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TWO CLOSED GEODESICS ON COMPACT BUMPY FINSLER MANIFOLDS*

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Abstract. In this paper, we prove there are at least two closed geodesics on any compact bumpy Finsler n-manifold with finite fundamental group and $n \geq 2$. Thus generically there are at least two closed geodesics on compact Finsler manifolds with finite fundamental group. Furthermore, there are at least two closed geodesics on any compact Finsler 2-manifold, and this lower bound is achieved by the Katok 2-sphere (S^2, F) and 2-real projective space $(S^2/\mathbf{Z}_2, F)$, cf. [Kat].

Key words. Closed geodesic, Finsler manifold, bumpy.

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1. Introduction and main results. It is well-known that there are at least two closed geodesics on all compact bumpy Riemannian manifolds M with $\dim M \geq 2$ except for some special type of manifolds, cf. [Fet] or Theorem 4.1.8 of [Kli2] and Remark 1.5 below. While its proof depends on the symmetric property for the Riemannian metric, and consequently the proof carries over to the symmetric Finsler case. But for non-symmetric Finsler case, the proof does not work, hence we must develop new methods to handle the problem in this case. This paper is devoted to do this.

Let us recall firstly the definition of the Finsler metrics.

DEFINITION 1.1 (cf. [BCS] or [She]). *Let M be a finite dimensional smooth manifold. A function $F : TM \to [0, +\infty)$ is a* Finsler metric *if it satisfies*

(F1) F is C^∞ on $TM \setminus \{0\}$,

(F2) $F(x, \lambda y) = \lambda F(x, y)$ for all $x \in M$, $y \in T_xM$ and $\lambda > 0$,

(F3) For every $y \in T_xM \setminus \{0\}$, the quadratic form

$$g_{x,y}(u, v) \equiv \frac{1}{2}\frac{\partial^2}{\partial s \partial t} F^2(x, y + su + tv)|_{t=s=0}, \qquad \forall u, v \in T_xM,$$

is positive definite.

In this case, (M, F) is called a Finsler manifold. *F is* symmetric *if $F(x, -y) = F(x, y)$ holds for all $x \in M$ and $y \in T_xM$. F is* Riemannian *if $F(x, y)^2 = \frac{1}{2}G(x)y \cdot y$ for some symmetric positive definite matrix function $G(x) \in GL(T_xM)$ depending on $x \in M$ smoothly.*

A closed curve on a Finsler manifold is a *closed geodesic* if it is locally the shortest path connecting any two nearby points on this curve (cf. [She]). As usual, on any Finsler manifold (M, F), a closed geodesic $c : S^1 = \mathbf{R}/\mathbf{Z} \to M$ is *prime* if it is not a multiple covering (i.e., iteration) of any other closed geodesics. Here the m-th iteration c^m of c is defined by $c^m(t) = c(mt)$, where $m \in \mathbf{N}$. The inverse curve c^{-1} of c is defined by $c^{-1}(t) = c(1 - t)$ for $t \in \mathbf{R}$. Note that unlike Riemannian manifold, the inverse curve c^{-1} of a closed geodesic c on a non-symmetric Finsler manifold need not be a geodesic. Two prime closed geodesics c and d are *distinct* if there is no

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$\theta \in (0,1)$ such that $c(t) = d(t+\theta)$ for all $t \in \mathbf{R}$. We shall omit the word *distinct* when we talk about more than one prime closed geodesic. On a symmetric Finsler (or Riemannian) manifold, two closed geodesics c and d are called *geometrically distinct* if $c(S^1) \neq d(S^1)$, i.e., their image sets in M are distinct.

For a closed geodesic c on $(M,\ F)$, denote by P_c the linearized Poincaré map of c. Then $P_c \in \mathrm{Sp}(2n-2)$ is symplectic. A closed geodesic c is called *non-degenerate* if 1 is not an eigenvalue of P_c. A Finsler manifold $(M,\ F)$ is called *bumpy* if all the closed geodesics on it are non-degenerate. Note that bumpy Finsler metrics are generic in the set of Finsler metrics, cf. [Rad4].

The following are the main results in this paper:

THEOREM 1.2. *There exist at least two prime closed geodesics on every compact bumpy Finsler manifold* (M,F) *with finite fundamental group and* $\dim M \geq 2$.

Furthermore, if $\dim M = 2$, the bumpy and finite fundamental group conditions are not needed, and we have the following:

THEOREM 1.3. *There exist at least two prime closed geodesics on every compact Finsler manifold* (M,F) *with* $\dim M = 2$.

