

EMBEDDINGS FROM NONCOMPACT SYMMETRIC SPACES TO THEIR COMPACT DUALS*

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Abstract. Every compact symmetric space M admits a dual noncompact symmetric space \check{M} . When M is a generalized Grassmannian, we can view \check{M} as a open submanifold of it consisting of space-like subspaces [4]. Motivated from this, we study the embeddings from noncompact symmetric spaces to their compact duals, including space-like embedding for generalized Grassmannians, Borel embedding for Hermitian symmetric spaces and the generalized embedding for symmetric R-spaces. We will compare these embeddings and describe their images using cut loci.

Key words. symmetric space, Grassmannian, embedding.

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1. Introduction. Riemannian symmetric spaces of semi-simple type come in pairs, the compact ones and their noncompact duals. The duality between the compact types and noncompact types not only provides two viewpoints of the classification problem of symmetric spaces, but also explains the formal analogy between spherical trigonometry and hyperbolic trigonometry.

In a recent paper [4] by Y.D. Huang and N.C. Leung, they give a uniform description of classical symmetric spaces (up to finite covers and abelian parts): compact classical symmetric spaces described as some kinds of Grassmannian and their noncompact duals as the Grassmannians of space-like subspaces. That means, for these generalized Grassmannians, we have the **space-like embedding p** from noncompact types to their compact duals.

It is also well-known that, for Hermitian symmetric spaces, we have the **Borel embedding b** from noncompact ones to their compact duals.

Are these two embeddings p and b compatible with each other? Can they be generalized to all the symmetric spaces? How to characterize their images? These are the questions we want to solve in this paper.

One of the key points to solve the first two questions is the fact that both the generalized Grassmannians and the Hermitian symmetric spaces have the following property: the compact symmetric space admits a connected Lie transformation group larger than the isometry group (definition of symmetric R-space), and hence can be described as a quotient of a semisimple Lie group by a parabolic subgroup [17]. Using this characterization, for all the symmetric R-spaces (which include all the compact generalized Grassmannian and compact Hermitian symmetric spaces), we can construct an open embedding g from its noncompact dual to it (section 3.3), which coincides with both p and b .

One of the key points to solve the third question is the fact that the **unit lattice** of a symmetric R-space is orthonormal (another characterization of symmetric R-spaces) [15]. With this property, it is easy to compute the cut loci of this symmetric

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R-space, which ensures us to construct an explicit embedding \mathbf{f} from its noncompact dual to it (section 4.3). From the definition of \mathbf{f} , we can easily see that its image is just “half of the interior points” of the compact symmetric spaces.

Comparing the embeddings \mathbf{f} and \mathbf{g} , notice that they both coincide with the space-like embedding \mathbf{p} and the Borel embedding \mathbf{b} , are they equal to each other? The answer is affirmative by using the characterization of a symmetric R-space that it is a compact Hermitian symmetric space or a real form of a compact Hermitian symmetric space.

The organization of this paper is as follows. Section 2 is a brief review of the duality between symmetric spaces. In section 3, we first study the space-like embedding \mathbf{p} for generalized Grassmannians, then review the Borel embedding \mathbf{b} for Hermitian symmetric spaces, and finally we study the embedding \mathbf{g} (which generalizes both \mathbf{p} and \mathbf{b}) for symmetric R-spaces. In section 4, we construct the embedding \mathbf{f} using cut loci. At last, in section 5, we compare these embeddings and show that $\mathbf{g}=\mathbf{f}$ for all the symmetric R-spaces except $SO(n+2)/SO(n)SO(2)$ and $SO(n+1) \times SO(m+1)/S(O(n) \times O(m))$.

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2. The duality for symmetric spaces. Let $M = G/K$ be an irreducible Riemannian symmetric space and $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ be the corresponding Cartan decomposition. For the Lie algebra \mathfrak{g} , we have the Killing form B , defined by $B(X, Y) := Tr(adX \cdot adY)$, for X, Y in \mathfrak{g} . Then \mathfrak{k} and \mathfrak{m} are orthogonal with respect to the Killing form B , and $B|_{\mathfrak{k}}$ is negative definite, i.e. $B|_{\mathfrak{k}} < 0$. The symmetric space $M = G/K$ is said to be of **compact** type, **noncompact** type, Euclidean type if $B|_{\mathfrak{m}} > 0$, $B|_{\mathfrak{m}} < 0$, $B|_{\mathfrak{m}} = 0$ respectively. Both the compact type and noncompact type are called semi-simple type.

There is a remarkable and important duality between Riemannian symmetric spaces of compact type and noncompact type. Given a noncompact symmetric space $M = G/K$ with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$. Define $\tilde{\mathfrak{g}} = \mathfrak{k} \oplus \sqrt{-1}\mathfrak{m}$, then the Lie algebra pair $(\tilde{\mathfrak{g}}, \mathfrak{k})$ gives us a finite number of compact symmetric spaces which have the same universal covering space. We call these compact symmetric spaces the compact duals of M . Conversely, given a compact symmetric space, we can get its noncompact dual in the similar way. The noncompact dual is unique since all the noncompact symmetric spaces are simply-connected.

The duality between the compact type and noncompact type not only provides two viewpoints of the classification problem of symmetric spaces, but also explains the formal analogy between spherical trigonometry and hyperbolic trigonometry.

EXAMPLE 2.1 (Page 242 in [3]). *The sphere $S^2 = SO(3)/SO(2)$ is dual to the two dimensional Lobachevsky space $H^2 = SO_0(2, 1)/SO(2)$.*

For the sphere, the formulas

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

hold for a geodesic triangle with angles A, B, C and sides of length a, b, c .

