcal{P} = \mathcal{P}(\eta_1, \eta_2)$, there exists a unique meromorphic $1-$form $\theta = \theta_\mathcal{P}$ such that, for every $\varpi \in \mathcal{P}$, it holds

$$d\varpi = \theta \wedge \varpi. \tag{13}$$

The germ of meromorphic $2-$form $d\theta$ is called *pencil curvature*. Note that, if we multiply $\varpi \in \mathcal{P}$ by a unity $h \in \mathcal{O}_n$, then

$$d(h\varpi) = \left(\theta + \frac{dh}{h} \right) \wedge (h\varpi), \tag{14}$$

so that $\theta_{h\mathcal{P}} = \theta + dh/h$ and the same *curvature* is associated to $\mathcal{P}$ and to $h\mathcal{P}$. The germ of holomorphic $2-$form $\eta_1 \wedge \eta_2$ is integrable and satisfies $\eta_1 \wedge \eta_2 \wedge \varpi = 0$ for every $\varpi \in \mathcal{P}$. This says that the codimension two holomorphic foliation $\mathcal{G}$ defined by $\eta_1 \wedge \eta_2$ is *tangent* to all foliations associated to $1-$forms in $\mathcal{P}$. The $2-$form $\eta_1 \wedge \eta_2$ — or the associated codimension two foliation — is called *axis* of $\mathcal{P}$.

The next result will be the main ingredient in the proof of Theorem 2. Its proof is contained, without an explicit mention, in Cerveau's paper [8] on the so-called "Brunella's conjecture" for foliations in $\mathbb{P}^3$. The arguments therein adapt to the local framework. Below, we include a sketch of the proof, which is carried out in a more thorough way in [13].

THEOREM 3.1. *Let $\mathcal{P}$ be an integrable pencil at $(\mathbb{C}^n, 0)$, $n \geq 3$. Then, at least one of the following conditions is satisfied:*
  (a) *There exists a closed meromorphic $1-$form θ such that $d\varpi = \theta \wedge \varpi$ for every $1-$form $\varpi \in \mathcal{P}$. In particular, if θ is holomorphic, all foliations in $\mathcal{P}$ admit holomorphic first integrals.*
  (b) *The axis of $\mathcal{P}$ is tangent to the levels of a non-constant meromorphic function.*

Proof. Let $\theta = \theta_\mathcal{P}$ as in (13). We have two cases to consider:
Case 1. $d\theta = 0$ (zero curvature). This is the first of the alternatives. In the particular situation where θ is holomorphic, we can write $\theta = dg$ for some $g \in \mathcal{O}_n$. Putting $h = \exp(g)$, we have that $dh/h = dg$ and thus $hd\varpi = dh \wedge \varpi$ for every $\varpi \in \mathcal{P}$, giving

$$d\left(\frac{\varpi}{h} \right) = \frac{1}{h^2}(hd\varpi - dh \wedge \varpi) = 0.$$

Thus, after multiplication by a same unity in $\mathcal{O}_n$, all $1-$forms in the pencil become closed. Their integration provide the holomorphic first integrals of the statement.
Case 2. $d\theta \neq 0$ (non-zero curvature). This shall give the second alternative. The arguments here are adapted from [8, Prop. 2]. Taking differentials in $d\varpi = \theta \wedge \varpi$, where $\varpi \in \mathcal{P}$, we get $d\theta \wedge \varpi = 0$. This is true, in particular, for $\varpi = \eta_1$ and η_2. Thus, at any point in a neighborhood of $0 \in \mathbb{C}^n$ outside $\mathrm{Sing}(\eta_1 \wedge \eta_2)$, the $2-$forms $d\theta$ and $\eta_1 \wedge \eta_2$ are collinear. Thus, outside $\mathrm{Sing}(\eta_1 \wedge \eta_2)$, we can find a non-zero holomorphic function α such that

$$d\theta = \alpha\eta_1 \wedge \eta_2. \tag{15}$$

Comparing coefficients in both sides, we conclude that α extends to a meromorphic function in a neighborhood of $0 \in \mathbb{C}^n$. We have two subcases:

Subcase 2.1. α is constant. Taking the exterior derivative of (15) and applying the pencil condition we conclude that $d\eta_1 \wedge \eta_2 = d\eta_2 \wedge \eta_1 = 0$. Thus, by (13), $\theta \wedge \eta_1 \wedge \eta_2 = 0$. Hence, there are meromorphic functions μ_1 and μ_2 such that

$$\theta = \mu_1 \eta_1 + \mu_2 \eta_2.$$

This applied to (13) gives

$$d\eta_1 = -\mu_2 \eta_1 \wedge \eta_2 \qquad \text{and} \qquad d\eta_2 = \mu_1 \eta_1 \wedge \eta_2. \tag{16}$$

If $\mu_1 = 0$, then $d\eta_2 = 0$ and there exists a non-constant $g \in \mathcal{O}_n$ such that $dg = \eta_2$. In particular, this g is a holomorphic first integral for the axis of $\mathcal{P}$. Let us suppose $\mu_1 \neq 0$. Write, from (16), $d\eta_1 = -(\mu_2/\mu_1)\,d\eta_2$, whose differentiation gives

$$\mu_1 d\left(\frac{\mu_2}{\mu_1}\right) \wedge \eta_1 \wedge \eta_2 = 0.$$

Then μ_2/μ_1 is a meromorphic first integral for the axis of $\mathcal{P}$, provided it is non-constant. However, if $\mu_2/\mu_1 = c$ is a constant, then $\varpi = \eta_1 + c\eta_2$ is a closed 1−form in $\mathcal{P}$. Again, there exists a non-constant $g \in \mathcal{O}_n$ such that $dg = \varpi$, giving, in particular, a holomorphic first integral for the axis of $\mathcal{P}$.

Subcase 2.2. α is non-constant. The exterior derivative of equation (15), along with (13), gives $(d\alpha + 2\alpha\theta) \wedge \eta_1 \wedge \eta_2 = 0$. This implies that there exist $k_1, k_2 \in \mathcal{M}_n$, such that

$$\frac{1}{2}\frac{d\alpha}{\alpha} + \theta = k_1\eta_1 + k_2\eta_2.$$

The same arguments of [8, Prop. 2] show that k_1^2/α, k_2^2/α and k_1/k_1 are constant along the leaves of the axis of $\mathcal{P}$, with at least one non constant. $\square$

4. Holomorphic foliations with real meromorphic first integrals. In this section we study germs of holomorphic foliations of codimension one at $(\mathbb{C}^n, 0)$ that are tangent to the levels of real meromorphic functions with real values. In the language of this paper, we are considering functions in $\mathcal{Q}_{n\mathbb{R}}$ whose levels define Levi-flat foliations. Our objective is to give a proof for Theorem 1.

We start with a consideration on the dynamics of subgroups of $\text{Diff}(\mathbb{C}, 0)$, the group of germs of biholomorphisms at $(\mathbb{C}, 0)$. We refer to [2, 15] for a more extensive treatment of the subject. Fixing an analytic coordinate z at $(\mathbb{C}, 0)$, an element $\phi \in \text{Diff}(\mathbb{C}, 0)$ has a Taylor series expansion in the form $\phi(z) = \lambda z + \cdots$, where $\lambda \in \mathbb{C}^*$ and the dots stand for the higher order terms. The number λ is the *multiplier* of ϕ and can be written as $\lambda = e^{2\pi i\alpha}$ for some $\alpha \in \mathbb{C}$, which is uniquely determined modulo the sum of integer numbers. An element $\phi \in \text{Diff}(\mathbb{C}, 0)$ can be:

- *Hyperbolic*, if $|\lambda| \neq 1$, corresponding to $\alpha \notin \mathbb{R}$. In this case, ϕ is analytically linearizable.
- *Elliptic*, if $|\lambda| = 1$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. Such an element is formally linearizable. When ϕ is analytically linearizable, we call it an *irrational rotation*.
- *Parabolic*, if $|\lambda| = 1$ and $\alpha \in \mathbb{Q}$. We also say that ϕ is *resonant* and, in the analytically linearizable case, that it is a *rational rotation*. In the particular case $\lambda = 1$ and $\phi \neq id$, we say that ϕ is *tangent to the identity*.