REMARK 1.4. In 1973, Katok in [Kat] found some non-symmetric Finsler metrics on CROSSs (compact rank one symmetric spaces) with only finitely many prime closed geodesics and all closed geodesics are non-degenerate. The number of closed geodesics on S^n that one obtains in these examples is $2[\frac{n+1}{2}]$, where $[a] = \max\{k \in \mathbf{Z} \,|\, k \leq a\}$ for $a \in \mathbf{R}$, cf. [Zil].

We are aware of a number of results concerning closed geodesics on Finsler manifolds. According to the classical theorem of Lyusternik-Fet [LyF] from 1951, there exists at least one closed geodesic on every compact Riemannian manifold. The proof of this theorem is variational and carries over to the Finsler case. In [BaL], V. Bangert and Y. Long proved that on any Finsler 2-sphere (S^2,F), there exist at least two closed geodesics. In [Rad3], H.-B. Rademacher studied the existence and stability of closed geodesics on positively curved Finsler manifolds. In [DuL1] of Duan and Long and in [Rad4] of Rademacher, they proved there exist at least two closed geodesics on any bumpy Finsler n-sphere independently. In [Rad5], Rademacher proved there exist at least two closed geodesics on any bumpy Finsler $\mathbf{CP^2}$. In [DLW], Duan, Long and Wang proved there exist at least two closed geodesics on any compact simply-connected bumpy Finsler manifold. In [DLX], Duan, Long and Xiao proved the existence of at least two non-contractible closed geodesics on any bumpy Finsler $\mathbf{RP}^3$. In [Tai2], Taimanov proved the existence of at least two non-contractible closed geodesics on any bumpy Finsler $\mathbf{RP}^2$. In [LiX], Liu and Xiao proved there exist at least two non-contractible closed geodesics on any bumpy Finsler RP^n.

REMARK 1.5. For the case $\pi_1(M)$ is infinite, as pointed out in [Tai1] of I. Taimanov, there are at least two prime closed geodesics on all compact Riemannian manifolds except M is an Eilenberg-MacLane complex $K(\pi,1)$ such that π is different from $\mathbf{Z}$ and contains an element $g \in \pi$ such that any element of π is conjugate to one power of g. cf. Theorem 4.1.8 of [Kli2] and [BaH] for a proof. Note that the same proof yields there are at least two prime closed geodesics on all compact Finsler manifolds with infinite fundamental group except M is an Eilenberg-MacLane complex $K(\pi,1)$ as above.

REMARK 1.6. In [LLX], Liu, Long and Xiao proved in every non-trivial homotopy class $\alpha \in \pi_1(S^n/\Gamma)$ with finite order of a bumpy Finsler S^n/Γ, where Γ is a finite group acts on S^n freely and isometrically, there exist at least two distinct closed geodesics. While we don't know whether the two closed geodesics obtained in Theorem 1.2 belongs to some identical homotopy class. Note also that the result of [LLX] does not imply there are more than two prime closed geodesics on S^n/Γ. For example, let (S^2, F) be the Katok 2-sphere, cf. [Kat] or [Zil], then $(S^2/\mathbf{Z}_2, F)$ has exactly two closed geodesics c_+ and c_- in $0 \neq \alpha \in \pi_1(S^2/\mathbf{Z}_2)$, and exactly two closed geodesics c_+^2 and c_-^2 in $0 = \alpha^2 \in \pi_1(S^2/\mathbf{Z}_2)$.

In this paper, let $\mathbf{N}$, $\mathbf{N}_0$, $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, and $\mathbf{C}$ denote the sets of natural integers, non-negative integers, integers, rational numbers, real numbers, and complex numbers respectively. We use only singular homology modules with $\mathbf{Q}$-coefficients. For terminologies in algebraic topology we refer to [GrH]. For $k \in \mathbf{N}$, we denote by $\mathbf{Q}^k$ the direct sum $\mathbf{Q} \oplus \cdots \oplus \mathbf{Q}$ of k copies of $\mathbf{Q}$ and $\mathbf{Q}^0 = 0$. For an S^1-space X, we denote by $\overline{X}$ the quotient space X/S^1.