For the two dimensional Lobachevsky space, the corresponding formulas are

$$\frac{\sinh a}{\sin A} = \frac{\sinh b}{\sin B} = \frac{\sinh c}{\sin C}$$

$$\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos A$$

Since $\sinh iz = i \sin z$ and $\cosh iz = \cos z$, the two sets of formulas correspond to each other under the substitution $a \rightarrow ia, b \rightarrow ib, c \rightarrow ic$.

Besides that, we can use the duality to study many aspects of Riemannian symmetric spaces such as polar actions on Riemannian symmetric spaces [7], Fourier transforms on Riemannian symmetric spaces [1] and so on.

3. The embeddings from Lie group aspect. In this section, we will first study the space-like embedding \mathfrak{p} for generalized Grassmannians, then review the Borel embedding \mathfrak{b} for Hermitian symmetric spaces, and finally show that the embedding \mathfrak{g} for symmetric R-spaces is a generalization of these two embeddings \mathfrak{p} and \mathfrak{b} .

3.1. Space-like embedding for generalized Grassmannians. In this subsection, we will first review the Grassmannian description of classical symmetric spaces given by Y.D. Huang and N.C. Leung in [4] and then study the corresponding space-like embedding.

For the classical compact symmetric spaces (Here classical means the corresponding Lie groups are matrix groups), we have a uniform description of them as generalizations of Real Grassmannian $O(n + m)/O(n)O(m) = \{L \subset \mathbb{R}^{n+m} : L \cong \mathbb{R}^n\}$ using two (associative) normed division algebras. That is, the four types of compact generalized Grassmannians: (i) Grassmannian, (ii) Lagrangian Grassmannian, (iii) double Lagrangian Grassmannian, (iv) maximum isotropic Grassmannian give us all possible types of irreducible classical compact symmetric spaces up to a finite cover and some abelian part [4]. We list all these compact symmetric spaces in the following:

First type: *Grassmannians*

$$Gr_{\mathbb{A}\mathbb{B}}(k, n) = \left\{ (\mathbb{A} \otimes \mathbb{B})^k \subset (\mathbb{A} \otimes \mathbb{B})^n \right\}.$$

$\mathbb{A} \setminus \mathbb{B}$	\mathbb{R}	\mathbb{C}	\mathbb{H}
\mathbb{R}	$\frac{O(n)}{O(k)O(n-k)}$	$\frac{U(n)}{U(k)U(n-k)}$	$\frac{Sp(n)}{Sp(k)Sp(n-k)}$
\mathbb{C}	$\frac{U(n)}{U(k)U(n-k)}$	$\frac{U(n)^2}{U(k)^2U(n-k)^2}$	$\frac{U(2n)}{U(2k)U(2n-2k)}$
\mathbb{H}	$\frac{Sp(n)}{Sp(k)Sp(n-k)}$	$\frac{U(2n)}{U(2k)U(2n-2k)}$	$\frac{O(4n)}{O(4k)O(4n-4k)}$

TABLE C1

Second type: *Lagrangian Grassmannians*

$$LGr_{\mathbb{A}\mathbb{B}}(n) = \left\{ \left(\frac{\mathbb{A}}{2} \otimes \mathbb{B} \right)^n \subset (\mathbb{A} \otimes \mathbb{B})^n \right\},$$

where $\frac{\mathbb{A}}{2}$ denotes \mathbb{R}, \mathbb{C} when \mathbb{A} is \mathbb{C}, \mathbb{H} respectively.

$\mathbb{A} \setminus \mathbb{B}$	\mathbb{R}	\mathbb{C}	\mathbb{H}
\mathbb{C}	$\frac{U(n)}{O(n)}$	$\frac{U(n)^2}{U(n)}$	$\frac{U(2n)}{Sp(n)}$
\mathbb{H}	$\frac{Sp(n)}{U(n)}$	$\frac{U(2n)}{(U(n)^2)}$	$\frac{SO(4n)}{U(2n)}$

TABLE C2

Third type: *Double Lagrangian Grassmannians*

$$LLGr_{\mathbb{A}\mathbb{B}}(n) = \left\{ \left(\frac{\mathbb{A}}{2} \otimes \frac{\mathbb{B}}{2} \right)^n \oplus \left(j_1 \frac{\mathbb{A}}{2} \otimes j_2 \frac{\mathbb{B}}{2} \right)^n \subset (\mathbb{A} \otimes \mathbb{B})^n \right\},$$

where $j_1 \in \mathbb{A}$ is i, j when \mathbb{A} is \mathbb{C}, \mathbb{H} respectively, and $j_2 \in \mathbb{B}$ is similar.

$\mathbb{A} \setminus \mathbb{B}$	\mathbb{C}	\mathbb{H}
\mathbb{C}	$\frac{U(n)^2}{O(n)^2}$	$\frac{U(2n)}{O(2n)}$
\mathbb{H}	$\frac{U(2n)}{O(2n)}$	$\frac{SO(4n)}{S(O(2n)^2)}$

TABLE C3

Fourth type: *Compact (semi-)simple Lie groups*

$$G_{\mathbb{A}\mathbb{B}}(n) = \{ (\mathbb{A} \otimes \mathbb{B})^n \subset (\mathbb{A} \otimes \mathbb{B})^n \oplus (\mathbb{A} \otimes \mathbb{B})^n \}^{\sigma_{g'}},$$

here $\sigma_{g'}$ is an involution induced by a canonical symmetric 2-tensor on $(\mathbb{A} \otimes \mathbb{B})^n \oplus (\mathbb{A} \otimes \mathbb{B})^n$.