Note that a germ of diffeomorphism $\phi \in \text{Diff}(\mathbb{C}, 0)$ of finite order is necessarily parabolic. If $G \subset \text{Diff}(\mathbb{C}, 0)$ is an abelian subgroup containing an element ϕ which

is either hyperbolic or elliptic, than G is linearizable in the same coordinate that linearize ϕ. In particular, if ϕ is analytically linearizable, then G is also analytically linearizable. A standard argument shows that a finite subgroup $G \subset \mathrm{Diff}(\mathbb{C}, 0)$ is analytically linearizable. In this case, it must be cyclic and generated by one of its elements of highest order.

A germ of real analytic one-dimensional foliation $\mathcal{F}$ at $(\mathbb{C}, 0)$ with an isolated singularity at $0 \in \mathbb{C}$ is *leafwise invariant* by $\phi \in \mathrm{Diff}(\mathbb{C}, 0)$ if, for every sufficiently small $z \in \mathbb{C} \setminus \{0\}$, z and $\phi(z)$ lie on the same leaf of $\mathcal{F}$. We say that $\mathcal{F}$ is *leafwise invariant* by a subgroup $G \subset \mathrm{Diff}(\mathbb{C}, 0)$ if $\mathcal{F}$ is *leafwise invariant* by every $\phi \in G$. In this case, we say that G is a subgroup of *center type* if $\mathcal{F}$ is a foliation of center type, meaning that all orbits outside $0 \in \mathbb{C}$ are closed.

We have the following characterization of center type subgroups of $\mathrm{Diff}(\mathbb{C}, 0)$:

LEMMA 4.1. *Let $G \subset \mathrm{Diff}(\mathbb{C}, 0)$ be a center type subgroup. Then G is an abelian group, without hyperbolic elements, that*
  - *either contains an irrational rotation and, thus, is analytically linearizable;*
  - *or is formed only by parabolic elements of finite order.*
Besides, G is a finite cyclic group if there is a second germ of real analytic foliation for which G is leafwise invariant.

Proof. Since G preserves closed curves around the origin, each orbit by G of $z \neq 0$ cannot accumulate to the origin. This gives at once the following facts:
  (i) G contains no hyperbolic elements.
  (ii) G contains no diffeomorphisms tangent to the identity, by the Flower Theorem [4].
  (iii) Thus, the commutators of G, having multiplier $\lambda = 1$, cannot be tangent to the identity. So they must be equal to the identity, implying that G is abelian.
  (iv) All elliptic elements are analytically linearizable. Indeed, by Dulac-Moussu's conjecture proved in [18], a non-linearizable elliptic germ of diffeomorphism has an orbit that accumulates to the origin.
Consequently, if G contains an elliptic element it is analytically linearizable. Otherwise, G is an abelian group containing only parabolic elements, all of them of finite order, since there are no elements tangent to the identity.

Consider now the last part of the statement. If there existed an elliptic element $\phi \in G$, then, in a coordinate z of $(\mathbb{C}, 0)$ that linearizes ϕ, for each $r > 0$ small, the orbits of G would be dense in the circles $z = r$, which are the leaves of the center type foliation leafwise invariant by G — and this should be the only foliation invariant by G. On the other hand, G cannot contain parabolic elements of arbitrarily high order, since this would imply that intersecting leaves of the two foliations leafwise invariant by G would have infinitely many points in common. This gives that G is finite cyclic, as we wished. $\square$

COROLLARY 4.2. *Let $G \subset \mathrm{Diff}(\mathbb{C}, 0)$ be a finitely generated center type subgroup. Then G is analytically linearizable.*

Proof. By Lemma 4.1, it is enough to consider the case where G is an abelian group whose elements are parabolic of finite order. However, the fact that G is finitely generated gives that it is finite and, thus, analytically linearizable. $\square$

Let $\mathcal{G}$ be a germ of singular holomorphic foliation at $(\mathbb{C}^2, 0)$. We recall that there exists a reduction of singularities for $\mathcal{G}$ [19]: a proper holomorphic map σ :

$(\tilde{M}, \mathcal{D}) \to (\mathbb{C}^2, 0)$, which is a composition of blow-ups, where $\mathcal{D} = \sigma^{-1}(0)$ is a divisor consisting of a finite union projective lines with normal crossings. It transforms $\mathcal{G}$ into a foliation $\tilde{\mathcal{G}} = \sigma^* \mathcal{G}$ — its *strict transform* — whose singularities are over $\mathcal{D}$ and are all *simple* or *reduced*. This means that, at each such point, $\tilde{\mathcal{G}}$ is induced by a vector field whose linear part is non-nilpotent and, when its eigenvalues λ_1, λ_2 are both non-zero, they satisfy $\lambda_1/\lambda_2 \notin \mathbb{Q}_+$. Such a singularity is said to be *non-degenerate*. A simple singularity is a *saddle-node* if one of the eigenvalues is zero. The reduction of singularities is not unique, but we can fix a minimal non-trivial reduction of singularities — two minimal reductions are isomorphic. The number of blow-ups performed is the *length* of the reduction process. Note that, if $\mathcal{G}$ has a simple singularity at $0 \in \mathbb{C}^2$, we have to blow-up it once in order to get a reduction of singularities and the associated length is one. We recall the following definition of [5]: $\mathcal{G}$ is a *generalized curve foliation* if there are no saddle-nodes in its reduction of singularities.

A component of $\mathcal{D}$ can be *non-dicritical*, if it is invariant by $\tilde{\mathcal{G}}$, or *dicritical*, otherwise. The foliation $\mathcal{G}$ itself is said to be *non-dicritical* is all components of $\mathcal{D}$ are non-dicritical. To each non-dicritical component $D \subset \mathcal{D}$ we associate a subgroup of $\mathrm{Diff}(\mathbb{C}, 0)$ in the following way. Take a point $q \in D \setminus \mathrm{Sing}(\tilde{\mathcal{G}})$ and a germ $(\Sigma, q) \simeq (\mathbb{C}, 0)$ of holomorphic section transversal to D at q. The *virtual holonomy group* is the subgroup $\mathrm{Hol}^{vir}(\tilde{\mathcal{G}}, D) \subset \mathrm{Diff}(\mathbb{C}, 0)$ formed by all germs of diffeomorphisms $\phi \in \mathrm{Diff}(\mathbb{C}, 0)$ such that z and $\phi(z)$ are in the same leaf of $\tilde{\mathcal{G}}$ for every $z \in \Sigma$ sufficiently small. In principle, $\mathrm{Hol}^{vir}(\tilde{\mathcal{G}}, D)$ depends on the point $q \in D \setminus \mathrm{Sing}(\tilde{\mathcal{G}})$ and on the section Σ at q, but we get rid of this dependence by considering it up to conjugation by a germ of diffeomorphism in $\mathrm{Diff}(\mathbb{C}, 0)$. The next two results regard the classification of a foliation by means of its virtual holonomy. We first have:

PROPOSITION 4.3. *Let $\mathcal{G}$ be a germ of non-dicritical holomorphic foliation at $(\mathbb{C}^2, 0)$ of generalized curve type. Suppose that, for every component D of its reduction divisor $\mathcal{D}$, the virtual holonomy $\mathrm{Hol}^{vir}(\tilde{\mathcal{G}}, D)$ is an abelian group with only finite order elements. Then $\mathcal{G}$ has a holomorphic first integral.*

