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STACKY GKM GRAPHS AND ORBIFOLD GROMOV-WITTEN THEORY*

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Abstract. A smooth GKM stack is a smooth Deligne-Mumford stack equipped with an action of an algebraic torus T, with only finitely many zero-dimensional and one-dimensional orbits.

(i) We define the stacky GKM graph of a smooth GKM stack, under the mild assumption that any one-dimensional T-orbit closure contains at least one T fixed point. The stacky GKM graph is a decorated graph which contains enough information to reconstruct the T-equivariant formal neighborhood of the 1-skeleton (union of zero-dimensional and one-dimensional T-orbits) as a formal smooth DM stack equipped with a T-action.

(ii) We axiomize the definition of a stacky GKM graph and introduce abstract stacky GKM graphs which are more general than stacky GKM graphs of honest smooth GKM stacks. From an abstract GKM graph we construct a formal smooth GKM stack.

(iii) We define equivariant orbifold Gromov-Witten invariants of smooth GKM stacks, as well as formal equivariant orbifold Gromov-Witten invariants of formal smooth GKM stacks. These invariants can be computed by virtual localization and depend only the stacky GKM graph or the abstract stacky GKM graph. Formal equivariant orbifold Gromov-Witten invariants of the stacky GKM graph of a smooth GKM stack $\mathcal{X}$ are refinements of equivariant orbifold Gromov-Witten invariants of $\mathcal{X}$.

Key words. Gromov-Witten invariants, Deligne-Mumford stacks, orbifolds, equivariant cohomology, localization.

Mathematics Subject Classification. Primary 14N35; Secondary 14D23, 14H10.

1. Introduction.

1.1. Background and motivation.

1.1.1. GKM manifolds and GKM graphs. An algebraic GKM manifold, named after Goresky, Kottwitz, and MacPherson, is a non-singular (complex) algebraic variety equipped with an action of an algebraic torus $T = (\mathbb{C}^*)^m$ such that there are finitely many zero-dimensional and one-dimensional orbits. The action T on X is equivariantly formal over a field $\mathbb{K}$ if the map from the T-equivariant cohomology $H^*_T(X;\mathbb{K})$ of X to the ordinary cohomology $H^*(X;\mathbb{K})$ of X is surjective. In [28], Goresky-Kottwitz-MacPherson showed that (when $\mathbb{K} = \mathbb{R}$) the T-equivariant cohomology and the ordinary cohomology of an equivariantly formal GKM manifold can be expressed in terms of a decorated graph known as the GKM graph. As an abstract graph, the vertices and edges of the GKM graph are in one-to-one correspondence with the zero-dimensional and one-dimensional orbits of the T-action on the GKM manifold. The additional decorations provide enough information to reconstruct the T-equivariant formal neighborhood of the 1-skeleton (the union of zero-dimensional and one-dimensional orbits of the torus action) of the GKM manifold. Toric manifolds are examples of GKM manifolds.

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1.1.2. Equivariant Gromov-Witten theory of GKM manifolds. The quantum cohomology of a projective manifold $\mathcal{X}$ is equal to the rational cohomology as a vector space over $\mathbb{Q}$, equipped with the quantum product which is a family of products parametrized by Novikov variables such that the classical cup product is recovered by setting all the Novikov variables to zero. The quantum product is determined by genus zero Gromov-Witten invariants which are virtual counts of rational curves in $\mathcal{X}$. More generally, genus g Gromov-Witten invariants are virtual counts of genus g stable maps to $\mathcal{X}$. When $\mathcal{X}$ is a GKM manifold which is equivariantly formal over $\mathbb{Q}$, Gromov-Witten invariants (which are rational numbers) can be lifted to equivariant Gromov-Witten invariants which take values in $H^*(\mathcal{B}T;\mathbb{Q}) \cong \mathbb{Q}[u_1,\dots,u_m]$, where $\mathcal{B}T$ is the classifying space of the torus T. Equivariant Gromov-Witten invariants can be computed by virtual localization on moduli of stable maps to $\mathcal{X}$ [5, 26, 27] and depend only on the GKM graph [50]; the formula also makes sense for non-compact GKM manifolds if one works over the fractional field $\mathbb{Q}(u_1,\dots,u_m)$ of $H^*(\mathcal{B}T;\mathbb{Q}) = \mathbb{Q}[u_1,\dots,u_m]$. In particular, equivariant and non-equivariant quantum cohomology of an equivariantly formal projective GKM manifold is an invariant of the GKM graph (and the semi-group of effective curve classes which determine the Novikov ring).

1.1.3. Generalization to orbifolds. The GKM theory and GKM graphs have been generalized to orbifolds. To our knowledge, GKM graphs have been defined for smooth orbifolds having presentation as a global quotient of a smooth manifold by action of a torus [30]. In the present paper we consider the most general possible definition that we can think of in the algebraic setting: we define a smooth GKM stack to be a smooth Deligne-Mumford (DM) stack equipped with an action by an algebraic torus, with finitely many zero-dimensional and one-dimensional orbits. In [69], Givental's quantization formula for all genus full descendant potential of equivariant Gromov-Witten (GW) theory of GKM manifolds is generalized to equivariant orbifold GW theory of smooth GKM stacks; the definition of a smooth GKM stack in this paper is the same as the definition of a GKM orbifold in [69]. One of the main goals of this paper is to provide details of localization computations of orbifold GW invariants in the generality needed in [69] which is beyond the toric stack case in [47].

1.1.4. Examples from geometric engineering and mirror symmetry. In [48], Li-Liu-Liu-Zhou introduced formal toric Calabi-Yau graphs and formal toric Calabi-Yau threefolds, which arise in geometric engineering in 5d and 6d (see e.g. Figure 1, 20-22, 25, 26, 29-32 of [31]) and Bryan's recent work on Donaldson-Thomas invariants of the banana manifold (Figure 2 of [10]). Formal toric Calabi-Yau threefolds are usually not equivariantly formal, but their Gromov-Witten invariants and Donaldson-Thomas invariants can be defined and computed by the topological vertex. In this paper, we introduce abstract stacky GKM graphs and formal smooth GKM stacks, which include (as a very special case) the orbifold version of formal toric Calabi-Yau graphs and formal toric Calabi-Yau threefolds, of which orbifold Donaldson-Thomas invariants can be defined and computed by the orbifold topological vertex [11].

Formal toric Calabi-Yau manifolds of arbitrary dimension arise in Strominger-Yau-Zaslow (SYZ) mirror symmetry for varieties of general type [34]. More precisely, such an example is obtained by taking a formal neighborhood of the 1-skeleton of a non-toric non-compact Kähler manifold equipped with a compact torus action which is not Hamiltonian. These examples naturally extend to orbifolds, which are special cases of formal smooth GKM stacks defined in this paper.

1.2. Summary of results. In this paper, algebraic varieties and algebraic stacks are defined over the field $\mathbb{C}$ of complex numbers. A brief review of smooth DM stacks is given in Section 2.

1.2.1. Smooth GKM stacks and stacky GKM graphs. In Section 3, we define the stacky GKM graph of a smooth GKM stack, under the mild assumption that any one-dimensional orbit contains at least one torus fixed point. As an abstract graph, the vertices and edges are in one-to-one correspondence with the zero-dimensional and one-dimensional orbits of the torus action, respectively.

(1) In the manifold case, a zero-dimensional orbit is a (scheme) point, and a one-dimensional orbit closure is isomorphic to the complex projective line $\mathbb{P}^1$ or a complex affine line $\mathbb{C}$.
(2) In the case of global quotients by torus action (such as smooth toric DM stacks), a zero-dimensional orbit is a stacky point $\mathcal{B}G = [\mathrm{point}/G]$ where G is a finite abelian group, and a one-dimensional orbit closure is a one-dimensional smooth toric DM stack.
(3) The general case studied in the present paper is significantly more complicated and subtle than the special case (2) studied in previous work: in the general case, a zero-dimensional orbit is of the form $\mathcal{B}G = [\mathrm{point}/G]$ where G is a possibly non-abelian finite group, and a proper one-dimensional orbit closure is a spherical DM curve in the sense of Behrend-Noohi [8].

Thanks to Behrend-Noohi's presentation of a general spherical DM curve $\mathfrak{l}$ as a quotient stack of the form $[(\mathbb{C}^2 - \{0\})/E]$, where E is a central extension of the fundamental group $\pi_1(\mathfrak{l})$ of the DM curve $\mathfrak{l}$ by $\mathbb{C}^*$ [8], we may express the total space of the normal bundle $N_{\mathfrak{l}/\mathcal{X}}$ of a proper one-dimensional orbit closure $\mathfrak{l}$ in a general smooth GKM stack $\mathcal{X}$ as a global quotient $[((\mathbb{C}^2 - \{0\}) \times \mathbb{C}^{r-1})/E]$, where r is the dimension of $\mathcal{X}$, though $\mathcal{X}$ itself is not necessary a global quotient. Such a presentation is crucial for the following two steps: (i) to identify the extra decorations that we need to include in the definition of a stacky GKM graph, so that the equivariant formal neighborhood of the 1-skeleton can be reconstructed; (ii) to describe the torus fixed locus in the moduli stack of twisted stable maps to $\mathcal{X}$, which is the first step of localization computations of equivariant orbifold GW invariants of $\mathcal{X}$.

Smooth toric DM stacks are examples of smooth GKM stacks; the stacky GKM graph of a smooth toric DM stack is determined by the stacky fan defining the smooth toric DM stack. See [9] for definitions of stacky fans and smooth toric DM stacks.

1.2.2. Abstract stacky GKM graphs and formal smooth GKM stacks. In Section 4, we axiomize the definition of a stacky GKM graph and introduce abstract stacky GKM graphs which are more general than stacky GKM graphs of honest smooth GKM stacks. From an abstract stacky GKM graph we construct a formal smooth GKM stack. Our definition of abstract stacky GKM graphs is inspired by the definition of abstract 1-skeleton in [30] and also generalizes it in several aspects.

Given a formal smooth GKM stack $\hat{\mathcal{X}}_{\vec{\Upsilon}}$ defined by an abstract GKM graph $\vec{\Upsilon}$, we define the set of effective classes $\mathrm{Eff}(\hat{\mathcal{X}}_{\vec{\Upsilon}})$ and a vector space $\mathcal{H}_{\vec{\Upsilon}}$ over the fractional field $\mathcal{Q}_T = \mathbb{Q}(u_1, \ldots, u_m)$ of $R_T := H^*(\mathcal{B}T; \mathbb{Q}) = \mathbb{Q}[u_1, \ldots, u_m]$. If $\hat{\mathcal{X}}_{\vec{\Upsilon}}$ is the stacky GKM graph of an honest smooth GKM stack $\mathcal{X}$, then there is a surjective map $j_* : \mathrm{Eff}(\hat{\mathcal{X}}_{\vec{\Upsilon}}) \to \mathrm{Eff}(\mathcal{X})$ and a $\mathcal{Q}_T$ linear map $j^* : H^*_T(\hat{\mathcal{X}}_{\vec{\Upsilon}}; \mathcal{Q}_T) \to \mathcal{H}_{\vec{\Upsilon}}$. If the T-action on $\mathcal{X}$ is equivariant formal then j^* is a linear isomorphism.

1.2.3. Equivariant orbifold Gromov-Witten theory. In Section 5, we define equivariant orbifold Gromov-Witten (GW) invariants of smooth GKM stacks and formal equivariant orbifold GW invariants of formal smooth GKM stacks. Let g, n, $a_1, \dots, a_n$ be non-negative integers.

(i) Let $\mathcal{X}$ be a smooth GKM stack and let T be the complex algebraic torus acting on $\mathcal{X}$. Given an effective curve class $\beta \in \mathrm{Eff}(\mathcal{X})$ and T-equivariant cohomology classes $\gamma_1^T, \dots, \gamma_n^T \in H_T^*(\mathcal{X}; \mathcal{Q}_T)$, where $\mathcal{Q}_T = \mathbb{Q}(u_1, \dots, u_m)$ is the fractional field of $H^*(\mathcal{B}T) = \mathbb{Q}[u_1, \dots, u_m]$, we define genus g, degree β, T-equivariant orbifold GW invariants

$$\langle \bar{\epsilon}_{a_1}(\gamma_1^T), \dots, \bar{\epsilon}_{a_n}(\gamma_n^T) \rangle_{g,\beta}^{\mathcal{X}_T} \in \mathcal{Q}_T$$

via localization. When the coarse moduli space of $\mathcal{X}$ is projective (so that non-equivariant orbifold Gromov-Witten invariants of $\mathcal{X}$ are defined) and the torus action on $\mathcal{X}$ is equivariantly formal over $\mathbb{Q}$ (in the sense that the map from T-equivariant orbifold Chen-Ruan cohomology to the non-equivariant orbifold Chen-Ruan cohomology is surjective), they are refinements of (non-equivariant) orbifold GW invariants of $\mathcal{X}$.

(ii) Given a formal smooth GKM stack $\hat{\mathcal{X}}_{\vec{\Upsilon}}$ defined by an abstract GKM graph $\vec{\Upsilon}$, we define the set of effective classes $\mathrm{Eff}(\hat{\mathcal{X}}_{\vec{\Upsilon}})$ and a vector space $\mathcal{H}_{\vec{\Upsilon}}$ over $\mathcal{Q}_T$. Given $\hat{\beta} \in \mathrm{Eff}(\hat{\mathcal{X}}_{\vec{\Upsilon}})$ and $\hat{\gamma}_1, \dots, \hat{\gamma}_n \in \mathcal{H}_{\vec{\Upsilon}}$, we define genus g, degree $\hat{\beta}$, formal T-equivariant orbifold GW invariants

$$\langle \bar{\epsilon}_{a_1}(\hat{\gamma}_1), \dots, \bar{\epsilon}_{a_n}(\hat{\gamma}_n) \rangle_{g,\hat{\beta}}^{\vec{\Gamma}}. \in \mathcal{Q}_T \tag{1.1}$$

via localization.

(iii) If $\vec{\Gamma}$ is the stacky GKM graph of $\mathcal{X}$ in (i) then there is a surjective map $j_* : \mathrm{Eff}(\hat{\mathcal{X}}_{\vec{\Upsilon}}) \longrightarrow \mathrm{Eff}(\mathcal{X})$ and a $\mathcal{Q}_T$-linear map $j^* : H_T^*(\mathcal{X}; \mathcal{Q}_T) \longrightarrow \mathcal{H}_{\vec{\Upsilon}}$ such that

$$\langle \bar{\epsilon}_{a_1}(\gamma_1^T), \dots, \bar{\epsilon}_{a_n}(\gamma_n^T) \rangle_{g,\beta}^{\mathcal{X}_T} = \sum_{\substack{\hat{\beta} \in \mathrm{Eff}(\hat{\mathcal{X}}_{\vec{\Upsilon}}) \\ j_*\hat{\beta} = \beta}} \bar{\epsilon}_{a_1}(j^*\gamma_1^T), \dots, \bar{\epsilon}_n(j^*\gamma_n^T) \rangle_{g,\hat{\beta}}^{\vec{\Gamma}}.$$

Therefore, formal equivariant orbifold GW invariants of $\vec{\Gamma}$ are refinements of equivariant orbifold GW invariants of the smooth GKM stack $\mathcal{X}$.

In Section 6, we derive explicit localization formula of equivariant orbifold GW invariants (1.1). The localization formula is stated as Theorem 6.9. In particular, we obtain localization formula of equivariant orbifold GW invariants of smooth GKM stacks.

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2. Smooth Deligne-Mumford Stacks. Let $\mathcal{X}$ be a smooth Deligne-Mumford (DM) stack, and let $\pi : \mathcal{X} \to X$ be the natural projection to the coarse moduli space X.

2.1. The inertia stack and its rigidification. The inertia stack $\mathcal{IX}$ associated to $\mathcal{X}$ is a smooth DM stack such that the following diagram is Cartesian:

$$\begin{array}{ccc} \mathcal{IX} & \longrightarrow & \mathcal{X} \\ \big\downarrow & & \big\downarrow \Delta \\ \mathcal{X} & \xrightarrow{\Delta} & \mathcal{X} \times \mathcal{X} \end{array}$$

where $\Delta : \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is the diagonal map. An object in the category $\mathcal{IX}$ is a pair (x, g), where x is an object in the category $\mathcal{X}$ and $g \in \mathrm{Aut}_{\mathcal{X}}(x)$:

$$\mathrm{Ob}(\mathcal{IX}) = \{(x, g) \mid x \in \mathrm{Ob}(\mathcal{X}), g \in \mathrm{Aut}_{\mathcal{X}}(x)\}.$$

The morphisms between two objects in the category $\mathcal{IX}$ are:

$$\mathrm{Hom}_{\mathcal{IX}}((x_1, g_1), (x_2, g_2)) = \{h \in \mathrm{Hom}_{\mathcal{X}}(x_1, x_2) \mid h \circ g_1 = g_2 \circ h\}.$$

In particular,

$$\mathrm{Aut}_{\mathcal{IX}}(x, g) = \{h \in \mathrm{Aut}_{\mathcal{X}}(x) \mid h \circ g = g \circ h\}.$$

The rigidified inertia stack $\overline{\mathcal{I}}\mathcal{X}$ satisfies

$$\mathrm{Ob}(\overline{\mathcal{I}}\mathcal{X}) = \mathrm{Ob}(\mathcal{IX}), \quad \mathrm{Aut}_{\overline{\mathcal{I}}\mathcal{X}}(x, g) = \mathrm{Aut}_{\mathcal{IX}}(x, g)/\langle g\rangle,$$

where $\langle g\rangle$ is the subgroup of $\mathrm{Aut}_{\mathcal{IX}}(x, g)$ generated by g.

There is a natural projection $q : \mathcal{IX} \to \mathcal{X}$ which sends (x, g) to x. There is a natural involution $\iota : \mathcal{IX} \to \mathcal{IX}$ which sends (x, g) to (x, g^{-1}). We assume that $\mathcal{X}$ is connected. Let

$$\mathcal{IX} = \bigsqcup_{i \in I} \mathcal{X}_i$$

be disjoint union of connected components. There is a distinguished connected component $\mathcal{X}_0$ whose objects are (x, id_x), where $x \in \mathrm{Ob}(\mathcal{X})$, and $\mathrm{id}_x \in \mathrm{Aut}(x)$ is the identity element; note that $\mathcal{X}_0 \cong \mathcal{X}$. The involution ι restricts to an isomorphism $\iota_i : \mathcal{X}_i \to \mathcal{X}_{\iota(i)}$. In particular, $\iota_0 : \mathcal{X}_0 \to \mathcal{X}_0$ is the identity functor.

Example 2.1 (classifying space). *Let G be a finite group. The stack $\mathcal{B}G =$ [point$/G$] is a category which consists of one object x, and* $\mathrm{Hom}(x, x) = G$. *The objects of its inertia stack $\mathcal{IB}G$ are*

$$\mathrm{Ob}(\mathcal{IB}G) = \{(x, g) \mid g \in G\}.$$

The morphisms between two objects are

$$\mathrm{Hom}((x,g_1),(x,g_2)) = \{g \in G \mid g_2 g = g g_1\} = \{g \in G \mid g_2 = g g_1 g^{-1}\}.$$

Therefore

$$\mathcal{IB}G \cong [G/G]$$

where G *acts on* G *by conjugation. We have*

$$\mathcal{IB}G = \bigsqcup_{c \in \mathrm{Conj}(G)} (\mathcal{B}G)_c$$

where $\mathrm{Conj}(G)$ *is the set of conjugacy classes in* G*, and* $(\mathcal{B}G)_c$ *is the connected component associated to the conjugacy class* $c \in \mathrm{Conj}(G)$*. We have*

$$(\mathcal{B}G)_c = [c/G] \cong [\{h\}/C_G(h)] \cong \mathcal{B}\left(C_G(h)\right).$$

for any element h *in the conjugacy class* c*, where* $C_G(h) = \{a \in G : ah = ha\}$ *is the centralizer of* h *in the group* G*.*

In particular, when G *is abelian, we have* $\mathrm{Conj}(G) = G$*, and*

$$\mathcal{IB}G = \bigsqcup_{h \in G} (\mathcal{B}G)_h$$

where $(\mathcal{B}G)_h = [\{h\}/G] \cong \mathcal{B}G$*.*

2.2. Age. Given a positive integer s, let μ_s denote the group of s-th roots of unity. It is a cyclic subgroup of $\mathbb{C}^*$ of order s, generated by

$$\zeta_s := e^{2\pi\sqrt{-1}/s}.$$

Given any object (x,g) in $\mathcal{IX}$, $g : T_x\mathcal{X} \to T_x\mathcal{X}$ is a linear isomorphism such that $g^s = \mathrm{id}$, where s is the order of g. The eigenvalues of $g : T_x\mathcal{X} \to T_x\mathcal{X}$ are $\zeta_s^{l_1}, \ldots, \zeta_s^{l_r}$, where $l_i \in \{0, 1, \ldots, s-1\}$ and $r = \dim_{\mathbb{C}} \mathcal{X}$. Define

$$\mathrm{age}(x,g) := \frac{l_1 + \cdots + l_r}{s}.$$

Then $\mathrm{age} : \mathcal{IX} \to \mathbb{Q}$ is constant on each connected component $\mathcal{X}_i$ of $\mathcal{IX}$. Define $\mathrm{age}(\mathcal{X}_i) = \mathrm{age}(x,g)$ where (x,g) is any object in the groupoid $\mathcal{X}_i$. Note that

$$\mathrm{age}(\mathcal{X}_i) + \mathrm{age}(\mathcal{X}_{\iota(i)}) = \dim_{\mathbb{C}} \mathcal{X} - \dim_{\mathbb{C}} \mathcal{X}_i.$$

2.3. The Chen-Ruan orbifold cohomology group. In [14], W. Chen and Y. Ruan introduced the orbifold cohomology group of a complex orbifold. See [1, Section 4.4] for a more algebraic version.

As a graded $\mathbb{Q}$ vector space, the Chen-Ruan orbifold cohomology group of $\mathcal{X}$ is defined to be

$$H^*_{\mathrm{CR}}(\mathcal{X};\mathbb{Q}) := \bigoplus_{a \in \mathbb{Q}_{\geq 0}} H^a_{\mathrm{CR}}(\mathcal{X};\mathbb{Q})$$

where

$$H^a_{\mathrm{CR}}(\mathcal{X};\mathbb{Q}) = \bigoplus_{i\in I} H^{a-2\mathrm{age}(\mathcal{X}_i)}(\mathcal{X}_i;\mathbb{Q}).$$

Suppose that $\mathcal{X}$ is proper. Then we have the following proper pushforward to a point:

$$\int_{\mathcal{X}} : H^*(\mathcal{X};\mathbb{Q}) \to H^*(\mathrm{point};\mathbb{Q}) = \mathbb{Q}.$$

The orbifold Poincaré pairing is defined by

$$(\alpha,\beta) := \begin{cases} \int_{\mathcal{X}_i} \alpha \cup \iota_i^*\beta, & j = \iota(i), \\ 0, & j \neq \iota(i), \end{cases}$$

where $\alpha \in H^*(\mathcal{X}_i;\mathbb{Q})$, $\beta \in H^*(\mathcal{X}_j;\mathbb{Q})$.

3. Smooth GKM stacks and stacky GKM graphs. In this section, we describe the geometry of smooth GKM stacks, and define the stacky GKM graph of a smooth GKM stack. In the algebraic setting, smooth GKM stacks are more general than the GKM orbifolds in Guillemin-Zara [29, 30] in the following ways:

(1) Guillemin-Zara consider compact GKM manifolds or orbifolds, whereas we consider smooth GKM stacks which are not necessarily compact.
(2) By orbifolds Guillemin-Zara mean orbifolds having a presentation of the form X/K, K being a torus and X being a manifold on which K acts in a faithful, locally free fashion [30, Section 1.2]. In particular, the inertia group of a point is a finite abelian group, and the generic inertia group is trivial. Our smooth GKM stacks do not have such a presentation in general; the inertia group of a point is a possibly non-abelian finite group, and the generic inertial group is not necessarily trivial.
(3) We do not assume the torus action on the smooth GKM stack is faithful.

On the other hand, Guillemin-Zara work in the C^∞ category and consider C^∞-action by a compact torus $U(1)^m$, while we restrict ourselves to smooth DM stacks and algebraic action by an algebraic torus $(\mathbb{C}^*)^m$ (which restricts to a $U(1)^m$-action).

3.1. Smooth GKM stacks. The following definition of a smooth GKM stack is the same as the definition of a GKM orbifold in [69].

Definition 3.1 (smooth GKM stacks). *Let $\mathcal{X}$ be a smooth DM stack. We say $\mathcal{X}$ is a smooth GKM stack if it is equipped with an action of an algebraic torus $T = (\mathbb{C}^*)^m$ with only finitely many zero-dimensional and one-dimensional orbits.*

The notion of a group action on a stack is discussed in [57].

Let $N = \mathrm{Hom}(\mathbb{C}^*, T) \cong \mathbb{Z}^m$ be the lattice of 1-parameter subgroups of T, and let $M = \mathrm{Hom}(T, \mathbb{C}^*)$ be the character lattice of T. Then $M = \mathrm{Hom}(N, \mathbb{Z})$ is the dual lattice of N. We introduce

$$N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R}, \quad M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}, \quad N_{\mathbb{Q}} := N \otimes_{\mathbb{Z}} \mathbb{Q}, \quad M_{\mathbb{Q}} := M \otimes_{\mathbb{Z}} \mathbb{Q}.$$

Then $M_{\mathbb{Q}}$ can be canonically identified with $H^2_T(\mathrm{point};\mathbb{Q}) = H^2(\mathcal{B}T;\mathbb{Q})$, where $\mathcal{B}T$ is the classifying space of T.

We make the following assumption on $\mathcal{X}$.

ASSUMPTION 3.2.
(1) *The set $\mathcal{X}^T$ of T fixed points in $\mathcal{X}$ is non-empty.*
(2) *The coarse moduli space of a one-dimensional orbit closure is either a complex projective line $\mathbb{P}^1$ or a complex affine line $\mathbb{C}$.*

Note that (1) and (2) hold when $\mathcal{X}$ is proper. Indeed, if $\mathcal{X}$ is proper then the coarse moduli space of any one-dimensional orbit closure is $\mathbb{P}^1$.

EXAMPLE 3.3. *If $\mathcal{X}$ is a smooth toric DM stack defined by a finite fan [9, 22], then $\mathcal{X}$ is a smooth GKM stack. In particular, any proper smooth toric DM stack is a smooth GKM stack.*

EXAMPLE 3.4 (footballs). *Given any positive integers m, n, define a subgroup $G_{m,n}$ of $(\mathbb{C}^*)^2$ by*

$$G_{m,n} = \{(t_1, t_2) \in (\mathbb{C}^*)^2 : t_1^n = t_2^m\}.$$

The football $\mathcal{F}(m, n)$ is defined as the quotient stack

$$\mathcal{F}(m, n) := [(\mathbb{C}^2 - \{0\})/G_{m,n}]$$

where $(t_1, t_2) \in G_{m,n}$ acts on $(z_1, z_2) \in \mathbb{C}^2 - \{0\}$ by $(t_1, t_2) \cdot (z_1, z_2) = (t_1 z_1, t_2 z_2)$. Then $\mathcal{F}(m, n)$ is a proper smooth toric DM stack, so it is a smooth GKM stack. The inertial groups of the two torus fixed points $[1, 0]$ and $[0, 1]$ are μ_m and μ_n, respectively; the inertial group of any other point is trivial.

EXAMPLE 3.5 (weighted projective lines). *Given any positive integers m, n, the weighted projective line $\mathbb{P}(m, n)$ is defined as the quotient stack*

$$\mathbb{P}(m, n) := [(\mathbb{C}^2 - \{0\})/\mathbb{C}^*]$$

where $t \in \mathbb{C}^*$ acts on $(z_1, z_2) \in \mathbb{C}^2 - \{0\}$ by $t \cdot (z_1, z_2) = (t^m z_1, t^n z_2)$. Then $\mathbb{P}(m, n)$ is a proper smooth toric DM stack, so it is a smooth GKM stack. The inertial group of the two torus fixed points $[1, 0]$ and $[0, 1]$ are μ_m and μ_n, respectively; the inertia group of any other point is μ_d, where $d = g.c.d.(m, n)$ is the greatest common divisor of m, n. The rigidification of $\mathbb{P}(m, n)$ is $\mathbb{P}(\frac{m}{d}, \frac{m}{d}) \cong \mathcal{F}(\frac{m}{d}, \frac{n}{d})$.