$\mathbb{A} \setminus \mathbb{B}$	\mathbb{R}	\mathbb{C}	\mathbb{H}
\mathbb{R}	$SO(n)$	$U(n)$	$Sp(n)$
\mathbb{C}	$U(n)$	$U(n)^2$	$U(2n)$
\mathbb{H}	$Sp(n)$	$U(2n)$	$SO(4n)$

TABLE C4

This uniform description give us many new insights to the intimate relationships among different symmetric spaces and to the geometries of special holonomy [8][9][10].

REMARK 3.1. From the above tables (C1) – (C4), we obtain all the classical compact symmetric spaces up to some abelian part and a finite cover. Note that in the rest of this paper, when we say **generalized Grassmannian**, we only mean those symmetric spaces in these tables and their noncompact duals.

In the generalized Grassmannian case, the noncompact symmetric spaces can be described as space-like Grassmannians (see the arguments below or [4]). That means they are open submanifolds of their compact duals consisting of space-like linear subspaces. Hence we can always embed the noncompact generalized Grassmannians to their compact duals. We will use \mathbf{p} to denote this space-like embedding in the generalized Grassmannian case.

EXAMPLE 3.1. Real Grassmannian manifold $M^c = O(n+m)/O(n)O(m) = \{L \subset \mathbb{R}^{n+m} : L \cong \mathbb{R}^n\}$ ($m \geq n$).

The noncompact dual of M^c is given by $M^n = O(n, m)/O(n)O(m)$.

Define a metric g on \mathbb{R}^{n+m} to be $g((u_1, \dots, u_{n+m})^t, (v_1, \dots, v_{n+m})^t) = u_1v_1 + \dots + u_nv_n - u_{n+1}v_{n+1} - \dots - u_{n+m}v_{n+m}$, denote $M^+ := \{L \in M^c : g|_L > 0\}$, such $L \in M^c$ with the property that $g|_L > 0$ is called a space-like subspace.

Fact 1: $M^+ \cong O(n, m)/O(n)O(m) = M^n$.

Proof. First we write the subspace L in the form of matrix: Choose a basis of L , namely $\{l_1, \dots, l_n\}$, then we have $L = [l_1, \dots, l_n]$, a $(n+m) \times n$ matrix.

The action of $O(n, m)$ on M^+ is given by $A \cdot L = AL$ as matrix product.

If $X \in L \in M^+$, i.e. $X^t \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} X > 0$, then $(AX)^t \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} (AX) = X^t A^t \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} AX = X^t \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} X > 0$, for any $A \in O(n, m)$.

Hence $A \cdot L \in M^+$, for any $A \in O(n, m)$, $L \in M^+$.

Transitive: Write $L_o = [e_1, \dots, e_n] = \begin{bmatrix} I_{n \times n} \\ 0_{m \times n} \end{bmatrix}$. For any $L \in M^+$, we can choose a basis of L , such that $L = \begin{bmatrix} I_{n \times n} \\ Y_{m \times n} \end{bmatrix}$. Since $L \in M^+$, the matrix $I - Y^tY$ is positive definite, in particular, it is invertible. Now we want to find $A \in O(n, m)$ such that $A \cdot L_o = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_3 \end{bmatrix} = \begin{bmatrix} I \\ Y \end{bmatrix}$. We only need to take $A = \begin{pmatrix} (I - Y^tY)^{-\frac{1}{2}} & Y^t(I - YY^t)^{-\frac{1}{2}} \\ Y(I - Y^tY)^{-\frac{1}{2}} & (I - YY^t)^{-\frac{1}{2}} \end{pmatrix}$, note the matrices here can take square roots since they are positive definite symmetric matrices.

Isotropic subgroup: By $A \cdot L_o = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix} \begin{bmatrix} I \\ 0 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_3 \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix}$, we have $A_3 = 0, A_2 = 0, A_1 \in O(n), A_4 \in O(m)$. Hence the isotropic subgroup is isomorphic to $O(n)O(m)$.

So we have $M^+ \cong O(n, m)/O(n)O(m) = M^n$. \square

From the above fact, we have the space-like embedding $\mathbf{p} : M^n \hookrightarrow M^c$ obtained by realized M^n as the space of all space-like subspaces in $R^{n,m}$.

Fact 2: The embedding \mathbf{p} is K -equivariant.

Proof. For any $L = \begin{bmatrix} I \\ Y \end{bmatrix} \in M^n$ and $k = \begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix} \in K = O(n)O(m)$,

$$k \cdot L = \begin{bmatrix} I \\ K_2 Y K_1^{-1} \end{bmatrix}$$

obviously, $\mathbf{p}(k \cdot L) = k \cdot \mathbf{p}(L)$. \square

Now we describe the embedding \mathbf{p} from the Lie group theoretical aspect. Since we have $M^c \cong GL(n + m, \mathbb{R})/P$, where $P = \left\{ \begin{pmatrix} U_{n \times n} & W_{n \times m} \\ 0_{m \times n} & V_{m \times m} \end{pmatrix} \in GL(n + m, \mathbb{R}) \right\}$, we can define an embedding $\mathbf{g} : O(n, m)/O(n)O(m) \hookrightarrow GL(n + m, \mathbb{R})/P \cong O(n + m)/O(n)O(m)$ by $AK \mapsto AP \mapsto ABK$, where $B \in P$ such that $AB \in O(n + m)$. We can use the Gram-Schmidt process to find such a B .

Fact 3: $\mathbf{g} = \mathbf{p}$.

Proof. We only need to show that for any $A \in O(n, m)$, $B \in P$, AK and ABK represent the same subspace, which is obvious by direct computations. \square

We generalize these properties for real Grassmannians to all generalized Grassmannians.