2. Critical point theory for closed geodesics. Let $M = (M, F)$ be a compact Finsler manifold, the space $\Lambda = \Lambda M$ of H^1-maps $\gamma : S^1 \to M$ has a natural structure of Riemannian Hilbert manifolds on which the group $S^1 = \mathbf{R}/\mathbf{Z}$ acts continuously by isometries, c.f. [Kli1]-[Kli3]. This action is defined by $(s\cdot\gamma)(t) = \gamma(t+s)$ for all $\gamma \in \Lambda$ and $s, t \in S^1$. For any $\gamma \in \Lambda$, the energy functional is defined by

$$E(\gamma) = \frac{1}{2}\int_{S^1} F(\gamma(t), \dot{\gamma}(t))^2 dt. \tag{2.1}$$

It is $C^{1,1}$ and invariant under the S^1-action, cf. [Mer]. The critical points of E of positive energies are precisely the closed geodesics $\gamma : S^1 \to M$. The index form of the functional E is well defined along any closed geodesic c on M, which we denote by $E''(c)$. As usual we define the index $i(c)$ of c as the maximal dimension of subspaces of $T_c\Lambda$ on which $E''(c)$ is negative definite, and the nullity $\nu(c)$ of c so that $\nu(c)+1$ is the dimension of the null space of $E''(c)$, cf. Definition 2.5.4 of [Kli3]. In the following, we denote by

$$\Lambda^\kappa = \{d \in \Lambda \mid E(d) \le \kappa\}, \quad \Lambda^{\kappa-} = \{d \in \Lambda \mid E(d) < \kappa\}, \quad \forall \kappa \ge 0. \tag{2.2}$$

For a closed geodesic c we set $\Lambda(c) = \{\gamma \in \Lambda \mid E(\gamma) < E(c)\}$.

For $m \in \mathbf{N}$ we denote the m-fold iteration map $\phi_m : \Lambda \to \Lambda$ by $\phi_m(\gamma)(t) = \gamma(mt)$, for all $\gamma \in \Lambda, t \in S^1$, as well as $\gamma^m = \phi_m(\gamma)$. If $\gamma \in \Lambda$ is not constant then the multiplicity $m(\gamma)$ of γ is the order of the isotropy group $\{s \in S^1 \mid s\cdot\gamma = \gamma\}$. For a closed geodesic c, the mean index $\hat{i}(c)$ is defined as usual by $\hat{i}(c) = \lim_{m\to\infty} i(c^m)/m$.

We call a closed geodesic satisfying the isolation condition, if the following holds:

(Iso) For all $m \in \mathbf{N}$ the orbit $S^1 \cdot c^m$ is an isolated critical orbit of E.

Note that if the number of prime closed geodesics on a Finsler manifold is finite, then all the closed geodesics satisfy (Iso).

Using singular homology with rational coefficients we consider the following critical $\mathbf{Q}$-module of a closed geodesic $c \in \Lambda$:

$$\overline{C}_*(E, c) = H_*\left((\Lambda(c) \cup S^1\cdot c)/S^1, \Lambda(c)/S^1\right). \tag{2.3}$$

PROPOSITION 2.1 (cf. Satz 6.11 of [Rad2] or Proposition 3.12 of [BaL]). *Let c be a prime closed geodesic on a Finsler manifold (M, F) satisfying (Iso). Then we have*

$$\begin{aligned}\overline{C}_q(E, c^m) &\equiv H_q\left((\Lambda(c^m) \cup S^1\cdot c^m)/S^1, \Lambda(c^m)/S^1\right)\\ &= \left(H_{i(c^m)}(U_{c^m}^- \cup \{c^m\}, U_{c^m}^-) \otimes H_{q-i(c^m)}(N_{c^m}^- \cup \{c^m\}, N_{c^m}^-)\right)^{+\mathbf{Z}_m}\end{aligned}$$

(i) When $\nu(c^m) = 0$*, there holds*

$$\overline{C}_q(E, c^m) = \begin{cases} \mathbf{Q}, & \text{if } \ i(c^m) - i(c) \in 2\mathbf{Z}, \text{ and } q = i(c^m), \\ 0, & \text{otherwise.} \end{cases}$$

(ii) When $\nu(c^m) > 0$*, there holds*

$$\overline{C}_q(E, c^m) = H_{q-i(c^m)}(N_{c^m}^- \cup \{c^m\}, N_{c^m}^-)^{(-1)^{i(c^m)-i(c)}\mathbf{Z}_m},$$

where N_{c^m} *is a local characteristic manifold at* c^m *and* $N_{c^m}^- = N_{c^m} \cap \Lambda(c^m)$*,* U_{c^m} *is a local negative disk at* c^m *and* $U_{c^m}^- = U_{c^m} \cap \Lambda(c^m)$*,* $H_*(X, A)^{\pm\mathbf{Z}_m} = \{[\xi] \in H_*(X, A) \mid T_*[\xi] = \pm[\xi]\}$ *where* T *is a generator of the* $\mathbf{Z}_m$*-action.*