EXAMPLE 3.6. *An algebraic GKM manifold (in the sense of [50]) is a smooth GKM stack.*

DEFINITION 3.7. *Let $\mathcal{X}$ be a smooth GKM stack. The* 0-skeleton *of $\mathcal{X}$ is defined to be $\mathcal{X}^0 := \mathcal{X}^T$ which is the union of zero-dimensional orbits of the T-action on $\mathcal{X}$. The* 1-skeleton *$\mathcal{X}^1$ of $\mathcal{X}$ is defined to be the union of zero-dimensional and one-dimensional orbits of the T-action on $\mathcal{X}$.*

3.2. A zero dimensional orbit and its normal bundle. Let $\mathcal{X}$ be an r-dimensional smooth GKM stack, so that $T = (\mathbb{C}^*)^m$ acts algebraically on $\mathcal{X}$. A zero-dimensional T orbit in $\mathcal{X}$ is a fixed (possibly stacky) point $\mathfrak{p} = \mathcal{B}G$ under the T-action on $\mathcal{X}$, where G is a finite group. The normal bundle of $\mathfrak{p}$ in $\mathcal{X}$ is the tangent space $T_{\mathfrak{p}}\mathcal{X}$ to $\mathcal{X}$ at $\mathfrak{p}$, which is a rank r vector bundle over $\mathcal{B}G$, or equivalently, a representation $\phi : G \to GL(r, \mathbb{C})$. The T-action on $\mathcal{X}$ induces a T-action on $T_{\mathfrak{p}}\mathcal{X} = [\mathbb{C}^r/G]$, which can be viewed as a T-equivariant vector bundle of rank r over $\mathcal{B}G$. The GKM assumption implies that $T_{\mathfrak{p}}\mathcal{X}$ is a direct sum of T-equivariant line bundles $L_1, \ldots, L_r$ over $\mathcal{B}G$, so that

$$\phi = \bigoplus_{i=1}^{r} \phi_i$$

is the direct sum of r one-dimensional representations $\phi_i : G \to GL(1, \mathbb{C}) = \mathbb{C}^*$. We may choose coordinates on $\mathbb{C}^r$ such that L_i corresponds to the i-th coordinate axis. The G-action on $\mathbb{C}^r$ is given by

$$g \cdot (z_1, \ldots, z_r) = (\phi_1(g)z_1, \ldots, \phi_r(g)z_r)$$

where $g \in G$ and $(z_1, \ldots, z_r) \in \mathbb{C}^r$. Let

$$\mathbf{w}_i := c_1^T(L_i) \in H_T^2(\mathcal{B}G; \mathbb{Q}) \cong H_T^2(\text{point}; \mathbb{Q}) = M_{\mathbb{Q}}.$$

The GKM condition implies that $\mathbf{w}_i$ and $\mathbf{w}_j$ are linearly independent if $i \neq j$. The tangent space $T_{\mathfrak{p}}\mathcal{X} = [\mathbb{C}^r/G]$, together with the T-action, is an affine smooth GKM stack characterized by the finite group G, $\phi_1, \ldots, \phi_r \in \text{Hom}(G, \mathbb{C}^*)$, and the weights $\mathbf{w}_1, \ldots, \mathbf{w}_r \in M_{\mathbb{Q}}$. The image of $\phi_i : G \to \mathbb{C}^*$ is a finite cyclic group μ_{r_i} of order $r_i > 0$. Let G_i be the kernel of ϕ_i. For each i, we have a short exact sequence of finite groups:

$$1 \to G_i \longrightarrow G \xrightarrow{\phi_i} \mu_{r_i} \to 1.$$

We define the stacky GKM graph of $[\mathbb{C}^r/G]$ as follows. The underlying graph has a single vertex σ and r rays $\epsilon_1, \ldots, \epsilon_r$ emanating from σ. The vertex is decorated by the group G; the edge ϵ_i is decorated by the group G_i; the flag (ϵ_i, σ) is decorated by ϕ_i, $\mathbf{w}_i$, and the injective group homomorphism $G_i \hookrightarrow G$.

The coarse moduli

$$\mathbb{C}^r/G = \text{Spec}\left(\mathbb{C}[z_1, \ldots, z_r]^G\right)$$

is an affine T scheme. Let $x_i := z_i^{r_i}$. The i-th axis

$$\ell_i = [\{(z_1, \ldots, z_r) \in \mathbb{C}^r : z_j = 0 \text{ for } j \neq i\}/G] = \text{Spec}\left(\mathbb{C}[z_i]^G\right) = \text{Spec}\mathbb{C}[x_i]$$

is a 1-dimension affine T subscheme of $\mathbb{C}^r/G$. The T-action on $\mathbb{C}^r$ restricts to a T-action on $\ell_i \cong \mathbb{C}$ with weight $r_i\mathbf{w}_i$, so

$$r_i\mathbf{w}_i \in M.$$

3.3. A proper one-dimensional orbit closure. The main reference of this subsection is [8]. We thank Behrend and Noohi for explaining results in their paper [8] to us.

Let $\mathfrak{l} \subset \mathcal{X}$ be a proper one-dimensional T orbit closure in $\mathcal{X}$. Then $\mathfrak{l}$ contains exactly two zero-dimensional T orbits x and y with inertia groups G_x and G_y, respectively. The representation of G_x (resp.G_y) on the tangent line $T_x\mathfrak{l}$ (resp. $T_y\mathfrak{l}$) determines a group homomorphism $\phi_x : G_x \to \mathbb{C}^*$ (resp. $\phi_y : G_y \to \mathbb{C}^*$) with image μ_{r_x} (resp. μ_{r_y}), where r_x and r_y are positive integers. Then $\mathfrak{l}$ is a G-gerbe over its rigidification $\mathfrak{l}^{\text{rig}}$, where $G \cong \text{Ker}(\phi_x) \cong \text{Ker}(\phi_y)$ and $\mathfrak{l}^{\text{rig}}$ is an orbifold DM curve isomorphic to the football $\mathcal{F}(r_x, r_y)$ (cf. Example 3.4); here an orbifold DM curve is a 1-dimensional smooth DM stack with a trivial generic inertia group. Let $\bar{x}$ and $\bar{y}$ be the images of x and y under the morphism $\mathfrak{l} \to \mathfrak{l}^{\mathrm{rig}}$. The inertia groups of $\bar{x}$ and $\bar{y}$ are μ_{r_x} and μ_{r_y}, respectively. The coarse moduli space ℓ of $\mathfrak{l}$ and $\mathfrak{l}^{\mathrm{rig}}$ is isomorphic to the projective line $\mathbb{P}^1$.

The DM curve $\mathfrak{l}$ is spherical in the sense of [8]. In the rest of this subsection, we recall some relevant facts from [8].

(1) The open embeddings

$$\iota_{\bar{x}} : \mathcal{U}_{\bar{x}} := \mathfrak{l}^{\mathrm{rig}} \setminus \{\bar{y}\} \cong [\mathbb{C}/\mu_{r_x}] \hookrightarrow \mathfrak{l}^{\mathrm{rig}}, \quad \iota_{\bar{y}} : \mathcal{U}_{\bar{y}} := \mathfrak{l}^{\mathrm{rig}} \setminus \{\bar{x}\} \cong [\mathbb{C}/\mu_{r_y}] \hookrightarrow \mathfrak{l}^{\mathrm{rig}}$$

induce surjective group homomorphisms

$$\iota_{\bar{x}*} : \pi_1(\mathcal{U}_{\bar{x}}) \cong \mu_{r_x} \to \pi_1(\mathfrak{l}^{\mathrm{rig}}) \cong \mu_a, \quad \iota_{\bar{y}*} : \pi_1(\mathcal{U}_{\bar{y}}) \cong \mu_{r_y} \to \pi_1(\mathfrak{l}^{\mathrm{rig}}) \cong \mu_a,$$

where $a = g.c.d.(r_x, r_y)$.

(2) The open embeddings

$$\iota_x : \mathcal{U}_x := \mathfrak{l} \setminus \{y\} \cong [\mathbb{C}/G_x] \hookrightarrow \mathfrak{l} \quad \iota_y : \mathcal{U}_y := \mathfrak{l} \setminus \{x\} \cong [\mathbb{C}/G_y] \hookrightarrow \mathfrak{l}$$

induce surjective group homomorphisms

$$\iota_{x*} : \pi_1(\mathcal{U}_x) \cong G_x \to \pi_1(\mathfrak{l}), \quad \iota_{y*} : \pi_1(\mathcal{U}_y) \cong G_y \to \pi_1(\mathfrak{l}).$$

(3) ι_{x*} and ι_{y*} restrict to the same group homomorphism $G \to \pi_1(\mathfrak{l})$, whose kernel is a cyclic group μ_d contained in the center $Z(G)$ of G, and whose cokernel is $\pi_1(\mathfrak{l}^{\mathrm{rig}}) \cong \mu_a$. In other words, we have the following exact sequence of finite groups:

$$1 \to \mu_d \to G \to \pi_1(\mathfrak{l}) \to \mu_a \to 1.$$

(4) We have a commutative diagram

$$\begin{array}{ccccccccc}
1 & \longrightarrow & G & \longrightarrow & G_x & \longrightarrow & \mu_{r_x} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \mathrm{id}_G} & & \downarrow{\scriptstyle \iota_{x*}} & & \downarrow{\scriptstyle \iota_{\bar{x}*}} & & \\
 & & G & \longrightarrow & \pi_1(\mathfrak{l}) & \longrightarrow & \mu_a & \longrightarrow & 1 \\
 & & \uparrow{\scriptstyle \mathrm{id}_G} & & \uparrow{\scriptstyle \iota_{y*}} & & \uparrow{\scriptstyle \iota_{\bar{y}*}} & & \\
1 & \longrightarrow & G & \longrightarrow & G_y & \longrightarrow & \mu_{r_y} & \longrightarrow & 1
\end{array}$$

where $\mathrm{id}_G : G \to G$ is the identity map and the rows are exact. The maps $\mu_{r_x} \to \mathrm{Out}(G)$ and $\mu_{r_y} \to \mathrm{Out}(G)$ factor through $\mu_a \to \mathrm{Out}(G)$.

(5) We have $(r_x, r_y) = (ap, aq)$, where $p, q \in \mathbb{Z}_{>0}$ are coprime. The universal cover of $\mathfrak{l}^{\mathrm{rig}}$ is $\mathcal{F}(p,q) = \mathbb{P}(p,q)$; the universal cover of $\mathfrak{l}$ is the weighted projective line $\mathbb{P}(dp, dq)$. (Recall that $\mathbb{P}(m,n)$ is simply connected for any positive integers m, n.)

(6) There exist
- a central extension E of the finite group $\pi_1(\mathfrak{l})$ by $\mathbb{C}^*$, so that we have a short exact sequence of groups

$$1 \to \mathbb{C}^* \xrightarrow{i} E \to \pi_1(\mathfrak{l}) \to 1$$

where $\mathbb{C}^*$ is contained in the center $Z(E)$ of E and is the connected component of the identity of E, and

- a representation $\rho : E \to GL(2,\mathbb{C})$, such that $\rho \circ i(t) = (t^{dp}, t^{dq})$ and

$$\mathfrak{l} \cong [(\mathbb{C}^2 - \{0\})/E]. \tag{3.1}$$

The inclusion $i : \mathbb{C}^* \hookrightarrow E$ induces a surjective morphism

$$\pi : \tilde{\mathfrak{l}} := \mathbb{P}(dp, dq) = [(\mathbb{C}^2 - \{0\})/\mathbb{C}^*] \longrightarrow \mathfrak{l} = [(\mathbb{C}^2 - \{0\})/E]$$

which is the universal covering map. Taking the rigidification yields

$$\pi^{\mathrm{rig}} : \tilde{\mathfrak{l}}^{\mathrm{rig}} = \mathbb{P}(p,q) = \mathcal{F}(p,q) \longrightarrow \mathfrak{l}^{\mathrm{rig}} = \mathcal{F}(r_x, r_y) = \mathcal{F}(ap, aq)$$

which is also the universal covering map.

The GKM condition implies the image of ρ in (6) lies in (up to conjugation) the subgroup $GL(1,\mathbb{C}) \times GL(1,\mathbb{C})$ of diagonal matrices, i.e. $\rho = \rho_x \oplus \rho_y$ is the direct sum of two 1-dimensional representations of E.

Under the isomorphism (3.1) we have the following identifications:

$$x = [1,0], \quad y = [0,1], \quad , G_x = \mathrm{Ker}(\rho_x), \quad G_y = \mathrm{Ker}(\rho_y), \quad G = \mathrm{Ker}(\rho),$$

$$\rho_y(G_x) = \mu_{r_x}, \quad \rho_x(G_y) = \mu_{r_y}.$$

3.4. Normal bundle of a proper one-dimensional orbit closure. Let

$$\mathfrak{l} = [(\mathbb{C}^2 - \{0\})/E]$$

be as in Section 3.3. The Picard group of $\mathfrak{l}$ is given by

$$\mathrm{Pic}(\mathfrak{l}) = \mathrm{Hom}(E, \mathbb{C}^*). \tag{3.2}$$

The normal bundle of $\mathfrak{l}$ in $\mathcal{X}$ is a direct sum of $(r-1)$-line bundles over $\mathfrak{l}$:

$$N_{\mathfrak{l}/\mathcal{X}} = L_1 \oplus \cdots \oplus L_{r-1}.$$

For $i = 1, \ldots, r-1$, let $\rho_i \in \mathrm{Hom}(E, \mathbb{C}^*)$ correspond to $L_i \in \mathrm{Pic}(\mathfrak{l})$ under the isomorphism (3.2). Then the total space of L_i is the quotient stack

$$L_i = [((\mathbb{C}^2 - \{0\}) \times \mathbb{C})/E]$$

where the action of E is given by

$$g \cdot (X, Y, Z) = (\rho_x(g)X, \rho_y(g)Y, \rho_i(g)Z).$$

If $t \in \mathbb{C}^* \subset E$ then

$$t \cdot (X, Y, Z) = (t^{dp}X, t^{dq}Y, t^{d_i}Z)$$

for some $d_i \in \mathbb{Z}$. Recall that for any positive integers m, n,

$$\mathrm{Pic}(\mathbb{P}(m,n)) \cong \mathbb{Z}$$

is generated by

$$\mathcal{O}_{\mathbb{P}(m,n)}(1) = [((\mathbb{C}^2 - \{0\}) \times \mathbb{C})/\mathbb{C}^*]$$

where $t \in \mathbb{C}^*$ acts by $t \cdot (X, Y, Z) = (t^m X, t^n Y, tZ)$. We have

$$\langle \mathcal{O}_{\mathbb{P}(m,n)}(1), [\mathbb{P}(m,n)^{\mathrm{rig}}] \rangle = \frac{1}{l.c.m.(m,n)}$$

where $l.c.m.(m,n)$ is the least common multiple of m, n.

$$\pi^* L_i = \mathcal{O}_{\mathbb{P}(dp,dq)}(d_i)$$

where $\pi : \mathbb{P}(dp, dq) \to \mathfrak{l}$ is the universal cover. Define

$$a_i := \langle c_1(L_i), [\mathfrak{l}^{\mathrm{rig}}] \rangle = \frac{d_i}{adpq}.$$

There is a map $\tilde{\pi}^{\mathrm{rig}} : \mathbb{P}(dp, dq) \to \mathbb{P}(p, q)$ from the universal cover $\mathbb{P}(dp, dq)$ of $\mathfrak{l}$ to the universal cover $\mathbb{P}(p, q)$ of $\mathfrak{l}^{\mathrm{rig}}$; this map can be identified with the map to rigidification, and is of degree $1/d$. We have

$$(\tilde{\pi}^{\mathrm{rig}})^* \mathcal{O}_{\mathbb{P}(p,q)}(1) = \mathcal{O}_{\mathbb{P}(dp,dq)}(d).$$

The map from $\mathfrak{l}$ to $\mathfrak{l}^{\mathrm{rig}}$ is of degree $1/|G|$. The universal covering maps $\mathbb{P}(dp, dq) \to \mathfrak{l}$ and $\mathbb{P}(p, q) \to \mathfrak{l}^{\mathrm{rig}} = \mathcal{F}(ap, aq)$ are of degrees $a|G|/d$ and a, respectively.

We have

- The G_x-actions on $T_x\mathfrak{l}$ and $(L_i)_x$ are given by $\rho_y|_{G_x}$ and $\rho_i|_{G_x}$, respectively;
- The G_y-actions on $T_y\mathfrak{l}$ and $(L_i)_y$ are given by $\rho_x|_{G_y}$ and $\rho_i|_{G_y}$, respectively.

For $i = 1, \ldots, r-1$, let $\mathbf{w}_{x,i}$ and $\mathbf{w}_{y,i}$ be the weights of the T-actions on $(L_i)_x$ and $(L_i)_y$, respectively; let $\mathbf{w}_{x,r}$ and $\mathbf{w}_{y,r}$ be the weights of the T-action on $T_x\mathfrak{l}$ and $T_y\mathfrak{l}$, respectively. Then

$$r_x \mathbf{w}_{x,r} + r_y \mathbf{w}_{y,r} = 0.$$

For $i = 1, \ldots, r$,

$$\mathbf{w}_{y,i} = \mathbf{w}_{x,i} - a_i r_x \mathbf{w}_{x,r} = \mathbf{w}_{x,i} + a_i r_y \mathbf{w}_{y,r}.$$

In particular, $a_r = \frac{1}{r_x} + \frac{1}{r_y}$.

The total space of $N_{\mathfrak{l}/\mathcal{X}}$ is the quotient stack

$$\left[\left((\mathbb{C}^2 - \{0\}) \times \mathbb{C}^{r-1}\right) / E\right]$$

where E acts on $(\mathbb{C}^2 - \{0\}) \times \mathbb{C}^{r-1}$ linearly by $\rho_x \oplus \rho_y \oplus \rho_1 \oplus \cdots \oplus \rho_{r-1}$.

REMARK 3.8. *If $\mathcal{X}$ is a smooth toric DM stack (in the sense of [9]) then $N_{\mathfrak{l}/\mathcal{X}}$ is a smooth toric DM substack, and the above presentation as a quotient stack can be constructed by the stacky fan, where E is abelian.*

We define the GKM graph of $N_{\mathfrak{l}/\mathcal{X}}$ as follows.

- The underlying abstract graph is a tree with two r-valent vertices σ_x, σ_y connected by a compact edge ϵ. There are $r-1$ rays $\epsilon_1, \ldots, \epsilon_{r-1}$ emanating from the vertex σ_x and $r-1$ rays $\epsilon'_1, \ldots, \epsilon'_{r-1}$ emanating from the vertex σ_y.
- The vertices σ_x and σ_y are decorated by finite groups G_x and G_y, respectively.
- The edge ϵ_i is decorated by the kernel G_i of $\phi_{x,i} := \rho_i|_{G_x}$. The edge ϵ'_i is decorated by the kernel G'_i of $\phi_{y,i} := \rho_i|_{G_y}$. The edge ϵ is decorated by the group G.

- The flag (σ_x, ϵ_i) is decorated by $\mathbf{w}_{x,i} \in M_{\mathbb{Q}}$, $\phi_{x,i} \in \mathrm{Hom}(G_x, \mathbb{C}^*)$, and the injection $G_i \hookrightarrow G_x$. The flag (σ_y, ϵ_i') is decorated by $\mathbf{w}_{y,i} \in M_{\mathbb{Q}}$, $\phi_{y,i} \in \mathrm{Hom}(G_y, \mathbb{C}^*)$, and the injection $G_i' \hookrightarrow G_y$. The flag (σ_x, ϵ) is decorated by $\mathbf{w}_x$, $\rho_y|_{G_x}$, and the injection $G \hookrightarrow G_x$. The flat (σ_y, ϵ) is decorated by $\mathbf{w}_y$, $\rho_x|_{G_y}$, and the injection $G \hookrightarrow G_y$.
- The unique compact edge ϵ is decorated by the central extension E of $\pi_1(\mathfrak{l})$ by $\mathbb{C}^*$ and $\rho_x, \rho_y, \rho_1, \ldots, \rho_{r-1} \in \mathrm{Hom}(E, \mathbb{C}^*)$ with isomorphisms $G_x \cong \mathrm{Ker}(\rho_x)$, $G_y \cong \mathrm{Ker}(\rho_y)$, $G \cong \mathrm{Ker}(\rho_x) \cap \mathrm{Ker}(\rho_y)$.

3.5. The stacky GKM graph of a smooth GKM stack. Let $\mathcal{X}$ be a smooth GKM stack of dimension r, so that $T = (\mathbb{C}^*)^m$ acts algebraically on $\mathcal{X}$. Similar to Guillemin-Zara [29, 30], we define the stacky GKM graph of $\mathcal{X}$. This generalizes the GKM graph of an algebraic GKM manifold in [50, Section 2.2] and the toric graph of a smooth toric DM stack in [47, Section 8.6].

Let $V(\Upsilon)$ (resp. $E(\Upsilon)$) denote the set of vertices (resp. edges) in Υ.

(1) (Vertices) We assign a vertex σ to each torus fixed point $\mathfrak{p}_\sigma$ in $\mathcal{X}$. Let p_σ be the corresponding torus fixed point in the coarse moduli space X.

(2) (Edges) We assign an edge ϵ to each one-dimensional orbit $\mathfrak{o}_\epsilon$ in X, and choose a point $\mathfrak{p}_\epsilon$ on $\mathfrak{o}_\epsilon$. Let $\mathfrak{l}_\epsilon$ be the closure of $\mathfrak{o}_\epsilon$, and let ℓ_ϵ be the coarse moduli space of $\mathfrak{l}_\epsilon$. Let $E(\Upsilon)_c := \{\epsilon \in E(\Upsilon) : \ell_\epsilon \cong \mathbb{P}^1\}$ be the set of compact edges in Υ. (Note that $E(\Upsilon)_c = E(\Upsilon)$ if $\mathcal{X}$ is proper.)

(3) (Flags) The set of flags in the graph Υ is given by

$$F(\Upsilon) = \{(\epsilon, \sigma) \in E(\Upsilon) \times V(\Upsilon) : \sigma \in \epsilon\} = \{(\epsilon, \sigma) \in E(\Upsilon) \times V(\Upsilon) : p_\sigma \in \ell_\epsilon\}.$$

(4) (Inertia) For each $\sigma \in V(\Upsilon)$, we assign a finite group G_σ which is the inertia group of $\mathfrak{p}_\sigma$, so that $\mathfrak{p}_\sigma = \mathcal{B}G_\sigma$. For each $\epsilon \in E(\Upsilon)$, we assign a finite group G_ϵ which is the inertial group of $\mathfrak{p}_\epsilon$ in item (2) above.

(5) For every flag $(\epsilon, \sigma) \in F(\Upsilon)$, we choose a path from $\mathfrak{p}_\epsilon$ to $\mathfrak{p}_\sigma$, which determines an injective group homomorphism $j_{(\epsilon,\sigma)} : G_\epsilon \to G_\sigma$. Let $\phi_{(\epsilon,\sigma)} : G_\sigma \to \mathbb{C}^*$ be the group homomorphism which corresponds to the 1-dimensional G_σ representation $T_{\mathfrak{p}_\sigma}\mathfrak{l}_\epsilon$. The image of $\phi_{(\epsilon,\sigma)}$ is a finite cyclic group; let $r_{(\epsilon,\sigma)}$ be the cardinality of this finite cyclic group. We have the following short exact sequence of finite groups:

$$1 \to G_\epsilon \stackrel{j_{(\epsilon,\sigma)}}{\longrightarrow} G_\sigma \stackrel{\phi_{(\epsilon,\sigma)}}{\longrightarrow} \mu_{r_{(\epsilon,\sigma)}} \to 1.$$

So

$$r_{(\epsilon,\sigma)} = \frac{|G_\sigma|}{|G_\epsilon|}.$$

(6) (fundamental groups) For each compact edge $\epsilon \in E(\Upsilon)_c$, there is a group homomorphism $G_\epsilon \to \pi_1(\mathfrak{l}_\epsilon)$ whose kernel is a cyclic subgroup of the center $Z(G_\epsilon)$ of G_ϵ. Let d_ϵ be the cardinality of this cyclic subgroup. Then we have an exact sequence of finite groups:

$$1 \to \mu_{d_\epsilon} \to G_\epsilon \to \pi_1(\mathfrak{l}_\epsilon) \to \pi_1(\mathfrak{l}_\epsilon^{\mathrm{rig}}) \to 1.$$

(7) (central extension of fundamental groups) For each compact edge $\epsilon \in E(\Upsilon)_c$, let $\sigma_x, \sigma_y \in V(\Upsilon)$ be the two ends of ϵ, and let

$$x = p_{\sigma_x}, \quad y = p_{\sigma_y}, \quad r_x = r_{(\epsilon,\sigma_x)}, \quad r_y = r_{(\epsilon,\sigma_y)}, \quad a_\epsilon = g.c.d.(r_x, r_y).$$

Then $\mathfrak{l}_\epsilon^{\mathrm{rig}}$ is the football $\mathcal{F}(r_x, r_y)$, so that $\pi_1(\mathfrak{l}_\epsilon^{\mathrm{rig}}) = \mu_{a_\epsilon}$. There is a triple $(i_\epsilon, E_\epsilon, \rho_\epsilon)$, where $i_\epsilon : \mathbb{C}^* \hookrightarrow E_\epsilon$ is a central injection and $E_\epsilon/\mathbb{C}^* \cong \pi_1(\mathfrak{l}_\epsilon)$, $\rho_\epsilon = (\rho_x, \rho_y) : E_\epsilon \to \mathbb{C}^* \times \mathbb{C}^*$ is a group homomorphism and

$$\rho_\epsilon \circ i_\epsilon = (t^{d_\epsilon r_x/a_\epsilon}, t^{d_\epsilon r_y/a_\epsilon}).$$

We have an isomorphism $[(\mathbb{C}^2 - \{(0,0)\})/E_\epsilon] \cong \mathfrak{l}_\epsilon$, and under this isomorphism

$$x = [1,0], \quad y = [0,1], \quad G_{\sigma_x} = \mathrm{Ker}(\rho_x), \quad G_{\sigma_y} = \mathrm{Ker}(\rho_y), \quad G_\epsilon = \mathrm{Ker}(\rho).$$

(8) (connection) Let $\epsilon \in E(\Upsilon)_c$, and let σ_x and σ_y be as above. Let E_x and E_y be the set of edges emanating from σ_x and σ_y, respectively. The normal bundle $N_{\mathfrak{l}_\epsilon/\mathcal{X}}$ of $\mathfrak{l}_\epsilon$ in $\mathcal{X}$ is a direct sum of line bundles

$$N_{\mathfrak{l}_\epsilon/\mathcal{X}} = L_1 \oplus \cdots \oplus L_{r-1}.$$

For $i = 1, \ldots, r-1$ there exist $\epsilon_i \in E_x$ and $\epsilon'_i \in E_y$ such that $(L_i)_x = T_x \mathfrak{l}_{\epsilon_i}$ and $(L_i)_y = T_y \mathfrak{l}_{\epsilon'_i}$. Then

$$E_x = \{\epsilon_1, \ldots, \epsilon_{r-1}, \epsilon\}, \quad E_y = \{\epsilon'_1, \ldots, \epsilon'_{r-1}, \epsilon\}.$$

Define a bijection $\theta_{(\epsilon,\sigma_x)} : E_x \to E_y$ by sending ϵ_i to ϵ'_i and sending ϵ to ϵ; let $\theta_{(\epsilon,\sigma_y)} : E_y \to E_x$ be the inverse map. We say $\{\epsilon_i, \epsilon'_i\}$ is a pair of edges related by the parallel transport along the compact edge ϵ. There exists $\rho_i \in \mathrm{Hom}(E_\epsilon, \mathbb{C}^*)$ such that $L_i = [((\mathbb{C}^2 - \{(0,0)\}) \times \mathbb{C})/E]$ where E acts by $\rho_x \oplus \rho_y \oplus \rho_i$. Then $\rho_i \circ i_\epsilon : \mathbb{C}^* \to \mathbb{C}^*$ is given by $t \mapsto t^{d_i}$ for some $d_i \in \mathbb{Z}$ and

$$a_i := \langle c_1(L_i), [\mathfrak{l}_\epsilon^{\mathrm{rig}}] \rangle = \frac{d_i}{l.c.m.(r_x, r_y) d_\epsilon} = \frac{d_i a_\epsilon}{r_x r_y d_\epsilon} \in \mathbb{Q}.$$

(Note that, if $\mathfrak{l}_\epsilon$ is the projective line $\mathbb{P}^1$ then $r_x = r_y = a_\epsilon = d_\epsilon = 1$, so $a_i = d_i \in \mathbb{Z}$.) Let Δ_ϵ be the set of pairs of edges related by the parallel transport along the compact edge ϵ. For each pair $\delta \in \Delta_\epsilon$, we get $\rho_\delta \in \mathrm{Hom}(E_\epsilon, \mathbb{C}^*)$ associated to a line bundle over $\mathfrak{l}_\epsilon$ which is a summand of $N_{\mathfrak{l}/\mathcal{X}}$.

(9) (compatibility along compact edges)

$$\rho_i|_{G_{\sigma_x}} = \phi_{(\epsilon_i,\sigma_x)}, \quad \rho_i|_{G_{\sigma_y}} = \phi_{(\epsilon'_i,\sigma_x)}, \quad \rho_y|_{G_{\sigma_x}} = \phi_{(\epsilon,\sigma_x)}, \quad \rho_x|_{G_{\sigma_y}} = \phi_{(\epsilon,\sigma_y)}.$$

Assumption 3.2 can be rephrased in terms of the graph Υ as follows.

Assumption 3.9.
(1) *$V(\Upsilon)$ is non-empty.*
(2) *Each edge in $E(\Upsilon)$ contains at least one vertex.*

Given a vertex $\sigma \in V(\Upsilon)$, we denote by E_σ the set of edges containing σ, i.e. $E_\sigma := \{e \in E : (\epsilon, \sigma) \in F(\Upsilon)\}$. Then $|E_\sigma| = r$ for all $\sigma \in V(\Upsilon)$, so Υ is an r-valent graph.