Proof. The proposition is obvious if $\mathcal{G}$ has a simple singularity at $0 \in \mathbb{C}^2$: we blow-up once in order to get a reduction of singularities, use the fact the virtual holonomy is finite and apply standard arguments in order to build a holomorphic first integral. Let us suppose the result true for foliations with reduction of singularities of length at most k. Let $\mathcal{G}$ be a foliation with the properties in the statement with reduction of singularities of length $k + 1$. We make a first blow-up $\pi \colon (M, D) \to (\mathbb{C}^2, 0)$. Let $p_1, \ldots, p_\ell \in D$ be the singularities of the strict transform foliation $\tilde{\mathcal{G}}_0 = \pi^* \mathcal{G}$ over D and denote by $\mathcal{G}_1, \ldots, \mathcal{G}_\ell$ the germs of $\tilde{\mathcal{G}}_0$ at $p_1, \ldots, p_\ell$, respectively. The induction hypothesis assures the existence of holomorphic first integrals for $\mathcal{G}_1, \ldots, \mathcal{G}_\ell$. Let (Σ, q) be a transversal section to D at $q \in D \setminus \{p_1, \ldots, p_\ell\}$. Clearly, we can calculate on Σ the virtual holonomy groups $H_j = \mathrm{Hol}^{vir}(\mathcal{G}_j, D)$, $j = 1 \ldots, \ell$, which are finite groups. They are subgroups of $\mathrm{Hol}^{vir}(\tilde{\mathcal{G}}_0, D) = \mathrm{Hol}^{vir}(\tilde{\mathcal{G}}, D)$, the same holding for $H = \langle H_1, \ldots, H_\ell \rangle$. Now, H is a finitely generated abelian group having only elements of finite order. Thus, it is a finite group. The result then follows from [16, Lem. 3]. $\square$

We say that a germ of foliation $\mathcal{G}$ of codimension one at $(\mathbb{C}^n, 0)$, $n \geq 2$, is *logarithmic* if there are germs of irreducible functions $f_1, \ldots, f_k \in \mathcal{O}_n$ and coefficients

$\lambda_1, \cdots, \lambda_k \in \mathbb{C}^*$ such that $\mathcal{G}$ is defined by the meromorphic $1-$form

$$\omega = \sum_{j=1}^{k} \lambda_j \frac{df_j}{f_j}. \qquad (17)$$

If there exists $\mu \in \mathbb{C}^*$ such that $\mu\lambda_i \in \mathbb{R}$ for every $i = 1, \ldots, k$, we say that $\mathcal{G}$ is *logarithmic with real residues*. In particular, if μ can be taken in such a way that $n_i = \mu\lambda_i \in \mathbb{Z}$ for every $i = 1, \ldots, k$, then $\mathcal{G}$ admits the meromorphic first integral $f = f_1^{n_1} \cdots f_k^{n_k}$, which is holomorphic if $n_i \in \mathbb{Z}_+$ for every $i = 1, \ldots, k$. The following conditions on the virtual holonomy groups are sufficient to conclude that a holomorphic foliation is logarithmic:

PROPOSITION 4.4. *Let $\mathcal{G}$ be a germ of non-dicritical holomorphic foliation at $(\mathbb{C}^2, 0)$ of generalized curve type. Suppose that:*
  (i) for every component $D \subset \mathcal{D}$, the virtual holonomy $\mathrm{Hol}^{vir}(\tilde{\mathcal{G}}, D)$ is analytically linearizable;
  (ii) for some $D \subset \mathcal{D}$, there exists an irrational rotation in $\mathrm{Hol}^{vir}(\tilde{\mathcal{G}}, D)$.
Then $\mathcal{G}$ is a logarithmic foliation. Besides, if there are no hyperbolic elements in some $\mathrm{Hol}^{vir}(\tilde{\mathcal{G}}, D)$, then $\mathcal{G}$ is logarithmic with real residues.

Proof. This result has been proved in [6] assuming the existence of a hyperbolic element in place of an irrational rotation. The proof of our version follows the very same steps, which we describe next. Essentially, the following fact is used: if $G \subset \mathrm{Diff}(\mathbb{C}, 0)$ is an abelian subgroup containing an irrational rotation ϕ, then the whole G is linear in the same coordinate that linearizes ϕ. We delineate a sketch of the proof in order to check how the argument works. First of all, each non-degenerate simple singularity $p \in \mathrm{Sing}(\tilde{\mathcal{G}})$ is linearizable since the holonomy of each separatrix contained in $\mathcal{D}$ is linearizable [16]. We have the following steps:

- as in [6, Lem. 3], the existence of an irrational rotation in $\mathrm{Hol}^{vir}(\tilde{\mathcal{G}}, D)$ for some $D \subset \mathcal{D}$ implies the existence of an irrational rotation in $\mathrm{Hol}^{vir}(\tilde{\mathcal{G}}, D)$ for every $D \subset \mathcal{D}$.
- Following [6, Prop. 1], for each $D \subset \mathcal{D}$, there exists a logarithmic $1-$form ω_D defining $\tilde{\mathcal{G}}$ in a neighborhood of D satisfying:
  Property $(*)$: if $(\Sigma, q) \simeq (\mathbb{C}, 0)$ is a transversal section of D at $q \in D \setminus \mathrm{Sing}(\tilde{\mathcal{G}})$ and

  z is an analytic coordinate at $(\mathbb{C}, 0)$ that linearizes $\mathrm{Hol}^{vir}(\tilde{\mathcal{G}}, D)$ at (Σ, q), then

  $$\omega_D|_\Sigma = dz/z.$$

  The construction of ω_D near the regular points of $\mathrm{Sing}(\tilde{\mathcal{G}})$ on D is based on the fact that the existence of an irrational rotation in $\mathrm{Hol}^{vir}(\tilde{\mathcal{G}}, D)$ implies that, given (Σ_1, q_1) and (Σ_2, q_2) transversal sections as above, with coordinates z_1 and z_2 which linearize $\mathrm{Hol}^{vir}(\tilde{\mathcal{G}}, D)$, then the germ of diffeomorphism $f_\gamma : (\Sigma_1, q_1) \to (\Sigma_2, q_2)$, obtained by the lifting of a path γ in $D \setminus \mathrm{Sing}(\tilde{\mathcal{G}})$ linking q_1 to q_2, is a linear map. The construction of ω_D in a neighborhood of a point $p \in \mathrm{Sing}(\tilde{\mathcal{G}})$ relies on the fact that $\tilde{\mathcal{G}}$ is linearizable at p.
- As in [6, Lem. 4], given two components $D_1, D_2 \subset \mathcal{D}$ intersecting at a point p, then there exists $c \in \mathbb{C}^*$ such that $\omega_{D_1} = c\,\omega_{D_2}$. This essentially follows from Property $(*)$ and from the fact that $\tilde{\mathcal{G}}$ is linearizable at p. We take coordinates (z_1, z_2) at p in which $\tilde{\mathcal{G}}$ is induced by the $1-$form $\omega = dz_1/z_1 - \lambda\,dz_2/z_2$, where $\lambda \in \mathbb{C}^* \setminus \mathbb{Q}_+$, such that $D_1 = \{z_1 = 0\}$, $D_2 = \{z_2 = 0\}$. For small $a_1, a_2 \in \mathbb{C}^*$, by taking sections

$\Sigma_1 = \{z_2 = a_2\}$, transversal to D_1, and $\Sigma_2 = \{z_1 = a_1\}$, transversal to D_2, then, if $\mathrm{Hol}^{vir}(\tilde{\mathcal{G}}, D_1)$ is linear in the coordinate z_1, an irrational rotation in $\mathrm{Hol}^{vir}(\tilde{\mathcal{G}}, D_1)$ is transferred to $\mathrm{Hol}^{vir}(\tilde{\mathcal{G}}, D_2)$ as a linear map in the coordinate z_2 of Σ_2. Thus, $\mathrm{Hol}^{vir}(\tilde{\mathcal{G}}, D_2)$ is linear in the coordinate z_2. This implies the result.

- A final adjustment of the constants $c \in \mathbb{C}^*$ along the components of the desingularization divisor $\mathcal{D}$ gives a logarithmic $1-$form $\tilde{\omega}$ that defines $\tilde{\mathcal{G}}$ in a neighborhood of $\mathcal{D}$. This corresponds to a logarithmic $1-$ form ω, as in (17), that defines $\mathcal{G}$ in a neighborhood of $0 \in \mathbb{C}^2$.