Definition 2.2. *The Euler characteristic* $\chi(c^m)$ *of* c^m *is defined by*

$$\begin{aligned} \chi(c^m) &\equiv \chi\left((\Lambda(c^m) \cup S^1 \cdot c^m)/S^1, \Lambda(c^m)/S^1\right), \\ &\equiv \sum_{q=0}^{\infty} (-1)^q \dim \overline{C}_q(E, c^m). \end{aligned} \tag{2.4}$$

Here $\chi(A, B)$ *denotes the usual Euler characteristic of the space pair* (A, B)*.*

The average Euler characteristic $\hat{\chi}(c)$ *of* c *is defined by*

$$\hat{\chi}(c) = \lim_{N\to\infty} \frac{1}{N} \sum_{1\le m\le N} \chi(c^m). \tag{2.5}$$

By Remark 5.4 of [Wan], $\hat{\chi}(c)$ is well-defined and is a rational number. In particular, if c^m are non-degenerate for $\forall m \in \mathbf{N}$, then

$$\hat{\chi}(c) = \begin{cases} (-1)^{i(c)}, & \text{if } \ i(c^2) - i(c) \in 2\mathbf{Z}, \\ \frac{(-1)^{i(c)}}{2}, & \text{otherwise.} \end{cases} \tag{2.6}$$

Set $\overline{\Lambda}^0 = \overline{\Lambda}^0 M = \{\text{constant point curves in } M\} \cong M$. Let (X, Y) be a space pair such that the Betti numbers $b_i = b_i(X, Y) = \dim H_i(X, Y; \mathbf{Q})$ are finite for all $i \in \mathbf{Z}$. As usual the *Poincaré series* of (X, Y) is defined by the formal power series $P(X, Y) = \sum_{i=0}^{\infty} b_i t^i$. We need the following results on Betti numbers.

For a compact and simply-connected Finsler manifold M with $H^*(M; \mathbf{Q}) \cong T_{d,h+1}(x)$ with the generator x of degree d and height $h + 1$, if d is odd, then $x^2 = 0$ and $h = 1$ in $T_{d,h+1}(x)$, thus M is rationally homotopy equivalent to S^d (cf. [Rad1] or [Hin]).

Proposition 2.3 (cf. Theorem 2.4, Remark 2.5 of [Rad1] and Lemma 2.5, 2.6 of [DuL2]). *Let* M *be a compact simply-connected manifold with* $H^*(M; \mathbf{Q}) \cong T_{d,h+1}(x)$*. Then the Betti numbers of the free loop space of* M *defined by* $b_q = \mathrm{rank} H_q(\Lambda M/S^1, \Lambda^0 M/S^1; \mathbf{Q})$ *for* $q \in \mathbf{Z}$ *are given by*

(i) If $h = 1$ *and* $d \in 2\mathbf{N} + 1$*, then we have*

$$b_q = \begin{cases} 2, & \text{if } \ q \in \mathcal{K} \equiv \{k(d-1) \mid 2 \le k \in \mathbf{N}\}, \\ 1, & \text{if } \ q \in \{d - 1 + 2k \mid k \in \mathbf{N}_0\} \setminus \mathcal{K}, \\ 0 & \text{otherwise.} \end{cases} \tag{2.7}$$

(ii) If $h = 1$ and $d \in 2\mathbf{N}$, then we have

$$b_q = \begin{cases} 2, & \text{if} \quad q \in \mathcal{K} \equiv \{k(d-1) \,|\, 3 \le k \in (2\mathbf{N}+1)\}, \\ 1, & \text{if} \quad q \in \{d-1+2k \,|\, k \in \mathbf{N}_0\} \setminus \mathcal{K}, \\ 0 & \text{otherwise}. \end{cases} \tag{2.8}$$

(iii) If $h \ge 2$ and $d \in 2\mathbf{N}$. Let $D = d(h+1) - 2$ and

$$\Omega(d,h) = \{k \in 2\mathbf{N}-1 \,|\, iD \le k-(d-1) = iD + jd \le iD + (h-1)d \quad \text{for some } i \in \mathbf{N} \text{ and } j \in [1, h-1]\}. \tag{2.9}$$

Then we have

$$b_q = \begin{cases} 0, & \text{if } q \in 2\mathbf{Z} \text{ or } q \le d-2, \\ [\frac{q-(d-1)}{d}] + 1, & \text{if } q \in 2\mathbf{N}-1 \text{ and } d-1 \le q < d-1+(h-1)d, \\ h+1, & \text{if } q \in \Omega(d,h), \\ h, & \text{otherwise}. \end{cases} \tag{2.10}$$