Given a flag $(\epsilon, \sigma) \in F(\Upsilon)$, let $\mathbf{w}_{(\epsilon,\sigma)} \in M_{\mathbb{Q}}$ be the weight of T-action on $T_{\mathfrak{p}_\sigma} \mathfrak{l}_\epsilon$, the tangent line to $\mathfrak{l}_\epsilon$ at the fixed point $\mathfrak{p}_\sigma = \mathcal{B}G_\sigma$, namely,

$$\mathbf{w}_{(\epsilon,\sigma)} := c_1^T(T_{\mathfrak{p}_\sigma} \mathfrak{l}_\epsilon) \in H_T^2(\mathfrak{p}_\sigma; \mathbb{Q}) \cong M_{\mathbb{Q}}.$$

This gives rise to a map

$$\mathbf{w} : F(\Upsilon) \longrightarrow M_{\mathbb{Q}}, \quad (\epsilon, \sigma) \mapsto \mathbf{w}_{(\epsilon,\sigma)}$$

satisfying the following properties.

(1) (GKM hypothesis) Given any $\sigma \in V(\Upsilon)$, and any two distinct edges $\epsilon, \epsilon' \in E_\sigma$, $\mathbf{w}_{(\epsilon,\sigma)}$ and $\mathbf{w}_{(\epsilon',\sigma)}$ are linearly independent in $M_{\mathbb{Q}}$.
(2) (integrality) For any flag $(\epsilon,\sigma) \in F(\Upsilon)$, $\overline{\mathbf{w}}_{(\epsilon,\sigma)} := r_{(\epsilon,\sigma)}\mathbf{w}_{(\epsilon,\sigma)} \in M$.
(3) Suppose that $\epsilon \in E(\Upsilon)_c$ is a compact edge and $\sigma_x, \sigma_y \in V(\Upsilon)$ are its two ends.
   (a) $r_{(\epsilon,\sigma_x)}\mathbf{w}_{(\epsilon,\sigma_x)} + r_{(\epsilon,\sigma_y)}\mathbf{w}_{(\epsilon,\sigma_y)} = 0$, i.e. $\overline{\mathbf{w}}_{(\epsilon,\sigma_x)} + \overline{\mathbf{w}}_{(\epsilon,\sigma_y)} = 0$.
   (b) Let $E_{\sigma_x} = \{\epsilon_1, \dots, \epsilon_r\}$, where $\epsilon_r = \epsilon$, and let $\epsilon_i' := \theta_{(\sigma_x,\epsilon)}(\epsilon_i) \in E_{\sigma_y}$. Then

$$\mathbf{w}_{(\epsilon_i',\sigma_y)} = \mathbf{w}_{(\epsilon_i,\sigma_x)} - a_i r_{(\epsilon,\sigma_x)}\mathbf{w}_{(\epsilon,\sigma_x)} = \mathbf{w}_{(\epsilon_i,\sigma_x)} + a_i r_{(\epsilon,\sigma_y)}\mathbf{w}_{(\epsilon,\sigma_y)}.$$

The normal bundle of the 1-dimensional smooth DM stack $\mathfrak{l}_\epsilon$ in $\mathcal{X}$ is given by

$$N_{\mathfrak{l}_\epsilon/\mathcal{X}} \cong L_1 \oplus \cdots \oplus L_{r-1}$$

where L_i is a degree a_i T-equivariant line bundle over $\mathfrak{l}_\epsilon$ such that the weights of the T-action on the fibers $(L_i)_z$ and $(L_i)_y$ are $\mathbf{w}_{(\epsilon_i,\sigma_x)}$ and $\mathbf{w}_{(\epsilon_i',\sigma_y)}$, respectively. The map $\mathbf{w} : F(\Upsilon) \to M_{\mathbb{Q}}$ is called the *axial function.*

We call $\vec{\Upsilon}$, which is the abstract graph Υ together with the above decorations and constraints, the *stacky GKM graph* of the smooth GKM stack $\mathcal{X}$ with the T-action.

If $\rho : T' \to T$ is a homomorphism between complex algebraic tori, then T' acts on X by $t' \cdot x = \rho(t') \cdot x$, where $t' \in T'$, $\rho(t') \in T$, $x \in X$. If the zero-dimensional and one-dimensional orbits of this T'-action coincide with those of the T-action, then the GKM graph with this T'-action is obtained by replacing $\mathbf{w}_{(\epsilon,\sigma)} \in M_{\mathbb{Q}}$ by $\rho^*\mathbf{w}_{(\epsilon,\sigma)} \in M'_{\mathbb{Q}}$, where

$$\rho^* : M_{\mathbb{Q}} = H^2(BT;\mathbb{Q}) \to M'_{\mathbb{Q}} := H^2(BT';\mathbb{Q}).$$

3.6. Equivariant Chen-Ruan orbifold cohomology group. Let $\mathcal{X}$ be a smooth GKM stack. The T-action on $\mathcal{X}$ induces a T-action on its inertia stack $\mathcal{IX} = \bigsqcup_{i \in I} \mathcal{X}_i$ and on each $\mathcal{X}_i$.

Let

$$R_T := H_T^*(\text{point};\mathbb{Q}) = H^*(BT;\mathbb{Q}) = \mathbb{Q}[u_1, \dots, u_m]$$

where $\deg(u_i) = 2$; let $Q_T = \mathbb{Q}(u_1, \dots, u_m)$ be its fractional field.

As a graded $\mathbb{Q}$-vector space, T-equivariant Chen-Ruan orbifold cohomology group of an smooth GKM stack is defined to be

$$H^*_{\mathrm{CR},T}(\mathcal{X};\mathbb{Q}) := \bigoplus_{a \in \mathbb{Q}_{\geq 0}} H^a_{\mathrm{CR},T}(\mathcal{X};\mathbb{Q})$$

where

$$H^a_{\mathrm{CR},T}(\mathcal{X};\mathbb{Q}) = \bigoplus_{i \in I} H_T^{a-2\mathrm{age}(\mathcal{X}_i)}(\mathcal{X}_i;\mathbb{Q}).$$

Suppose that $\mathcal{X}$ is proper. For each $i \in I$, we have the following proper pushforward to a point:

$$\int_{\mathcal{X}} : H_T^*(\mathcal{X}_i;\mathbb{Q}) \longrightarrow H_T^*(\text{point};\mathbb{Q}) = R_T$$

which is R_T-linear. The T-equivariant orbifold Poincaré pairing is defined by

$$(\alpha,\beta)_T := \begin{cases} \int_{\mathcal{X}_i} \alpha\cup\iota_i^*\beta, & j=\iota(i),\\ 0, & j\neq \iota(i), \end{cases} \tag{3.3}$$

where $\alpha\in H_T^*(\mathcal{X}_i;\mathbb{Q})$, $\beta\in H_T^*(\mathcal{X}_j;\mathbb{Q})$.

When $\mathcal{X}$ is not proper, we define a T-equivariant Poincaré pairing on

$$H^*_{\mathrm{CR},T}(\mathcal{X};Q_T) = H^*_{\mathrm{CR},T}(\mathcal{X};\mathbb{Q})\otimes_{R_T} Q_T$$

as follows:

$$(\alpha,\beta)_T := \begin{cases} \displaystyle\int_{\mathcal{X}_i^T} \frac{(\alpha\cup\iota_i^*\beta)|_{\mathcal{X}_i}}{e_T(N_{\mathcal{X}_i^T/\mathcal{X}_i})}, & j=\iota(i),\\ 0, & j\neq\iota(i), \end{cases} \tag{3.4}$$

where $\mathcal{X}_i^T\subset\mathcal{X}_i$ is the T fixed substack, and $e_T(N_{\mathcal{X}_i^T/\mathcal{X}_i})$ is the T-equivariant Euler class of the normal bundle $N_{\mathcal{X}_i^T/\mathcal{X}_i}$ of $\mathcal{X}_i^T$ in $\mathcal{X}_i$. Each $\mathcal{X}_i^T$ is a disjoint of union of finitely many (stacky) points.

EXAMPLE 3.10 (affine smooth GKM stack). *Let $\mathcal{X}=[\mathbb{C}^r/G]$ be an affine smooth GKM stack. Let $\phi_i: G\to\mathbb{C}^*$, $\mathbf{w}_i\in M_{\mathbb{Q}}$, and r_i be defined as in Section 3.2. Given $h\in G$, let $c_i(h)$ be the unique element in*

$$\left\{0,\frac{1}{r_i},\dots,\frac{r_i-1}{r_i}\right\}$$

such that

$$e^{2\pi\sqrt{-1}c_i(h)}=\phi_i(h).$$

Then

$$\mathcal{I}\mathcal{X}=\bigsqcup_{c\in\mathrm{Conj}(G)}\mathcal{X}_c,$$

where

$$\mathcal{X}_c\cong[(\mathbb{C}^r)^h/C_G(h)]$$

for any $h\in c$. We have

$$\mathrm{age}(\mathcal{X}_c)=\sum_{i=1}^r c_i(h)$$

where h is any element in the conjugacy class c.

Let $\mathbb{1}_c$ denote the identity element of $H_T^*(\mathcal{X}_c;\mathbb{Q})$. Then

$$H^*(\mathcal{X}_c;\mathbb{Q})=\mathbb{Q}\mathbb{1}_c,\quad H_T^*(\mathcal{X}_c;\mathbb{C})=R_T\mathbb{1}_c.$$

So

$$H^*_{\mathrm{CR}}(\mathcal{X};\mathbb{Q})=\bigoplus_{c\in\mathrm{Conj}(G)}\mathbb{Q}\mathbb{1}_c$$

as a $\mathbb{Q}$ vector space, and

$$H^*_{\mathrm{CR},T}(\mathcal{X};\mathbb{Q}) = \bigoplus_{c\in \mathrm{Conj}(G)} R_T \mathbb{1}_c$$

as an R_T-module.

Given $c \in \mathrm{Conj}(G)$, define

$$\mathbf{e}_c := e_T(T_{[0/G]}(\mathbb{C}^r)^h) = \prod_{i=1}^r \mathbf{w}_i^{\delta_{c_i(h),0}}$$

where h is any element in c. Note that the right hand side of the above equation does not depend on the choice of $h \in c$.

Given $h \in G$, let $[h] = \{aha^{-1} : a \in G\}$ be the conjugacy class of h.

The T-equivariant Poincaré pairing on

$$H^*_{\mathrm{CR},T}(\mathcal{X};Q_T) = \bigoplus_{c\in \mathrm{Conj}(G)} Q_T \mathbb{1}_c$$

is given by

$$(\mathbb{1}_{[h]}, \mathbb{1}_{[h']})_T = \frac{1}{|C_G(h)|}\frac{\delta_{[h^{-1}],[h']}}{\mathbf{e}_{[h]}}.$$

Definition 3.11 (equivariant formality). *Let $\mathcal{X}$ be a smooth GKM stack, so that $T = (\mathbb{C}^*)^m$ acts algebraically on $\mathcal{X}$. We say $\mathcal{X}$ is equivariantly formal if*

$$H^*_{\mathrm{CR},T}(\mathcal{X};\mathbb{Q}) \to H^*_{\mathrm{CR}}(\mathcal{X};\mathbb{Q})$$

is surjective.

Smooth toric DM stacks and affine smooth GKM stacks are equivariantly formal smooth GKM stacks.

3.7. Cup product. In this section, we describe the cup product on

$$H^*_{\mathrm{CR},T}(\mathcal{X};Q_T),$$

first for an affine smooth GKM stack, and then for any equivariantly formal smooth GKM stacks.

Given $c, c' \in \mathrm{Conj}(G)$, define

$$c_i(c,c') := c_i(h) + c_i(h') - c_i(hh') \in \{0,1\}.$$

where $h \in c$ and $h' \in c'$; note that the right hand side of the above equation does not depend on choice of $h \in c$ and $h' \in c'$.

- Let $\mathcal{X} = [\mathbb{C}^r/G]$ be an affine smooth GKM stack as in Example 3.10. The cup product on $H^*_{\mathrm{CR},T}(\mathcal{X};\mathbb{Q})$ is given by

$$\mathbb{1}_c \star \mathbb{1}_{c'} = \prod_{i=1}^r \mathbf{w}_i^{c_i(c,c')} \sum_{h\in c, h'\in c'} \frac{|C_G(hh')|}{|G|}\mathbb{1}_{[hh']}.$$

- Let $\mathcal{X}$ be an equivariantly formal smooth GKM stack, and let $\vec{\Upsilon}$ be the stacky GKM graph of $\mathcal{X}$. Then we have an isomorphism of Q_T-algebras

$$H^*_{\mathrm{CR},T}(\mathcal{X}; Q_T) \cong \bigoplus_{\sigma \in V(\Upsilon)} H^*_{\mathrm{CR},T}(T_{\mathfrak{p}_\sigma}\mathcal{X}; Q_T) \tag{3.5}$$

which preserves the T-equivariant Poincaré pairing; the isomorphism (3.5) is an isomorphism of Frobenius algebras over the field Q_T.

4. Abstract stacky GKM graphs and formal smooth GKM stacks. Let $\mathcal{X}$ be a smooth GKM stack equipped with a T-action. The formal completion $\hat{\mathcal{X}}$ of $\mathcal{X}$ along its 1-skeleton $\mathcal{X}^1$, together with the T-action inherited from $\mathcal{X}$, can be reconstructed from the stacky GKM graph of $\mathcal{X}$. In this section, we will define abstract stacky GKM graphs which are generalization of stacky GKM graphs of smooth GKM stacks. Given an abstract stacky GKM graph, we will construct a formal smooth GKM stack, which is a formal smooth DM stack together with an action by an algebraic torus $T = (\mathbb{C}^*)^m$. The construction of a formal smooth GKM stack from an abstract stacky GKM graph can be viewed as generalization of the reconstruction of $\hat{\mathcal{X}}$ from the stacky GKM graph of a smooth GKM stack $\mathcal{X}$, and is inspired by the construction of a formal toric Calabi-Yau (FTCY) threefold from an FTCY graph in [48, Section 3].

4.1. Abstract stacky GKM graphs. We fix $T = (\mathbb{C}^*)^m$ and a positive integer r. An *abstract stacky GKM graph* is a decorated graph consisting of the following data.

(1) (graph) The underlying graph Γ is a connected r-valent graph Γ with finitely many vertices and edges. Let $V(\Upsilon)$ (resp. $E(\Upsilon)$) denote the set of vertices (resp. edges) in Γ. Each edge in $E(\Upsilon)$ is either a compact edge connecting two vertices or a ray emanating from one vertex. Let $E(\Upsilon)_c \subset E(\Upsilon)$ be the set of compact edges. Let

$$F(\Upsilon) = \{(\epsilon, \sigma) \in E(\Upsilon) \times V(\Upsilon) : \sigma \in \epsilon\}$$

be the set of flags in Γ. Given a vertex $\sigma \in V(\Upsilon)$, let

$$E_\sigma := \{\epsilon \in E(\Upsilon) : (\epsilon, \sigma) \in F(\Upsilon)\}$$

be the set of edges emanating from the vertex v. By the r-valent condition, $|E_\sigma| = r$ for all $r \in V(\Upsilon)$.

(2) (inertia and tangent representations) Each vertex $\sigma \in V(\Upsilon)$ (resp. edge $\epsilon \in E(\Upsilon)$) is decorated by a finite group G_σ (resp. G_ϵ). Each flag $(\epsilon, \sigma) \in F(\Upsilon)$ is decorated by
   - an injective group homomorphism $j_{(\epsilon,\sigma)} : G_\epsilon \hookrightarrow G_\sigma$, and
   - a one-dimensional representation $\phi_{(\epsilon,\sigma)} : G_\sigma \to GL(1, \mathbb{C}) = \mathbb{C}^*$,

   such that $\mathrm{Im}(j_{(\epsilon,\sigma)}) = \mathrm{Ker}(\phi_{(\epsilon,\sigma)})$.
   Note that the image of $\phi_{(\epsilon,\sigma)}$ is a finite cyclic group; let $r_{(\epsilon,\sigma)}$ be the cardinality of this finite cyclic group. Then we have a short exact sequence of finite groups:

$$1 \to G_\epsilon \stackrel{j_{(\epsilon,\sigma)}}{\longrightarrow} G_\sigma \stackrel{\phi_{(\epsilon,\sigma)}}{\longrightarrow} \mu_{r_{(\epsilon,\sigma)}} \to 1.$$

(3) (fundamental groups and central extensions) Let $\epsilon \in E(\Upsilon)_c$ be a compact edge, and let $\sigma_x, \sigma_y \in V(\Upsilon)$ be the two ends of ϵ. Let $a_\epsilon = g.c.d.(r_{(\epsilon,\sigma_x)}, r_{(\epsilon,\sigma_y)})$. In addition to G_ϵ, ϵ is decorated by:

- Another finite group π_ϵ together with a group homomorphism $G_\epsilon \to \pi_\epsilon$ such that we have the following exact sequence of finite groups:

$$1 \to \mu_{d_\epsilon} \to G_\epsilon \to \pi_\epsilon \to \mu_{a_\epsilon} \to 1$$

where μ_{d_ϵ} is contained in the center of G_ϵ.

- A central extension $1 \to \mathbb{C}^* \xrightarrow{i_\epsilon} E_\epsilon \to \pi_\epsilon \to 1$ of π_ϵ by $\mathbb{C}^*$, and a group homomorphism $\rho_\epsilon = (\rho_x, \rho_y) : E_\epsilon \to \mathbb{C}^* \times \mathbb{C}^*$.

(4) (connection) Let $\epsilon \in E(\Upsilon)_c$ be a compact edge, and let $\sigma_x, \sigma_y \in V(\Upsilon)$ be the two ends of ϵ. There are bijections $\theta_{(\epsilon,\sigma_x)} : E_{\sigma_x} \to E_{\sigma_y}$ and $\theta_{(\epsilon,\sigma_y)} : E_{\sigma_y} \to E_{\sigma_x}$ which are inverses of each other and send ϵ to ϵ.

(5) (normal representations) Suppose that $\epsilon \in E(\Upsilon)_c$ is a compact edge, $\sigma_x, \sigma_y \in V(\Upsilon)$ are two ends of ϵ, and $\delta = \{\epsilon_x, \epsilon_y\}$ is a pair of edges such that $\epsilon_x \in E_{\sigma_x} - \{\epsilon\}$ and $\epsilon_y = \theta_{(\epsilon,\sigma_x)}(\epsilon_x) \in E_{\sigma_y} - \{\epsilon\}$. Such a pair is decorated by a one-dimensional representation $\rho_\delta : E_\epsilon \to GL(1,\mathbb{C}) = \mathbb{C}^*$.

(6) (compatibility along compact edges) In the notation of (3), (4), (5) above, $\mathrm{Ker}(\rho_x) = G_{\sigma_x}$, $\mathrm{Ker}(\rho_y) = G_{\sigma_y}$, $\mathrm{Ker}(\rho_\epsilon) = G_\epsilon$,

$$\rho_y|_{G_{\sigma_x}} = \phi_{(\epsilon,\sigma_x)}, \quad \rho_x|_{G_{\sigma_y}} = \phi_{(\epsilon,\sigma_y)}, \quad \rho_\delta|_{G_{\sigma_x}} = \phi_{(\epsilon_x,\sigma_x)}, \quad \rho_\delta|_{G_{\sigma_y}} = \phi_{(\epsilon_y,\sigma_y)}.$$

(7) (axial function) There is a map

$$\mathbf{w} : F(\Upsilon) \to M_{\mathbb{Q}}, \quad (\epsilon, \sigma) \mapsto \mathbf{w}_{(\epsilon,\sigma)}$$

satisfying the following properties.

(a) (GKM hypothesis) Given any $\sigma \in V(\Upsilon)$ and any two distinct edges $\epsilon, \epsilon' \in E_\sigma$, $\mathbf{w}_{(\epsilon,\sigma)}$ and $\mathbf{w}_{(\epsilon',\sigma)}$ are linearly independent vectors in $M_{\mathbb{Q}}$.

(b) (integrality) For any $(\epsilon, \sigma) \in F(\Upsilon)$, $\overline{\mathbf{w}}_{(\epsilon,\sigma)} := r_{(\epsilon,\sigma)}\mathbf{w}_{(\epsilon,\sigma)} \in M$.

(c) For any compact edge $\epsilon \in E(\Upsilon)_c$, let $\sigma_x, \sigma_y \in V(\Upsilon)$ be its two ends. Then the following properties hold.

(i) $r_{(\epsilon,\sigma_x)}\mathbf{w}_{(\epsilon,\sigma_x)} + r_{(\epsilon,\sigma_y)}\mathbf{w}_{(\epsilon,\sigma_y)} = 0$, i.e., $\overline{\mathbf{w}}_{(\epsilon,\sigma_x)} + \overline{\mathbf{w}}_{(\epsilon,\sigma_y)} = 0$.

(ii) Suppose that $E_\sigma = \{\epsilon_1, \dots, \epsilon_r\}$, where $\epsilon_r = \epsilon$. Let $\epsilon_i' := \theta_{(\epsilon,\sigma)}(\epsilon_i) \in E_{\sigma'}$, so that $E_{\sigma'} = \{\epsilon_1', \dots, \epsilon_r'\}$. Let

$$a_i = \frac{d_i a_\epsilon}{r_{(\epsilon,\sigma_x)} r_{(\epsilon,\sigma_y)} d_\epsilon}$$

where $d_i \in \mathbb{Z}$ is determined by by $\rho_{\{\epsilon_i,\epsilon_i'\}} \circ i_\epsilon(t) = t^{d_i}$ for $t \in \mathbb{C}^*$. Then

$$\mathbf{w}_{(\epsilon_i',\sigma_y)} = \mathbf{w}_{(\epsilon_i,\sigma_x)} - a_i r_{(\epsilon,\sigma_x)}\mathbf{w}_{(\epsilon,\sigma_x)} = \mathbf{w}_{(\epsilon_i,\sigma_i)} + a_i r_{(\epsilon,\sigma_y)}\mathbf{w}_{(\epsilon,\sigma_y)},$$

or equivalently,

$$\mathbf{w}_{(\epsilon_i',\sigma_y)} = \mathbf{w}_{(\epsilon_i,\sigma_x)} - a_i\overline{\mathbf{w}}_{(\epsilon,\sigma_x)} = \mathbf{w}_{(\epsilon_i,\sigma_i)} + a_i\overline{\mathbf{w}}_{(\epsilon,\sigma_y)},$$

In particular, $\epsilon_r' = \epsilon_r = \epsilon$ and $a_r = \frac{1}{r_{(\epsilon,\sigma_x)}} + \frac{1}{r_{(\epsilon,\sigma_y)}}$.

Let $\vec{\Upsilon}$ denote the underlying abstract graph Υ together with all the above decorations.

REMARK 4.1. *We may also define abstract GKM graphs by the following specialization.*

- *All the finite groups* G_σ, G_ϵ, π_ϵ *are trivial, and we always have* $E_\epsilon = \mathbb{C}^*$ *and* $\rho_x, \rho_y : \mathbb{C}^* \to \mathbb{C}^*$ *are identity maps. So we do not need (2), (3), (6) above.*
- *In (7), the axial function* $\mathbf{w}$ *takes value in* M *instead of* $M_{\mathbb{Q}}$, *and*

$$r_{(\epsilon,\sigma)} = r_{(\epsilon,\sigma')} = 1, \quad a_\epsilon = d_\epsilon = 1, \quad a_i = d_i \in \mathbb{Z}.$$

- *The normal characters in (5) are determined by the axial function.*

Abstract GKM graphs are generalization of GKM graphs of algebraic GKM manifolds [50, Section 2.2].

4.2. Formal smooth GKM stacks. Given an abstract stacky GKM graph $\vec{\Upsilon}$ defined as in the previous subsection, we will construct a formal smooth DM stack $\hat{\mathcal{X}}_{\vec{\Upsilon}}$ of dimension r equipped with an action of $T = (\mathbb{C}^*)^m$.

4.2.1. The stacky affine line associated to a flag. For any flag $(\epsilon, \sigma) \in F(\Upsilon)$, define a "stacky" affine line

$$D_{(\epsilon,\sigma)} := [\mathrm{Spec}\mathbb{C}[z_{(\epsilon,\sigma)}]/G_\sigma] \cong [\mathbb{A}^1/G_\sigma]$$

where G_σ acts on $\mathbb{A}^1$ via the group homomorphism $\phi_{(\epsilon,\sigma)} : G_\sigma \to \mathbb{C}^*$. The coarse moduli space of $D_{(\epsilon,\sigma)}$ is

$$\mathrm{Spec}(\mathbb{C}[z_{(\epsilon,\sigma)}]^{G_\sigma}) = \mathrm{Spec}(\mathbb{C}[z_{(\epsilon,\sigma)}^{r_{(\epsilon,\sigma)}}]) = \mathrm{Spec}(\mathbb{C}[x_{(\epsilon,\sigma)}]) \cong \mathbb{A}^1$$

where $x_{(\epsilon,\sigma)} = (z_{(\epsilon,\sigma)})^{r_{(\epsilon,\sigma)}}$.

4.2.2. The formal smooth DM stack associated to a vertex. For any vertex $\sigma \in V(\Upsilon)$, define an r-dimensional affine smooth GKM stack

$$\mathcal{X}_\sigma = [\mathrm{Spec}\mathbb{C}[z_{(\epsilon,\sigma)} : \epsilon \in E_\sigma]/G_\sigma] = [\mathbb{A}^{E_v}/G_\sigma].$$

The T-action on $z_{(\epsilon,\sigma)}$ is determined by $\mathbf{w}_{(\epsilon,\sigma)} \in M_{\mathbb{Q}}$. Let $\hat{\mathcal{X}}_\sigma$ be the formal completion of $\mathcal{X}_\sigma$ along its 1-skeleton.

4.2.3. The formal smooth DM stack associated to a compact edge. For any compact edge $\epsilon \in E(\Upsilon)_c$, define

$$\mathfrak{l}_\epsilon := [(\mathbb{C}^2 - \{0\})/E_\epsilon]$$

where the action of E_ϵ is given by the group homomorphism $\rho_\epsilon : E_\epsilon \to \mathbb{C}^* \times \mathbb{C}^*$. Let $\sigma_x, \sigma_y \in V(\Upsilon)$ be its two ends. Suppose that $E_{\sigma_x} = \{\epsilon_1, \ldots, \epsilon_{r-1}, \epsilon\}$, and let $\epsilon_i' = \theta_{(\epsilon,\sigma_x)}(\epsilon_i) \in E_{\sigma_y} - \{\epsilon\}$. Let $\rho_i = \rho_{\{\epsilon_i,\epsilon_i'\}} \in \mathrm{Hom}(E_\epsilon, \mathbb{C}^*)$. Let L_i be the line bundle over the smooth DM curve $\mathfrak{l}_\epsilon$ defined by

$$L_i = [\left((\mathbb{C}^2 - \{0\}) \times \mathbb{C}\right)/E_\epsilon]$$

where the action on the last factor $\mathbb{C}$ is given by the group homomorphism $\rho_i : E_\epsilon \to \mathbb{C}^*$. Let $\mathcal{X}_\epsilon$ be the total space of $L_1 \oplus \cdots \oplus L_{r-1}$, which is a smooth GKM stack. Let $\hat{\mathcal{X}}_\epsilon$ be the formal completion of $\mathcal{X}_\epsilon$ along its 1-skeleton $\mathcal{X}_\epsilon^1$. By compatibility along compact edges, there are T-equivariant open embeddings of formal smooth DM stacks:

$$i_{(\epsilon,\sigma_x)} : \hat{\mathcal{X}}_{\sigma_x} \hookrightarrow \hat{\mathcal{X}}_\epsilon, \quad i_{(\epsilon,\sigma_y)} : \hat{\mathcal{X}}_{\sigma_y} \hookrightarrow \hat{\mathcal{X}}_\epsilon.$$

4.2.4. Gluing. We will construct the r-dimensional formal smooth DM T-stack $\hat{\mathcal{X}}_{\vec{\Upsilon}}$ by induction on the number $|E(\Upsilon)_c|$ of compact edges in the underlying graph Υ. (Note that if $r = 0$ then $|E(\Upsilon)_c| = 0$; if $r = 1$ then $|E(\Upsilon)_c| \leq 1$.)

If $|E(\Upsilon)_c| = 0$ then $V(\Upsilon)$ consists of a single vertex σ. In this case, we define $\hat{\mathcal{X}}_{\vec{\Upsilon}} := \hat{\mathcal{X}}_\sigma$.

If $|E(\Upsilon)_c| = 1$, let ϵ be the unique element in $E_c(\Upsilon)$. In this case, we define $\hat{\mathcal{X}}_{\vec{\Upsilon}} := \hat{\mathcal{X}}_\epsilon$.

Let $s \geq 1$ be a positive integer. Suppose that for any abstract stacky GKM graph $\vec{\Upsilon}$ with $|E(\Upsilon)_c| \leq s$, we have constructed an r-dimensional formal smooth DM T-stack $\hat{\mathcal{X}}_{\vec{\Upsilon}}$ with the following properties.

(i) For any vertex $\sigma \in V(\Upsilon)$, there is a T-equivariant open embedding $i_\sigma^{\vec{\Upsilon}} : \hat{\mathcal{X}}_\sigma \hookrightarrow \hat{\mathcal{X}}_{\vec{\Gamma}}$.

(ii) For any compact edge $\epsilon \in E(\Upsilon)_c$, there is a T-equivariant open embedding $i_\epsilon^{\vec{\Upsilon}} : \hat{\mathcal{X}}_\epsilon \hookrightarrow \hat{\mathcal{X}}_{\vec{\Gamma}}$.