Recall that, the compact (resp. noncompact) generalized Grassmannians can be viewed as fixed point sets in the compact (resp. noncompact) Real Grassmannians of some isometric involutions [4]. Namely, they are (intersections of) reflective submanifolds of some real Grassmannians. Reflective submanifolds of symmetric spaces are studied by S.P. Leung in [11][12][13][14]. Since the duality between compact symmetric spaces and noncompact symmetric spaces is invariant under taking reflective submanifold and the noncompact real Grassmannian consists of space-like subspaces (Fact 1 of Example 3.1), we know that if the compact generalized Grassmannian consists of subspaces invariant under some isometric involutions, then its noncompact dual consists of space-like subspaces which are also invariant under the same isometric involutions (generalization of Fact 1).

Hence we have the space-like embedding \mathbf{p} defined the same way with the real Grassmannian case. In fact, this space-like embedding is just the restriction of the space-like embedding for the corresponding real Grassmannians. From Fact 2 of Example 3.1, we have the embedding \mathbf{p} is K -equivariant for all the generalized Grassmannians (generalization of Fact 2).

And also for each of these compact generalized Grassmannians G^c/K , we can find a connected transformation group L larger than the isometry group G^c such that $G^c/K \cong L/P$ where P is a parabolic subgroup of L and $G^c \cap P = K$ [6][17]. At this time, for its noncompact dual G^n/K , G^n is also a subgroup of L and $G^n \cap P = K$. The following natural embeddings

$$G^n \subset L \text{ and } G^c \subset L$$

induce the natural embeddings

$$G^n/K \hookrightarrow L/P \text{ and } G^c/K \hookrightarrow L/P.$$

The latter embedding is in fact an isomorphism, hence we have the following map

$$\mathbf{g} : G^n/K \hookrightarrow L/P \cong G^c/K$$

$$AK \mapsto AP \mapsto ABK$$

where $B \in P$ such that $AB \in G^c$ and such B is unique up to K .

Comparing these two embeddings \mathfrak{p} and \mathfrak{g} , they can both be viewed as the restrictions of \mathfrak{p} and \mathfrak{g} respectively for the corresponding real Grassmannians. From Fact 3 of Example 3.1, we have $\mathfrak{p}=\mathfrak{g}$ for all the generalized Grassmannians (generalization of Fact 3).

3.2. Borel embedding for Hermitian symmetric spaces. In this subsection, we will review the Borel embedding for Hermitian symmetric spaces. We refer to [16] for basics about Hermitian symmetric spaces. Hermitian symmetric spaces are all simply-connected. There is a one-one correspondence between compact Hermitian symmetric spaces and noncompact Hermitian symmetric spaces. Let $M^c = G^c/K$ be a compact Hermitian symmetric space and $M^n = G^n/K$ its noncompact dual. The corresponding Cartan decompositions are $\mathfrak{g}^c = \mathfrak{k} + \mathfrak{m}_c$ and $\mathfrak{g}^n = \mathfrak{k} + \mathfrak{m}_n$. Complexify \mathfrak{g}^c and \mathfrak{g}^n , we get $\mathfrak{g}^c \otimes \mathbb{C} = \mathfrak{g}^n \otimes \mathbb{C} = \mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} + \mathfrak{m}^{\mathbb{C}}$. Denote by $G^{\mathbb{C}}$ the corresponding Lie group of $\mathfrak{g}^{\mathbb{C}}$. Decompose $\mathfrak{m}^{\mathbb{C}}$: $\mathfrak{m}^{\mathbb{C}} = \mathfrak{m}^+ \oplus \mathfrak{m}^-$, where \mathfrak{m}^{\pm} are $(\pm\sqrt{-1})$ -eigenspaces of the complex structure J . Since the group K of isometries on M^n and M^c preserves J , \mathfrak{m}^+ and \mathfrak{m}^- are invariant under the adjoint action of K , i.e. $[\mathfrak{k}, \mathfrak{m}^+] \subset \mathfrak{m}^+$, $[\mathfrak{k}, \mathfrak{m}^-] \subset \mathfrak{m}^-$, we have the following lemma [16]:

LEMMA 3.1. \mathfrak{m}^+ and \mathfrak{m}^- are abelian subalgebras of $\mathfrak{g}^{\mathbb{C}}$. Moreover, the complex vector subspace $\mathfrak{p} = \mathfrak{k}^{\mathbb{C}} + \mathfrak{m}^- \subset \mathfrak{g}^{\mathbb{C}}$ is a complex Lie subalgebra of $\mathfrak{g}^{\mathbb{C}}$.

Denote $P = \exp(\mathfrak{p})$ to be the real Lie subgroup of $G^{\mathbb{C}}$ corresponding to \mathfrak{p} .

THEOREM 3.1 (Borel embedding theorem). *The embedding $G^c \hookrightarrow G^{\mathbb{C}}$ induces a biholomorphism $M^c = G^c/K \cong G^{\mathbb{C}}/P$ onto the complex homogeneous manifold $G^{\mathbb{C}}/P$. The embedding $G^n \hookrightarrow G^{\mathbb{C}}$ induces an open embedding $M^n = G^n/K \hookrightarrow G^{\mathbb{C}}/P \cong M^c$, realizing M^n as an open subset of its compact dual M^c .*

We will use \mathfrak{b} to denote the Borel embedding in the Hermitian symmetric space case. Since we have $M^c \cong G^{\mathbb{C}}/P$, we can define an embedding $\mathfrak{g} : G^n/K \hookrightarrow G^{\mathbb{C}}/P \cong G^c/K$ by $AK \mapsto AP \mapsto ABK$, where $B \in P$ such that $AB \in G^c$. It is obviously that $\mathfrak{b}=\mathfrak{g}$ and