For the last part of the statement, we first remark that, invoking again [6, Lem. 3], the groups $\mathrm{Hol}^{vir}(\tilde{\mathcal{G}}, D)$ are devoid of hyperbolic elements for all components $D \subset \mathcal{D}$. By construction, the meromorphic $1-$form $\tilde{\omega} = \sigma^*\omega$, that induces $\tilde{\mathcal{G}} = \sigma^*\mathcal{G}$, has simple poles over the normal crossings divisor $(\tilde{\omega})_\infty = \mathcal{D} \cup \tilde{S}_1 \cup \cdots \cup \tilde{S}_k$, where $\tilde{S}_i = \sigma^* S_i$ is the strict transform of $S_i = \{f_i = 0\}$. Evidently, the residue of $\tilde{\omega}$ along each curve $\tilde{S}_i$ is λ_i. For a simple singularity $p \in \mathcal{D}$ of $\tilde{\mathcal{G}}$, which is non-degenerate since $\mathcal{G}$ is of generalized curve type, the ratio of eigenvalues is the negative of the ratio of residues associated to the two components of $(\tilde{\omega})_\infty$ intersecting at p. This ratio must be real, since otherwise the holonomy of the two separatrices of $\tilde{\mathcal{G}}$ at p would engender hyperbolic elements in the virtual holonomy. This fact, considered along $(\tilde{\omega})_\infty$, leads to the conclusion that ω has real residues. □

We now have the elements to propose a proof for Theorem 1.

Proof of Theorem 1. Let $\mathcal{G}$ be a germ of holomorphic foliation as in the assertion, tangent to the levels of $f \in \mathcal{Q}_{n\mathbb{R}}$. By taking a smooth two-dimensional section $\rho : (\mathbb{C}^2, 0) \to (\mathbb{C}^n, 0)$ generically transversal to $\mathcal{G}$ and to the indeterminacy set of f (that exists by [16]), we can suppose that $n = 2$ and that $0 \in \mathbb{C}^2$ is an isolated singularity of $\mathcal{G}$ (see, for instance, [9, Sec. 3.3]). Note that if a leaf of $\mathcal{G}$ accumulates to the origin, the same holds for the level of f that contains it. If this level has real codimension one, it defines a $\mathcal{G}$-invariant real analytic Levi-flat hypersurface at $(\mathbb{C}^2, 0)$. As a consequence of Cerveau-Lins Neto's Theorem [9], $\mathcal{G}$ admits a meromorphic first integral, proving the result. On the other hand, if this level has real codimension two, it must be a complex analytic curve lying in the singular locus of f. This curve is also a separatrix for $\mathcal{G}$. Thus, we are reduced to the following situation: there are finitely many leaves of $\mathcal{G}$ accumulating to the origin, all of them separatrices of $\mathcal{G}$ contained in the singular locus of f. This means, in particular, that $\mathcal{G}$ is a non-dicritical generalized curve foliation. Take a reduction of singularities $\sigma : (\tilde{M}, \mathcal{D}) \to (\mathbb{C}^2, 0)$ for $\mathcal{G}$ and set $\tilde{\mathcal{G}} = \sigma^*\mathcal{G}$. For every component $D \subset \mathcal{D}$, the virtual holonomy groups $\mathrm{Hol}_{vir}(\tilde{\mathcal{G}}, D)$ are subgroups of center type. Thus, by Lemma 4.1, they are all abelian, either analytically linearizable containing no hyperbolic elements or formed by finite order resonant elements. In the latter case, Proposition 4.3 gives a holomorphic first integral for $\mathcal{G}$. In the former, by Proposition 4.4, we find that $\mathcal{G}$ is a logarithmic foliation with real residues. Actually, in this case, all simple singularities of $\tilde{\mathcal{G}}$ must have ratio of eigenvalues in $\mathbb{Q}_+$ (see example bellow). This implies that the residues of ω can be taken in $\mathbb{Z}_+$, giving a holomorphic first integral for $\mathcal{G}$. □

EXAMPLE 4.5. Suppose that $\mathcal{G}$ has a simple non-degenerate linearizable singularity at $0 \in \mathbb{C}^2$, being defined by the $1-$form

$$\omega = z_2 dz_1 - \lambda z_1 dz_2, \quad \text{with} \ \ \lambda \in \mathbb{C}^* \setminus \mathbb{Q}_+.$$

If $\mathcal{G}$ has a real meromorphic first integral $f \in \mathcal{Q}_{n\mathbb{R}}$ then $\lambda \in \mathbb{Q}_-$. Indeed, by Cerveau-Lins Neto's Theorem, as in the above proof, we are reduced to the case where the only

leaves of $\mathcal{G}$ accumulating to the origin are the separatrices $z_1 = 0$ and $z_2 = 0$. This discards the options $\lambda \in \mathbb{C}^* \setminus \mathbb{R}$ and $\lambda \in \mathbb{R}_+ \setminus \mathbb{Q}_+$ (nodal case). If $\lambda \in \mathbb{R}_- \setminus \mathbb{Q}_-$, the foliation $\mathcal{G}$ has the multivalued first integral $z_1 z_2^{-\lambda}$, so that the real analytic 1−form

$$|z_1 z_2|^2 \frac{d|z_1 z_2^{-\lambda}|^2}{|z_1 z_2^{-\lambda}|^2} = \bar{z}_1 |z_2|^2 dz_1 + z_1 |z_2|^2 d\bar{z}_1 - \lambda |z_1|^2 \bar{z}_2 dz_2 - \lambda |z_1|^2 z_2 d\bar{z}_2$$

defines a Levi-flat foliation tangent to $\mathcal{G}$, which is independent of the Levi-flat foliation defined by the levels of f. Therefore, an application of Lemma 4.1 gives that the holonomy group of each separatrix of $\mathcal{G}$ is finite, which is a contradiction. The only remaining case is $\lambda \in \mathbb{Q}_-$.

EXAMPLE 4.6. We illustrate Theorem 1 with an example of a foliation with holomorphic first integral tangent to the levels of a non-analytic real meromorphic function with real levels. Consider coordinates (z_1, z_2) in $\mathbb{C}^2$, where $z_j = x_j + iy_j$ for $j = 1, 2$, and the function $f \in \mathcal{Q}_{2\mathbb{R}} \setminus \mathcal{A}_{2\mathbb{R}}$ defined by

$$f(z_1, z_2) = \frac{x_1^2 + y_1^2}{x_1^4 + y_1^4}.$$

The only level of f accumulating to $0 \in \mathbb{C}^2$ is the z_2-axis. Evidently, the non-singular vertical foliation $\mathcal{G}$ defined by $\omega = dz_1$ is tangent to the levels of f. In order to get a truly singular holomorphic foliation with holomorphic first integral, it suffices to take the strict transform of $\mathcal{G}$ by the quadratic map $\pi(z, z_2) = (zz_2, z_2)$, where $z = x + iy$, getting a foliation $\tilde{\mathcal{G}}$ defined by $\tilde{\omega} = z_2 dz + z dz_2$ which is tangent to the levels of

$$\tilde{f}(z, z_2) = \frac{(xx_2 - yy_2)^2 + (yx_2 + xy_2)^2}{(xx_2 - yy_2)^4 + (yx_2 + xy_2)^4}.$$

5. Holomorphic foliations tangent to Levi-flat foliations. Let $\mathcal{F}$ be a germ of real analytic Levi-flat foliation at $(\mathbb{C}^n, 0)$, defined by a germ of real analytic 1−form ω. Keeping the notation of Section 2, we denote by $\mathcal{L} = \mathcal{L}(\mathcal{F})$ the Levi foliation, which we suppose to be holomorphic. In this situation, Theorem 2 asserts that there are two possible models for $\mathcal{F}$, described in the following examples.

EXAMPLE 5.1. Let τ be a closed meromorphic 1−form at $(\mathbb{C}^n, 0)$. We include here the case $\tau = d\rho$, where ρ is either in $\mathcal{O}_n$ or in $\mathcal{M}_n$. Let $\psi \in \mathcal{O}_n$ be an equation for the poles of τ. If $h \in \mathcal{A}_{n\mathbb{R}}$ is a unity, then

$$\omega = h\psi\bar{\psi} \frac{\tau + \bar{\tau}}{2} = h|\psi|^2 \text{Re}(\tau)$$

is a real analytic 1−form that satisfies the integrability condition. It defines a Levi-flat foliation whose Levi foliation $\mathcal{L}$ is holomorphic defined by τ.