(iii) If $\epsilon \in E(\Upsilon)_c$ is a compact edge and $(\epsilon, \sigma) \in F(\Upsilon)$ is a flag, then $i_\sigma^{\vec{\Upsilon}} = i_\epsilon^{\vec{\Upsilon}} \circ i_{(\epsilon,\sigma)}$, where $i_{(\epsilon,\sigma)} : \hat{\mathcal{X}}_\sigma \hookrightarrow \hat{\mathcal{X}}_\epsilon$ is as in Section 4.2.3.

Let $\vec{\Upsilon}$ be an abstract stacky GKM graph such that $E(\Upsilon)_c = s + 1$. We cut the $\vec{\Upsilon}$ along a compact edge $\epsilon_0 \in E(\Upsilon)_c$. More precisely, we replace ϵ_0 by two rays ϵ_1, ϵ_2 emanating from the two ends $\sigma_1, \sigma_2 \in V(\Upsilon)$ of ϵ_0. For $i = 1, 2$, we define (cf. item (2) in Section 4.1)

$$G_{\epsilon_i} = G_{\epsilon_0}, \quad j_{(\epsilon_i,\sigma_i)} = j_{(\epsilon_0,\sigma_i)}, \quad \phi_{(\epsilon_i,\sigma_i)} = \phi_{(\epsilon_0,\sigma_i)}.$$

Recall that the underlying graph of an abstract stack GKM graph is connected, so there are two cases.

Case 1. We obtain a union of two abstract stacky GKM graphs $\vec{\Upsilon}_1$ and $\vec{\Upsilon}_2$, where $\sigma_i \in V(\Upsilon_i)$ and $\epsilon_i \in E(\Upsilon_i)$. Let $s_i := |E(\Upsilon_i)_c|$. Then $s_1 + s_2 = s$. In particular, $s_i \leq s$, so $\hat{\mathcal{X}}_{\vec{\Upsilon}_1}$ and $\hat{\mathcal{X}}_{\vec{\Upsilon}_2}$ have been constructed by the induction hypothesis. Let $\hat{\mathcal{X}}_0$ be the disjoint union of $\hat{\mathcal{X}}_{\sigma_1}$ and $\hat{\mathcal{X}}_{\sigma_2}$; let $\hat{\mathcal{X}}_{\vec{\Upsilon}_1 \sqcup \vec{\Upsilon}_2}$ be the disjoint union of $\hat{\mathcal{X}}_{\vec{\Upsilon}_1}$ and $\hat{\mathcal{X}}_{\vec{\Upsilon}_2}$.

- There is a T-equivariant open embedding $f : \hat{\mathcal{X}}_0 \longrightarrow \hat{\mathcal{X}}_{\vec{\Upsilon}_1 \sqcup \vec{\Upsilon}2}$ such that such that
$$f|_{\hat{\mathcal{X}}_{\sigma_i}} = i_{\sigma_i}^{\vec{\Upsilon}_i} : \hat{\mathcal{X}}_{\sigma_i} \hookrightarrow \hat{\mathcal{X}}_{\vec{\Upsilon}_i}, \quad i = 1, 2.$$
- Let $g : \hat{\mathcal{X}}_0 \longrightarrow \hat{\mathcal{X}}_{\epsilon_0}$ be the unique T-equivariant morphism such that
$$g|_{\hat{\mathcal{X}}_{\sigma_i}} = i_{(\epsilon_0,\sigma_i)} : \hat{\mathcal{X}}_{\sigma_i} \hookrightarrow \hat{\mathcal{X}}_{\epsilon_0}, \quad i = 1, 2.$$

We define $\hat{\mathcal{X}}_{\vec{\Gamma}}$ to be the fiber coproduct of f and g:

$$\begin{array}{ccc} \hat{\mathcal{X}}_0 & \xrightarrow{\quad g \quad} & \hat{\mathcal{X}}_{\epsilon_0} \\ \Big\downarrow f & & \Big\downarrow \\ \hat{\mathcal{X}}_{\vec{\Upsilon}_1 \sqcup \vec{\Upsilon}_2} & \longrightarrow & \hat{\mathcal{X}}_{\vec{\Upsilon}} = \hat{\mathcal{X}}_{\vec{\Upsilon}_1 \sqcup \vec{\Upsilon}_2} \coprod_{\hat{\mathcal{X}}_0} \hat{\mathcal{X}}_{\epsilon_0} \end{array}$$

Case 2. We obtain an abstract stacky GKM graph $\vec{\Upsilon}'$ with

$$V(\Upsilon') = V(\Upsilon), \quad E(\Upsilon') = (E(\Upsilon) \setminus \{\epsilon_0\}) \cup \{\epsilon_1, \epsilon_2\}, \quad E(\Upsilon')_c = E(\Upsilon)_c \setminus \{\epsilon_0\}.$$

In particular $|E(\Upsilon')_c| = s$, so $\hat{\mathcal{X}}_{\vec{\Upsilon}'}$ has been constructed by the induction hypothesis. Let $g : \hat{\mathcal{X}}_0 \longrightarrow \hat{\mathcal{X}}_{\epsilon_0}$ be defined as in Case 1, and let $f : \hat{\mathcal{X}}_0 \longrightarrow \hat{\mathcal{X}}_{\vec{\Upsilon}'}$ be the unique T-equivariant morphism such that

$$f|_{\hat{\mathcal{X}}_{\sigma_i}} = i^{\vec{\Upsilon}'}_{\sigma_i} : \hat{\mathcal{X}}_{\sigma_i} \hookrightarrow \hat{\mathcal{X}}_{\vec{\Upsilon}'}, \quad i = 1, 2.$$

We define $\hat{\mathcal{X}}_{\vec{\Gamma}}$ to be the fiber coproduct of f and g:

$$\begin{array}{ccc} \hat{\mathcal{X}}_0 & \xrightarrow{\quad g \quad} & \hat{\mathcal{X}}_{\epsilon_0} \\ \Big\downarrow f & & \Big\downarrow \\ \hat{\mathcal{X}}_{\vec{\Upsilon}'} & \longrightarrow & \hat{\mathcal{X}}_{\vec{\Upsilon}} = \hat{\mathcal{X}}_{\vec{\Upsilon}'} \coprod_{\hat{\mathcal{X}}_0} \hat{\mathcal{X}}_{\epsilon_0} \end{array}$$

In both cases, $\hat{\mathcal{X}}_{\vec{\Upsilon}}$ is an r-dimensional formal smooth DM T-stack. Different choices of $\epsilon_0 \in E(\Upsilon)_c$ produce isomorphic formal smooth DM T-stacks. Properties (i), (ii), and (iii) hold by construction and the induction hypothesis.

If $\vec{\Upsilon}$ is the stacky GKM graph of an smooth GKM stack $\mathcal{X}$ then $\hat{\mathcal{X}}_{\vec{\Upsilon}}$ is the formal completion of $\mathcal{X}$ along its 1-skeleton $\mathcal{X}^1$.

4.3. Equivariant Chen-Ruan orbifold cohomology of an abstract stacky GKM graph. Given a stacky GKM graph $\vec{\Upsilon}$,we define

$$\mathcal{H}_{\vec{\Upsilon}} := \bigoplus_{\sigma \in V(\Upsilon)} H^*_{\mathrm{CR},T}(\mathcal{X}_\sigma; Q_T)$$

as a Frobenius algebra over the field Q_T. By Section 3.7, if $\vec{\Upsilon}$ is the stacky GKM graph of an equivariantly formal smooth GKM stack $\mathcal{X}$ then

$$\mathcal{H}_{\vec{\Upsilon}} = H^*_{\mathrm{CR},T}(\mathcal{X}; Q_T).$$

5. Orbifold Gromov-Witten theory. In [15], Chen-Ruan developed Gromov-Witten theory for symplectic orbifolds. The algebraic counterpart, Gromov-Witten theory for smooth DM stacks, was developed by Abramovich-Graber-Vistoli [1, 2]. In this section, we give a brief review of algebraic orbifold Gromov-Witten theory, following [2].

5.1. Twisted curves and their moduli. An n-pointed, genus g twisted curve is a connected proper one-dimensional DM stack $\mathcal{C}$ together with n disjoint closed substacks $\mathfrak{x}_1, \ldots, \mathfrak{x}_n$ of $\mathcal{C}$, such that

(1) $\mathcal{C}$ is étale locally a nodal curve;
(2) formally locally near a node, $\mathcal{C}$ is isomorphic to the quotient stack
$$[\mathrm{Spec}(\mathbb{C}[x, y]/(xy))/\mu_r],$$
where the action of $\zeta \in \mu_r$ is given by $\zeta \cdot (x, y) = (\zeta x, \zeta^{-1} y)$;
(3) each $\mathfrak{x}_i \subset \mathcal{C}$ is contained in the smooth locus of $\mathcal{C}$;
(4) each stack $\mathfrak{x}_i$ is an étale gerbe over $\mathrm{Spec}\mathbb{C}$ *with a section* (hence trivialization);
(5) $\mathcal{C}$ is a scheme outside the twisted points $\mathfrak{x}_1, \ldots, \mathfrak{x}_n$ and the singular locus;
(6) the coarse moduli space C is a nodal curve of arithmetic genus g.

Let $\pi : \mathcal{C} \to C$ be the projection to the coarse moduli space, and let $x_i = \pi(\mathfrak{x}_i)$. Then $x_1, \ldots, x_n$ are distinct smooth points of C, and $(C, x_1, \ldots, x_n)$ is an n-pointed, genus g prestable curve.

Let $\mathcal{M}^{\mathrm{tw}}_{g,n}$ be the moduli of n-pointed, genus g twisted curves. Then $\mathcal{M}^{\mathrm{tw}}_{g,n}$ is a smooth algebraic stack, locally of finite type [55].

5.2. Riemann-Roch theorem for twisted curves. Let $(\mathcal{C}, \mathfrak{x}_1, \ldots, \mathfrak{x}_n)$ be an n-pointed, genus g twisted curve, and let $(C, x_1, \ldots, x_n)$ be the coarse curve, which is an n-pointed, genus g prestable curve. Let $\mathcal{E} \to \mathcal{C}$ be a vector bundle over $\mathcal{C}$. Then $\mathfrak{x}_i \cong \mathcal{B}\mu_{r_i}$. There is a unique generator ζ of the cyclic group μ_{r_i} such that ζ acts on the tangent line $T_{\mathfrak{x}_i}\mathcal{C}$ with eigenvalue $e^{\frac{2\pi\sqrt{-1}}{r_i}}$. Then ζ acts on $\mathcal{E}|_{\mathfrak{x}_i}$ with eigenvalues $e^{\frac{2\pi\sqrt{-1}}{r_i}l_1}, \ldots, e^{\frac{2\pi\sqrt{-1}}{r_i}l_N}$, where $N = \mathrm{rank}\mathcal{E}$ and $l_i \in \{0, 1, \ldots, r_i - 1\}$. In other words,

$$\mathcal{E}|_{\mathfrak{x}_i} = \bigoplus_{i=1}^{N} (T_{\mathfrak{x}_i}\mathcal{C})^{\otimes l_i}$$

as vector bundles over $\mathfrak{x}_i = \mathcal{B}\mu_{r_i}$. Note that $l_1, \ldots, l_N$ are unique up to permutation, so

$$\mathrm{age}_{x_i}(\mathcal{E}) := \frac{l_1 + \cdots + l_N}{r_i} \in \mathbb{Q}$$

is well-defined. The Riemann-Roch theorem for twisted curves says

$$\chi(\mathcal{E}) = \int_{\mathcal{C}} c_1(\mathcal{E}) + \mathrm{rank}(\mathcal{E})(1-g) - \sum_{i=1}^{n} \mathrm{age}_{x_i}(\mathcal{E}). \tag{5.1}$$

5.3. Moduli of twisted stable maps. Let $\mathcal{X}$ be a smooth DM stack with a quasi-projective coarse moduli space X, and let $\beta \in H_2(X; \mathbb{Z})$ be an effective curve class in X. An n-pointed, genus g, degree β twisted stable map to $\mathcal{X}$ is a representable morphism $f : \mathcal{C} \to \mathcal{X}$, where the domain $\mathcal{C}$ is an n-pointed, genus g twisted curve, and the induced morphism $C \to X$ between the coarse moduli spaces is an n-pointed, genus g, degree β stable map to X.

Let $\overline{\mathcal{M}}_{g,n}(\mathcal{X}, \beta)$ be the moduli stack of n-pointed, genus g, degree β twisted stable maps to $\mathcal{X}$. Then $\overline{\mathcal{M}}_{g,n}(\mathcal{X}, \beta)$ is a DM stack; it is proper if X is projective.

For $j = 1, \ldots, n$, there are evaluation maps

$$\mathrm{ev}_j : \overline{\mathcal{M}}_{g,n}(\mathcal{X}, \beta) \to \mathcal{I}\mathcal{X} = \bigsqcup_{i \in I} \mathcal{X}_i$$

where $\{\mathcal{X}_i : i \in I\}$ are connected components of $\mathcal{I}\mathcal{X}$. Given $\vec{i} = (i_1, \ldots, i_n)$, where $i_j \in I$, define

$$\overline{\mathcal{M}}_{g,\vec{i}}(\mathcal{X}, \beta) := \bigcap_{j=1}^{n} \mathrm{ev}_j^{-1}(\mathcal{X}_{i_j}).$$

Then $\overline{\mathcal{M}}_{g,\vec{i}}(\mathcal{X}, \beta)$ is a union of connected components of $\overline{\mathcal{M}}_{g,n}(\mathcal{X}, \beta)$, and

$$\overline{\mathcal{M}}_{g,n}(\mathcal{X}, \beta) = \bigsqcup_{\vec{i} \in I^n} \overline{\mathcal{M}}_{g,\vec{i}}(\mathcal{X}, \beta).$$

Remark 5.1. *In the definition of twisted curves in Section 5.1, if we replace (4) by*

(4)' each stack $\mathfrak{x}_i$ is an étale gerbe over $\mathrm{Spec}\mathbb{C}$*;*

i.e. without a section, then the resulting moduli space is $\mathcal{K}_{g,n}(\mathcal{X}, \beta)$ in [2], and the evaluation maps take values in the rigidified inertial stack $\overline{\mathcal{I}}\mathcal{X}$ instead of the inertia stack $\mathcal{I}\mathcal{X}$.

5.4. Obstruction theory and virtual fundamental classes. The tangent space T^1_ξ and the obstruction space T^2_ξ at a moduli point $\xi = [f : (\mathcal{C}, \mathfrak{x}_1, \dots, \mathfrak{x}_n) \to \mathcal{X}] \in \overline{\mathcal{M}}_{g,n}(\mathcal{X}, \beta)$ fit in the *tangent-obstruction exact sequence*:

$$\begin{aligned} 0 \to &\mathrm{Ext}^0_{\mathcal{O}_\mathcal{C}}(\Omega_\mathcal{C}(\mathfrak{x}_1 + \cdots + \mathfrak{x}_n), \mathcal{O}_\mathcal{C}) \to H^0(\mathcal{C}, f^*T_\mathcal{X}) \to T^1_\xi \\ \to &\mathrm{Ext}^1_{\mathcal{O}_\mathcal{C}}(\Omega_\mathcal{C}(\mathfrak{x}_1 + \cdots + \mathfrak{x}_n), \mathcal{O}_\mathcal{C}) \to H^1(\mathcal{C}, f^*T_\mathcal{X}) \to T^2_\xi \to 0 \end{aligned} \tag{5.2}$$

where

- $\mathrm{Ext}^0_{\mathcal{O}_\mathcal{C}}(\Omega_\mathcal{C}(\mathfrak{x}_1 + \cdots + \mathfrak{x}_n), \mathcal{O}_\mathcal{C})$ is the space of infinitesimal automorphisms of the domain $(\mathcal{C}, \mathfrak{x}_1, \dots, \mathfrak{x}_n)$,
- $\mathrm{Ext}^1_{\mathcal{O}_\mathcal{C}}(\Omega_\mathcal{C}(\mathfrak{x}_1 + \cdots + \mathfrak{x}_n), \mathcal{O}_\mathcal{C})$ is the space of infinitesimal deformations of the domain $(\mathcal{C}, \mathfrak{x}_1, \dots, \mathfrak{x}_n)$,
- $H^0(\mathcal{C}, f^*T_\mathcal{X})$ is the space of infinitesimal deformations of the morphism f for a fixed domain, and
- $H^1(\mathcal{C}, f^*T_\mathcal{X})$ is the space of obstructions to deforming the morphism f for a fixed domain.

T^1_ξ and T^2_ξ are fibers of coherent sheaves T^1 and T^2 on the moduli space $\overline{\mathcal{M}}_{g,\vec{\imath}}(\mathcal{X}, \beta)$. The moduli space $\overline{\mathcal{M}}_{g,\vec{\imath}}(\mathcal{X}, \beta)$ is equipped with a perfect obstruction theory of virtual dimension $d^{\mathrm{vir}}_{g,\vec{\imath},\beta}$, where

$$d^{\mathrm{vir}}_{g,\vec{\imath},\beta} = \int_\beta c_1(T_\mathcal{X}) + (\dim \mathcal{X} - 3)(1 - g) + n - \sum_{j=1}^n \mathrm{age}(\mathcal{X}_{i_j}). \tag{5.3}$$

Locally there is a two term complex $[E \to F]$ of locally free sheaves such that

$$\mathrm{rank} E - \mathrm{rank} F = d^{\mathrm{vir}}_{g,\vec{\imath},\beta}$$

and T^1 and T^2 are the kernel and cokernel of $E \to F$, i.e.,

$$0 \to T^1 \to E \to F \to T^2 \to 0 \tag{5.4}$$

is an exact sequence of sheaves of $\mathcal{O}_{\overline{\mathcal{M}}_{g,\vec{\imath}}(\mathcal{X},\beta)}$-modules. This determines a virtual fundamental class (constructed in the algebraic setting in [6, 45]):

$$[\overline{\mathcal{M}}_{g,\vec{\imath}}(\mathcal{X}, \beta)]^{\mathrm{vir}} \in A_{d^{\mathrm{vir}}_{g,\vec{\imath},\beta}}(\overline{\mathcal{M}}_{g,\vec{\imath}}(\mathcal{X}, \beta); \mathbb{Q}).$$

Given a pair $(x, g) \in Ob(\mathcal{IX})$, where $x \in Ob(\mathcal{X})$ and $g \in \mathrm{Aut}_\mathcal{X}(x)$, define $r(x, g) = |\langle g \rangle|$. Then $(x, g) \mapsto |\langle g \rangle|$ defines a map $r : \mathcal{IX} \to \mathbb{Z}_{>0}$ which is constant on each connected component $\mathcal{X}_i$ of $\mathcal{IX}$. Let $r_i = r(\mathcal{X}_i)$. The *weighted virtual fundamental class* is defined by

$$[\overline{\mathcal{M}}_{g,\vec{\imath}}(\mathcal{X}, \beta)]^w := \Big(\prod_{j=1}^n r_{i_j}\Big) [\overline{\mathcal{M}}_{g,\vec{\imath}}(\mathcal{X}, \beta)]^{\mathrm{vir}}.$$

5.5. Moduli of twisted stable maps to a formal smooth GKM stack. Let $\hat{\mathcal{X}}_{\vec{\Upsilon}}$ be the formal smooth GKM stack defined by an abstract stacky GKM graph $\vec{\Upsilon}$, and let $\hat{X}_{\vec{\Upsilon}}$ be its coarse moduli space. Then

$$H_2(\hat{X}_{\vec{\Upsilon}}; \mathbb{Z}) = \bigoplus_{\epsilon \in E(\Upsilon)_c} \mathbb{Z}[\ell_\epsilon].$$

Let

$$\mathrm{Eff}(\hat{\mathcal{X}}_{\vec{\Upsilon}}) = \Big\{ \sum_{\epsilon \in E(\Upsilon)_c} d_\epsilon [\ell_\epsilon] : d_\epsilon \in \mathbb{Z}_{\geq 0} \Big\} \subset H_2(\hat{X}_{\vec{\Upsilon}}; \mathbb{Z})$$

be the set of effective classes. Given $g, n \in \mathbb{Z}_{\geq 0}$ and $\hat{\beta} \in \mathrm{Eff}(\hat{\mathcal{X}}_{\vec{\Upsilon}})$, let $\overline{\mathcal{M}}_{g,n}(\hat{\mathcal{X}}_{\vec{\Upsilon}}, \hat{\beta})$ be the moduli of genus g, n-pointed, degree $\hat{\beta}$ twisted stable maps to $\hat{\mathcal{X}}_{\vec{\Upsilon}}$, which is the analogue of $\overline{\mathcal{M}}_{g,n}(X, \beta)$ defined in Section 5.3. Let

$$\mathcal{I}\hat{\mathcal{X}}_{\vec{\Upsilon}} = \bigsqcup_{i \in I_{\vec{\Upsilon}}} (\hat{\mathcal{X}}_{\vec{\Upsilon}})_i$$

be disjoint union of connected components. Let $\overline{\mathcal{M}}_{g,\vec{i}}(\hat{\mathcal{X}}_{\vec{\Upsilon}}, \hat{\beta})$ be the analogue of $\overline{\mathcal{M}}_{g,\vec{i}}(\mathcal{X}, \beta)$. Then we have a disjoint union

$$\overline{\mathcal{M}}_{g,n}(\hat{\mathcal{X}}_{\vec{\Upsilon}}, \hat{\beta}) = \bigsqcup_{\vec{i} \in I_{\vec{\Upsilon}}{}^n} \overline{\mathcal{M}}_{g,\vec{i}}(\hat{\mathcal{X}}_{\vec{\Upsilon}}, \hat{\beta}).$$

Each $\overline{\mathcal{M}}_{g,\vec{i}}(\hat{\mathcal{X}}_{\vec{\Upsilon}}, \hat{\beta})$ is equipped with a T-action together with a T-equivariant perfect obstruction theory of virtual dimension $d^{\mathrm{vir}}_{g,\vec{i},\hat{\beta}}$, where

$$d^{\mathrm{vir}}_{g,\vec{i},\hat{\beta}} = \int_{\hat{\beta}} c_1(T_{\hat{\mathcal{X}}_{\vec{\Upsilon}}}) + (\dim \hat{\mathcal{X}}_{\vec{\Upsilon}} - 3)(1-g) + n - \sum_{j=1}^{n} \mathrm{age}((\hat{\mathcal{X}}_{\vec{\Upsilon}})_{i_j}).$$

Let $\mathcal{X}$ be a smooth GKM stack, and let $\vec{\Upsilon}$ be its stacky GKM graph. There is a T-equivariant morphism $j : \hat{\mathcal{X}}_{\vec{\Upsilon}} \to \mathcal{X}$, which induces a T-equivariant morphism

$$\mathcal{I}\hat{\mathcal{X}}_{\vec{\Upsilon}} = \bigsqcup_{i \in I_{\vec{\Upsilon}}} (\hat{\mathcal{X}}_{\vec{\Upsilon}})_i. \longrightarrow \mathcal{I}\mathcal{X} = \bigsqcup_{i \in I} \mathcal{X}_i.$$

and a surjective group homomorphism $j_* : H_2(\hat{X}_{\vec{\Upsilon}}; \mathbb{Z}) \to H_2(X; \mathbb{Z})$. Define $j : I_{\vec{\Upsilon}} \to I$ such that $j((\hat{\mathcal{X}}_{\vec{\Upsilon}})_i) = \mathcal{X}_{j(i)}$. We have T-equivariant morphisms

$$\overline{\mathcal{M}}_{g,n}(\hat{\mathcal{X}}_{\vec{\Upsilon}}, \hat{\beta}) \to \overline{\mathcal{M}}_{g,n}(\mathcal{X}, j_*\hat{\beta}), \qquad \overline{\mathcal{M}}_{g,(i_1,\ldots,i_n)}(\hat{\mathcal{X}}_{\vec{\Upsilon}}, \hat{\beta}) \to \overline{\mathcal{M}}_{g,(j(i_1),\ldots,j(i_n))}(\mathcal{X}, j_*\hat{\beta}).$$

5.6. Hurwitz-Hodge integrals. By Example 2.1, when $\mathcal{X} = \mathcal{B}G$ we have

$$\mathcal{I}\mathcal{B}G = \bigsqcup_{c \in \mathrm{Conj}(G)} (\mathcal{B}G)_c$$

where $\mathrm{Conj}(G)$ is the set of conjugacy classes of G. Give $\vec{c} = (c_1, \ldots, c_n) \in \mathrm{Conj}(G)^n$, let $\overline{\mathcal{M}}_{g,\vec{c}}(\mathcal{B}G) = \overline{\mathcal{M}}_{g,\vec{c}}(\mathcal{B}G, \beta = 0)$. Then $\overline{\mathcal{M}}_{g,\vec{c}}(\mathcal{B}G)$ is a union of connected components of $\overline{\mathcal{M}}_{g,n}(\mathcal{B}G) := \overline{\mathcal{M}}_{g,n}(\mathcal{B}G, 0)$, and

$$\overline{\mathcal{M}}_{g,n}(\mathcal{B}G) = \bigsqcup_{\vec{c} \in \mathrm{Conj}(G)^n} \overline{\mathcal{M}}_{g,\vec{c}}(\mathcal{B}G).$$

We now fix a genus g and n conjugacy classes $\vec{c} = (c_1, \ldots, c_n) \in \mathrm{Conj}(G)^n$. Let $\pi : \mathcal{U} \to \overline{\mathcal{M}}_{g,\vec{c}}(\mathcal{B}G)$ be the universal curve, and let $f : \mathcal{U} \to \mathcal{B}G$ be the universal map. Let $\rho : G \to GL(V)$ be an irreducible representation of G, where V is a finite dimensional vector space over $\mathbb{C}$. Then $\mathcal{E}_\rho := [V/G]$ is a vector bundle over $\mathcal{B}G = [\text{point}/G]$. We have

$$\pi_* f^* \mathcal{E}_\rho = \begin{cases} \mathcal{O}_{\overline{\mathcal{M}}_{g,\vec{c}}(\mathcal{B}G)}, & \text{if } \rho : G \to GL(1,\mathbb{C}) \text{ is the trivial representation,} \\ 0, & \text{otherwise.} \end{cases}$$

The ρ-twisted Hurwitz-Hodge bundle $\mathbb{E}_\rho$ can be defined as the dual of the vector bundle $R^1\pi_* f^*\mathcal{E}_\rho$. If $\rho = 1$ is the trivial representation, then $\mathbb{E}_1 = \epsilon^*\mathbb{E}$, where $\epsilon : \overline{\mathcal{M}}_{g,\vec{c}}(\mathcal{B}G) \to \overline{\mathcal{M}}_{g,n}$, and $\mathbb{E} \to \overline{\mathcal{M}}_{g,n}$ is the Hodge bundle of $\overline{\mathcal{M}}_{g,n}$. So $\text{rank}\mathbb{E}_1 = g$. If ρ is a nontrivial irreducible representation, it follows from the Riemann-Roch theorem for twisted curves (see Section 5.2) that

$$\text{rank}\mathbb{E}_\rho = \text{rank}(\mathcal{E}_\rho)(g-1) + \sum_{j=1}^{n} \text{age}_{c_j}(\mathcal{E}_\rho), \tag{5.5}$$

where $\text{age}_{c_j}(\mathcal{E}_\rho)$ is given as follows. Choose $g \in c_j$. Let $s > 0$ be the order of g in G, let $N = \text{rank}\mathcal{E}_\rho = \dim V$. If the eigenvalues of $\rho(g) \in GL(V) = GL(N,\mathbb{C})$ are $\{e^{\frac{2\pi\sqrt{-1}}{s} l_i} : 1 \le i \le N\}$, where $l_i \in \{0, 1, \ldots, s-1\}$, then

$$\text{age}_{c_j}(\mathcal{E}_\rho) = \frac{l_1 + \cdots + l_N}{s}.$$

The definition is independent of choice of $g \in c_j$. Note that

$$\sum_{j=1}^{n} \text{age}_{c_j}(\mathcal{E}_\rho) \in \mathbb{Z}.$$

(If G is *abelian* then any irreducible representation of G is 1-dimensional, so in this case $\text{rank}(\mathcal{E}_\rho) = 1$ for any irreducible representation ρ of G.)