LEMMA 3.2. *The embedding $\mathfrak{b}=\mathfrak{g}$ is K -equivariant.*

Proof. For any $AK \in G^n/K$, $\mathfrak{g}(AK) = ABK$, where $B \in P$ such that $AB \in G^c$. Now for any $k \in K$, $\mathfrak{g}(k \cdot AK) = kAB'K$, where $B' \in P$ such that $kAB' \in G^c$. We can just choose $B' = B$. So $\mathfrak{g}(k \cdot AK) = kABK = k \cdot \mathfrak{g}(AK)$. \square

EXAMPLE 3.2. *For the Riemann sphere $S^2 = SU(2)/S(U(1)^2)$, its noncompact dual is the unit disk $SU(1,1)/S(U(1)^2)$ in the complex line \mathbb{C} . The Borel embedding is defined through the manifold $SL(2, \mathbb{C})/P$ which is isomorphic to $SU(2)/S(U(1)^2)$, where P is a parabolic subgroup of $SL(2, \mathbb{C})$. Since we can embed the complex line into the sphere by stereographic projection, the unit disk is the lower hemisphere of the sphere, this is just the Borel embedding. In this case, the image of the Borel embedding consists of those points with distance less than half of the diameter (least upper bound of the length of an arbitrary minimal geodesic) of the sphere from the south pole.*

For Hermitian symmetric spaces, We also have polydisc theorem and Harish-Chandra embedding theorem which we will not discuss in this paper.

3.3. The generalized embedding for symmetric R-spaces. From the above two subsections, we have space-like embedding \mathbf{p} for the generalized Grassmannians and Borel embedding \mathbf{b} for the Hermitian symmetric spaces. Though these two embeddings seem very different from their definitions, their formulas are the same (the same \mathbf{g}) when we write them from the Lie group theoretical aspect. The definition of \mathbf{g} depends on the fact that the compact symmetric space G/K admits a connected transformation group larger than the isometry group G , which is just the definition for symmetric R-spaces.

Symmetric R-spaces, introduced by Takeuchi and Nagano in the 1960s, form a distinguished subclass of compact Riemannian symmetric spaces. A symmetric R-space is a compact symmetric space G/K which admits a connected transformation group larger than the isometry group G . Let L be the largest such group, then L is semisimple and $G/K \cong L/P$ where P is a parabolic subgroup of L and $G \cap P = K$. Irreducible symmetric R-spaces have been first classified by Kobayashi and Nagano in [6][17]:

- (i) all Hermitian symmetric spaces of compact type;
- (ii) Grassmannian manifolds $O(n + m)/O(n)O(m)$, $Sp(n + m)/Sp(n)Sp(m)$;
- (iii) the classical groups $SO(m)$, $U(m)$, $Sp(m)$;
- (iv) $U(2m)/Sp(m)$, $U(m)/O(m)$;
- (v) $SO(n + 1) \times SO(m + 1)/S(O(n) \times O(m))$;
- (vi) the Cayley projective plane and three exceptional spaces.

From the above list, we can see that all the compact generalized Grassmannians and compact Hermitian symmetric spaces are symmetric R-spaces.

REMARK 3.2. *Note that there are three types irreducible classical symmetric R-spaces which are not generalized Grassmannian: $SO(n + 2)/SO(n)SO(2)$, $SO(n + 1) \times SO(m + 1)/S(O(n) \times O(m))$, and $SO(2n)/U(n)$ (n odd). However, we can also describe them as some kind of “Grassmannian”.*

*For $SO(n + 2)/SO(n)SO(2)$, it is a Hermitian symmetric space, consisting of **oriented** subspaces $L \cong \mathbb{R}^2$ in \mathbb{R}^{n+2} .*

*For $SO(n + 1) \times SO(m + 1)/S(O(n) \times O(m))$, it is a real form of $SO(n + m + 2)/SO(n + m)SO(2)$, hence consisting of **oriented** subspaces $L \cong \mathbb{R}^2$ in \mathbb{R}^{n+m+2} which are invariant under the corresponding isometric involution.*

For $SO(2n)/U(n)$ (n odd), it is a reflective submanifold of $U(2n)/U(n)U(n)$, hence consisting of subspace $L \cong \mathbb{C}^n$ in \mathbb{C}^{2n} which are invariant under the corresponding isometric involution.

REMARK 3.3. *From Remark 3.2, we can also describe these three types of classical symmetric R-spaces as some kind of “Grassmannians”. The natural question is whether their noncompact duals are the corresponding “space-like Grassmannians”? However, this is not true in general.*

*For $SO(n + 2)/SO(n)SO(2)$, consisting of **oriented** subspaces $L \cong \mathbb{R}^2$ in \mathbb{R}^{n+2} , its noncompact dual is $SO_o(2, n)/SO(n)SO(2)$, consisting of space-like $L \cong \mathbb{R}^2$ in \mathbb{R}^{n+2} without the oriented condition.*

For $SO(n + 1) \times SO(m + 1)/S(O(n) \times O(m))$, it is the same as the $SO(n + 2)/SO(n)SO(2)$ case, since it is a real form of $SO(n + m + 2)/SO(n + m)SO(2)$.

For $SO(2n)/U(n)$ (n odd), it is a reflective submanifold of $U(2n)/U(n)U(n)$, hence its noncompact dual is the corresponding space-like Grassmannian. That means the space-like embedding \mathbf{p} can be generalized to $SO(2n)/U(n)$ (n odd).

From this definition or characterization for a symmetric R-space, we can easily construct an open embedding from its noncompact dual to it. Let $M^c = G^c/K \cong L/P$ be a **symmetric R-space**, and $M^n = G^n/K$ its noncompact dual, then G^n is a subgroup of L and $G^n \cap P = K$. We define

$$g : G^n/K \hookrightarrow L/P \cong G^c/K$$

$$AK \mapsto AP \mapsto ABK$$

where $B \in P$ such that $AB \in G^c$ and such B is unique up to K . That is, this map generalizes space-like embedding p for the generalized Grassmannians and Borel embedding b for the Hermitian symmetric spaces. Obviously, we have

LEMMA 3.3. *The embedding g is K -equivariant.*

Since the duality between compact symmetric space and noncompact symmetric space is invariant under taking reflective submanifold, we have the following proposition.