By a similar proof of Theorem 5.5 of [Wan], we have the following mean index identity:

PROPOSITION 2.4 (cf. Theorem 3.1 of [Rad1] and Satz 7.9 of [Rad2]). *Let (M, F) be a compact simply-connected Finsler manifold with $H^*(M, \mathbf{Q}) = T_{d,h+1}(x)$ and possess finitely many prime closed geodesics. Denote the prime closed geodesics on (M, F) with positive mean indices by $\{c_j\}_{1 \le j \le q}$ for some $q \in \mathbf{N}$. Then the following identity holds*

$$\sum_{j=1}^{q} \frac{\hat{\chi}(c_j)}{\hat{i}(c_j)} = B(d,h) = \begin{cases} -\frac{h(h+1)d}{2d(h+1)-4}, & d \text{ is even}, \\ \frac{d+1}{2d-2}, & d \text{ is odd (then } h = 1), \end{cases} \tag{2.11}$$

where $\dim M = hd$.

We have the following version of the Morse inequality.

THEOREM 2.5 (Theorem 6.1 of [Rad2]). *Suppose that there exist only finitely many prime closed geodesics $\{c_j\}_{1 \le j \le p}$ on (M, F), and $0 \le a < b \le \infty$ are regular values of the energy functional E. Define for each $q \in \mathbf{Z}$,*

$$M_q(\overline{\Lambda}^b, \overline{\Lambda}^a) = \sum_{1 \le j \le p,\ a < E(c_j^m) < b} \mathrm{rank}\overline{C}_q(E, c_j^m),$$
$$b_q(\overline{\Lambda}^b, \overline{\Lambda}^a) = \mathrm{rank} H_q(\overline{\Lambda}^b, \overline{\Lambda}^a).$$

Then there holds

$$M_q(\overline{\Lambda}^b, \overline{\Lambda}^a) - M_{q-1}(\overline{\Lambda}^b, \overline{\Lambda}^a) + \cdots + (-1)^q M_0(\overline{\Lambda}^b, \overline{\Lambda}^a) \ge b_q(\overline{\Lambda}^b, \overline{\Lambda}^a) - b_{q-1}(\overline{\Lambda}^b, \overline{\Lambda}^a) + \cdots + (-1)^q b_0(\overline{\Lambda}^b, \overline{\Lambda}^a), \tag{2.12}$$

$$M_q(\overline{\Lambda}^b, \overline{\Lambda}^a) \ge b_q(\overline{\Lambda}^b, \overline{\Lambda}^a). \tag{2.13}$$

3. Proof of main theorems. In this section, we give the proofs of the main theorems. We prove by contradiction, by [LyF] we suppose the following holds:

(C) There is only one prime closed geodesic c on (M, F).

Proof of Theorem 1.2. Let $p : \tilde{M} \to M$ be the universal covering of M and $\tilde{F} = p^*(F)$. Then $(\tilde{M}, \tilde{F})$ is a compact Finsler manifold and it is locally isometric to (M, F). In fact, $\tilde{M}$ is a compact manifold without boundary follows from the property for covering spaces (cf. Theorem 6.7 and Corollary 6.8 of [GrH] or Section 2.6 of [Spa]): Since M is a compact manifold without boundary, the universal covering space $\tilde{M}$ of M exists by Theorem 6.7 and Corollary 6.8 of [GrH]. By the definition of covering space (cf. p. 21 of [GrH]) and Theorem 5.8 of [GrH], $\forall x \in M$ there exists a open neighborhood U_x of x which is homeomorphic to the open disk $\{p \in \mathbf{R}^n : ||p|| < 1\}$ in $\mathbf{R}^n$ and $\overline{U_x}$ being compact such that $p^{-1}(U_x) = \cup_{\alpha \in \pi_1(M)} V_{x,\alpha}$ with $V_{x,\alpha} \cap V_{x,\beta} = \emptyset$ for $\alpha \neq \beta$ and each $V_{x,\alpha}$ is homeomorphic to U_x for $\alpha \in \pi_1(M)$. Since M is compact, we can find finitely many $\{x_i\}_{1 \leq i \leq k}$ such that $M = \cup_{1 \leq i \leq k} U_{x_i}$, thus $\tilde{M} = \cup_{\alpha \in \pi_1(M)} V_{x_i,\alpha}$. Since $\pi_1(M)$ is finite, $\tilde{M}$ is a compact manifold without boundary.