EXAMPLE 5.2. Let ρ be a non-constant meromorphic (or holomorphic) function at $(\mathbb{C}^n, 0)$ whose levels define a holomorphic foliation $\mathcal{L}$. Let $\kappa \in \mathcal{Q}_n$ be a real meromorphic function, with $\kappa/\bar{\kappa}$ constant along the leaves of $\mathcal{L}$, such that the 1−form $\eta = \kappa d\rho$ is real analytic without non-trivial factors in $\mathcal{A}_{n\mathbb{R}}$. Define the real analytic 1−form

$$\omega = \frac{1}{2} (\kappa d\rho + \bar{\kappa} d\bar{\rho}) = \text{Re}(\kappa d\rho).$$

Observe that the integrability condition,

$$\omega \wedge d\omega = \frac{1}{4} \left(\bar{\kappa} d\kappa - \kappa d\bar{\kappa} \right) \wedge d\rho \wedge d\bar{\rho} = 0,$$

is equivalent to $\kappa/\bar{\kappa}$ being constant along the leaves of $\mathcal{L}$. Thus, ω defines a Levi-flat foliation whose Levi foliation is $\mathcal{L}$. Note that $\mathcal{L}$ admits real meromorphic first integrals in $\mathcal{Q}_{n\mathbb{R}}$ (for instance, $\mathrm{Re}(\rho)$ and $\mathrm{Im}(\rho)$, as well as $\mathrm{Re}(\kappa/\bar{\kappa})$ and $\mathrm{Im}(\kappa/\bar{\kappa})$, if these are non-constant).

As seen in Subsect. 2.2, the Levi foliation $\mathcal{L}$ is tangent to the distribution of complex hyperplanes defined by the real analytic $1-$form η of type $(1,0)$ such that $\omega = \mathrm{Re}(\eta)$ (equation (3)). By Lemma 2.1, the condition of $\mathcal{L}$ being holomorphic is equivalent to existence of $\phi \in \mathcal{A}_n$ such that $\eta = \phi\sigma$, where σ is an integrable holomorphic $1-$form at $(\mathbb{C}^n, 0)$ that defines $\mathcal{L}$. Writing $\sigma = \sum_{j=1}^n \alpha_j(z) dz_j$, where $\alpha_j \in \mathcal{O}_n$, for $j = 1, \ldots, n$, are without common factors, we get, by taking mirrors, the integrable holomorphic $1-$form $\sigma^* = \sum_{j=1}^n \alpha_j^*(w) dw_j$ that defines the foliation $\mathcal{L}^*$ in $\mathbb{C}^{n*}$. Their complexifications produce two product foliations at $(\mathbb{C}^n \times \mathbb{C}^{n*}, 0) \simeq (\mathbb{C}^{2n}, 0)$:

- $\mathcal{L} \times \mathbb{C}^{n*}$, defined by $\sigma_{\mathbb{C}}$, whose leaves are vertical cylinders over the leaves of $\mathcal{L}$;
- $\mathbb{C}^n \times \mathcal{L}^*$, defined by $\sigma_{\mathbb{C}}^*$, whose leaves are horizontal cylinders over the leaves of the mirror foliation $\mathcal{L}^*$.

Now, the complexification of ω is a germ of integrable holomorphic $1-$form $\omega_{\mathbb{C}}$ at $(\mathbb{C}^{2n}, 0)$ which defines the holomorphic foliation of codimension one $\mathcal{F}_{\mathbb{C}}$, the complexification of $\mathcal{F}$. We have

$$\omega_{\mathbb{C}} = \frac{1}{2}(\eta + \bar{\eta})_{\mathbb{C}} = \frac{1}{2}(\eta_{\mathbb{C}} + \eta_{\mathbb{C}}^*) = \frac{1}{2}\phi_{\mathbb{C}}\sigma_{\mathbb{C}} + \frac{1}{2}\phi_{\mathbb{C}}^*\sigma_{\mathbb{C}}^*. \tag{18}$$

In particular, $\omega_{\mathbb{C}}$ is $(*)$-symmetric, that is $\omega_{\mathbb{C}}^* = \omega_{\mathbb{C}}$. On the other hand, the complexification $\mathcal{L}_{\mathbb{C}}$ of $\mathcal{L}$ is the holomorphic foliation of codimension two at $(\mathbb{C}^{2n}, 0)$ defined by $(\eta \wedge \bar{\eta})_{\mathbb{C}} = \eta_{\mathbb{C}} \wedge \eta_{\mathbb{C}}^*$ — that is to say, $\mathcal{L}_{\mathbb{C}}$ is the product foliation $\mathcal{L} \times \mathcal{L}^*$. Since $\mathcal{L}$ is tangent to $\mathcal{F}$, we have that $\mathcal{L}_{\mathbb{C}}$ is tangent to $\mathcal{F}_{\mathbb{C}}$ — and also to $\mathcal{L} \times \mathbb{C}^{n*}$ and to $\mathbb{C}^n \times \mathcal{L}^*$.

Observe that both $\eta_{\mathbb{C}}$ and $\eta_{\mathbb{C}}^*$ may have codimension one components in their singular sets, given by $\phi_{\mathbb{C}} = 0$ and $\phi_{\mathbb{C}}^* = 0$. However, $\phi_{\mathbb{C}}$ and $\phi_{\mathbb{C}}^*$ are relatively prime in $\mathcal{O}_{2n}$, since otherwise, by Proposition 2.3, the $1-$form ω would have a non-trivial factor in $\mathcal{A}_{n\mathbb{R}}$. Hence, equation (18) says that $\eta_{\mathbb{C}} = \phi_{\mathbb{C}}\sigma_{\mathbb{C}}$ and $\eta_{\mathbb{C}}^* = \phi_{\mathbb{C}}^*\sigma_{\mathbb{C}}^*$ define an integrable pencil $\mathcal{P} = \mathcal{P}(\eta_{\mathbb{C}}, \eta_{\mathbb{C}}^*)$ that contains $\omega_{\mathbb{C}}$, whose axis is the codimension two foliation $\mathcal{L}_{\mathbb{C}} = \mathcal{L} \times \mathcal{L}^*$. This geometric fact is the core of the proof of Theorem 2, which we present next.

Proof of Theorem 2. We apply Theorem 3.1 to the integrable pencil $\mathcal{P} = \mathcal{P}(\eta_{\mathbb{C}}, \eta_{\mathbb{C}}^*)$. The alternatives therein are in direct correspondence with those in the theorem's statement. Let us examine them.

Alternative *(a)*. There exists a closed meromorphic $1-$form θ such that $d\varpi = \theta \wedge \varpi$ for every $1-$form ϖ in $\mathcal{P}$. In particular, we have the equations

$$d\omega_{\mathbb{C}} = \theta \wedge \omega_{\mathbb{C}} \quad \text{and} \quad d\eta_{\mathbb{C}} = \theta \wedge \eta_{\mathbb{C}} \quad \text{and} \quad d\eta_{\mathbb{C}}^* = \theta \wedge \eta_{\mathbb{C}}^*, \tag{19}$$

which, by mirroring, turn into

$$d\omega_{\mathbb{C}} = \theta^* \wedge \omega_{\mathbb{C}} \quad \text{and} \quad d\eta_{\mathbb{C}}^* = \theta^* \wedge \eta_{\mathbb{C}}^* \quad \text{and} \quad d\eta_{\mathbb{C}} = \theta^* \wedge \eta_{\mathbb{C}}.$$

Taking into account these sets of equations, the closed $1-$form $\vartheta = \theta - \theta^*$ satisfies

$$\vartheta \wedge \omega_{\mathbb{c}} = \vartheta \wedge \eta_{\mathbb{c}} = \vartheta \wedge \eta_{\mathbb{c}}^* = 0.$$

Since $\omega_{\mathbb{c}}$, $\eta_{\mathbb{c}}$ and $\eta_{\mathbb{c}}^*$ are independent, this is possible if and only if $\vartheta = 0$. Hence θ is $(*)$-symmetric: $\theta = \theta^*$.