- *Hodge classes.* Given an irreducible representation ρ of G, define

$$\lambda_i^\rho = c_i(\mathbb{E}_\rho) \in H^{2i}(\overline{\mathcal{M}}_{g,\vec{c}}(\mathcal{B}G); \mathbb{Q}), \quad i = 1, \ldots, \text{rank}\mathbb{E}_\rho.$$

- *Descendant classes.* There is a forgetful map $\epsilon : \overline{\mathcal{M}}_{g,\vec{c}}(\mathcal{B}G) \to \overline{\mathcal{M}}_{g,n}$. Define

$$\bar{\psi}_j = \epsilon^* \psi_j \in H^2(\overline{\mathcal{M}}_{g,\vec{c}}(\mathcal{B}G)), \quad j = 1, \ldots, n.$$

Hurwitz-Hodge integrals are top intersection numbers of Hodge classes λ_i^ρ and descendant classes $\bar{\psi}_j$:

$$\int_{\overline{\mathcal{M}}_{g,\vec{c}}(\mathcal{B}G)} \bar{\psi}_1^{a_1} \cdots \bar{\psi}_n^{a_n} (\lambda_1^{\rho_1})^{k_1} \cdots (\lambda_g^{\rho_g})^{k_g}. \tag{5.6}$$

In [68], Jian Zhou describes an algorithm of computing Hurwitz-Hodge integrals, as follows. By Tseng's orbifold quantum Riemann-Roch theorem [64], Hurwitz-Hodge integrals can be reconstructed from descendant integrals on $\overline{\mathcal{M}}_{g,\vec{c}}(\mathcal{B}G)$:

$$\int_{\overline{\mathcal{M}}_{g,\vec{c}}(\mathcal{B}G)} \bar{\psi}_1^{a_1} \cdots \bar{\psi}_n^{a_n}. \tag{5.7}$$

Jarvis-Kimura relate the descendant integrals on $\overline{\mathcal{M}}_{g,\vec{c}}(\mathcal{B}G)$ to those on $\overline{\mathcal{M}}_{g,n}$ [32]. We now state their result. Given $g \in \mathbb{Z}_{\geq 0}$ and $\vec{c} = (c_1, \dots, c_n) \in \mathrm{Conj}(G)^n$, let

$$V^G_{g,\vec{c}} := \Big\{(a_1, b_1, \dots, a_g, b_g, e_1, \dots, e_n) \in G^{2g+n} \mid a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} = \prod_{j=1}^n e_j, \quad e_j \in c_j\Big\}$$

which is a finite set. The moduli of flat G-bundles over a compact Riemann surface of genus g with markings $c_1, \dots, c_n$ is the quotient stack

$$[V^G_{g,\vec{c}}/G]$$

where G acts on $V^G_{g,\vec{c}}$ by diagonal conjugation:

$$\begin{aligned} & h \cdot (a_1, b_1, \dots, a_g, b_g, e_1, \dots, e_n) \\ = & (ha_1h^{-1}, hb_1h^{-1}, \dots, ha_gh^{-1}, hb_gh^{-1}, he_1h^{-1}, \dots, he_nh^{-1}). \end{aligned}$$

EXAMPLE 5.2. *If $h \in G$ then*

$$[V^G_{0,[h],[h^{-1}]}/G] \cong [\{(h, h^{-1})\}/C_G(h)] = \mathcal{B}C_G(h)$$

where $C_G(h)$ is the centralizer of h in G.

If G is *abelian* then each c_i consists of a single element $h_i \in G$, and

$$V^G_{g,\vec{c}} = \begin{cases} G^{2g} \times \{h_1\} \times \cdots \{h_n\} & \text{if } h_1 \cdots h_n = 1, \\ \emptyset \quad \text{(the empty set)} & \text{if } h_1 \cdots h_n \neq 1; \end{cases}$$

so $[V^g_{g,\vec{c}}/G] \cong G^{2g} \times \mathcal{B}G$ if $h_1 \cdots h_n = 1$, and is empty otherwise. In general, $\overline{\mathcal{M}}_{g,\vec{c}}(\mathcal{B}G)$ is non-empty if and only if $2g - 2 + n > 0$ and $V^G_{g,\vec{c}}$ is non-empty. In this case, the forgetful map $\epsilon : \overline{\mathcal{M}}_{g,\vec{c}}(\mathcal{B}G) \to \overline{\mathcal{M}}_{g,n}$ is of degree $|V^G_{g,\vec{c}}|/|G|$.

THEOREM 5.3 (Jarvis-Kimura [32, Proposition 3.4]). *Suppose that $2g-2+n > 0$ and $V^G_{g,\vec{c}}$ is nonempty. Then*

$$\int_{\overline{\mathcal{M}}_{g,\vec{c}}(\mathcal{B}G)} \bar{\psi}_1^{a_1} \cdots \bar{\psi}_n^{a_n} = \frac{|V^G_{g,\vec{c}}|}{|G|} \int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n}.$$

5.7. Orbifold GW invariants. There is a morphism $\epsilon : \overline{\mathcal{M}}_{g,\vec{i}}(\mathcal{X}, \beta) \to \overline{\mathcal{M}}_{g,n}(X, \beta)$. Define $\bar{\psi}_i = \epsilon^* \psi_i$.

Suppose that the coarse moduli space X is *projective.* Then $\overline{\mathcal{M}}_{g,\vec{i}}(\mathcal{X}, \beta)$ is proper. Let

$$\gamma_j \in H^{d_j}(\mathcal{X}_{i_j}; \mathbb{Q}) \subset H^{d_j + 2\mathrm{age}(\mathcal{X}_{i_j})}_{\mathrm{CR}}(\mathcal{X}; \mathbb{Q}),$$

Define orbifold Gromov-Witten invariants

$$\langle \bar{\epsilon}_{a_1}\gamma_1, \dots, \bar{\epsilon}_{a_n}\gamma_n \rangle^{\mathcal{X}}_{g,\beta} := \int_{[\overline{\mathcal{M}}_{g,\vec{i}}(\mathcal{X},\beta)]^w} \prod_{j=1}^n \left(\mathrm{ev}_j^* \gamma_j \cup \bar{\psi}_j^{a_j}\right) \tag{5.8}$$

which is zero unless

$$\sum_{j=1}^n (d_j + 2\mathrm{age}(\mathcal{X}_{i_j}) + 2a_j) = 2\left(\int_\beta c_1(T_{\mathcal{X}}) + (1-g)(\dim \mathcal{X} - 3) + n\right).$$

5.8. Equivariant orbifold GW invariants. Suppose that $\mathcal{X}$ is equipped with a T-action, which induces a T-action on $\overline{\mathcal{M}}_{g,\vec{i}}(\mathcal{X},\beta)$ and on the perfect obstruction theory. Then there is a T-equivariant virtual fundamental class

$$[\overline{\mathcal{M}}_{g,\vec{i}}(\mathcal{X},\beta)]^{\mathrm{vir},T} \in H^T_{2d^{\mathrm{vir}}_{g,\vec{i},\beta}}(\overline{\mathcal{M}}_{g,\vec{i}}(\mathcal{X},\beta);\mathbb{Q}).$$

The weighted T-equivariant virtual fundamental class is defined by

$$[\overline{\mathcal{M}}_{g,\vec{i}}(\mathcal{X},\beta)]^{w,T} = \Big(\prod_{j=1}^n r_{i_j}\Big)[\overline{\mathcal{M}}_{g,\vec{i}}(\mathcal{X},\beta)]^{\mathrm{vir},T}.$$

Suppose that $\overline{\mathcal{M}}_{g,\vec{i}}(\mathcal{X},\beta)$ is *proper*. (If the coarse moduli space X is projective then $\overline{\mathcal{M}}_{g,\vec{i}}(\mathcal{X},\beta)$ is proper for any $g,\vec{i},\beta$.) Given $\gamma_j^T \in H_T^{d_j}(\mathcal{X}_{i_j};\mathbb{Q}) \subset H_{\mathrm{CR},T}^{d_j+2\mathrm{age}(\mathcal{X}_{i_j})}(\mathcal{X};\mathbb{Q})$ and $a_j \in \mathbb{Z}_{\geq 0}$, we define T-equivariant orbifold Gromov-Witten invariants

$$\begin{aligned}\langle \bar{\epsilon}_{a_1}(\gamma_1^T),\cdots,\bar{\epsilon}_{a_n}(\gamma_n^T)\rangle^{\mathcal{X}_T}_{g,\beta} &:= \int_{[\overline{\mathcal{M}}_{g,\vec{i}}(\mathcal{X},\beta)]^{w,T}} \prod_{j=1}^n \big(\mathrm{ev}_j^*\gamma_j^T \cup (\bar{\psi}_1^T)^{a_j}\big) \\ &\in \mathbb{Q}[u_1,\ldots,u_m]\big(\sum_{j=1}^n (d_j+2a_j) - 2d^{\mathrm{vir}}_{\vec{i}}\big).\end{aligned} \tag{5.9}$$

where $\mathbb{Q}[u_1,\ldots,u_m](2k)$ is the space of degree k homogeneous polynomials in $u_1,\ldots,u_l$ with rational coefficients, and $\mathbb{Q}[u_1,\ldots,u_m](2k+1)=0$. In particular,

$$\langle \bar{\epsilon}_{a_1}(\gamma_1^T),\cdots,\bar{\epsilon}_{a_n}(\gamma_n^T)\rangle^{\mathcal{X}_T}_{g,\beta} = \begin{cases} 0, & \sum_{j=1}^n (d_j+2a_j) < 2d^{\mathrm{vir}}_{g,\vec{i},\beta}, \\ \langle \bar{\epsilon}_{a_1}(\gamma_1),\cdots,\bar{\epsilon}_{a_n}(\gamma_n)\rangle^{\mathcal{X}}_{g,\beta} \in \mathbb{Q}, & \sum_{j=1}^n (d_j+2a_j) = 2d^{\mathrm{vir}}_{g,\vec{i},\beta}. \end{cases}$$

where $\gamma_j \in H^{d_j}(\mathcal{X}_{i_j};\mathbb{Q})$ is the image of γ_j^T under the map $H_T^{d_j}(\mathcal{X}_{i_j};\mathbb{Q}) \to H^{d_j}(\mathcal{X}_{i_j};\mathbb{Q})$.

5.9. Virtual localization. Let $\mathcal{F} = \overline{\mathcal{M}}_{g,\vec{i}}(\mathcal{X},\beta)^T \subset \overline{\mathcal{M}}_{g,\vec{i}}(\mathcal{X},\beta)$ be the substack of T fixed points. The restriction of the exact sequence (5.4) to $\mathcal{F}$ splits into two exact sequences of $\mathcal{O}_{\mathcal{F}}$-modules:

$$0 \to T^{1,f} \to E^f \to F^f \to T^{2,f} \to 0, \tag{5.10}$$

$$0 \to T^{1,m} \to E^m \to F^m \to T^{2,m} \to 0, \tag{5.11}$$

where (5.10) and (5.11) are the fixed and moving parts of (5.4), respectively. The 2-term complex $[(F^f)^\vee \to (E^f)^\vee]$ defines a perfect obstruction theory on $\mathcal{F}$; in other words, $\mathcal{F}$ is equipped with a virtual tangent bundle

$$T^{\mathrm{vir}}_{\mathcal{F}} = T^{1,f} - T^{2,f} = E^f - F^f.$$

which might have different ranks on different connected components of $\mathcal{F}$. This defines a virtual fundamental class [6, 45]

$$[\mathcal{F}]^{\mathrm{vir}} \in A_*(\mathcal{F}).$$

The virtual normal bundle of $\mathcal{F}$ in $\overline{\mathcal{M}}_{g,\vec{i}}(\mathcal{X},\beta)$ is

$$N^{\mathrm{vir}} = T^{1,m} - T^{2,m} = E^m - F^m.$$

which might also have different ranks on different connected components of $\mathcal{F}$, but

$$\mathrm{rank}(T_{\mathcal{F}}^{\mathrm{vir}}) + \mathrm{rank}(N^{\mathrm{vir}}) = d_{g,\vec{i},\beta}$$

is constant on $\mathcal{F}$.

By virtual localization [5, 27],

$$\int_{[\overline{\mathcal{M}}_{g,\vec{i}}(\mathcal{X},\beta)]^{w,T}} \prod_{j=1}^{n} \big(\mathrm{ev}_j^*\gamma_j^T \cup (\bar{\psi}_j^T)^{a_j}\big) = \int_{[\mathcal{F}]^w} \frac{i_T^*\left(\prod_{j=1}^n \left(\mathrm{ev}_j^*\gamma_j^T \cup (\bar{\psi}_1^T)^{a_j}\right)\right)}{e^T(N^{\mathrm{vir}})} \tag{5.12}$$

where

$$[\mathcal{F}]^w = \Big(\prod_{j=1}^n r_{i_j}\Big)[\mathcal{F}]^{\mathrm{vir}}.$$

Suppose that $\overline{\mathcal{M}}_{g,\vec{i}}(\mathcal{X},\beta)$ is not proper, but $\mathcal{F} = \overline{\mathcal{M}}_{g,\vec{i}}(\mathcal{X},\beta)^T$ is proper. (If $\mathcal{X}$ is a smooth GKM stack then $\overline{\mathcal{M}}_{g,\vec{i}}(\mathcal{X},\beta)^T$ is proper for any $g, \vec{i}, \beta$.) We *define*

$$\langle \bar{\epsilon}_{a_1}(\gamma_1^T), \ldots, \bar{\epsilon}_{a_n}(\gamma_n^T)\rangle_{g,\beta}^{\mathcal{X}_T} = \int_{[\mathcal{F}]^w} \frac{i_T^*\left(\prod_{j=1}^n \left(\mathrm{ev}_j^*\gamma_j^T \cup (\bar{\psi}_j^T)^{a_j}\right)\right)}{e^T(N^{\mathrm{vir}})}. \tag{5.13}$$

When $\overline{\mathcal{M}}_{g,\vec{i}}(\mathcal{X},\beta)$ is not proper, the right hand side of (5.13) is a rational function (instead of a polynomial) in $u_1, \ldots, u_m$. It can be nonzero when $\sum_{j=1}^n (d_j + 2a_j) < 2d_{\vec{i}}^{\mathrm{vir}}$, and does not have a nonequivariant limit ($u_i \to 0$) in general.

5.10. Formal equivariant orbifold GW invariants. Let $\hat{\mathcal{X}}_{\vec{\Upsilon}}$ be the formal smooth GKM stack defined by an abstract stacky GKM graph $\vec{\Upsilon}$. Then there is a T-equivariant virtual fundamental class

$$[\overline{\mathcal{M}}_{g,\vec{i}}(\hat{\mathcal{X}}_{\vec{\Upsilon}},\hat{\beta})]^{\mathrm{vir},T} \in H^T_{2d^{\mathrm{vir}}_{g,\vec{i},\beta}}(\overline{\mathcal{M}}_{g,\vec{i}}(\hat{\mathcal{X}}_{\vec{\Upsilon}},\hat{\beta});\mathbb{Q}).$$

The weighted T-equivariant virtual fundamental class is defined by

$$[\overline{\mathcal{M}}_{g,\vec{i}}(\hat{\mathcal{X}}_{\vec{\Upsilon}},\hat{\beta})]^{w,T} = \Big(\prod_{j=1}^n r_{i_j}\Big)[\overline{\mathcal{M}}_{g,\vec{i}}(\hat{\mathcal{X}}_{\vec{\Upsilon}},\hat{\beta})]^{\mathrm{vir},T}.$$

Define

$$[\overline{\mathcal{M}}_{g,n}(\hat{\mathcal{X}}_{\vec{\Upsilon}},\hat{\beta})]^{w,T} = \sum_{\vec{i}\in(I_{\vec{\Upsilon}})^n} [\overline{\mathcal{M}}_{g,\vec{i}}(\hat{\mathcal{X}}_{\vec{\Upsilon}},\hat{\beta})]^{w,T}.$$

Let $\overline{\mathcal{M}}_{g,\vec{i}}(\hat{\mathcal{X}}_{\vec{\Upsilon}},\hat{\beta})^T \subset \overline{\mathcal{M}}_{g,\vec{i}}(\hat{\mathcal{X}}_{\vec{\Upsilon}},\hat{\beta})$ be the substack of T fixed points. Then $\overline{\mathcal{M}}_{g,\vec{i}}(\hat{\mathcal{X}}_{\vec{\Upsilon}},\hat{\beta})^T$ is a proper DM stack equipped with a perfect obstruction theory which is the T fixed the part of the restriction of the perfect obstruction theory on $\overline{\mathcal{M}}_{g,\vec{i}}(\hat{\mathcal{X}}_{\vec{\Upsilon}},\hat{\beta})$, so we have

$$[\overline{\mathcal{M}}_{g,\vec{i}}(\hat{\mathcal{X}}_{\vec{\Upsilon}},\hat{\beta})^T]^{\mathrm{vir}} \in H_*(\overline{\mathcal{M}}_{g,\vec{i}}(\hat{\mathcal{X}}_{\vec{\Upsilon}},\hat{\beta})^T)$$

and

$$[\overline{\mathcal{M}}_{g,\vec{i}}(\hat{\mathcal{X}}_{\vec{\Upsilon}},\hat{\beta})^T]^w := \Big(\prod_{j=1}^n r_{i_j}\Big)[\overline{\mathcal{M}}_{g,\vec{i}}(\hat{\mathcal{X}}_{\vec{\Upsilon}},\hat{\beta})^T]^{\mathrm{vir}} \in H_*(\overline{\mathcal{M}}_{g,\vec{i}}(\hat{\mathcal{X}}_{\vec{\Upsilon}},\hat{\beta})^T).$$

Define

$$[\overline{\mathcal{M}}_{g,n}(\hat{\mathcal{X}}_{\vec{\Upsilon}},\hat{\beta})^T]^w = \sum_{\vec{i}\in(I_{\vec{\Upsilon}})^n}[\overline{\mathcal{M}}_{g,\vec{i}}(\hat{\mathcal{X}}_{\vec{\Upsilon}},\hat{\beta})^T]^w.$$

Given $\hat{\gamma}_1^T,\dots,\hat{\gamma}_n^T \in \mathcal{H}_{\vec{\Upsilon}}$, we define

$$\langle \bar{\epsilon}_{a_1}(\hat{\gamma}_1^T),\dots,\bar{\epsilon}_{a_n}(\hat{\gamma}_n^T)\rangle_{g,\hat{\beta}}^{\vec{\Upsilon}} = \int_{[\overline{\mathcal{M}}_{g,n}(\hat{\mathcal{X}}_{\vec{\Upsilon}},\hat{\beta})^T]^w} \frac{i_T^*\Big(\prod_{j=1}^n\big(\mathrm{ev}_j^*\hat{\gamma}_j^T\cup(\bar{\psi}_j^T)^{a_j}\big)\Big)}{e^T(N^{\mathrm{vir}})}. \tag{5.14}$$

In the remainder of this subsection, we relate the above formal equivariant orbifold GW invariants to the equivariant orbifold GW invariants defined in the previous subsection (Section 5.8). Let $\mathcal{X}$ be a smooth GKM stack and let $\vec{\Upsilon}$ be its stacky GKM graph. We define a surjective map $j_*:\mathrm{Eff}(\hat{\mathcal{X}}_{\vec{\Upsilon}})\longrightarrow \mathrm{Eff}(\mathcal{X})$ and an injective map $j^*: H^*_{\mathrm{CR},T}(\mathcal{X};\mathcal{Q}_T)\to\mathcal{H}_\Upsilon$ as follows.

(1) Let I, $I_{\vec{\Upsilon}}$, $j: I_{\vec{\Upsilon}}\to I$ be defined as in Section 5.5. The surjective group homomorphism

$$j_*: H_2(\hat{\mathcal{X}}_{\vec{\Upsilon}};\mathbb{Z}) = \bigoplus_{\epsilon\in E(\Upsilon)_c}\mathbb{Z}[\ell_\epsilon]\longrightarrow H_2(\mathcal{X};\mathbb{Z})$$

restricts to a *surjective* map

$$j_*:\mathrm{Eff}(\hat{\mathcal{X}}_{\vec{\Upsilon}})\longrightarrow\mathrm{Eff}(\mathcal{X})$$

where $\mathrm{Eff}(\mathcal{X})$ is the set of effective classes in $\mathcal{X}$. Note that given $\beta\in\mathrm{Eff}(\mathcal{X})$, $\{\hat{\beta}\in\mathrm{Eff}(\hat{\mathcal{X}}_{\vec{\Upsilon}}): j_*\hat{\beta}=\beta\}$ is a finite set.

(2) There is a $\mathcal{Q}_T$-linear map

$$j^* = \bigoplus_{\sigma\in V(\Upsilon)} j_\sigma^*: H^*_{\mathrm{CR},T}(\mathcal{X};\mathcal{Q}_T)\to\mathcal{H}_{\vec{\Upsilon}} = \bigoplus_{\sigma\in V(\Upsilon)} H^*_{\mathrm{CR},T}(\mathcal{X}_\sigma;\mathcal{Q}_T)$$

where j_σ^* is induced by the inclusion $j_\sigma:\mathcal{X}_\sigma\hookrightarrow\mathcal{X}$.

The following identity follows from the localization computations in Section 6.

PROPOSITION 5.4. *Given nonnegative integers $g,a_1,\dots,a_n$ an effective class $\beta\in\mathrm{Eff}(\mathcal{X})$, and $\gamma_1^T,\dots,\gamma_n^T\in H^*_{\mathrm{CR},T}(\mathcal{X};\mathcal{Q}_T)$, we have*

$$\langle\bar{\epsilon}_{a_1}(\gamma_1^T),\dots,\bar{\epsilon}_{a_n}(\gamma_n^T)\rangle_{g,\beta}^{\mathcal{X}} = \sum_{\substack{\hat{\beta}\in\mathrm{Eff}(\hat{\mathcal{X}}_{\vec{\Upsilon}})\\ j_*\hat{\beta}=\beta}}\langle\bar{\epsilon}_{a_1}(j^*\gamma_1^T),\dots,\bar{\epsilon}_{a_n}(j^*\gamma_n^T)\rangle_{g,\hat{\beta}}^{\vec{\Upsilon}}.$$

Therefore, equivariant orbifold GW invariants of $\mathcal{X}$ can be expressed in terms of the more refined formal equivariant orbifold GW invariants of its stacky GKM graph $\vec{\Upsilon}$.

6. Torus fixed points and virtual localization.

6.1. The fundamental group of a one-dimensional orbit. The union of 1-dimensional T orbits in $\mathcal{X}$ is

$$\mathcal{X}^1 \setminus \mathcal{X}^T = \bigcup_{\epsilon \in E(\Upsilon)} \mathfrak{o}_\epsilon$$

where each 1-dimensional T orbit $\mathfrak{o}_\epsilon$ is a G_ϵ-gerbe over its coarse moduli $o_\epsilon \cong \mathbb{C}^*$. Let $\mathfrak{p}_\epsilon \cong \mathcal{B}G_\epsilon$ be a point in $\mathfrak{o}_\epsilon$ chosen as in Section 3.5, and let

$$H_\epsilon := \pi_1(\mathfrak{o}_\epsilon, \mathfrak{p}_\epsilon)$$

be the fundamental group of $\mathfrak{o}_\epsilon$. The projection $\mathfrak{o}_\epsilon \to o_\epsilon$ induces a surjective group homomorphism

$$H_\epsilon = \pi_1(\mathfrak{o}_\epsilon, \mathfrak{p}_\epsilon) \longrightarrow \pi_1(o_\epsilon, p_\epsilon) \cong \mathbb{Z}$$

whose kernel is G_ϵ. In other words, we have a short exact sequence of groups

$$1 \to G_\epsilon \xrightarrow{j_\epsilon} H_\epsilon \xrightarrow{\phi_\epsilon} \mathbb{Z} \to 1. \tag{6.1}$$

Let $\epsilon \in E_c(\Upsilon)$ be a compact edge, so that $\ell_\epsilon \cong \mathbb{P}^1$. Let $\sigma_x, \sigma_y \in V(\Upsilon)$ be the two ends of the edge ϵ, and let $x = \mathfrak{p}_{\sigma_x}$ and $y = \mathfrak{p}_{\sigma_y}$ be the two torus fixed point corresponding to σ_x and σ_y, respectively. Then $x = \mathcal{B}G_{\sigma_x}$, $y = \mathcal{B}G_{\sigma_y}$, and

$$\mathcal{U}_x := \mathfrak{l}_\epsilon \setminus \{y\} \cong [\mathbb{C}/G_{\sigma_x}], \quad \mathcal{U}_y := \mathfrak{l}_\epsilon \setminus \{x\} \cong [\mathbb{C}/G_{\sigma_y}], \quad \mathcal{U}_x \cap \mathcal{U}_y = \mathfrak{o}_\epsilon.$$

The open embeddings $\mathfrak{o}_\epsilon \hookrightarrow \mathcal{U}_x$ and $\mathfrak{o}_\epsilon \hookrightarrow \mathcal{U}_y$ induce surjective group homomorphisms

$$H_\epsilon = \pi_1(\mathfrak{o}_\epsilon) \xrightarrow{\pi_{(\epsilon,\sigma_x)}} \pi_1(\mathcal{U}_x) \cong G_{\sigma_x}, \quad H_\epsilon = \pi_1(\mathfrak{o}_\epsilon) \xrightarrow{\pi_{(\epsilon,\sigma_y)}} \pi_1(\mathcal{U}_y) \cong G_{\sigma_y}. \tag{6.2}$$

Recall that we also have the following two short exact sequences of groups:

$$1 \to G_\epsilon \xrightarrow{j_{(\epsilon,\sigma_x)}} G_{\sigma_x} \xrightarrow{\phi_{(\epsilon,\sigma_x)}} \mu_{r_{(\epsilon,\sigma_x)}} \to 1, \tag{6.3}$$

$$1 \to G_\epsilon \xrightarrow{j_{(\epsilon,\sigma_y)}} G_{\sigma_y} \xrightarrow{\phi_{(\epsilon,\sigma_y)}} \mu_{r_{(\epsilon,\sigma_y)}} \to 1. \tag{6.4}$$

Equations (6.1)-(6.4) fit into the following commutative diagram:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & G_\epsilon & \xrightarrow{j_{(\epsilon,\sigma_x)}} & G_{\sigma_x} & \xrightarrow{\phi_{(\epsilon,\sigma_x)}} & \mu_{r_{(\epsilon,\sigma_x)}} & \longrightarrow & 1 \\
 & & \uparrow{\scriptstyle \mathrm{id}_{G_\epsilon}} & & \uparrow{\scriptstyle \pi_{(\epsilon,\sigma_x)}} & & \uparrow & & \\
1 & \longrightarrow & G_\epsilon & \xrightarrow{j_\epsilon} & H_\epsilon & \xrightarrow{\phi_\epsilon} & \mathbb{Z} & \longrightarrow & 1 \\
 & & \downarrow{\scriptstyle \mathrm{id}_{G_\epsilon}} & & \downarrow{\scriptstyle \pi_{(\epsilon,\sigma_y)}} & & \downarrow & & \\
1 & \longrightarrow & G_\epsilon & \xrightarrow{j_{(\epsilon,\sigma_y)}} & G_{\sigma_y} & \xrightarrow{\phi_{(\epsilon,\sigma_y)}} & \mu_{r_{(\epsilon,\sigma_y)}} & \longrightarrow & 1
\end{array}$$

where $\mathbb{Z} \longrightarrow \mu_{r_{(\epsilon,\sigma_x)}}$ and $\mathbb{Z} \longrightarrow \mu_{r_{(\epsilon,\sigma_y)}}$ are given by $d \mapsto e^{2\pi\sqrt{-1}d/r_{(\epsilon,\sigma_x)}}$ and $d \mapsto e^{2\pi\sqrt{-1}d/r_{(\epsilon,\sigma_y)}}$, respectively.

6.2. Torus fixed points in the moduli spaces. Let $f : (\mathcal{C}, \mathfrak{x}_1, \dots, \mathfrak{x}_n) \to \mathcal{X}$ be a twisted stable morphism which represents a T-fixed point in $\overline{\mathcal{M}}_{g,n}(\mathcal{X}, \beta)$. Then there exists a surjective group homomorphism $p : \widetilde{T} \cong (\mathbb{C}^*)^m \longrightarrow T \cong (\mathbb{C}^*)^m$ with finite kernel and a group homomorphism $\phi : \widetilde{T} \longrightarrow \mathrm{Aut}(C, x_1, \dots, x_n)$ such that $p(t) \cdot f(z) = f(\phi(t) \cdot z)$ for all $z \in C$. The image of f lies in the 1-skeleton $\mathcal{X}^1$, the union of zero-dimensional and one-dimensional T orbits in $\mathcal{X}$. In particular f defines a twisted stable morphism with target $\hat{\mathcal{X}}_{\vec{\Upsilon}}$ which represents a T-fixed point in $\overline{\mathcal{M}}_{g,n}(\hat{\mathcal{X}}_{\vec{\Upsilon}}, \hat{\beta})$ where $\hat{\beta} \in \mathrm{Eff}(\hat{\mathcal{X}}_{\vec{\Upsilon}})$ satisfies $j_* \hat{\beta} = \beta$.

If $\mathcal{C}_v$ is a connected component of $f^{-1}(\mathcal{X}^T)$ then the image of $\mathcal{C}_v$ is a T fixed point $\mathfrak{p}_\sigma \cong \mathcal{B}G_\sigma$ for some $\sigma \in V(\Upsilon)$. If O_e is a connected component of $f^{-1}(\mathcal{X}^1 \setminus \mathcal{X}^T)$ then $O_e \cong \mathbb{C}^*$, and the image of O_e is a 1-dimensional T orbit $\mathfrak{o}_\epsilon$ for some $\epsilon \in E_c(\Upsilon)$. The maps

$$O_e \xrightarrow{f|_{O_e}} \mathfrak{o}_\epsilon \to o_\epsilon$$

induce

$$\pi_1(O_e) = \mathbb{Z} \xrightarrow{(f|_{O_e})_*} \pi_1(\mathfrak{o}_\epsilon) = H_\epsilon \xrightarrow{\phi_\epsilon} \pi_1(o_\epsilon) = \mathbb{Z}.$$

Let $\gamma_e \in H_\epsilon$ be the image of the generator of $\pi_1(O_e) = \mathbb{Z}$ under $(f|_{O_e})_*$, and let $d_e = \phi_\epsilon(\gamma_e) \in \mathbb{Z}$. Then $d_e > 0$ is the degree of the map $O_e = \mathbb{C}^* \longrightarrow o_\epsilon = \mathbb{C}^*$.