If $\tilde{d}$ is a closed geodesic on $(\tilde{M}, \tilde{F})$, then $d = p(\tilde{d})$ is a closed geodesic on (M, F) and $P_{\tilde{d}} = P_d$. In fact, $P_{\tilde{d}} = P_d$ follows from the fact that the Jacobi field equations along d and $\tilde{d}$ are the same, while P_γ for a closed geodesic γ can be defined by using the Jacobi field along γ, cf. Section 3 of [Rad4]. Since (M, F) is bumpy, d is non-degenerate, i.e. $1 \notin \sigma(P_d)$, therefore $\tilde{d}$ is non-degenerate also, hence $(\tilde{M}, \tilde{F})$ is bumpy. Now we have the following two cases:

Case 1. $H^*(\tilde{M}; \mathbf{Q}) \neq T_{d,h+1}(x)$.

In this case, there must be infinitely many prime closed geodesics on $(\tilde{M}, \tilde{F})$ by [ViS] of M. Vigué-Poirrier and D. Sullivan. In fact, the Betti numbers of the free loop space $\Lambda\tilde{M}$ are unbounded and the theorem in [GrM] of Gromoll-Meyer can be applied to the Finsler manifolds as well. Hence there must be infinitely many prime closed geodesics on (M, F) since $\pi_1(M)$ is finite. This proves Theorem 1.2 in this case.

Case 2. $H^*(\tilde{M}; \mathbf{Q}) = T_{d,h+1}(x)$.

In this case, as in Case 1, it is sufficient to consider the case there are finitely many prime closed geodesics on $(\tilde{M}, \tilde{F})$.

Denote by $\{\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_k\}$ the prime closed geodesics on $(\tilde{M}, \tilde{F})$, then we have $k \geq 2$ by [DLW]. If $\pi_1(M) = 0$, then Theorem 1.2 holds since $\tilde{M} = M$ in this case, thus it remains to consider the case that $\pi_1(M) \neq 0$. For $1 \leq i \leq k$, we have $p(\tilde{d}_i) = c^{m_i}$ for some $m_i \in \mathbf{N}$ by the assumption (C). By translating the parameters if necessary, we may assume $p(\tilde{d}_j(0)) = c^{m_j}(0) = c(0)$ for $1 \leq j \leq k$.

Note that for each $i \in \{2, \ldots, k\}$, there exists a covering transformation $f_i : (\tilde{M}, \tilde{F}) \to (\tilde{M}, \tilde{F})$ such that $f_i(\tilde{d}_i(0)) = \tilde{d}_1(0)$. By the definition of the Finsler metric $\tilde{F}$ on $\tilde{M}$, the map f_i is an isometry on $(\tilde{M}, \tilde{F})$. Therefore $f_i(\tilde{d}_i)$ is a closed geodesic started at $f_i(\tilde{d}_i(0)) = \tilde{d}_1(0)$. By the property of covering transformation, we have

$$p(f_i(\tilde{d}_i(t))) = p(\tilde{d}_i(t)) = c^{m_i}(t) = c(m_i t), \quad \forall t \in \mathbf{R}. \tag{3.1}$$

On the other hand,

$$p(\tilde{d}_1(t)) = c^{m_1}(t) = c(m_1 t), \quad \forall t \in \mathbf{R}. \tag{3.2}$$

Hence we have $f_i(\tilde{d}_i(t)) = \tilde{d}_1(\frac{m_i}{m_1} t)$ for $\forall t \in \mathbf{R}$. Since $f_i(\tilde{d}_i)$ is a closed geodesic and $\tilde{d}_1$ is a prime closed geodesic, we have $m_1 | m_i$. Exchanging $\tilde{d}_1$ and $\tilde{d}_i$, we obtain $m_i | m_1$, and then $m_i = m_1$. This yields $f_i(\tilde{d}_i) = \tilde{d}_1$. Since f_i is an isometry on $(\tilde{M}, \tilde{F})$, it preserves the energy functional, i.e., $E(\gamma) = E(f_i(\gamma))$ for any $\gamma \in \Lambda(\tilde{M})$. This implies $i(\tilde{d_i}^m) = i(\tilde{d_1}^m)$ for any $m \in \mathbf{N}$. By Proposition 2.1, we have

$$\begin{aligned} M_q(\tilde{M}) \equiv M_q(\overline{\Lambda\tilde{M}}, \overline{\Lambda\tilde{M}}^0) &= \sum_{1\le j\le k,\ m\in\mathbf{N}} \mathrm{rank}\overline{C}_q(E, \tilde{d}_j^m) \\ &= \sum_{1\le j\le k} {}^{\#}\{m \,|\, i(\tilde{d}_j^m) - i(\tilde{d}_j) \in 2\mathbf{Z} \ \ and \ \ q = i(\tilde{d}_j^m)\} \\ &= k\,{}^{\#}\{m \,|\, i(\tilde{d}_1^m) - i(\tilde{d}_1) \in 2\mathbf{Z} \ \ and \ \ q = i(\tilde{d}_1^m)\}. \end{aligned} \quad (3.3)$$