If θ is holomorphic then $\theta = dG$, for some $G \in \mathcal{O}_{2n}$. Since $\theta = \theta^*$, we also have that $\theta = d(G + G^*)/2$. Thus, replacing G by $(G + G^*)/2$, we can suppose that G is $(*)$-symmetric. Then the $(*)$-symmetric function $H = \exp(G)$ is a unity in $\mathcal{O}_{2n}$ such that $d(\varpi/H) = 0$ for every $\varpi \in \mathcal{P}$. In particular $d(\eta_{\mathbb{c}}/H) = 0$, so we can find $F \in \mathcal{O}_{2n}$ such that $\eta_{\mathbb{c}}/H = dF$. Since $\eta_{\mathbb{c}}$ is a $1-$form of type $(1, 0)$, the same is true for dF. This means that, in the variables $(z, w) \in \mathbb{C}^{2n}$, F is a function only in z. We have

$$\frac{\omega_{\mathbb{c}}}{H} = \frac{1}{H} \left(\frac{\eta_{\mathbb{c}} + \eta_{\mathbb{c}}^*}{2} \right) = \frac{dF + dF^*}{2}.$$

Decomplexifying this expression, we get

$$\frac{\omega}{h} = \frac{df + d\bar{f}}{2} = \mathrm{Re}(df),$$

where $h \in \mathcal{A}_{n\mathbb{R}}$ and $f \in \mathcal{O}_n$ are such that $h_{\mathbb{c}} = H$ and $f_{\mathbb{c}} = F$. This a particular case of item (a) in the statement, with $\tau = df$ and $\psi = 1$ (or of item (b) with $\rho = f$ and $\kappa = h$).

Now, when the $1-$form θ is purely meromorphic, by [10] it can be written as

$$\theta = \sum_{j=1}^{\ell} \lambda_j \frac{dF_j}{F_j} + d \left(\frac{G}{F_1^{k_1} \cdots F_k^{k_\ell}} \right), \tag{20}$$

where $F_1, \ldots, F_\ell \in \mathcal{O}_{2n}$ are irreducible equations for the components of the polar set, $G \in \mathcal{O}_{2n}$, $\lambda_1, \ldots, \lambda_\ell \in \mathbb{C}$ and $k_1, \ldots, k_\ell \in \mathbb{Z}_{\geq 0}$. Using that θ is $(*)$-symmetric, we can refine this writing in the following way:

$$\theta = \sum_{j=1}^{r} \mu_j \frac{dh_j}{h_j} + \sum_{j=1}^{s} \left(\lambda_j \frac{df_j}{f_j} + \bar{\lambda}_j \frac{df_j^*}{f_j^*} \right) + d \left(\frac{G}{h_1^{m_1} \cdots h_r^{m_r} (f_1 f_1^*)^{n_1} \cdots (f_s f_s^*)^{n_s}} \right), \tag{21}$$

where $h_1, \ldots, h_r, G \in \mathcal{O}_{2n}$ are $(*)$-symmetric, $f_1, \ldots, f_s \in \mathcal{O}_{2n}$ are not $(*)$-symmetric, $\mu_1, \ldots, \mu_r \in \mathbb{R}$, $\lambda_1, \ldots, \lambda_s \in \mathbb{C}$ and $m_1, \ldots, m_r, n_1, \ldots, n_s \in \mathbb{Z}_{\geq 0}$.

Using $d\eta_{\mathbb{c}} = \theta \wedge \eta_{\mathbb{c}}$ and $\eta_{\mathbb{c}} = \phi_{\mathbb{c}} \sigma_{\mathbb{c}}$, we have

$$d\phi_{\mathbb{c}} \wedge \sigma_{\mathbb{c}} + \phi_{\mathbb{c}} d\sigma_{\mathbb{c}} = \phi_{\mathbb{c}} \theta \wedge \sigma_{\mathbb{c}},$$

which gives

$$d\sigma_{\mathbb{c}} = \left(\theta - \frac{d\phi_{\mathbb{c}}}{\phi_{\mathbb{c}}} \right) \wedge \sigma_{\mathbb{c}}. \tag{22}$$

This expression, when mirrored, becomes

$$d\sigma_{\mathbb{c}}^* = \left(\theta - \frac{d\phi_{\mathbb{c}}^*}{\phi_{\mathbb{c}}^*} \right) \wedge \sigma_{\mathbb{c}}^*. \tag{23}$$

The following fact is well known: if ϖ is an integrable holomorphic 1$-$form, with $\mathrm{Sing}(\varpi)$ of codimension at least two, and ϑ is a closed meromorphic 1-form such that $d\varpi = \vartheta \wedge \varpi$, then the poles of ϑ are invariant by the foliation induced by ϖ. Applying this to (22) and to (23), we find that the poles of $\theta - d\phi_{\mathbb{C}}/\phi_{\mathbb{C}}$ are invariant by the vertical foliation $\mathcal{L} \times \mathbb{C}^{n*}$ induced by $\sigma_{\mathbb{C}}$, while the poles of $\theta - d\phi_{\mathbb{C}}^*/\phi_{\mathbb{C}}^*$ are invariant by the horizontal foliation $\mathbb{C}^n \times \mathcal{L}^*$ induced by $\sigma_{\mathbb{C}}^*$.

Now, regarding the expression (21) of θ, we can say the following:

- there are no poles of the form $h_j = 0$, for $h_j \in \mathcal{O}_{2n}$ $(*)$-symmetric. Indeed, otherwise h_j would be a factor of both $\phi_{\mathbb{C}}$ and $\phi_{\mathbb{C}}^*$, since $h_j = 0$ is invariant by neither $\mathcal{L} \times \mathbb{C}^{n*}$ nor $\mathbb{C}^n \times \mathcal{L}^*$. The decomplexification of h_j would thus engender a non-trivial factor of ω in $\mathcal{A}_{n\mathbb{R}}$.
- In other words, the poles of θ are either horizontal or vertical. Thus, in coordinates $(z, w) \in \mathbb{C}^{2n}$, we can suppose that, for each $j = 1, \ldots, s$, there exists a unity $h_j \in \mathcal{O}_{2n}$ such that $f_j = h_j \tilde{f}_j$, where $\tilde{f}_j = \tilde{f}_j(z) \in \mathcal{O}_{2n}$ is a function only in the variables z. As a consequence, $f_j^* = h_j^* \tilde{f}_j^*$, where $\tilde{f}_j^* = \tilde{f}_j^*(w) \in \mathcal{O}_{2n}$ is a function only in the variables w.
- All poles must then be cancelled by either $d\phi_{\mathbb{C}}/\phi_{\mathbb{C}}$ or by $d\phi_{\mathbb{C}}^*/\phi_{\mathbb{C}}^*$. Thus, they are all simple and $n_1 = \cdots = n_s = 0$.
- For this same reason, there exist a unity $g_0 \in \mathcal{O}_{2n}$ and $r_1, \ldots, r_s \in \mathbb{Z}^*$ such that
$$\phi_{\mathbb{C}} = g_0(\tilde{f}_1^*)^{r_1} \cdots (\tilde{f}_s^*)^{r_s} = g_0 \, \Psi^* \qquad \text{and} \qquad \phi_{\mathbb{C}}^* = g_0^* \tilde{f}_1^{r_1} \cdots \tilde{f}_s^{r_s} = g_0^* \, \Psi,$$
where $\Psi = \tilde{f}_1^{r_1} \cdots \tilde{f}_s^{r_s} \in \mathcal{O}_{2n}$ is a function in the variables z only. Besides, $\lambda_j = r_j \in \mathbb{Z}^*$ for all $j = 1, \ldots, s$.
- The exact part can be written in the form $dG = dh_0/h_0$, where $h_0 = \exp(G)$ is a $(*)$-symmetric unity in $\mathcal{O}_{2n}$.