PROPOSITION 3.1. *For any symmetric R-space M , if N is a reflective submanifold of M which is also a symmetric R-space, then $g_N = g_M|_N$.*

Since this embedding g generalizes the embeddings p and b , we will call it the generalized embedding in the rest of the paper.

As a conclusion of this section, the three embeddings p , b and g are compatible with each other, they are all K -equivariant and satisfy Proposition 3.1.

4. The embeddings from Lie algebra aspect. In the above section, we have a generalized embedding g for symmetric R-spaces which generalizes both p and b , defined from Lie group theoretical aspect. How to characterize the image of this embedding? To answer this question, we will first construct an embedding for symmetric R-spaces from Lie algebra theoretical aspect, so the image is easy to describe, and then show this new embedding is just our g .

In this section, we will first give further characterizations and properties of symmetric R-spaces, then review the cut loci of symmetric R-spaces, and finally construct an embedding from the Lie algebra aspect using the cut loci.

4.1. Unit lattices of symmetric R-spaces. We will list some important properties of symmetric R-spaces in this subsection.

In 1984, Takeuchi [23] showed that irreducible symmetric R-spaces are either irreducible Hermitian symmetric spaces of compact type or compact connected real forms of them and vice-versa.

In 1985, O. Loos [15] gave another intrinsic characterization of symmetric R-spaces among all compact symmetric spaces by the property that its unit lattice is **orthonormal** (i.e. the unit lattice possesses an orthonormal \mathbb{Z} -basis). Recall that for a compact symmetric space G/K with the corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, choose a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{m}$ in \mathfrak{m} (called Cartan subalgebra of (G, K)), then the unit lattice is:

$$\Gamma(G, K) := \{A \in \mathfrak{a} : \exp A \in K\}.$$

The unit lattice $\Gamma(G, K)$ of G/K is said to be **orthonormal** if there exists a basis $\{A_1, \dots, A_r\}$ of \mathfrak{a} (where $r = \text{rank}(M) := \dim(\mathfrak{a})$) with the properties

- (i) $A_i \perp A_j$ if $i \neq j$
- (ii) $|A_i| = |A_j|$
- (iii) $\Gamma(G, K) = \text{span}_{\mathbb{Z}}(A_1, \dots, A_r) = \{\sum_{i=1}^{i=r} m_i A_i; m_i \in \mathbb{Z}\}$

We will compute some examples of unit lattices of compact symmetric spaces below.

EXAMPLE 4.1. *The Real Grassmannian $M = O(n + m)/O(n)O(m)$ ($m \geq n$), then $M \cong \{L \subset \mathbb{R}^{n+m} : L \cong \mathbb{R}^n\}$ and $\text{rank}(M) = n$.*

For the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$, we have

$$\begin{aligned} \mathfrak{g} &= \mathfrak{o}(n + m) = \left\{ \begin{pmatrix} A & B \\ -B^t & D \end{pmatrix} : A^t + A = 0, D^t + D = 0 \right\}, \\ \mathfrak{k} &= \mathfrak{o}(n) \times \mathfrak{o}(m) = \left\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : A^t + A = 0, D^t + D = 0 \right\}, \\ \mathfrak{m} &= \left\{ \begin{pmatrix} 0 & B \\ -B^t & 0 \end{pmatrix} \right\}. \end{aligned}$$

Denote R_{ij} ($1 \leq i \leq n, n + 1 \leq j \leq n + m$) to be the matrix with 1 at (i, j) -entry, -1 at (j, i) -entry, and 0 otherwise. Then $\{R_{ij}\}$'s form an orthonormal basis of \mathfrak{m} with respect to the inner product $\langle \cdot, \cdot \rangle$, where $\langle X, Y \rangle := -\frac{1}{2}\text{tr}(XY)$. By computing the Lie bracket, we know that $\mathfrak{a} := \text{span}_{\mathbb{R}}(R_{1,n+1}, R_{2,n+2}, \dots, R_{n,2n})$ is a Cartan subalgebra of (G, K) , then $\Gamma(G, K) = \{A \in \mathfrak{a} | \exp A \in O(n) \times O(m)\} = \{\sum_{i=1}^n m_i \pi R_{i,n+i}, m_i \in \mathbb{Z}\}$. The unit lattice $\Gamma(G, K)$ is generated by $\{A_1 = \pi R_{1,n+1}, \dots, A_n = \pi R_{n,2n}\}$ with $A_i \perp A_j$ and $|A_i| = |A_j|, \forall i \neq j \in \{1, 2, \dots, n\}$, hence orthonormal.

Similar to the above example, we can check that the unit lattices of all the classical irreducible symmetric R-spaces are orthonormal.

EXAMPLE 4.2. *For the special unitary group $SU(3)$ (not a symmetric R-space), its unit lattice is just its integral lattice, the picture of this lattice can be found in page 227 of [2]. Obviously, this lattice is not orthonormal.*

For the exceptional symmetric spaces, because the explicit description of the corresponding Lie algebras as matrix algebras is too complicated, we can't determine whether these unit lattices are orthonormal or not by direct computations as above. However for the five exceptional compact Lie groups, we can use the relationship between integral lattices and inverse roots to get the answer (page 223 of [2]). That is, for the five simply-connected exceptional compact Lie groups, the integral lattice is the same with the abelian group generated by inverse roots, hence they are not orthonormal (the list of the root system of exceptional Lie algebras can be found in page 686-692 of [5]).