The map $f|_{O_e} : O_e \to \mathfrak{o}_\epsilon$ is of degree $d_e|G_\epsilon|$. We have

$$\mathrm{Aut}(f|_{O_e}) \cong C_{H_\epsilon}(\gamma_e)/\langle \gamma_e \rangle.$$

In particular, if G_ϵ is trivial then $H_\epsilon = \mathbb{Z}$ and $\mathrm{Aut}(f|_{O_e}) = \mathbb{Z}/d_e\mathbb{Z}$; if H_ϵ is abelian then $\mathrm{Aut}(f|_{O_e}) = H_\epsilon/\langle \gamma_e \rangle$ and $|\mathrm{Aut}(f|_{O_e})| = d_e|G_\epsilon|$.

Let $\mathcal{C}_e$ be the closure of O_e in $\mathcal{C}$. Then $\mathcal{C}_e$ is a football $\mathcal{F}(r_u, r_v)$ and $f_e := f|_{\mathcal{C}_e} : \mathcal{C}_e \to \mathfrak{l}_\epsilon$ is determined by $\gamma_e \in H_\epsilon$. Suppose that $\sigma_x, \sigma_y \in V(\Upsilon)$ are the two ends of the edge ϵ. We define

$$k_{(\epsilon,\sigma_x)} := \pi_{(\epsilon,\sigma_x)}(\gamma_\epsilon) \in G_{\sigma_x}, \quad k_{(\epsilon,\sigma_y)} := \pi_{(\epsilon,\sigma_y)}(\gamma_\epsilon) \in G_{\sigma_y}.$$

The map $f_e : \mathcal{C}_e = \mathcal{F}(r_u, r_v) \to \mathfrak{l}_\epsilon$ is representable, so r_u and r_v are the orders of $k_{(\epsilon,\sigma_x)} \in G_{\sigma_x}$ and $k_{(\epsilon,\sigma_y)} \in G_{\sigma_y}$, respectively. In particular, the domain $\mathcal{C}_e$ of f_e is also determined by γ_e. Let $\bar{f}_e : C_e = \mathbb{P}^1 \to \ell_\epsilon = \mathbb{P}^1$ be the map between coarse moduli spaces. Then $f_e([x, y]) = [x^{d_e}, y^{d_e}]$ in terms of homogeneous coordinates on $\mathbb{P}^1$.

6.3. Torus fixed points and decorated graphs. Given a smooth GKM stack $\mathcal{X}$, let

$$\overline{\mathcal{M}}(\mathcal{X}) := \bigsqcup_{\substack{g,n \in \mathbb{Z}_{\geq 0} \\ \beta \in \mathrm{Eff}(\mathcal{X})}} \overline{\mathcal{M}}_{g,n}(\mathcal{X}, \beta)$$

and let $\overline{\mathcal{M}}(\mathcal{X})^T$ be the T fixed substack.

Given an abstract stacky GKM graph $\vec{\Gamma}$, let

$$\overline{\mathcal{M}}(\hat{\mathcal{X}}_{\vec{\Upsilon}}) := \bigsqcup_{\substack{g,n \in \mathbb{Z}_{\geq 0} \\ \hat{\beta} \in \mathrm{Eff}(\hat{\mathcal{X}}_{\vec{\Upsilon}})}} \overline{\mathcal{M}}_{g,n}(\hat{\mathcal{X}}_{\vec{\Upsilon}}, \hat{\beta})$$

and let $\overline{\mathcal{M}}(\hat{\mathcal{X}}_{\vec{\Upsilon}})^T$ denote the T fixed substack. By the discussion in Section 6.2, if $\vec{\Gamma}$ is the stacky GKM graph of a smooth GKM stack $\mathcal{X}$ then the morphism $\overline{\mathcal{M}}(\hat{\mathcal{X}}_{\vec{\Upsilon}}) \to \overline{\mathcal{M}}(\mathcal{X})$ restricts to an isomorphism $\overline{\mathcal{M}}(\hat{\mathcal{X}}_{\vec{\Upsilon}})^T \to \overline{\mathcal{M}}(\mathcal{X})^T$. In this subsection, we will describe $\overline{\mathcal{M}}(\hat{\mathcal{X}}_{\vec{\Upsilon}})^T$ for a general abstract stacky GKM graph $\vec{\Gamma}$; in particular, this gives a description of $\overline{\mathcal{M}}(\mathcal{X})^T$ for any smooth GKM stack $\mathcal{X}$.

We fix an abstract stacky GKM graph $\vec{\Gamma}$, which defines a formal GKM stack $\hat{\mathcal{X}}_{\vec{\Upsilon}}$. Let $\hat{\mathcal{X}}^1_{\vec{\Upsilon}} = \bigcup_{\epsilon\in E(\Upsilon)} \mathfrak{l}_\epsilon$ be the 1-skeleton of $\hat{\mathcal{X}}_{\vec{\Upsilon}}$. Given a twisted stable map $f : (\mathcal{C}, \mathfrak{x}_1, \dots, \mathfrak{x}_n) \to \hat{\mathcal{X}}_{\vec{\Upsilon}}$ which represents a point in $\overline{\mathcal{M}}(\hat{\mathcal{X}}_{\vec{\Upsilon}})^T$, we define a decorated graph $\vec{\Gamma} = (\Gamma, \vec{f}, \vec{\gamma}, \vec{g}, \vec{s}, \vec{c})$ as follows.

(1) (graph) Γ is a compact, connected 1 dimensional CW complex. We denote the set of vertices (resp. edges) in Γ by $V(\Gamma)$ (resp. $E(\Gamma)$). For each connected component $\mathcal{C}_v$ of $f^{-1}(\hat{\mathcal{X}}^T_{\vec{\Upsilon}}) = f^{-1}(\{\mathfrak{p}_\sigma : \sigma \in V(\Upsilon)\})$, we associate a vertex $v \in V(\Gamma)$. For each connected component $O_e \cong \mathbb{C}^*$ of $f^{-1}(\hat{\mathcal{X}}^1_{\vec{\Upsilon}}) \setminus f^{-1}(\hat{\mathcal{X}}^T_{\vec{\Upsilon}})$, we associate an edge $e \in E(\Gamma)$; the closure $\mathcal{C}_e$ of O_e in $\mathcal{C}$ is a football. The set of flags of Γ is defined to be

$$\begin{aligned} F(\Gamma) &= \{(e, v) \in E(\Gamma) \times V(\Gamma) \mid v \in e\} \\ &= \{(e, v) \in E(\Gamma) \times V(\Gamma) \mid \mathcal{C}_v \cap \mathcal{C}_e \text{ is nonempty}\}. \end{aligned}$$

(2) (label) For each vertex $v \in V(\Gamma)$ let C_v denote the coarse moduli of $\mathcal{C}_v$. Then C_v is a curve (with at most nodal singularities) or a point, and $f(\mathcal{C}_v) = \mathfrak{p}_{\sigma_v}$ for some $\sigma_v \in V(\Upsilon)$. For each edge $e \in E(\Gamma)$, $f(\mathcal{C}_e) = \mathfrak{l}_{\epsilon_e}$ for some $\epsilon_e \in E(\Upsilon)$. The *label map* $\vec{f} : V(\Gamma) \cup E(\Gamma) \to V(\Upsilon) \cup E(\Upsilon)_c$ sends a vertex $v \in V(\Gamma)$ to the vertex $\sigma_v \in V(\Upsilon)$ and an edge $e \in E(\Gamma)$ to the edge edge $\epsilon_e \in E(\Upsilon)_c$. Moreover, $\vec{f}$ defines a map from the graph Γ to the graph Υ: if $(e, v) \in F(\Gamma)$ then $(\epsilon_e, \sigma_v) \in F(\Upsilon)$.

(3) (degree) The *degree map* $\vec{\gamma}$ sends an edge $e \in E(\Gamma)$ to the conjugacy class $[\gamma_e] \in \mathrm{Conj}(H_{\epsilon_e})$, where $\gamma_e \in H_{\epsilon_e}$ is defined as in Section 6.2. We call $[\gamma_e]$ the degree of the map $f_e = f|_{\mathcal{C}_e} : \mathcal{C}_e \to \mathfrak{l}_{\epsilon_e}$. The positive integer $d_e := \phi_\epsilon(\gamma_e)$ is the degree of the map $\bar{f}_e : C_e = \mathbb{P}^1 \to \ell_{\epsilon_e} = \mathbb{P}^1$ between coarse moduli spaces. (Note that $\phi_\epsilon(\gamma_e)$ depends only on the conjugacy class $[\gamma_e]$ of γ_e.)

(4) (genus) The *genus map* $\vec{g} : V(\Gamma) \to \mathbb{Z}_{\geq 0}$ sends a vertex $v \in V(\Gamma)$ to a nonnegative integer g_v, where $g_v = 0$ if C_v is a point, and $g_v = h^1(C_v, \mathcal{O}_{C_v})$ if C_v is a curve.

(5) (marking) The *marking map* $\vec{s} : \{1, 2, \dots, n\} \to V(\Gamma)$ sends j to v if $x_j \in C_v$.

(6) (monodromy) For any $v \in V(\Gamma)$ we define $G_v = G_{\sigma_v}$. Suppose that $j \in \{1, \dots, n\}$ and $v \in \vec{s}(j) \in V(\Gamma)$. Let r_j be the cardinality of the inertia group $\mathrm{Aut}(\mathfrak{x}_j)$ of the j-th marked point $\mathfrak{x}_j$, and let ξ_j be the generator of $\mathrm{Aut}(\mathfrak{x}_j) \cong \mu_{r_j}$ which acts on the tangent line $T_{\mathfrak{x}_j}\mathcal{C}$ by $e^{2\pi\sqrt{-1}/r_j}$. Let $k_j \in G_v$ be the image of $\xi_j \in \mathrm{Aut}(\mathfrak{x}_j)$ under the group homomorphism $\mathrm{Aut}(\mathfrak{x}_j) \to \mathrm{Aut}(\mathfrak{p}_{\sigma_v}) = G_v$. The representability of f implies $\mathrm{Aut}(\mathfrak{x}_j) \to \mathrm{Aut}(\mathfrak{p}_{\sigma_v})$ is injective, so r_j is equal to the order of $k_j \in G_v$. The *monodromy map* $\vec{c}$ sends a marking $j \in \{1, \dots, n\}$ to the conjugacy class $c_j := [k_j] \in \mathrm{Conj}(G_v)$ where $v = \vec{s}(j)$,

The map $e \in E(\Gamma) \mapsto [\gamma_e] \in \mathrm{Conj}(H_{\epsilon_e})$ determines a map

$$(e, v) \in F(\Gamma) \mapsto c_{(e,v)} := [\pi_{(\epsilon_e,\sigma_v)}(\gamma_e)] \in \mathrm{Conj}(G_{\sigma_v}).$$

Given a flag (e, v), let $\mathfrak{y}_{(e,v)}$ be the intersection point of $\mathcal{C}_v$ and $\mathcal{C}_e$. (If C_v is a point then $\mathcal{C}_v = \{\mathfrak{y}_{(e,v)}\}$; if C_v is a curve then $\mathfrak{y}_{(e,v)}$ is a node.) Let $r_{(e,v)}$ be the cardinality of the inertia group of $\mathfrak{y}_{(e,v)}$, and let $\xi_{(e,v)}$ be the generator of $\mathrm{Aut}(\mathfrak{y}_{(e,v)}) \cong \mu_{r_{(e,v)}}$ which acts on the tangent line $T_{\mathfrak{y}_{(e,v)}}\mathcal{C}_e$ by $e^{2\pi\sqrt{-1}/r_{(e,v)}}$. Then the image of $\xi_{(e,v)}$ under the injective group homomorphism $\mathrm{Aut}(\mathfrak{y}_{(e,v)}) \to \mathrm{Aut}(\mathfrak{p}_{\sigma_v}) = G_v$ is an element $k_{(e,v)}$ in the conjugacy class $c_{(e,v)}$. The representability of f implies $r_{(e,v)}$ is equal to the order of $k_{(e,v)}$.

Given $v \in V(\Gamma)$, we define $E_v \subset E(\Gamma)$ and $S_v \subset \{1,\dots,n\}$ by

$$\begin{aligned} E_v &= \{e \in E(\Gamma) : (e,v) \in F(\Gamma)\} \\ S_v &= \{j \in \{1,\cdots,n\} : x_j \in C_v\}. \end{aligned} \tag{6.5}$$

Given a conjugacy class $c \in \mathrm{Conj}(G_v)$, let $\bar{c}$ denote the conjugacy class $\bar{c} = \{k^{-1} : k \in c\}$. Define $\vec{c}_v : E_v \cup S_v \to \mathrm{Conj}(G_v)$ by $\vec{c}_v(e) = \overline{c_{(e,v)}}$ if $e \in E_v$, and $\vec{c}_v(j) = c_j$ if $j \in S_v$. Then $V^{G_v}_{g_v,\vec{c}_v}$ is non-empty. Here we view $\vec{c}_v$ as an element in $\mathrm{Conj}(G_v)^{E_v \cup S_v} = \mathrm{Conj}(G_v)^{n_v}$, where $n_v := |E_v| + |S_v|$.

The inertia stack of $\hat{\mathcal{X}}^T_{\vec{\Upsilon}}$ is

$$\mathcal{I}(\hat{\mathcal{X}}^T_{\vec{\Upsilon}}) = \bigsqcup_{\sigma \in V(\Upsilon)} \mathcal{I}\mathfrak{p}_\sigma \cong \bigsqcup_{(\sigma,c) \in I^T_{\vec{\Upsilon}}} (\mathcal{B}G_\sigma)_c$$

where

$$I^T_{\vec{\Upsilon}} = \{(\sigma,c) : \sigma \in V(\Upsilon), c \in \mathrm{Conj}(G_\sigma)\}.$$

Connected components of $\mathcal{I}(\hat{\mathcal{X}}^T_{\vec{\Upsilon}})$ are in one-to-one correspondence with pairs $(\sigma,c) \in I^T_{\vec{\Upsilon}}$. The inclusion $\mathcal{I}(\hat{\mathcal{X}}^T_{\vec{\Upsilon}}) \hookrightarrow \mathcal{I}(\hat{\mathcal{X}}_{\vec{\Upsilon}})$ induces a surjective map $j_0 : I^T_{\vec{\Upsilon}} \to I_{\vec{\Upsilon}}$ such that the image of $(\mathcal{B}G_\sigma)_c$ under j_0 is contained in $(\hat{\mathcal{X}}_{\vec{\Upsilon}})_{j_0(\sigma,c)}$. Let $G(\vec{\Upsilon})$ be the set of decorated graphs associated to some T fixed twisted stable maps to $\hat{\mathcal{X}}_{\vec{\Upsilon}}$. Then $G(\vec{\Upsilon})$ is a countable infinite set. We have a map $\overline{\mathcal{M}}(\hat{\mathcal{X}}_{\vec{\Upsilon}})^T \to G(\vec{\Upsilon})$; let $\mathcal{F}_{\vec{\Gamma}} \subset \overline{\mathcal{M}}(\hat{\mathcal{X}}_{\vec{\Upsilon}})^T$ be the preimage of $\vec{\Gamma} \in G(\vec{\Upsilon})$. Then

$$\overline{\mathcal{M}}(\hat{\mathcal{X}}_{\vec{\Upsilon}})^T = \bigsqcup_{\vec{\Gamma} \in G(\vec{\Upsilon})} \mathcal{F}_{\vec{\Gamma}}.$$

where each $\mathcal{F}_{\vec{\Gamma}}$ is a union of connected components.

DEFINITION 6.1. *Let $\vec{\Gamma} = (\Gamma, \vec{f}, \vec{\gamma}, \vec{g}, \vec{s}, \vec{c}) \in G(\vec{\Upsilon})$. We define the genus of $\vec{\Gamma}$ to be*

$$g(\vec{\Gamma}) := b_1(\Gamma) + \sum_{v \in V(\Gamma)} g_v = |E(\Gamma)| - |V(\Gamma)| + 1 + \sum_{v \in V(\Gamma)} g_v \tag{6.6}$$

and define the degree of $\vec{\Gamma}$ to be

$$\hat{\beta}(\vec{\Gamma}) := \sum_{e \in E(\Gamma)} d_e[\ell_{\epsilon_e}] = \sum_{\epsilon \in E(\Upsilon)_c} \Big(\sum_{e \in \vec{f}^{-1}(\epsilon)} d_e \Big) [\ell_\epsilon] \in \mathrm{Eff}(\hat{\mathcal{X}}_{\vec{\Upsilon}}). \tag{6.7}$$

If the domain of the marking map $\vec{s}$ is $\{1,\dots,n\}$, we define $n(\vec{\Gamma}) = n$, and define

$$\vec{i}(\vec{\Gamma}) := (j_0(\sigma_1,c_1),\dots,j_0(\sigma_n,c_n)) \in (I_{\vec{\Upsilon}})^{n(\vec{\Gamma})},$$

where $\sigma_j = \vec{f} \circ \vec{s}(j) \in V(\Upsilon)$ *and* $c_j \in \mathrm{Conj}(G_{\sigma_j})$.

Given nonnegative integers g, n and an effective class $\hat{\beta} \in \mathrm{Eff}(\hat{\mathcal{X}}_{\vec{\Upsilon}})$, define

$$G_{g,n}(\vec{\Upsilon}, \hat{\beta}) := \{\vec{\Gamma} \in G(\vec{\Upsilon}) : g(\vec{\Gamma}) = g, n(\vec{\Gamma}) = n, \hat{\beta}(\vec{\Gamma}) = \hat{\beta}\}.$$

Then $G_{g,n}(\vec{\Upsilon}, \hat{\beta})$ is a finite set, and

$$\overline{\mathcal{M}}_{g,n}(\hat{\mathcal{X}}_{\vec{\Upsilon}}, \hat{\beta})^T = \bigsqcup_{\vec{\Gamma} \in G_{g,n}(\vec{\Upsilon}, \hat{\beta})} \mathcal{F}_{\vec{\Gamma}}. \tag{6.8}$$

Given $\vec{i} = (i_1, \ldots, i_n) \in (I_{\vec{\Upsilon}})^n$, define

$$G_{g,\vec{i}}(\vec{\Upsilon}, \hat{\beta}) := \{\vec{\Gamma} \in G(\vec{\Upsilon}) : g(\vec{\Gamma}) = g, \vec{i}(\vec{\Gamma}) = \vec{i}, \hat{\beta}(\vec{\Gamma}) = \hat{\beta}\},$$

which is a subset of $G_{g,n}(\vec{\Upsilon}, \hat{\beta})$. Then

$$\overline{\mathcal{M}}_{g,\vec{i}}(\hat{\mathcal{X}}_{\vec{\Upsilon}}, \hat{\beta})^T = \bigsqcup_{\vec{\Gamma} \in G_{g,\vec{i}}(\vec{\Upsilon}, \hat{\beta})} \mathcal{F}_{\vec{\Gamma}}.$$

In the remainder of this section, we give an explicit description of $\mathcal{F}_{\vec{\Gamma}}$ for each decorated graph $\vec{\Gamma} \in G(\vec{\Upsilon})$. We first introduce some notation. Let

$$\begin{aligned}
V^S(\vec{\Gamma}) &= \{v \in V(\Gamma) : 2g_v - 2 + n_v > 0\} = \{v \in V(\Gamma) : \mathcal{C}_v \text{ is a curve}\},\\
V^{0,1}(\vec{\Gamma}) &= \{v \in V(\Gamma) : g_v = 0, |S_v| = 0, |E_v| = 1\},\\
V^{1,1}(\vec{\Gamma}) &= \{v \in V(\Gamma) : g_v = 0, |S_v| = |E_v| = 1\},\\
V^{0,2}(\vec{\Gamma}) &= \{v \in V(\Gamma) : g_v = 0, |S_v| = 0, |E_v| = 2\}.
\end{aligned}$$

Then $V(\Gamma)$ is a disjoint union of $V^S(\vec{\Gamma}), V^{0,1}(\vec{\Gamma}), V^{1,1}(\vec{\Gamma})$, and $V^{0,2}(\vec{\Gamma})$. We say a vertex v is stable if $v \in V^S(\vec{\Gamma})$; otherwise we call it unstable. Let

$$F^S(\vec{\Gamma}) = \{(e, v) \in F(\Gamma) : v \in V^S(\vec{\Gamma})\}$$

be the set of stable flags. Note that $V(\Gamma)$, $E(\Gamma)$, $F(\Gamma)$ depend only on the underlying graph Γ, while $V^S(\vec{\Gamma})$, $V^{0,1}(\vec{\Gamma})$, $V^{1,1}(\vec{\Gamma})$, $V^{0,2}(\vec{\Gamma})$, $F^S(\vec{\Gamma})$ depend on the decorated graph $\vec{\Gamma}$.

Given an edge $e \in E(\Gamma)$, let $v, v' \in V(\Gamma)$ be its two ends. By the discussion in Section 6.2, the map $f_e := f|_{\mathcal{C}_e} : \mathcal{C}_e \to \mathfrak{l}_\epsilon$, where $\epsilon = \vec{f}(e)$, is determined by $\gamma_e \in H_{\epsilon_e}$. The automorphism group of f_e is a finite group

$$\mathrm{Aut}(f_e) = \mathrm{Aut}(f|_{O_e}) \cong c_{H_\epsilon}(\gamma_e)/\langle \gamma_e \rangle.$$

The moduli space of f_e is

$$\mathcal{M}_e = \mathcal{B}(\mathrm{Aut}(f_e)).$$

Given a stable vertex $v \in V^S(\vec{\Gamma})$, the map $f_v := f|_{\mathcal{C}_v} : \mathcal{C}_v \to \mathfrak{p}_\sigma = \mathcal{B}G_v$, where $\sigma = \vec{f}(v)$, represents a point in $\overline{\mathcal{M}}_{g_v, \vec{c}_v}(\mathcal{B}G_v)$, where $\vec{c}_v \in \mathrm{Conj}(G_v)^{E_v \cup S_v}$. To obtain a T fixed point $[f : (\mathcal{C}, \mathfrak{x}_1, \ldots, \mathfrak{x}_n) \to \hat{\mathcal{X}}_{\vec{\Upsilon}}]$, we glue the the maps

$$\{f_v : \mathcal{C}_v \to \mathfrak{p}_{\sigma_v} \mid v \in V^S(\vec{\Gamma})\}, \quad \{f_e : \mathcal{C}_e \to \mathfrak{l}_{\epsilon_e} \mid e \in E(\Gamma)\}$$

along the nodes

$$\{\mathfrak{y}_{(e,v)} = \mathcal{C}_e \cap \mathcal{C}_v : (e,v) \in F^S(\vec{\Gamma})\} \cup \{\mathfrak{y}_v = \mathcal{C}_v \mid v \in V^{0,2}(\vec{\Gamma})\}.$$

We define $\widetilde{\mathcal{M}}_{\vec{\Gamma}}$ by the following 2-cartesian diagram

$$\begin{array}{ccc} \widetilde{\mathcal{M}}_{\vec{\Gamma}} & \xrightarrow{f_E} & \prod_{e\in E(\Gamma)} \mathcal{M}_e \\ f_V \downarrow & & \mathrm{ev}_E \downarrow \\ \mathcal{M}_{\vec{\Gamma}} := \prod_{v\in V^S(\vec{\Gamma})} \overline{\mathcal{M}}_{g_v,\vec{c}_v}(\mathcal{B}G_v) & \xrightarrow{\mathrm{ev}_V} & \prod_{(e,v)\in F^S(\vec{\Gamma})} \overline{\mathcal{I}}\mathcal{B}G_v \times \prod_{v\in V^{0,2}(\vec{\Gamma})} \overline{\mathcal{I}}\mathcal{B}G_v \end{array}$$

where ev_V and ev_E are given by evaluation at nodes, and $\overline{\mathcal{I}}\mathcal{B}G_v$ is the rigidified inertia stack. More precisely:

- For every stable flag $(e,v) \in F^S(\vec{\Gamma})$, let $\mathrm{ev}_{(e,v)}$ be the evaluation map at the node $\mathfrak{y}_{(e,v)}$,
- For every $v \in V(\Gamma)$, let ι be the involution on $\mathcal{I}\mathcal{B}G_v$ induced by the involution $G_v \to G_v$ given by $h \mapsto h^{-1}$.
- Define

$$\mathrm{ev}_V = \prod_{(e,v)\in F^S(\vec{\Gamma})} \mathrm{ev}_{(e,v)}$$

$$\mathrm{ev}_E = \prod_{(e,v)\in F^S(\vec{\Gamma})} \big(\iota \circ \mathrm{ev}_{(e,v)}\big) \times \prod_{\substack{v \in V^{0,2}(\vec{\Gamma}) \\ E_v = \{e_1, e_2\}}} \mathrm{ev}_{(e_1,v)} \times \big(\iota \circ \mathrm{ev}_{(e_2,v)}\big)$$

- If $v \in V^{0,2}(\vec{\Gamma})$ and $E_v = \{e_1, e_2\}$, we define $r_v = r_{(e_1,v)} = r_{(e_2,v)}$, and define $c_v = c_{(e_1,v)} = \overline{c_{(e_2,v)}}$.

We have

$$\mathcal{F}_{\vec{\Gamma}} = \widetilde{\mathcal{M}}_{\vec{\Gamma}}/\mathrm{Aut}(\vec{\Gamma})$$

which is a proper smooth DM stack of dimension

$$d_{\vec{\Gamma}} = \sum_{v\in V^S(\vec{\Gamma})} (3g_v - 3 + n_v).$$

It has a fundamental class

$$[\mathcal{F}_{\vec{\Gamma}}] = c_{\vec{\Gamma}}[\mathcal{M}_{\vec{\Gamma}}] \in= A_{d_{\vec{\Gamma}}}(\mathcal{F}_{\vec{\Gamma}};\mathbb{Q}) = A_{d_{\vec{\Gamma}}}(\mathcal{M}_{\vec{\Gamma}};\mathbb{Q})$$

where

$$[\mathcal{M}_{\vec{\Gamma}}] = \prod_{v\in V^S(\vec{\Gamma})} \big[\overline{\mathcal{M}}_{g_v,\vec{c}_v}(\mathcal{B}G_v)\big], \tag{6.9}$$

and

$$\begin{aligned} c_{\vec{\Gamma}} &= \frac{1}{|\mathrm{Aut}(\vec{\Gamma})| \prod_{e\in E(\Gamma)} |\mathrm{Aut}(f_e)|} \cdot \prod_{(e,v)\in F^S(\vec{\Gamma})} \frac{|G_v|}{r_{(e,v)}|c_{(e,v)}|} \cdot \prod_{v\in V^{0,2}(\vec{\Gamma})} \frac{|G_v|}{r_v|c_v|} \\ &= \frac{1}{|\mathrm{Aut}(\vec{\Gamma})| \prod_{e\in E(\Gamma)} |\mathrm{Aut}(f_e)|} \cdot \prod_{(e,v)\in F^S(\vec{\Gamma})} \frac{|C_{G_v}(k_{(e,v)})|}{r_{(e,v)}} \cdot \prod_{v\in V^{0,2}(\vec{\Gamma})} \frac{|C_{G_v}(k_v)|}{r_v}. \end{aligned} \tag{6.10}$$

In the second line in Equation (6.10) above, $k_{(e,v)}$ (resp. k_v) is any element in the conjugacy class $c_{(e,v)}$ (resp. c_v), and $C_{G_v}(k)$ denotes the centralizer of k in G_v.