Taking into account all these comments, we can write

$$\theta = \sum_{j=1}^{s} r_j \left(\frac{d(h_j \tilde{f}_j)}{h_j \tilde{f}_j} + \frac{d(h_j^* \tilde{f}_j^*)}{h_j^* \tilde{f}_j^*} \right) + \frac{dh_0}{h_0} = \frac{d(\Psi \Psi^*)}{\Psi \Psi^*} + \frac{dH}{H},$$

where $H = h_0(h_1 h_1^*)^{r_1} \cdots (h_s h_s^*)^{r_s}$ is a $(*)$-symmetric unity in $\mathcal{O}_{2n}$. Thus, by considering the integrable pencil $(1/H)\mathcal{P}$ instead of $\mathcal{P}$ — which corresponds to performing the above calculations to the Levi-flat 1$-$form $(1/h)\omega$, where $h \in \mathcal{A}_{n\mathbb{R}}$ is a unity such that $h_{\mathbb{C}} = H$ — we can suppose that

$$\theta = \theta_{\mathcal{P}} = \frac{d(\Psi \Psi^*)}{\Psi \Psi^*}.$$

Inserting this in equations (19), we find

$$d\left(\frac{1}{\Psi \Psi^*} \omega_{\mathbb{C}} \right) = d\left(\frac{1}{\Psi \Psi^*} \eta_{\mathbb{C}} \right) = d\left(\frac{1}{\Psi \Psi^*} \eta_{\mathbb{C}}^* \right) = 0. \tag{24}$$

Let us define $\zeta_{\mathbb{C}} = \eta_{\mathbb{C}}/(g_0 \Psi \Psi^*) = \sigma_{\mathbb{C}}/\Psi$ and, symmetrically, $\zeta_{\mathbb{C}}^* = \eta_{\mathbb{C}}^*/(g_0^* \Psi \Psi^*) = \sigma_{\mathbb{C}}^*/\Psi^*$. Observe that, with respect to the decomposition $\mathbb{C}^n \times \mathbb{C}^{n*} \simeq \mathbb{C}^{2n}$, the 1$-$form $\zeta_{\mathbb{C}}$ is of type $(1, 0)$, expressed only in the variable z, whereas $\zeta_{\mathbb{C}}^*$, of type $(0, 1)$, is written only in the variable w. The second closed differential form in (24) then reads

$$0 = d\left(g_0 \zeta_{\mathbb{C}} \right) = dg_0 \wedge \zeta_{\mathbb{C}} + g_0 \, d\zeta_{\mathbb{C}}.$$

Decomposing $d = \partial_z + \partial_w$ into exterior derivatives with respect to the variables z and w and assembling $2-$forms of the same type, we find that

$$\partial_w g_0 \wedge \zeta_{\mathbb{c}} = 0.$$

This gives $\partial_w g_0 = 0$, implying that $g_0 = g_0(z)$ depends exclusively on the variables z.

Let us define $\tau_{\mathbb{c}} = g_0 \zeta_{\mathbb{c}} = \eta_{\mathbb{c}}/(\Psi\Psi^*) = g_0 \sigma_{\mathbb{c}}/\Psi$. It is a meromorphic $1-$form in the variable z, closed by (24). Returning to formula (18), we get

$$\omega_{\mathbb{c}} = \frac{\eta_{\mathbb{c}} + \eta_{\mathbb{c}}^*}{2} = \Psi\Psi^* \frac{\tau_{\mathbb{c}} + \tau_{\mathbb{c}}^*}{2}.$$

Decomplexifying this expression, we find

$$\omega = \psi\bar{\psi}\frac{\tau + \bar{\tau}}{2} = |\psi|^2 \mathrm{Re}(\tau),$$

where $\psi \in \mathcal{O}_n$ is such that $\psi_{\mathbb{c}} = \Psi$ and τ is a germ of closed meromorphic $1-$form at $(\mathbb{C}^n, 0)$ whose complexification is $\tau_{\mathbb{c}}$. Observe that τ, being a meromorphic multiple of σ, induces the Levi foliation $\mathcal{L}$ and also that ψ is an equation for the poles of τ. Finally, reincorporating the unity $h \in \mathcal{A}_{n\mathbb{R}}$, we get that ω has the desired shape.

Alternative (b). The axis $\mathcal{L} \times \mathcal{L}^*$ of the pencil $\mathcal{P}$ has a meromorphic first integral. This means that there exists a non-constant germ of meromorphic function F at $(\mathbb{C}^n \times \mathbb{C}^{n*}, 0)$ such that

$$dF \wedge \eta_{\mathbb{c}} \wedge \eta_{\mathbb{c}}^* = 0.$$

We can suppose that F is $(*)$-symmetric. If this is not so, we apply the $(*)$-operator, finding

$$dF^* \wedge \eta_{\mathbb{c}} \wedge \eta_{\mathbb{c}}^* = 0.$$

Thus

$$d\left(F + F^*\right) \wedge \eta_{\mathbb{c}} \wedge \eta_{\mathbb{c}}^* = 0.$$

If $F + F^*$ is non-constant, replacing F by $F + F^*$, we have a $(*)$-symmetric meromorphic first integral for $\mathcal{L} \times \mathcal{L}^*$. On the other hand, $F + F^*$ is constant if and only if $F + F^* = c$ for some $c \in \mathbb{R}$. Thus, putting $F - c/2$ in place of F, we can suppose that $F + F^* = 0$. Thus, the non-constant meromorphic function iF is such that $(iF)^* = (-i)F^* = iF$.

Now, by Proposition 2.2, we can write $F = G/H$, where $G, H \in \mathcal{O}_{2n}$ are relatively prime $(*)$-symmetric functions. Let $g, h \in \mathcal{A}_{n\mathbb{R}}$ be real analytic functions such that $g_{\mathbb{c}} = G$ and $h_{\mathbb{c}} = H$. Thus, $f = g/h$ is a non-constant real meromorphic function in $\mathcal{Q}_{n\mathbb{R}}$ such that

$$df \wedge \eta \wedge \bar{\eta} = 0,$$

which means that $\mathcal{L}$ is tangent to the levels of f. By Theorem 1, $\mathcal{L}$ has a meromorphic first integral $\rho \in \mathcal{M}_n$. Therefore, there exists a real meromorphic function $\kappa \in \mathcal{Q}_n$ such that $\eta = \kappa d\rho$. Consequently

$$\omega = \frac{1}{2}\left(\kappa d\rho + \bar{\kappa}d\bar{\rho}\right) = \mathrm{Re}(\kappa d\rho).$$

As seen in Example 5.2, the integrability for ω implies that $\kappa/\bar{\kappa}$ is constant along the leaves of $\mathcal{L}$. This finishes the proof. $\square$

6. The global case: algebraic Levi-flat foliations. Throughout this section we consider the projective space $\mathbb{P}^n = \mathbb{P}^n_{\mathbb{C}}$ obtained by the usual identification of points of $\mathbb{C}^{n+1} \setminus \{0\}$ lying in the same line through the origin. We identify geometric objects in $\mathbb{P}^n$ with their cones in $\mathbb{C}^{n+1} \setminus \{0\}$ and work in coordinates $z = (z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$.

In this way, a real algebraic foliation $\mathcal{F}$ of codimension one in $\mathbb{P}^n$ is defined in $\mathbb{C}^{n+1}$ by an integrable real $1-$form ω with relatively prime polynomial coefficients satisfying the following conditions:

(i) $i_{\boldsymbol{r}}\omega = i_{\boldsymbol{r}}\omega^{\sharp} = 0$, where $i_{\boldsymbol{r}}$ denotes the contraction by the real radial vetor field;

(ii) $(\Lambda_\lambda)^* \omega = |\lambda|^{2d}\omega$ for some $d \geq 1$ and for every $\lambda \in \mathbb{C}^*$, where Λ_λ is the homothety of $\mathbb{C}^n$ given by $\Lambda_\lambda(z) = \lambda z$.

Employing the notation of Subsection 2.2, we write

$$\omega = \sum_{j=0}^{n} (A_j dx_j + B_j dy_j) \qquad \text{and} \qquad \omega^{\sharp} = \sum_{j=0}^{n} (-B_j dx_j + A_j dy_j),$$

where $A_j, B_j \in \mathbb{C}[z, \bar{z}]$ are such that $A_j = \bar{A}_j$ and $B_j = \bar{B}_j$ for every $j = 0, \ldots, n$. We also take the canonical decomposition $\omega = (\eta + \bar{\eta})/2$, where

$$\eta = \sum_{j=0}^{n} \frac{A_j - iB_j}{2} dz_j = \sum_{j=0}^{n} \varepsilon_j dz_j.$$

Condition (i) expresses that $\mathcal{F}$ contains all complex lines through the origin of $\mathbb{C}^{n+1}$. In coordinates, $\boldsymbol{r} = \sum_{j=0}^{n} (x_j \partial/\partial x_j + y_j \partial/\partial y_j)$ and we have

$$\sum_{j=0}^{n} (x_j A_j + y_j B_j) = 0 \qquad \text{and} \qquad \sum_{j=0}^{n} (-x_j B_j + y_j A_j) = 0.$$

This is the same of asking the vanishing of the contraction of η by the complex radial vetor field $\boldsymbol{R} = \sum_{j=0}^{n} z_j \partial/\partial z_j$, that is, $\sum_{j=0}^{n} z_j \varepsilon_j = 0$.