In 2012, P. Quast and M.S. Tanaka [19] gave the following important property of symmetric R-spaces: reflective submanifolds of symmetric R-spaces are (geodesically) convex. That means any shortest geodesic arc in the reflective submanifolds is still the shortest in the symmetric R-spaces.

Further characterizations and properties of symmetric R-spaces can be found in [24].

4.2. Cut loci of symmetric R-spaces. This subsection is mainly based on T.Sakai's work [20][21][22]. More about cut loci, Cartan polyhedrons and injective radii for symmetric spaces can be found in [26][27]. Let (M, g) be a complete Riemannian manifold with a fixed point x in M , then we have an exponential map

$\exp_x : T_xM \rightarrow M$. For any $X \in T_xM, |X| = 1, \gamma_X(t) := \exp_x(tX)$ is a geodesic parametrized by arc length.

DEFINITION 4.1. t_0X (resp. $\exp_x(t_0X)$) is called a *tangent conjugate point* (resp. *conjugate point*) of x along a geodesic γ_X if there exists a nonzero Jacobi field $J(t)$ along γ_X such that $J(0) = J(t_0) = 0$.

\bar{t}_0X (resp. $\exp_x(\bar{t}_0X)$) is called a **tangent cut point** (resp. **cut point**) of x along γ_X if $\gamma_X|_{[0, \bar{t}_0]}$ is minimal but $\gamma_X|_{[0, s]}$ is not minimal for any $s > \bar{t}_0$.

Denote $\text{Cut}(x)$ to be the set of all cut points of x along any geodesic, it is called the **cut loci** of x . The following standard result about tangent cut points is Proposition 2.1 in [20].

PROPOSITION 4.1. If \bar{t}_0X is a tangent cut point of x along γ_X , then either (a) \bar{t}_0X is the first tangent conjugate point of x along γ_X , or (b) there exists a geodesic $\gamma_Y \neq \gamma_X$ joining x to $\gamma_X(\bar{t}_0)$ such that $l(\gamma_Y|_{[0, \bar{t}_0]}) = l(\gamma_X|_{[0, \bar{t}_0]})$. Here l is the length of the geodesic.

Conversely, if (a) or (b) is satisfied, then there exists $\bar{t} \in (0, \bar{t}_0]$, such that $\bar{t}X$ is a tangent cut point of x along γ_X .

EXAMPLE 4.3. The three space forms, i.e. simply connected Riemannian manifold with constant sectional curvature K ,

$K = +1$, unit sphere S^n has $\text{Cut}(x) = \{-x\}$.

$K = 0$, Euclidean space \mathbb{R}^n has $\text{Cut}(x) = \emptyset$.

$K = -1$, hyperbolic space \mathbb{H}^n has $\text{Cut}(x) = \emptyset$.

EXAMPLE 4.4. Real projective space $\mathbb{R}P^n$ has $\text{Cut}(x) \cong \mathbb{R}P^{n-1}$.

Complex projective space $\mathbb{C}P^n$ with Fubini-study metric has $\text{Cut}(x) \cong \mathbb{C}P^{n-1}$.

EXAMPLE 4.5. Flat torus $T^2 = S^1 \times S^1, \text{Cut}(x) \cong S^1 \times \{0\} \cup \{0\} \times S^1$.

REMARK 4.1. All the examples above are homogeneous spaces, hence the cut loci of two different points look the same.

Let M be a symmetric space. If M is noncompact, then the sectional curvature of M is non-positive, by Hadamard-Cartan Theorem, the exponential map is a covering map. For any $x, y \in M$, there exists a unique minimizing geodesic joining x and y . So $\text{Cut}(x) = \emptyset$ for any $x \in M$. If M is compact, its diameter is finite, hence there exists a cut point for any point $x \in M$ along any geodesic starting from x . So $\text{Cut}(x) \neq \emptyset$ for any $x \in M$.

Now given a compact symmetric space M and a fixed point o in $M = G/K$, we want to determine the tangent cut loci of o . For the corresponding Lie algebra $\mathfrak{g}, \mathfrak{k}$ of G, K , we have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{m}$, where $T_oM \cong \mathfrak{m}$. Let \mathfrak{a} be a maximal abelian subspace of \mathfrak{m} , since $\text{Ad}K(\mathfrak{a}) = \mathfrak{m}$ and $\text{Ad}K$ acts on \mathfrak{m} as isometries, it suffices to compute $\bar{t}_0(X)$ for $X \in \mathfrak{a}$. From Proposition 4.1, obviously, we have

PROPOSITION 4.2. Let $M = G/K$ be a compact symmetric space and \mathfrak{a} be a Cartan subalgebra of (G, K) . For a unit vector $X \in \mathfrak{a}$, assume that \bar{t}_0X is a tangent cut point of o along γ_X . Then either \bar{t}_0X is the first tangent conjugate point of o along γ_X or there exists a unit vector $Y \in \mathfrak{a}, Y \neq X$ such that $\exp_o \bar{t}_0X = \exp_o \bar{t}_0Y$ does hold.

Now given a unit tangent vector $X \in \mathfrak{a}$, $|X| = 1$, we want to compute $\bar{t}_0(X)$ to determine the cut point of o along the geodesic γ_X . By the above proposition, $\bar{t}_0(X) = \min\{t_0(X), \tilde{t}_0(X)\}$, where $t_0(X)$ is given by the first conjugate point of o along γ_X , $\tilde{t}_0(X)$ is the minimum positive value such that $\exp_o \tilde{t}_0(X)X = \exp_o \tilde{t}_0(X)Y$ holds for some unit tangent vector $Y \in \mathfrak{a}$, $Y \neq X$.