6.4. Virtual tangent and normal bundles. Given $\vec{\Gamma} \in G(\vec{\Upsilon})$ and a twisted stable map $f : (\mathcal{C}, \mathfrak{x}_1, \ldots, \mathfrak{x}_n) \to \hat{\mathcal{X}}_{\vec{\Upsilon}}$ which represents a point ξ in $\mathcal{F}_{\vec{\Gamma}} \subset \overline{\mathcal{M}}_{g,n}(\hat{\mathcal{X}}_{\vec{\Upsilon}}, \hat{\beta})$, the tangent space T^1_ξ and obstruction space T^2_ξ of $\overline{\mathcal{M}}_{g,n}(\hat{\mathcal{X}}_{\vec{\Upsilon}}, \hat{\beta})$ at ξ fits in an following exact sequence of T-representations

$$0 \to B_1 \to B_2 \to T^1_\xi \to B_4 \to B_5 \to T^2_\xi \to 0 \tag{6.11}$$

where

$$B_1 = \mathrm{Ext}^0(\Omega_{\mathcal{C}}(\mathfrak{x}_1 + \cdots + \mathfrak{x}_n), \mathcal{O}_{\mathcal{C}}), \quad B_2 = H^0(\mathcal{C}, f^*T\mathcal{X})$$
$$B_4 = \mathrm{Ext}^1(\Omega_{\mathcal{C}}(\mathfrak{x}_1 + \cdots + \mathfrak{x}_n), \mathcal{O}_{\mathcal{C}}), \quad B_5 = H^1(\mathcal{C}, f^*T\mathcal{X})$$

Let B_i^m and B_i^f be the moving and fixed parts of B_i, respectively; let $T^{i,m}_\xi$ and $T^{i,f}_\xi$ be the moving and fixed parts of T^i_ξ, respectively. The exact sequence (6.11) splits into the following two exact sequences:

$$0 \to B_1^f \to B_2^f \to T^{1,f}_\xi \to B_4^f \to B_5^f \to T^{2,f}_\xi \to 0, \tag{6.12}$$

$$0 \to B_1^m \to B_2^m \to T^{1,m}_\xi \to B_4^m \to B_5^m \to T^{2,m}_\xi \to 0. \tag{6.13}$$

Varying ξ in the fixed locus $\mathcal{F}_{\vec{\Gamma}}$ gives rise to the following two exact sequences of sheaves of $\mathcal{O}_{\mathcal{F}_{\vec{\Gamma}}}$-modules on $\mathcal{F}_{\vec{\Gamma}}$:

$$0 \to B_1^f \to B_2^f \to T^{1,f} \to B_4^f \to B_5^f \to T^{2,f} \to 0, \tag{6.14}$$

$$0 \to B_1^m \to B_2^m \to T^{1,m} \to B_4^m \to B_5^m \to T^{2,m} \to 0. \tag{6.15}$$

Here we abuse notation: B_i^f (resp. B_i^m) are complex vector spaces in (6.12) (resp. (6.13)) and are sheaves over $\mathcal{F}_{\vec{\Gamma}}$ in (6.14) (resp. (6.15)). The restriction of the exact sequence (5.4) to $\mathcal{F}_{\vec{\Gamma}}$ also splits into two exact sequences of $\mathcal{O}_{\mathcal{F}_{\vec{\Gamma}}}$-modules:

$$0 \to T^{1,f} \to E^f \to F^f \to T^{2,f} \to 0. \tag{6.16}$$

$$0 \to T^{1,m} \to E^m \to F^m \to T^{2,m} \to 0. \tag{6.17}$$

The dual complex of $[E^f \to F^f]$ is a perfect obstruction theory on the smooth proper DM stack $\mathcal{F}_{\vec{\Gamma}}$; in other words, $\mathcal{F}_{\vec{\Gamma}}$ is equipped with a virtual tangent bundle

$$T^{\mathrm{vir}}_{\vec{\Gamma}} = T^{1,f} - T^{2,f}.$$

As we will see in Section 6.5-6.7 below, $T^{1,f} = T\mathcal{F}_{\vec{\Gamma}}$ is the tangent bundle of the smooth DM stack $\mathcal{F}_{\vec{\Gamma}}$, whereas $T^{2,f} = 0$, so the virtual tangent bundle is the tangent bundle. By [6, Proposition 5.5],

Theorem 6.2.

$$[\mathcal{F}_{\vec{\Gamma}}]^{\mathrm{vir}} = [\mathcal{F}_{\vec{\Gamma}}] = c_{\vec{\Gamma}} \prod_{v \in V^S(\vec{\Gamma})} [\overline{\mathcal{M}}_{g_v, \vec{c}_v}(\mathcal{B}G_v)].$$

The virtual normal bundle of $\mathcal{F}_{\vec{\Gamma}}$ in $\overline{\mathcal{M}}(\hat{\mathcal{X}}_{\vec{\Upsilon}})$ is

$$N^{\mathrm{vir}}_{\vec{\Gamma}} = T^{1,m} - T^{2,m}.$$

So

$$\frac{1}{e_T(N^{\mathrm{vir}}_{\vec{\Gamma}})} = \frac{e_T(T^{2,m})}{e_T(T^{1,m})} = \frac{e_T(B_1^m)}{e_T(B_4^m)} \frac{e_T(B_5^m)}{e_T(B_2^m)}. \tag{6.18}$$

We will compute $\frac{e_T(B_1^m)}{e_T(B_4^m)}$ and $\frac{e_T(B_5^m)}{e_T(B_2^m)}$ in Section 6.5 and Section 6.6, respectively.

6.5. Deformation of the domain. Recall that the nodes of $\mathcal{C}$ are

$$\{\mathfrak{y}_{(e,v)} = \mathcal{C}_e \cap \mathcal{C}_v : (e,v) \in F^S(\vec{\Gamma})\} \cup \{\mathfrak{y}_v = \mathcal{C}_v : (e,v) \in V^{0,2}(\vec{\Gamma})\}.$$

6.5.1. Infinitesimal automorphisms of the domain.

$$\begin{aligned} B_1^f &= \bigoplus_{\substack{e \in E(\Gamma) \\ (e,v),(e,v') \in F(\Gamma)}} \mathrm{Hom}(\Omega_{\mathcal{C}_e}(\mathfrak{y}_{(e,v)} + \mathfrak{y}_{(e,v')}), \mathcal{O}_{\mathcal{C}_e}) \\ &= \bigoplus_{\substack{e \in E(\Gamma) \\ (e,v),(e,v') \in F(\Gamma)}} H^0(\mathcal{C}_e, T\mathcal{C}_e(-\mathfrak{y}_{(e,v)} - \mathfrak{y}_{(e,v')})), \\ B_1^m &= \bigoplus_{\substack{v \in V^{0,1}(\vec{\Gamma}) \\ (e,v) \in F(\Gamma)}} T_{\mathfrak{y}_{(e,v)}} \mathcal{C}_e. \end{aligned}$$

We define

$$w_{(e,v)} := e^T(T_{\mathfrak{y}_{(e,v)}} \mathcal{C}_e) = \frac{r_{(\epsilon_e,\sigma_v)} \mathbf{w}_{(\epsilon_e,\sigma_v)}}{r_{(e,v)} d_e} \in H_T^2(\mathfrak{y}_{(e,v)}) = M_{\mathbb{Q}}.$$

6.5.2. Inifinitesimal deformations of the domain. Given any $v \in V^S(\Gamma)$, define a divisor $\mathbf{x}_v$ of $\mathcal{C}_v$ by

$$\mathbf{x}_v = \sum_{i \in S_v} \mathfrak{x}_i + \sum_{e \in E_v} \mathfrak{y}_{(e,v)}.$$

Then

$$\begin{aligned} B_4^f &= \bigoplus_{v \in V^S(\vec{\Gamma})} \mathrm{Ext}^1(\Omega_{\mathcal{C}_v}(\mathbf{x}_v), \mathcal{O}_{\mathcal{C}}) = \bigoplus_{v \in V^S(\vec{\Gamma})} T_{[(\mathcal{C}_v,\mathbf{x}_v)]} \overline{\mathcal{M}}_{g_v, \vec{i}_v}(\mathcal{B}G_v) \\ B_4^m &= \bigoplus_{\substack{v \in V^{0,2}(\vec{\Gamma}) \\ E_v = \{e,e'\}}} T_{\mathfrak{y}_v} \mathcal{C}_e \otimes T_{\mathfrak{y}_v} \mathcal{C}_{e'} \oplus \bigoplus_{(e,v) \in F^S(\vec{\Gamma})} T_{\mathfrak{y}(e,v)} \mathcal{C}_v \otimes T_{\mathfrak{y}_{(e,v)}} \mathcal{C}_e \end{aligned}$$

where $e^T(T_{\mathfrak{y}_v}\mathcal{C}_e \otimes T_{\mathfrak{y}_v}\mathcal{C}_{e'}) = w_{(e,v)} + w_{(e',v)}$ if $v \in V^{0,2}(\vec{\Gamma})$ and $E_v = \{e,e'\}$, and $e^T(T_{\mathfrak{y}_{(e,v)}}\mathcal{C}_v \otimes T_{\mathfrak{y}_{(e,v)}}\mathcal{C}_e) = w_{(e,v)} - \frac{\bar{\psi}_{(e,v)}}{r_{(e,v)}}$ if $(e,v) \in F^S(\vec{\Gamma})$.

6.5.3. Unifying stable and unstable vertices. From the discussion in Section 6.5.1 and Section 6.5.2,

$$\begin{aligned}\frac{e^T(B_1^m)}{e^T(B_4^m)} = &\prod_{\substack{v\in V^{0,1}(\vec{\Gamma})\\(e,v)\in F(\Gamma)}} w_{(e,v)} \prod_{\substack{v\in V^{0,2}(\vec{\Gamma})\\E_v=\{e,e'\}}} \frac{1}{w_{(e,v)}+w_{(e',v)}}\\ &\cdot \prod_{v\in V^S(\vec{\Gamma})} \frac{1}{\prod_{e\in E_v}\left(w_{(e,v)}-\frac{\bar{\psi}_{(e,v)}}{r_{(e,v)}}\right)}.\end{aligned} \tag{6.19}$$

To unify the stable and unstable vertices, we use the following convention for the empty sets $\overline{\mathcal{M}}_{0,(\{1\})}(\mathcal{B}G)$ and $\overline{\mathcal{M}}_{0,([h],[h^{-1}])}(\mathcal{B}G)$, where $1 \in G$ is the identity element, and $[h]$ is the conjugacy class of $h \in G$. Let G be a finite group and let w_1, w_2 be formal variables.

- $\overline{\mathcal{M}}_{0,(\{1\})}(\mathcal{B}G)$ is a -2 dimensional space, and
$$\int_{\overline{\mathcal{M}}_{0,(\{1\})}(\mathcal{B}G)} \frac{1}{w_1-\bar{\psi}_1} = \frac{w_1}{|G|} \tag{6.20}$$
- $\overline{\mathcal{M}}_{0,([h],[h^{-1}])}(\mathcal{B}G)$ is a -1 dimensional space, and
$$\int_{\overline{\mathcal{M}}_{0,([h],[h^{-1}])}(\mathcal{B}G)} \frac{1}{(w_1-\bar{\psi}_1)(w_2-\bar{\psi}_2)} = \frac{1}{(w_1+w_2)\cdot|C_G(h)|} \tag{6.21}$$
$$\int_{\overline{\mathcal{M}}_{0,([h],[h^{-1}])}(\mathcal{B}G)} \frac{1}{w_1-\bar{\psi}_1} = \frac{1}{|C_G(h)|} \tag{6.22}$$

From (6.20), (6.21), (6.22), we obtain the following identities for non-stable vertices:

(i) If $v \in V^{0,1}(\vec{\Gamma})$ and $(e,v) \in F(\Gamma)$, then $r_{(e,v)} = 1$, and
$$|G_v| \int_{\overline{\mathcal{M}}_{0,(\{1\})}(\mathcal{B}G_v)} \frac{1}{w_{(e,v)}-\bar{\psi}_{(e,v)}} = w_{(e,v)}.$$

(ii) If $v \in V^{0,2}(\vec{\Gamma})$, $E_v = \{e,e'\}$, and $k_v \in c_{(e,v)}$, then $c_{(e',v)} = \overline{c_{(e,v)}} = [k_v^{-1}]$ and
$$\begin{aligned}&\frac{|C_{G_v}(k_v)|}{r_v}\cdot\frac{|C_{G_v}(k_v)|}{r_v}\cdot\int_{\overline{\mathcal{M}}_{0,([k_v],[k_v^{-1}])}(\mathcal{B}G_v)} \frac{1}{\left(w_{(e,v)}-\frac{\bar{\psi}_{(e,v)}}{r_v}\right)\left(w_{(e',v)}-\frac{\bar{\psi}_{(e',v)}}{r_v}\right)}\\ &= \frac{|C_{G_v}(k_v)|}{r_v}\cdot\frac{1}{w_{(e,v)}+w_{(e',v)}}.\end{aligned}$$

(iii) If $v \in V^{1,1}(\vec{\Gamma})$ and $(e,v) \in F(\Gamma)$, then
$$\frac{|C_{G_v}(h)|}{r_{(e,v)}} \int_{\overline{\mathcal{M}}_{0,([h],[h^{-1}])}(\mathcal{B}G_v)} \frac{1}{w_{(e,v)}-\frac{\bar{\psi}_1}{r_{(e,v)}}} = 1.$$

We then redefine $\mathcal{M}_{\vec{\Gamma}}$ and $c_{\vec{\Gamma}}$ as follows:

$$\mathcal{M}_{\vec{\Gamma}} = \prod_{v\in V(\Gamma)} \overline{\mathcal{M}}_{g_v,\vec{i}_v}(\mathcal{B}G_v), \quad [\mathcal{F}_{\vec{\Gamma}}] = c_{\vec{\Gamma}}[\mathcal{M}_{\vec{\Gamma}}], \tag{6.23}$$

$$c_{\vec{\Gamma}} = \frac{1}{|\mathrm{Aut}(\vec{\Gamma})|\prod_{e\in E(\Gamma)}|\mathrm{Aut}(f_e)|}\prod_{(e,v)\in F(\Gamma)}\frac{|c_{G_v}(k_{(e,v)})|}{r_{(e,v)}}, \tag{6.24}$$

where $k_{(e,v)}$ is an element in the conjugacy class $c_{(e,v)}$.

With the above conventions (6.20)–(6.24), we may rewrite (6.19) in the following form.

Proposition 6.3.

$$\frac{e^T(B_1^m)}{e^T(B_4^m)} = \prod_{v\in V(\Gamma)}\frac{1}{\prod_{e\in E_v}\left(w_{(e,v)} - \frac{\bar{\psi}_{(e,v)}}{r_{(e,v)}}\right)}.$$

The following lemma shows that the conventions (6.20), (6.21), and (6.22) are consistent with the stable case $\overline{\mathcal{M}}_{0,(c_1,\dots,c_n)}(\mathcal{B}G)$, $n \geq 3$.

Lemma 6.4. *Let G be a finite group. Let $\vec{c} = (c_1,\dots,c_n) \in Conj(G)^n$. Let $w_1,\dots,w_n$ be formal variables. Then*

(a) $$\int_{\overline{\mathcal{M}}_{0,\vec{c}}(\mathcal{B}G)}\frac{1}{\prod_{i=1}^n(w_i-\bar{\psi}_i)} = \frac{|V_{0,\vec{c}}^G|}{|G|\cdot w_1\cdots w_n}\Big(\frac{1}{w_1}+\cdots\frac{1}{w_n}\Big)^{n-3}.$$

(b) $$\int_{\overline{\mathcal{M}}_{0,\vec{c}}(\mathcal{B}G)}\frac{1}{w_1-\bar{\psi}_1} = \frac{|V_{0,\vec{c}}^G|}{|G|}w_1^{2-n}.$$

Proof. The unstable cases $n = 1$ and $n = 2$ follow from the definitions (6.20) and (6.21), respectively. The stable case ($n \geq 3$) follows from Theorem 5.3 and the well known identity below.

$$\int_{\overline{\mathcal{M}}_{0,n}}\psi_1^{a_1}\cdots\psi_n^{a_n} = \frac{(n-3)!}{a_1!\cdots a_n!}.$$

□

6.6. Deformation of the map. We first introduce some notation. Given $\sigma \in V(\Upsilon)$ and $c \in \mathrm{Conj}(G_\sigma)$, let $\big(T_{\mathfrak{p}_\sigma}\mathcal{X}\big)^c$ denote the subspace of $T_{\mathfrak{p}_\sigma}\mathcal{X}$ which is invariant under the action of any $k \in c$ (or equivalently, of some $k \in c$). Then $\big(T_{\mathfrak{p}_\sigma}\mathcal{X}\big)^c = \big(T_{\mathfrak{p}_\sigma}\mathcal{X}\big)^{\bar{c}}$, where $\bar{c} = \{k^{-1} : k \in c\}$.

Consider the normalization sequence

$$\begin{aligned} 0 \to \mathcal{O}_{\mathcal{C}} \to &\bigoplus_{v\in V^S(\vec{\Gamma})}\mathcal{O}_{\mathcal{C}_v} \oplus \bigoplus_{e\in E(\Gamma)}\mathcal{O}_{\mathcal{C}_e} \\ &\to \bigoplus_{v\in V^{0,2}(\vec{\Gamma})}\mathcal{O}_{\mathfrak{y}_v} \oplus \bigoplus_{(e,v)\in F^S(\vec{\Gamma})}\mathcal{O}_{\mathfrak{y}_{(e,v)}} \to 0. \end{aligned} \tag{6.25}$$

We twist the above short exact sequence of sheaves by $f^*T\mathcal{X}$. The resulting short exact sequence gives rise a long exact sequence of cohomology groups

$$\begin{aligned} 0 \to B_2 \to &\bigoplus_{v\in V^S(\vec{\Gamma})}H^0(\mathcal{C}_v) \oplus \bigoplus_{e\in E(\Gamma)}H^0(\mathcal{C}_e) \\ \to &\bigoplus_{\substack{v\,\in\,V^{0,2}(\vec{\Gamma})\\ E_v\,=\,\{e,e'\}}}(T_{f(\mathfrak{y}_v)}\mathcal{X})^{c_{(e,v)}} \oplus \bigoplus_{(e,v)\in F^S(\vec{\Gamma})}\big(T_{f(\mathfrak{y}_{(e,v)})}\mathcal{X}\big)^{c_{(e,v)}} \\ \to B_5 \to &\bigoplus_{v\in V^S(\vec{\Gamma})}H^1(\mathcal{C}_v) \oplus \bigoplus_{e\in E(\Gamma)}H^1(\mathcal{C}_e) \to 0. \end{aligned}$$

where $H^i(\mathcal{C}_v) = H^i(\mathcal{C}_v, f_v^*T\mathcal{X})$ and $H^i(\mathcal{C}_e) = H^i(\mathcal{C}_e, f_e^*T\mathcal{X})$ for $i = 0, 1$.

$f(\mathfrak{y}_v) = \mathfrak{p}_{\sigma_v} = f(\mathfrak{y}_{(e,v)})$. Given $(e, v) \in F(\Gamma)$, define

$$\mathbf{h}(e,v) = e^T\big((T_{\mathfrak{p}_\sigma}\mathcal{X})^{c_{(e,v)}}\big) = \prod_{\substack{\epsilon\in E_{\sigma_v}\\ \phi_{(\epsilon,\sigma_v)}(c_{(e,v)})=1}} \mathbf{w}_{(\epsilon,\sigma_v)}. \tag{6.26}$$

The map $B_1 \to B_2$ sends $H^0(\mathcal{C}_e, T\mathcal{C}_e(-\mathfrak{y}_{(e,v)} - \mathfrak{y}_{(e',v)}))$ isomorphically to $H^0(\mathcal{C}_e, f_e^*T\mathfrak{l}_{\epsilon_e})^f$, the fixed part of $H^0(\mathcal{C}_e, f_e^*T\mathfrak{l}_{\epsilon_e})$.

It remains to compute

$$\mathbf{h}(v) := \frac{e^T(H^1(\mathcal{C}_v, f_v^*T\mathcal{X})^m)}{e^T(H^0(\mathcal{C}_v, f_v^*T\mathcal{X})^m)}, \quad \mathbf{h}(e) := \frac{e^T(H^1(\mathcal{C}_e, f_e^*T\mathcal{X})^m)}{e^T(H^0(\mathcal{C}_e, f_e^*T\mathcal{X})^m)}.$$

The formulae of $\mathbf{h}(v)$ and $\mathbf{h}(e)$ will be given by Equation (6.27) and Equation (6.30) below. Before deriving these formulae , we introduce some notation.

- If $v \in V^S(\vec{\Gamma})$, then there is a cartesian diagram

$$\begin{array}{ccc} \widetilde{\mathcal{C}}_v & \xrightarrow{\widetilde{f}_v} & \mathrm{pt} \\ \downarrow & & \downarrow \\ \mathcal{C}_v & \xrightarrow{f_v} & \mathcal{B}G_v. \end{array}$$

  Let $\widehat{G}_v$ denote the subgroup of G_v generated by the monodromies of the G_v-cover $\widetilde{\mathcal{C}}_v \to \mathcal{C}_v$. Then the number of connected components of $\widetilde{\mathcal{C}}_v$ is $|G_v/\widehat{G}_v|$, and each connected component is a $\widehat{G}_v$-cover of $\mathcal{C}_v$.
- Given a 1-dimensional representation $\phi : G_v \to \mathbb{C}^*$, we ϕ of G_v, we define

$$\Lambda^\vee_\phi(u) = \sum_{i=0}^{\mathrm{rank}\mathbb{E}_\phi} (-1)^i \lambda_i^\phi u^{\mathrm{rank}\mathbb{E}_\phi - i},$$

  where $\lambda_i^\phi \in A^i(\overline{\mathcal{M}}_{g_v,\vec{c}_v}(\mathcal{B}G_v))$ are Hurwitz-Hodge classes associated to ϕ, and $\mathrm{rank}\mathbb{E}_\rho$ is the rank of the ϕ-twisted Hurwitz-Hodge bundle $\mathbb{E}_\rho \to \overline{\mathcal{M}}_{g_v,\vec{c}_v}(\mathcal{B}G_v)$ (see Section 5.6).
- Given a G_v representation V, let V^{G_v} denote the subspace on which G_v acts trivially.

LEMMA 6.5. *Suppose that $v \in V^S(\vec{\Gamma})$ and $\vec{f}(v) = \sigma \in V(\Upsilon)$. Then*

$$\mathbf{h}(v) = \frac{\prod_{\epsilon\in E_\sigma} \Lambda^\vee_{\phi(\epsilon,\sigma)}(\mathbf{w}_{(\epsilon,\sigma)})}{\prod_{\epsilon\in E_\sigma, \widehat{G}_v\subset G_\epsilon} \mathbf{w}_{(\epsilon,\sigma)}}. \tag{6.27}$$

Proof. Let $\mathbb{C}_\phi$ denote the 1-dimensional representation associated to $\phi : G_v \to \mathbb{C}^*$. Then

$$H^i(\mathcal{C}_v, f_v^*T\mathcal{X}) = \Big(H^i(\widetilde{\mathcal{C}}_v, \mathcal{O}_{\widetilde{\mathcal{C}}_v}) \otimes T_\sigma\mathcal{X}\Big)^{G_v} \cong \bigoplus_{\epsilon\in E_\sigma} \Big(H^i(\widetilde{\mathcal{C}}_v, \mathcal{O}_{\widetilde{\mathcal{C}}_v}) \otimes \mathbb{C}_{\phi_{(\epsilon,\sigma)}}\Big)^{G_v},$$

where $H^0(\widetilde{\mathcal{C}}_v, \mathcal{O}_{\widetilde{\mathcal{C}}_v})$ is the regular representation of $G_v/\widehat{G}_v$. The surjective group homomorphism $G_v \to G_v/\widehat{G}_v$ induces an injective map $\mathrm{Hom}(G_v/\widehat{G}_v, \mathbb{C}^*) \to \mathrm{Hom}(G_v, \mathbb{C}^*)$, whose image contains $\phi^{-1}_{(\epsilon,\sigma)}$ iff $\widehat{G}_v \subset G_\epsilon$. So

$$e_T\Big(\big(H^0(\widetilde{\mathcal{C}}_v, \mathcal{O}_{\widetilde{\mathcal{C}}_v}) \otimes \mathbb{C}_{\phi_{(\epsilon,\sigma)}}\big)^{G_v}\Big) = \begin{cases} \mathbf{w}_{(\epsilon,\sigma)}, & \widehat{G}_v \subset G_\epsilon, \\ 1, & \widehat{G}_v \not\subset G_\epsilon. \end{cases}$$

Therefore,

$$e_T(H^0(\mathcal{C}_v, f_v^* T\mathcal{X})^m) = e_T(H^0(\mathcal{C}_v, f_v^* T\mathcal{X})) = \prod_{\epsilon \in E_\sigma, \widehat{G}_v \subset G_\epsilon} \mathbf{w}_{(\epsilon,\sigma)} \tag{6.28}$$

$$\big(H^1(\widetilde{\mathcal{C}}_v, \mathcal{O}_{\mathcal{C}_v}) \otimes \mathbb{C}_{\phi(\epsilon,\sigma)}\big)^{G_v} = \mathbb{E}^\vee_{\phi(\epsilon,\sigma)},$$

so

$$e_T(H^1(\mathcal{C}_v, f_v^* T\mathcal{X})^m) = e_T(H^1(\mathcal{C}_v, f_v^* T\mathcal{X})) = \prod_{\epsilon \in E_\sigma} \Lambda^\vee_{\phi_{(\epsilon,\sigma)}}(\mathbf{w}_{(\epsilon,\sigma)}). \tag{6.29}$$

Equation (6.27) follows from (6.28) and (6.29). □

Given any real number x, let $\lfloor x \rfloor$ denote the greatest integer which is less or equal to x, and let $\langle x \rangle = x - \lfloor x \rfloor \in [0,1)$.

LEMMA 6.6. *Suppose that $e \in E(\Gamma)$. Let $d = d_e \in \mathbb{Z}_{>0}$, and let $\epsilon = \vec{f}(e) \in E(\Upsilon)_c$. Define $\sigma, \sigma', \epsilon_i, \epsilon_i', a_i$ as in Section 3.5. Suppose that $(e,v), (e,v') \in F(\Gamma)$, $\vec{f}(v) = \sigma$, $\vec{f}(v') = \sigma'$. Then any element in the conjugacy class $c_{(e,v)} \in \mathrm{Conj}(G_\sigma)$ acts on $T_{\mathfrak{p}_\sigma}\mathfrak{l}_\epsilon$ by multiplication by $e^{2\pi\sqrt{-1}\langle d/r_{(\epsilon,\sigma)}\rangle}$, and acts on $T_{\mathfrak{p}_\sigma}\mathfrak{l}_{\epsilon_i}$ by $e^{2\pi\sqrt{-1}\alpha_i}$, where*

$$\langle \frac{d}{r_{(\epsilon,\sigma)}} \rangle, \alpha_1, \ldots, \alpha_{r-1} \in \Big\{0, \frac{1}{r_{(e,v)}}, \ldots, \frac{r_{(e,v)}-1}{r_{(e,v)}}\Big\}.$$

Define

$$\mathbf{u} = r_{(\epsilon,\sigma)} \mathbf{w}_{(\epsilon,\sigma)} = -r_{(\epsilon,\sigma')} \mathbf{w}_{(\epsilon,\sigma')}.$$

Then

$$\mathbf{h}(e) = \frac{(\frac{d}{\mathbf{u}})^{\lfloor \frac{d}{r_{(\epsilon,\sigma)}} \rfloor}}{\lfloor \frac{d}{r_{(\epsilon,\sigma)}} \rfloor !} \frac{(-\frac{d}{\mathbf{u}})^{\lfloor \frac{d}{r_{(\epsilon,\sigma')}} \rfloor}}{\lfloor \frac{d}{r_{(\epsilon,\sigma')}} \rfloor !} \prod_{i=1}^{r-1} \mathbf{b}_i \tag{6.30}$$

where

$$\mathbf{b}_i = \begin{cases} \displaystyle\prod_{j=0}^{\lfloor da_i - \alpha_i \rfloor} (\mathbf{w}_{(\epsilon_i,\sigma)} - (j+\alpha_i)\frac{\mathbf{u}}{d})^{-1}, & a_i \geq 0, \\ \displaystyle\prod_{j=1}^{\lceil \alpha_i - da_i - 1 \rceil} (\mathbf{w}_{(\epsilon_i,\sigma)} + (j-\alpha_i)\frac{\mathbf{u}}{d}), & a_i < 0. \end{cases} \tag{6.31}$$

Proof. Let

$$\mathbf{w}_i = \mathbf{w}_{(\epsilon_i,\sigma)}, \quad i = 1, \ldots, r-1.$$

We have

$$N_{\mathfrak{l}_\epsilon/\mathcal{X}} = L_1 \oplus \cdots \oplus L_{r-1}.$$

- The weights of T-actions on $(L_i)_{\mathfrak{p}_\sigma}$ and $(L_i)_{\mathfrak{p}_{\sigma'}}$ are $\mathbf{w}_i$ and $\mathbf{w}_i - a_i\mathbf{u}$, respectively.
- The weights of T-action on $T_{\mathfrak{p}_\sigma}\mathfrak{l}_\epsilon$ and $T_{\mathfrak{p}_{\sigma'}}\mathfrak{l}_\epsilon$ are $\dfrac{\mathbf{u}}{r_{(\epsilon,\sigma)}}$ and $\dfrac{-\mathbf{u}}{r_{(\epsilon,\sigma')}}$, respectively.
- Let $\mathfrak{p}_v = f_e^{-1}(\mathfrak{p}_\sigma), \mathfrak{p}_{v'} = f_e^{-1}(\mathfrak{p}_{\sigma'})$ be the two torus fixed points in $\mathcal{C}_e$. Then the weights of T-action on $T_{\mathfrak{p}_v}\mathcal{C}_e$ and $T_{\mathfrak{p}_{v'}}\mathcal{C}_e$ are $\dfrac{\mathbf{u}}{dr_{(e,v)}}$ and $\dfrac{-\mathbf{u}}{dr_{(e,v')}}$, respectively.

By [47, Example 8.5],

$$\mathrm{ch}_T(H^1(\mathcal{C}_e, f_e^*L_i) - H^0(\mathcal{C}_e, f_e^*L_i)) = \begin{cases} -\displaystyle\sum_{j=0}^{\lfloor da_i-\alpha_i\rfloor} e^{\mathbf{w}_i-(j+\alpha_i)\frac{\mathbf{u}}{d}}, & a_i \geq 0, \\ \displaystyle\sum_{j=1}^{\lceil \alpha_i - da_i - 1\rceil} e^{\mathbf{w}_i+(j-\alpha_i)\frac{\mathbf{u}}{d}}, & a_i < 0. \end{cases}$$

Note that $\mathbf{w}_i - (j+\alpha_i)\mathbf{u}$ and $\mathbf{w}_i + (j-\alpha_i)\mathbf{u}$ are nonzero for any $j \in \mathbb{Z}$ since $\mathbf{w}_i$ and $\mathbf{u}$ are linearly independent for $i = 1, \ldots, r-1$. So

$$\frac{e^T\left(H^1(\mathcal{C}_e, f_e^*L_i)^m\right)}{e^T\left(H^0(\mathcal{C}_e, f_e^*L_i)^m\right)} = \frac{e^T\left(H^1(\mathcal{C}_e, f_e^*L_i)\right)}{e^T\left(H^0(\mathcal{C}_e, f_e^*L_i)\right)} = \mathbf{b}_i$$

where $\mathbf{b}_i$ is defined by (6.31).