In turn, condition (ii) asserts that the distribution of real hyperplanes induced by ω descends, in a well defined way, to $\mathbb{P}^n$. Suppose, initially, that the coefficients of η, $\varepsilon_j \in \mathbb{C}[z, \bar{z}]$, are bihomogeneous of bidegree $(d-1, d)$, for some $d \geq 1$. Then, for $\lambda \in \mathbb{C}^*$, we have

$$(\Lambda_\lambda)^* \eta = \lambda \sum_{j=0}^{n} \lambda^{d-1} \bar{\lambda}^d \varepsilon_j(z, \bar{z}) dz_j = |\lambda|^{2d}\eta. \tag{25}$$

Similarly, $(\Lambda_\lambda)^* \bar{\eta} = |\lambda|^{2d}\bar{\eta}$, which gives

$$(\Lambda_\lambda)^* \omega = (\Lambda_\lambda)^* \left(\frac{\eta + \bar{\eta}}{2} \right) = |\lambda|^{2d}\omega.$$

Reciprocally, if condition (ii) holds, it is straightforward that the coefficients ε_j of η must be bihomogeneous of bidegree $(d-1, d)$.

Let us suppose that $\mathcal{F}$ is an algebraic foliation in $\mathbb{P}^n$ that is Levi-flat, meaning that it is a local Levi-flat foliation at each point of $\mathbb{P}^n$. Thus, the Levi foliation $\mathcal{L} = \mathcal{L}(\mathcal{F})$ is a real analytic complex foliation in $\mathbb{P}^n$ of complex codimension one. In consonance with the local study of the previous sections, we suppose that $\mathcal{L}$ is a holomorphic

foliation. Thus, we associate to $\mathcal{L}$ a degree $d_0 \geq 0$, which counts its tangencies, with multiplicities, with a generic line in $\mathbb{P}^n$. In coordinates $z = (z_0, \ldots, z_n) \in \mathbb{C}^{n+1}$, the foliation $\mathcal{L}$ is induced by an integrable polinomial 1-form $\sigma = \sum_{j=0}^n \alpha_j(z) dz_j$, whose coefficients $\alpha_j \in \mathbb{C}[z]$ are relatively prime homogeneous polynomials of degree $d_0 + 1$, which contracts to zero by the complex radial vector field, that is, $\sum_{j=0}^n z_j \alpha_j = 0$. Since η defines the same distribution of complex hyperplanes as σ, there exists a bihomogeneous polynomial $\varphi \in \mathbb{C}[z, \bar{z}]$ of bidegree $(d - d_0 - 2, d)$ such that

$$\eta = \varphi \sigma.$$

In particular, $d_0 \leq d - 2$.

Now we analyze the two alternatives of Theorem 2 for $\mathcal{F}$, considered as a local foliation at $0 \in \mathbb{C}^{n+1}$:

Alternative *(a)*. $\omega = h|\psi|^2 \mathrm{Re}(\tau)$, where τ is a closed meromorphic 1−form that induces $\mathcal{G}$, $\psi \in \mathcal{O}_{n+1}$ is an equation for polar set of τ and $h \in \mathcal{A}_{n+1,\mathbb{R}}$ is a unity. The 1−form τ goes down to $\mathbb{P}^n$, that is, for every $\lambda \in \mathbb{C}^*$ the 1−forms τ and $(\Lambda_\lambda)^* \tau$ define the same distribution of complex hyperplanes and $i_{\boldsymbol{R}} \tau = 0$, where $\boldsymbol{R}$ is the complex radial vector field. Hence, τ has a writing as in (20) (we borrow the notation, putting τ instead of θ), where $F_1, \ldots, F_\ell, G \in \mathbb{C}[z]$ are homogeneous polynomials satisfying:

- $\deg G = k_1 \deg F_1 + \cdots + k_\ell \deg F_\ell$;
- $\lambda_1 \deg F_1 + \cdots + \lambda_\ell \deg F_\ell = 0$ (Residue Theorem).

The function ψ is an equation of the poles of τ, thus it is a homogeneous polynomial in $\mathbb{C}[z]$. Finally, the homogeneity of ω, ψ and τ gives that h must be a constant, supposed to be 1.

Alternative *(b)*. $\omega = \mathrm{Re}(\kappa d\rho)$, where $\rho \in \mathcal{M}_{n+1}$ is a complex meromorphic function, $\kappa \in \mathcal{Q}_{n+1}$ is a real meromorphic function such that $\kappa/\bar{\kappa}$ is constant along the leaves of $\mathcal{G}$. Again, since ρ is constant along the leaves of a foliation that goes down to $\mathbb{P}^n$, all leaves of $\mathcal{L}$ are algebraic and ρ must be a complex rational function in $\mathbb{P}^n$, given in homogeneous coordinates as the quotient of relatively prime homogeneous polynomials in $\mathbb{C}[z]$ of the same degree. From (25), considering that $\eta = \kappa d\rho$, we see that κ must be a real rational function of bidegree (d, d) defined as the quotient of relatively prime homogeneous polynomials in $\mathbb{C}[z, \bar{z}]$.

We summarize this discussion in the following global version of Theorem 2:

THEOREM 3. *Let $\mathcal{F}$ be a real algebraic Levi-flat foliation on $\mathbb{P}^n$, $n \geq 2$, induced in homogeneous coordinates $z = (z_0, \cdots, z_n) \in \mathbb{C}^{n+1}$ by an integrable 1−form ω with polynomial coefficients satisfying conditions (i) and (ii) above. If the Levi foliation $\mathcal{L} = \mathcal{L}(\mathcal{F})$ is holomorphic, then at least one of the following two possibilities occurs:*

(a) $\omega = |\psi|^2 \mathrm{Re}(\tau)$, where τ is closed complex rational 1−form that defines $\mathcal{L}$ and $\psi \in \mathbb{C}[z]$ is a homogeneous equation for polar set of τ.

(b) $\omega = \mathrm{Re}(\kappa d\rho)$, where ρ is a complex rational function that is a first integral for $\mathcal{L}$, κ is a real rational function of bidegree (d, d), for some $d \geq 1$, defined as a quotient of relatively prime homogeneous polynomials in $\mathbb{C}[z, \bar{z}]$, such that $\kappa/\bar{\kappa}$ is constant along the leaves of $\mathcal{G}$.

We finish with an illustration on how to produce examples of the situation (b) in the theorem:

EXAMPLE 6.1. Let $\rho = F/G$ be a complex rational function, where $F, G \in \mathbb{C}[z]$ are relatively prime homogeneous polynomials of the same degree, whose levels define a holomorphic foliation $\mathcal{G}$ on $\mathbb{P}^n$. Let $\upsilon = R/S$ be a complex rational function in the

variables $u = (u_1, u_2) \in \mathbb{C}^2$, defined as a quotient of relatively prime homogeneous polynomials $R, S \in \mathbb{C}[u]$. Then $v \circ \rho = R(F, G)/S(F, G)$ is also a rational first integral for $\mathcal{G}$. Suppose also that u_2^2 factors R, so that G^2 divides $R(F, G)$, making $R(F, G)d\rho$ a polynomial $1-$form. Let $\kappa = R(F, G) \overline{S(F, G)}$. It is a homogeneous polynomial in $\mathbb{C}[z, \bar{z}]$ of bidegree (d, d), where $d = \deg(R(F, G)) = \deg(S(F, G))$, such that $\kappa/\bar{\kappa} = v \circ \rho / \overline{v \circ \rho}$ is constant along the leaves of $\mathcal{G}$. Thus, up to cancelling factors of the form $|\varphi|^2$, where $\varphi \in \mathbb{C}[z]$ is a homogeneous factor of F or G, the $1-$form $\omega = \mathrm{Re}(\kappa d\rho)$ defines a Levi-flat foliation on $\mathbb{P}^n$ whose underlying foliation is $\mathcal{G}$.

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