By Bott formula, c.f. [Bot], $i(\tilde{d}_1^m) \ge i(\tilde{d}_1)$ for $m \in \mathbf{N}$, then we have $M_q(\tilde{M}) = 0$ for $q < i(\tilde{d}_1)$ by (3.3). By Proposition 2.1 and (3.3), we have $M_{i(\tilde{d}_1)}(\tilde{M}) = k\,{}^{\#}\{m \,|\, i(\tilde{d}_1^m) = i(\tilde{d}_1)\}$ and $M_{i(\tilde{d}_1)+1}(\tilde{M}) = 0$. We claim that $i(\tilde{d}_1) = d-1$. In fact, by Proposition 2.3 and Theorem 2.5, $M_{d-1}(\tilde{M}) \ge b_{d-1}(\tilde{M}) = 1$, this implies $i(\tilde{d}_1) \le d-1$. By (3.3), there exists $m \in \mathbf{N}$ such that $d-1 = i(\tilde{d}_1^m)$ and $i(\tilde{d}_1^m) - i(\tilde{d}_1) \in 2\mathbf{Z}$, this implies $i(\tilde{d}_1) - (d-1) \in 2\mathbf{Z}$. Thus if $i(\tilde{d}_1) < d-1$, we must have $i(\tilde{d}_1) < d-2$ holds. Hence by Theorem 2.5 and Proposition 2.3,

$$\begin{aligned} &-k\,{}^{\#}\{m \,|\, i(\tilde{d}_1^m) = i(\tilde{d}_1)\} \\ =&M_{i(\tilde{d}_1)+1}(\tilde{M}) - M_{i(\tilde{d}_1)}(\tilde{M}) + \cdots + (-1)^{i(\tilde{d}_1)+1} M_0(\tilde{M}) \\ \ge&b_{i(\tilde{d}_1)+1}(\tilde{M}) - b_{i(\tilde{d}_1)}(\tilde{M}) + \cdots + (-1)^{i(\tilde{d}_1)+1} b_0(\tilde{M}) \\ =&0. \end{aligned} \quad (3.4)$$

This contradiction proves that $i(\tilde{d}_1) = d-1$. By Theorem 2.5 and Proposition 2.3 again,

$$\begin{aligned} &-k\,{}^{\#}\{m \,|\, i(\tilde{d}_1^m) = i(\tilde{d}_1) = d-1\} \\ =&M_d(\tilde{M}) - M_{d-1}(\tilde{M}) + \cdots + (-1)^d M_0(\tilde{M}) \\ \ge&b_d(\tilde{M}) - b_{d-1}(\tilde{M}) + \cdots + (-1)^d b_0(\tilde{M}) \\ =&-1. \end{aligned} \quad (3.5)$$

This contradicts to $k \ge 2$ and proves Theorem 1.2. □

Proof of Theorem 1.3. Let $\tilde{M}$ be the orientable double cover of M if M is not orientable and $\tilde{M} = M$ otherwise. Let $p : \tilde{M} \to M$ be the projection and $\tilde{F} = p^*(F)$. Then $(\tilde{M}, \tilde{F})$ is a compact orientable Finsler manifold without boundary of dimension 2 as proved in Theorem 1.2. Due to the classification of surfaces, cf. Chapter 9 of [Hir], $\tilde{M}$ is either a sphere or a torus of genus g with $g \in \mathbf{N}$. In fact, by Section 9.1 of [Hir], if M is a surface with genus g, then its Euler characteristic $\chi(M) = 2-2g$ if M is orientable, while $\chi(M) = 2-g$ if M is not orientable; therefore, by Section 5.2 of [Hir], $\chi(\tilde{M}) = 2-2g$ if M is orientable, while $\chi(\tilde{M}) = 2\chi(M) = 4-2g$ if M is not orientable.