By [47, Example 8.5] again,

$$\begin{aligned} &\mathrm{ch}_T(H^1(\mathcal{C}_e, f_e^*T\mathfrak{l}_\epsilon) - H^0(\mathcal{C}_e, f_e^*T\mathfrak{l}_\epsilon)) \\ =& \sum_{\substack{j\in\mathbb{Z} \\ -\langle\frac{d}{r_{(\epsilon,\sigma)}}\rangle \leq j \leq \frac{d}{r_{(\epsilon,\sigma)}} + \frac{d}{r_{(\epsilon,\sigma')}} - \langle\frac{d}{r_{(\epsilon,\sigma)}}\rangle}} e^{\frac{\mathbf{u}}{r_{(\epsilon,\sigma)}} - (j+\langle\frac{d}{r_{(\epsilon,\sigma)}}\rangle)\frac{\mathbf{u}}{d}} \\ =& = 1 + \sum_{j=1}^{\lfloor\frac{d}{r_{(\epsilon,\sigma)}}\rfloor} e^{j\frac{\mathbf{u}}{d}} + \sum_{j=1}^{\lfloor\frac{d}{r_{(\epsilon,\sigma')}}\rfloor} e^{-j\frac{\mathbf{u}}{d}}. \end{aligned}$$

So

$$\frac{e^T(H^1(\mathcal{C}_e, f_e^*T\mathfrak{l}_\epsilon)^m)}{e^T(H^0(\mathcal{C}_e, f_e^*T\mathfrak{l}_\epsilon)^m)} = \prod_{j=1}^{\lfloor\frac{d}{r_{(\epsilon,\sigma)}}\rfloor} \frac{1}{j\frac{\mathbf{u}}{d}} \prod_{j=1}^{\lfloor\frac{d}{r_{(\epsilon,\sigma')}}\rfloor} \frac{1}{-j\frac{\mathbf{u}}{d}} = \frac{\left(\frac{d}{\mathbf{u}}\right)^{\lfloor\frac{d}{r_{(\epsilon,\sigma)}}\rfloor}}{\lfloor\frac{d}{r_{(\epsilon,\sigma)}}\rfloor!} \frac{\left(-\frac{d}{\mathbf{u}}\right)^{\lfloor\frac{d}{r_{(\epsilon,\sigma')}}\rfloor}}{\lfloor\frac{d}{r_{(\epsilon,\sigma')}}\rfloor!}.$$

Therefore,

$$\begin{aligned} \frac{e^T(H^1(\mathcal{C}_e, f_e^*T\mathcal{X})^m)}{e^T(H^0(\mathcal{C}_e, f_e^*T\mathcal{X})^m)} &= \frac{e^T(H^1(\mathcal{C}_e, f_e^*T\mathfrak{l}_\epsilon)^m)}{e^T(H^0(\mathcal{C}_e, f_e^*T\mathfrak{l}_\epsilon)^m)} \cdot \prod_{i=1}^{r-1} \frac{e^T(H^1(\mathcal{C}_e, f_e^*\mathcal{L}_i)^m)}{e^T(H^0(\mathcal{C}_e, f_e^*\mathcal{L}_i)^m)} \\ &= \frac{\left(\frac{d}{\mathbf{u}}\right)^{\lfloor\frac{d}{r_{(\epsilon,\sigma)}}\rfloor}}{\lfloor\frac{d}{r_{(\epsilon,\sigma)}}\rfloor!} \frac{\left(-\frac{d}{\mathbf{u}}\right)^{\lfloor\frac{d}{r_{(\epsilon,\sigma')}}\rfloor}}{\lfloor\frac{d}{r_{(\epsilon,\sigma')}}\rfloor!} \prod_{i=1}^{r-1} \mathbf{b}_i. \end{aligned}$$

□

From the above derivation, we conclude that

$$\frac{e^T(B_5^m)}{e^T(B_2^m)} = \prod_{\substack{v\in V^{0,2}(\vec{\Gamma}) \\ E_v=\{e,e'\}}} \mathbf{h}(e,v) \cdot \prod_{(e,v)\in F^S(\vec{\Gamma})} \mathbf{h}(e,v) \cdot \prod_{v\in V^S(\vec{\Gamma})} \mathbf{h}(v) \cdot \prod_{e\in E(\Gamma)} \mathbf{h}(e) \qquad (6.32)$$

where $\mathbf{h}(e,v)$, $\mathbf{h}(v)$, and $\mathbf{h}(e)$ are defined by (6.26), (6.27), (6.30), respectively. To unify the stable and unstable vertices, we define

$$\mathbf{h}(v) := \begin{cases} \dfrac{1}{\mathbf{h}(e,v)}, & v \in V^{0,1}(\vec{\Gamma}) \cup V^{1,1}(\Gamma), \quad E_v = \{e\}, \\ \dfrac{1}{\mathbf{h}(e,v)} = \dfrac{1}{\mathbf{h}(e',v)}, & v \in V^{0,2}(\vec{\Gamma}), \quad E_v = \{e, e'\}. \end{cases}$$

In the above notation, (6.32) can be written as follows.

PROPOSITION 6.7.

$$\frac{e^T(B_5^m)}{e^T(B_2^m)} = \prod_{v \in V(\Gamma)} \mathbf{h}(v) \cdot \prod_{(e,v) \in F(\Gamma)} \mathbf{h}(e,v) \cdot \prod_{e \in E(\Gamma)} \mathbf{h}(e).$$

6.7. Contribution from each graph.

6.7.1. Virtual tangent bundle. We have $B_1^f = B_2^f$, $B_5^f = 0$. So

$$T^{1,f} = B_4^f = \bigoplus_{v \in V^S(\vec{\Gamma})} T_{[(\mathcal{C}_v, \mathbf{x}_v)]} \overline{\mathcal{M}}_{g_v, n_v}, \quad T^{2,f} = 0.$$

This completes the proof of Theorem 6.2.

6.7.2. Virtual normal bundle. Equation (6.18), Proposition 6.3, and Proposition 6.7 imply

$$\frac{1}{e_T(N_{\vec{\Gamma}}^{\mathrm{vir}})} = \prod_{v \in V(\Gamma)} \frac{\mathbf{h}(v)}{\prod_{e \in E_v} \left(w_{(e,v)} - \frac{\bar{\psi}_{(e,v)}}{r_{(e,v)}} \right)} \prod_{(e,v) \in F(\Gamma)} \mathbf{h}(e,v) \cdot \prod_{e \in E(\Gamma)} \mathbf{h}(e). \tag{6.33}$$

6.7.3. Integrand. Given $\sigma \in V(\Upsilon)$, let $i_\sigma^* : \mathcal{H}_\Upsilon \to H^*_{\mathrm{CR},T}(\mathfrak{p}_\sigma; Q_T)$ be the composition

$$\mathcal{H}_\Upsilon = \bigoplus_{\sigma \in V(\Upsilon)} H^*_{\mathrm{CR},T}(\mathcal{X}_\sigma; Q_T) \to H^*_{CR,T}(\mathcal{X}_\sigma; Q_T) \xrightarrow{i_\sigma^*} H^*_{\mathrm{CR},T}(\mathfrak{p}_\sigma; Q_T)$$

where the first arrow is projection to a direct summand, and the second arrow is induced by the inclusion $i_\sigma : \mathfrak{p}_\sigma \to \mathcal{X}_\sigma$. Given $\vec{\Gamma} \in G_{g,\vec{i}}(\Upsilon, \hat{\beta})$, let

$$i_{\vec{\Gamma}}^* : A_T^*(\overline{\mathcal{M}}_{g,\vec{i}}(\mathcal{X}, \beta)) \to A_T^*(\mathcal{F}_{\vec{\Gamma}}) \cong A_T^*(\mathcal{M}_{\vec{\Gamma}})$$

be induced by the inclusion $i_{\vec{\Gamma}} : \mathcal{F}_{\vec{\Gamma}} \to \overline{\mathcal{M}}_{g,\vec{i}}(\hat{\mathcal{X}}_{\vec{\Upsilon}}, \hat{\beta})$. Then

$$\begin{aligned} & i_{\vec{\Gamma}}^* \prod_{j=1}^n \left(\mathrm{ev}_j^* \gamma_j^T \cup (\bar{\psi}_j^T)^{a_j} \right) \\ = & \prod_{\substack{v \in V^{1,1}(\vec{\Gamma}) \\ S_v = \{j\},\, E_v = \{e\}}} i_{\sigma_v}^* \gamma_j^T (-w_{(e,v)})^{a_j} \cdot \prod_{v \in V^S(\vec{\Gamma})} \Big(\prod_{j \in S_v} i_{\sigma_v}^* \gamma_j^T \prod_{e \in E_v} \bar{\psi}_{(e,v)}^{a_j} \Big). \end{aligned} \tag{6.34}$$

To unify the stable vertices in $V^S(\vec{\Gamma})$ and the unstable vertices in $V^{1,1}(\vec{\Gamma})$, we use the following convention: for $a \in \mathbb{Z}_{\geq 0}$ and $h \in G$, we define

$$\int_{\overline{\mathcal{M}}_{0,([h],[h^{-1}])}(\mathcal{B}G)} \frac{\bar{\psi}_2^a}{w_1 - \bar{\psi}_1} = \frac{(-w_1)^a}{|C_G(h)|}. \tag{6.35}$$

In particular, (6.22) is obtained by setting $a = 0$. With the convention (6.35), we may rewrite (6.34) as

$$i_{\vec{\Gamma}}^* \prod_{j=1}^n \big(\mathrm{ev}_j^* \gamma_j^T \cup (\bar{\psi}_j^T)^{a_j}\big) = \prod_{v \in V(\Gamma)} \Big(\prod_{j \in S_v} i_{\sigma_v}^* \gamma_j^T \prod_{e \in E_v} \bar{\psi}_{(e,v)}^{a_j} \Big). \tag{6.36}$$

The following lemma shows that the convention (6.35) is consistent with the stable case $\overline{\mathcal{M}}_{0,(c_1,\dots,c_n)}(\mathcal{B}G)$, $n \geq 3$.

LEMMA 6.8. *Let n, a be integers, $n \geq 2$, $a \geq 0$. Let $\vec{c} = (c_1, \dots, c_n) \in \mathrm{Conj}(G)^n$. Then*

$$\int_{\overline{\mathcal{M}}_{0,\vec{c}}(\mathcal{B}G)} \frac{\bar{\psi}_2^a}{w_1 - \bar{\psi}_1} = \begin{cases} \dfrac{\prod_{i=0}^{a-1}(n-3-i)}{a!} w_1^{a+2-n} \dfrac{|V_{0,\vec{c}}^G|}{|G|}, & n = 2 \text{ or } 0 \leq a \leq n-3. \\ 0, & \text{otherwise.} \end{cases}$$

Proof. The case $n = 2$ follows from (6.35). For $n \geq 3$,

$$\int_{\overline{\mathcal{M}}_{0,\vec{c}}(\mathcal{B}G)} \frac{\psi_2^a}{w_1 - \bar{\psi}_1} = \frac{1}{w_1} \int_{\overline{\mathcal{M}}_{0,\vec{c}}(\mathcal{B}G)} \frac{\bar{\psi}_2^a}{1 - \frac{\bar{\psi}_1}{w_1}} = w_1^{a+2-n} \int_{\overline{\mathcal{M}}_{0,\vec{c}}(\mathcal{B}G)} \bar{\psi}_1^{n-3-a} \bar{\psi}_2^a$$
$$= w_1^{a+2-n} |V_{0,\vec{c}}^G| \cdot \frac{1}{|G|} \cdot \frac{(n-3)!}{(n-3-a)! a!} = \frac{\prod_{i=0}^{a-1}(n-3-i)}{a!} w_1^{a+2-n} \frac{|V_{0,\vec{c}}^G|}{|G|}.$$

∎

6.7.4. The integral. Define

$$I_{\vec{\Gamma}} := \int_{[\mathcal{F}_{\vec{\Gamma}}]^{\mathrm{vir}}} \frac{i_{\vec{\Gamma}}^* \prod_{j=1}^n (\mathrm{ev}_j^* \hat{\gamma}_j^T \cup (\bar{\psi}_j^T)^{a_i})}{e^T(N_{\vec{\Gamma}}^{\mathrm{vir}})}. \tag{6.37}$$

By Theorem 6.2, Equation (6.33), and Equation (6.34),

$$\begin{aligned} I_{\vec{\Gamma}} = c_{\vec{\Gamma}} \prod_{e \in E(\Gamma)} \mathbf{h}(e) \prod_{(e,v) \in F(\Gamma)} \mathbf{h}(e,v) \\ \cdot \prod_{v \in V(\Gamma)} \int_{[\overline{\mathcal{M}}_{g_v, \vec{c}_v}(\mathcal{B}G_v)]} \frac{\mathbf{h}(v) \cdot \prod_{j \in S_v} \big(i_{\sigma_v}^* \hat{\gamma}_j^T \cup \bar{\psi}_j^{a_j} \big)}{\prod_{e \in E_v} \Big(w_{(e,v)} - \frac{\bar{\psi}_{(e,v)}}{r_{(e,v)}} \Big)}. \end{aligned} \tag{6.38}$$

(Recall that $c_{\vec{\Gamma}} \in \mathbb{Q}$ is defined by Equation (6.24).)

6.8. Sum over graphs. Let

$$i_T^* : H_T^*(\overline{\mathcal{M}}_{g,\vec{i}}(\hat{\mathcal{X}}_{\vec{\Upsilon}}, \hat{\beta})) \to H_T^*(\overline{\mathcal{M}}_{g,\vec{i}}(\hat{\mathcal{X}}_{\vec{\Upsilon}}, \hat{\beta})^T)$$

be induced by the inclusion $i_T : \overline{\mathcal{M}}_{g,\vec{i}}(\hat{\mathcal{X}}_{\vec{\Upsilon}}, \hat{\beta})^T \to \overline{\mathcal{M}}_{g,\vec{i}}(\hat{\mathcal{X}}_{\vec{\Upsilon}}, \hat{\beta})$. Then

$$\int_{[\overline{\mathcal{M}}_{g,\vec{i}}(\hat{\mathcal{X}}_{\vec{\Upsilon}}, \hat{\beta})^T]^{\mathrm{vir},T}} \frac{i_T^* \prod_{j=1}^n (\mathrm{ev}_j^* \hat{\gamma}_j^T \cup (\bar{\psi}_j^T)^{a_j})}{e^T(N^{\mathrm{vir}})} = \sum_{\vec{\Gamma} \in G_{g,n}(\vec{\Upsilon}, \hat{\beta})} I_{\vec{\Gamma}},$$

where the contribution $I_{\vec{\Gamma}}$ from the decorated graph $\vec{\Gamma}$ is given in Section 6.7.4 above. We obtain:

THEOREM 6.9.

$$\langle \bar{\epsilon}_{a_1}(\hat{\gamma}_1^T)\cdots\bar{\epsilon}_{a_n}(\hat{\gamma}_n^T)\rangle^{\Upsilon}_{g,\hat{\beta}} = \sum_{\vec{\Gamma}\in G_{g,\vec{i}}(\mathcal{X},\beta)} c_{\vec{\Gamma}} \prod_{e\in E(\Gamma)} \mathbf{h}(e) \prod_{(e,v)\in F(\Gamma)} \mathbf{h}(e,v) \cdot \prod_{v\in V(\Gamma)} \int_{[\overline{\mathcal{M}}_{g,\vec{i}_v}(\mathcal{B}G_v)]^w} \frac{\mathbf{h}(v)\prod_{j\in S_v}\left(i^*_{\sigma_v}\hat{\gamma}_j^T\bar{\psi}_j^{a_j}\right)}{\prod_{e\in E_v}\left(w_{(e,v)}-\frac{\bar{\psi}_{(e,v)}}{r_{(e,v)}}\right)} \tag{6.39}$$

where $\mathbf{h}(e)$, $\mathbf{h}(e,v)$, $\mathbf{h}(v)$ *are given by* (6.30), (6.26), (6.27), *respectively, and we have the following convention for the* $v\notin V^S(\vec{\Gamma})$*:*

$$\int_{\overline{\mathcal{M}}_{0,(\{1\})}(\mathcal{B}G_v)} \frac{1}{w_1-\bar{\psi}_1} = \frac{w_1}{|G|},$$

$$\int_{\overline{\mathcal{M}}_{0,([h],[h^{-1}])}(\mathcal{B}G_v)} \frac{1}{(w_1-\bar{\psi}_1)(w_2-\bar{\psi}_2)} = \frac{1}{|C_{G_v}(h)|\cdot(w_1+w_2)},$$

$$\int_{\overline{\mathcal{M}}_{0,([h],[h^{-1}]}(\mathcal{B}G_v)} \frac{\bar{\psi}_2^a}{w_1-\bar{\psi}_1} = \frac{(-w_1)^a}{|C_{G_v}(h)|}, \quad a\in\mathbb{Z}_{\geq 0}$$

where $h\in G_v$, $[h]\in \mathrm{Conj}(G_v)$, *and* $C_{G_v}(h)=\{a\in G_v : aha^{-1}=h\}$ *is the centralizer of* h *in* G_v.

REFERENCES

[1] D. ABRAMOVICH, T. GRABER AND A. VISTOLI, *Algebraic orbifold quantum products*, Orbifolds in mathematics and physics (Madison, WI, 2001), pp. 1–24, Contemp. Math., 310, Amer. Math. Soc., Providence, RI, 2002.

[2] D. ABRAMOVICH, T. GRABER AND A. VISTOLI, *Gromov-Witten theory of Deligne-Mumford stacks,* Amer. J. Math., 130:5 (2008), pp. 1337–1398.

[3] M. F. ATIYAH AND R. BOTT, *The moment map and equivariant cohomology,* Topology, 23:1 (1984), pp. 1–28.

[4] K. BEHREND, *Gromov-Witten invariants in algebraic geometry,* Invent. Math., 127:3 (1997), pp. 601–617.

[5] K. BEHREND, *Localization and Gromov-Witten invariants,* Quantum cohomology, pp. 3–38, Lecture Notes in Math., 1776, Springer, Berlin, 2002.

[6] K. BEHREND AND B. FANTECHI, *Intrinsic normal cone,* Invent. Math., 128:1 (1997), pp. 45–88.

[7] K. BEHREND AND Y. MANIN, *Stacks of stable maps and Gromov-Witten invariants,* Duke Math. J., 85:1 (1996), pp. 1–60.

[8] K. BEHREND AND B. NOOHI, *Uniformization of Deligne-Mumford curves,* J. Reine Angew. Math., 599:559 (2006), pp. 111–153.

[9] L. BORISOV, L. CHEN AND G. G. SMITH, *The orbifold Chow ring of toric Deligne-Mumford stacks,* J. Amer. Math. Soc., 18:1 (2005), pp. 193–215.

[10] J. BRYAN, *The Donaldson-Thomas partition function of the banana manifold,* with an appendix coauthored with Stephen Pietromonaco, Algebr. Geom., 8:1 (2021), pp. 133–170.

[11] J. BRYAN, C. CADMAN AND B. YOUNG, *The orbifold topological vertex,* Adv. Math., 229:1 (2012), pp. 531–595.

[12] H.-L. CHANG, Y.-H. KIEM AND J. LI, *Torus localization and wall crossing for cosection localized virtual cycles,* Adv. Math., 308 (2017), pp. 964–986.

[13] L. CHEN, Y. LI AND K. LIU, *Localization, Hurwitz numbers and the Witten conjecture,* Asian J. Math., 12:4 (2008), pp. 511–518.

[14] W. CHEN AND Y. RUAN, *A new cohomology theory of orbifold,* Comm. Math. Phys., 248:1 (2004), pp. 1–31.

[15] W. CHEN AND Y. RUAN, *Orbifold Gromov-Witten theory,* Orbifolds in mathematics and physics (Madison, WI, 2001), pp. 25–85, Contemp. Math., 310, Amer. Math. Soc., Providence, RI, 2002.

[16] D. Cheong, I. Ciocan-Fontanine and B. Kim, *Orbifold quasimap theory*, Math. Ann., 363:3-4 (2015), pp. 777–816.

[17] P. Deligne and D. Mumford, *The irreducibility of the space of curves of given genus,* Inst. Hautes Études Sci. Publ. Math., 36 (1969), pp. 75–109.

[18] D. Edidin and W. Graham, *Equivariant intersection theory,* Invent. Math., 131:3 (1998), pp. 595–634.

[19] D. Edidin and W. Graham, *Localization in equivariant intersection theory and the Bott residue formula,* Amer. J. Math., 120:3 (1998), pp. 619–636.

[20] C. Faber, *Algorithms for computing intersection numbers on moduli spaces of curves, with an application to the class of the locus of Jacobians,* New trends in algebraic geometry (Warwick, 1996), pp. 93–109, London Math. Soc. Lecture Note Ser., 264, Cambridge Univ. Press, Cambridge, 1999.

[21] C. Faber and R. Pandharipande, *Hodge integrals and Gromov-Witten theory,* Invent. Math., 139:1 (2000), pp. 173–199.

[22] B. Fantechi, E. Mann and F. Nironi, *Smooth toric Deligne-Mumford stacks,* J. Reine Angew. Math., 648 (2010), pp. 201–244.

[23] W. Fulton, *Intersection theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete, Springer-Verlag, Berlin, 1984.

[24] W. Fulton, *Equivariant Cohomology in Algebraic Geometry,* Eilenberg lectures at Columbia University, Spring 2007, notes by Dave Anderson are available at `https://people.math.osu.edu/anderson.2804/eilenberg`.

[25] R. Goldin, T. Holm and A. Knutson, *Orbifold cohomology of torus quotients,* Duke Math. J., 139:1 (2007), pp. 89–139.

[26] T. Graber and R. Pandharipande, *Localization of virtual classes*, preprint 1998, available at `http://www.math.ethz.ch/~rahul/loc.ps`.

[27] T. Graber and R. Pandharipande, *Localization of virtual classes,* Invent. Math., 135:2 (1999), pp. 487–518.

[28] M. Goresky, R. Kottwitz and R. MacPherson, *Equivariant cohomology, Koszul duality, and the localization theorem,* Invent. Math., 131:1 (1998), pp. 25–83.

[29] V. Guillemin and C. Zara, *Equivariant de Rham theory and graphs,* Sir Michael Atiyah: a great mathematician of the twentieth century. Asian J. Math., 3:1 (1999), pp. 49–76.

[30] V. Guillemin and C. Zara, *1-skeleta, Betti numbers, and equivariant cohomology,* Duke Math. J., 107:2 (2001), pp. 283–349.

[31] T. Hollowood, A. Iqbal and C. Vafa, *Matrix models, geometric engineering and elliptic genera,* J. High Energy Phys. 2008, no. 3, 069, 81 pp.

[32] T. J. Jarvis and T. Kimura, *Orbifold quantum cohomology of the classifying space of a finite group,* Orbifolds in mathematics and physics (Madison, WI, 2001), pp. 123–134, Contemp. Math., 310, Amer. Math. Soc., Providence, RI, 2002.

[33] P. Johnson, *Equivariant GW Theory of Stacky Curves,* Comm. Math. Phys., 327:2 (2014), pp. 333–386.

[34] A. Kanazawa and S.-C. Lau, *Local Calabi-Yau manifolds of type $\tilde{A}$ via SYZ mirror symmetry,* J. Geom. Phys., 139 (2019), pp. 103–138.

[35] T. Kawasaki, *The Riemann-Roch theorem for complex V-manifolds*, Osaka Math. J., 16:1 (1979), pp. 151–159.

[36] M. E. Kazarian, *KP hierarchy for Hodge integrals,* Adv. Math., 221:1 (2009), pp. 1–21.

[37] M. E. Kazarian and S. K. Lando, *An algebro-geometric proof of Witten's conjecture,* J. Amer. Math. Soc., 20:4 (2007), pp. 1079–1089.

[38] Y.-S. Kim and K. Liu, *Virasoro constraints and Hurwitz numbers through asymptotic analysis,* Pacific J. Math., 241:2 (2009), pp. 275–284.

[39] F. Knudsen, *The projectivity of the moduli space of stable curves. II. The stacks $M_{g,n}$,* Math. Scand., 52:2 (1983), pp. 161–199.

[40] F. Knudsen, *The projectivity of the moduli space of stable curves. III. The line bundles on $M_{g,n}$, and a proof of the projectivity of $\overline{M}_{g,n}$ in characteristic* 0, Math. Scand., 52:2 (1983), pp. 200–212.

[41] F. Knudsen and D. Mumford, *The projectivity of the moduli space of stable curves. I. Preliminaries on "det" and "Div",* Math. Scand., 39:1 (1976), pp. 19–55.

[42] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function,* Comm. Math. Phys., 147:1 (1992), pp. 1–23.

[43] M. Kontsevich, *Enumeration of rational curves via torus actions,* The moduli space of curves (Texel Island, 1994), pp. 335–368, Progr. Math., 129, Birkhäuser Boston, Boston, MA, 1995.

[44] A. Kresch, *Cycle groups for Artin stacks,* Invent. Math., 138:3 (1999), pp. 495–536.

[45] J. Li and G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of algebraic varieties,* J. Amer. Math. Soc., 11:1 (1998), pp. 119–174.
[46] J. Li and G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of general symplectic orbifolds,* Topics in symplectic 4-orbifolds (Irvine, CA, 1996), pp. 47–83, First Int. Press Lect. Ser., I, Int. Press, Cambridge, MA, 1998.
[47] C.-C. M. Liu, *Localization in Gromov-Witten theory and orbifold Gromov-Witten theory,* Handbook of Moduli, Volume II, pp. 353–425, Adv. Lect. Math., (ALM) 25, International Press and Higher Education Press, 2013.
[48] J. Li, C.-C. M. Liu, K. Liu and J. Zhou, *A mathematical theory of the topological vertex,* Geom. Topol., 13:1 (2009), pp. 527–621.
[49] J. Li, K. Liu and J. Zhou, *Topological string partition functions as equivariant indices,* Asian J. Math., 10:1 (2006), pp. 81–114.
[50] C.-C. M. Liu and A. Sheshmani, *Equivariant Gromov-Witten Invariants of Algebraic GKM Manifolds,* SIGMA, 13 (2017), 048, 21 pages.
[51] Y. I. Manin, *Frobenius orbifolds, quantum cohomology, and moduli spaces*, American Mathematical Society Colloquium Publications, 47, American Mathematical Society, Providence, RI, 1999.
[52] M. Mirzakhani, *Weil-Petersson volumes and intersection theory on the moduli space of curves,* J. Amer. Math. Soc., 20:1 (2007), pp. 1–23.
[53] M. Mulase and N. Zhang, *Polynomial recursion formula for linear Hodge integrals,* Commun. Number Theory Phys., 4:2 (2010), pp. 267–293.
[54] D. Mumford, *Towards an enumerative geometry of the moduli space of curves,* Arithmetic and geometry, Vol. II, pp. 271–328, Progr. Math., 36, Birkhäuser Boston, Boston, MA, 1983.
[55] M. C. Olsson, *(Log) twisted curves,* Compos. Math., 143:2 (2007), pp. 476–494.
[56] A. Okounkov and R. Pandharipande, *Gromov-Witten theory, Hurwitz numbers, and matrix models,* Algebraic geometry—Seattle 2005. Part 1, pp. 325–414, Proc. Sympos. Pure Math., 80, Part 1, Amer. Math. Soc., Providence, RI, 2009.
[57] M. Romagny, *Group actions on stacks and applications,* Michigan Math. J., 53:1 (2005), pp. 209–236.
[58] Y. Ruan, *Topological sigma model and Donaldson-type invariants in Gromov theory,* Duke Math. J., 83:2 (1996), pp. 461–500.
[59] Y. Ruan, *Virtual neighborhoods and pseudo-holomorphic curves,* Proceedings of 6th Gökova Geometry-Topology Conference, Turkish J. Math., 23:1 (1999), pp. 161–231.
[60] Y. Ruan and G. Tian, *A mathematical theory of quantum cohomology,* J. Differential Geom., 42:2 (1995), pp. 259–367.
[61] Y. Ruan and G. Tian, *Higher genus symplectic invariants and sigma models coupled with gravity,* Invent. Math., 130:3 (1997), pp. 455–516.
[62] B. Siebert, *Gromov-Witten invariants of general symplectic manifolds,* `arXiv:dg-ga/9608005`.
[63] H. Spielberg, *A formula for the Gromov-Witten invariants of toric varieties,* Thése, Université Louis Pasteur (Strasbourg I), Strasbourg, 1999, 103 pp. "The Gromov-Witten invariants of symplectic manifold," `arXiv:math/0006156`.
[64] H.-H. Tseng, *Orbifold quantum Riemann-Roch, Lefschetz and Serre,* Geom. Topol., 14:1 (2010), pp. 1–81.
[65] A. Vistoli, *Intersection theory on algebraic stacks and on their moduli spaces,* Invent. Math., 97:3 (1989), pp. 613–670.
[66] E. Witten, *Two-dimensional gravity and intersection theory on moduli space,* Surveys in differential geometry (Cambridge, MA, 1990), pp. 243–310, Lehigh Univ., Bethlehem, PA, 1991.
[67] E. Zaslow, *Topological orbifold models and quantum cohomology rings,* Comm. Math. Phys., 156:2 (1993), pp. 301–331
[68] J. Zhou, *On computations of Hurwitz-Hodge integrals*, `arXiv:0710.1679`.
[69] Z. Zong, *Equivariant Gromov-Witten theory of GKM orbifolds,* `arXiv:1604.07270`.