We have the following two cases:

Case 1. *$\tilde{M}$ is a torus of genus g with $g \in \mathbf{N}$.*

In this case, we have $b_1(\tilde{M}) = 2g \ge 2$. Choose two linearly independent generators α, β of $H_1(\tilde{M})$ and $c_m \in \Lambda\tilde{M}$ such that $[c_m] = \alpha\beta^m$ and c_m being a closed geodesic for $m \in \mathbf{N}$. In fact, since $H_1(\tilde{M}) = \pi_1(\tilde{M})/[\pi_1(\tilde{M}), \pi_1(\tilde{M})]$, one can choose some $c'_m \in \Lambda\tilde{M}$ such that $[c'_m] = \alpha\beta^m$. Since the energy functional E satisfies the Palais-Smale condition (cf. Theorem 4.6 of [Mer]), the sequence $\{\phi_s(c'_m)\}_{s\ge 0}$ has a limit point c_m, where ϕ_s is the negative gradient flow of E. Thus c_m is a closed geodesic and c_m is homotopic to $\phi_s(c'_m)$ for some s large enough, and then $[c_m] = [\phi_s(c'_m)] = [c'_m] \in H_1(\tilde{M})$. Clearly c_ms are distinct prime closed geodesics. Therefore there are infinitely many prime closed geodesics on $\tilde{M}$. Since $p : \tilde{M} \to M$ is k-fold cover with $k \leq 2$, there are infinitely many prime closed geodesics on M also. This proves Theorem 1.3 in this case.

Case 2. $\tilde{M} = S^2$.

As above it is sufficient to consider the case that $(S^2, \tilde{F})$ possess finitely many prime closed geodesics. Denote by $\{\tilde{d}_1, \tilde{d}_2, \ldots, \tilde{d}_k\}$ the prime closed geodesics on $(S^2, \tilde{F})$. Thus for $1 \leq i \leq k$, we have $p(\tilde{d}_i) = c^{m_i}$ for some $m_i \in \mathbf{N}$ by the assumption (C). As in Theorem 1.2, we have $m_i = m_1$ and $i(\tilde{d}_i{}^m) = i(\tilde{d}_1{}^m)$ and $k_l(\tilde{d}_i{}^m)^{\pm 1} = k_l(\tilde{d}_1{}^m)^{\pm 1}$ for any $m \in \mathbf{N}, l \in \mathbf{Z}$, where $k_l(\tilde{d}_i{}^m)^{\pm 1} = \mathrm{rank} H_{q-i(\tilde{d}_i{}^m)}(N^-_{\tilde{d}_i{}^m} \cup \{\tilde{d}_i{}^m\}, N^-_{\tilde{d}_i{}^m})^{\pm 1}$. In fact, each covering transformation f_i is an isometry on $(S^2, \tilde{F})$, it preserves the energy functional, i.e., $E(\gamma) = E(f_i(\gamma))$ for any $\gamma \in \Lambda(S^2)$, therefore f_i induces a homeomorphism between characteristic manifolds at $\tilde{d}_1{}^m$ and $\tilde{d}_i{}^m$. Hence $i(\tilde{d}_i{}^m) = i(\tilde{d}_1{}^m)$, $\nu(\tilde{d}_i{}^m) = \nu(\tilde{d}_1{}^m)$ and $k_l(\tilde{d}_i{}^m)^{\pm 1} = k_l(\tilde{d}_1{}^m)^{\pm 1}$ for any $m \in \mathbf{N}$. Therefore we obtain $\hat{\chi}(\tilde{d}_i) = \hat{\chi}(\tilde{d}_1)$. By Lemma 4.3 and Theorem 4.4 of [LoW], we have $\hat{i}(\tilde{d}_1) > 0$. By Proposition 2.4 or Theorem 4.4 of [LoW], we have

$$k\frac{\hat{\chi}(\tilde{d}_1)}{\hat{i}(\tilde{d}_1)} = \sum_{j=1}^{k} \frac{\hat{\chi}(\tilde{d}_j)}{\hat{i}(\tilde{d}_j)} = B(2,1) = 1. \tag{3.6}$$

Since $\hat{\chi}(\tilde{d}_1) \in \mathbf{Q}$, this implies $\hat{i}(\tilde{d}_1) \in \mathbf{Q}$ also. Thus there is no irrationally elliptic closed geodesics on $(S^2, \tilde{F})$ since such a closed geodesic d must satisfy $\hat{i}(d) \notin \mathbf{Q}$, cf. Section 3 of [LoW]. This contradicts to [LoW] which claims there exist at least two irrationally elliptic prime closed geodesics on every Finsler 2-sphere possessing only finitely many prime closed geodesics. This proves Theorem 1.3. □

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