REMARK 4.2. *For simply-connected Riemannian symmetric spaces, the first conjugate loci and the cut loci coincide. Without the simply-connectedness condition, we have $\tilde{t}_0(X) \leq t_0(X)$ (see the proof of Theorem 2.5 in [20]).*

LEMMA 4.1. *Let $X = Adk(H) \in \mathfrak{m}$ with $H \in \mathfrak{a}$, $k \in K$, $|H| = 1$. Then*

$$\bar{t}_0(X) = \bar{t}_0(H) = \min_{\{A \in \Gamma(G, K) - \{0\}\}} \frac{\langle A, A \rangle}{2|\langle H, A \rangle|},$$

where $\Gamma(G, K) := \{A \in \mathfrak{a} : \exp A \in K\}$ is the **unit lattice**.

Now let H be a compact Lie group, if we put $G = H \times H$, $K = \{(x, x) | x \in H\}$ and the involution σ of G : $\sigma(x, y) = (y, x)$, then (G, K) is a compact Riemannian symmetric pair corresponding to $H \cong G/H$. Let \mathfrak{h} be the Lie algebra of H and $\mathfrak{a}^* \subset \mathfrak{h}$ be a maximal abelian subalgebra of \mathfrak{h} , then $\mathfrak{m} = \{(X, -X) : X \in \mathfrak{h}\}$ and $\mathfrak{a} := \{(Y, -Y) : Y \in \mathfrak{a}^*\}$ is a Cartan subalgebra of (G, K) .

LEMMA 4.2. *Let $X \in \mathfrak{a}^*$ with $|X| = 1$, then $\bar{t}_0(X) = \min_{\{A \in \Gamma(H) - \{0\}\}} \frac{\langle A, A \rangle}{2|\langle X, A \rangle|}$, where $\Gamma(H) := \{A \in \mathfrak{a}^* : \exp A = e\}$.*

Note that the unit lattice $\Gamma(H) := \{A \in \mathfrak{a}^* : \exp A = e\}$ is just the **integral lattice** of the compact Lie group H .

By the above lemmas, we have the following theorem.

THEOREM 4.1. *Let $M = G/K$ be a compact symmetric space. Then the tangent cut loci of o is given by $AdK(\bigcup\{\bar{t}_0(X)X; X \in \mathfrak{a}, |X| = 1\})$, i.e. the tangent cut loci of o is determined by the tangent cut loci of o in the flat torus $\mathfrak{a}/\Gamma(G, K)$.*

If the unit lattice $\Gamma(G, K)$ is orthonormal, then for any unit vector $X = \sum x_i A_i$ in \mathfrak{a} , the tangent cut point along direction X is determined by $\bar{t}_0(X) = 1/(2 \max |x_i|)$. In fact, if we put $\alpha^2 = \alpha_i^2 = \langle A_i, A_i \rangle$, then $\sum (x_i)^2 (\alpha_i)^2 = 1$. For any non-zero $A = \sum m_i A_i \in \Gamma(G, K)$, we have

$$\begin{aligned} \frac{\langle A, A \rangle}{2|\langle X, A \rangle|} &= \frac{\alpha^2(m_1^2 + \dots + m_r^2)}{2\alpha^2|m_1x_1 + \dots + m_r x_r|} \\ &\geq \frac{(m_1^2 + \dots + m_r^2)}{2 \max |x_i| (|m_1| + \dots + |m_r|)} \\ &\geq \frac{1}{2 \max |x_i|} \end{aligned}$$

and if $\max |x_i| = |x_{i_0}|$, then for taking $m_i = \delta_{i, i_0}$, the " = " holds. Since the unit lattice of a symmetric R-space is always orthonormal, we have the following theorem [18]:

THEOREM 4.2. *Let $M = G/K$ be a symmetric R-space, then the tangent cut loci of o is given by $AdK(\bigcup\{1/(2 \max |x_i|)X; X = \sum x_i A_i \in \mathfrak{a}, |X| = 1\})$.*

4.3. The embedding using cut loci. Let $M^c = G^c/K$ be a **symmetric R-space** and $M^n = G^n/K$ be its noncompact dual. We want to construct a natural embedding $M^n \hookrightarrow M^c$. The exponential map of any noncompact symmetric space is a diffeomorphism. Let $R := \{tX \in T_oM^c \mid |X| = 1, t < \bar{t}_0(X)\} \subset T_oM^c$, where $\bar{t}_0(X)$ is a positive number such that $\bar{t}_0(X)X$ is the tangent cut point of o along γ_X . Then $\text{Int}(o) := M^c \setminus \text{Cut}(o) = \exp_o(R) \subset M^c$ is an open cell, with the exponential map a diffeomorphism from R to $\text{Int}(o)$. Now we only need to construct an injective map $h : T_oM^n \rightarrow T_oM^c$ such that $\text{Im}(h) \subset R$, then the map $f = \exp_o^c \circ h \circ (\exp_o^n)^{-1} : M^n \rightarrow T_oM^n \rightarrow T_oM^c \rightarrow M^c$ is an open embedding.

EXAMPLE 4.6. *The injective map from positive line to circle.*

For any $\exp(\sqrt{-1}\theta) \in S^1, \exp(t) \in \mathbb{R}^+,$ where $\theta, t \in \mathbb{R},$ the stereographic projection maps $\exp(\sqrt{-1}\theta)$ to $\exp(t)$ if and only if $\tan(\frac{\theta}{2}) = \tanh(\frac{t}{2}).$ So we can define an injective map from \mathbb{R}^+ to S^1 as follows:

$$i : \mathbb{R}^+ \rightarrow S^1.$$

$$\exp(t) \mapsto \exp(2\sqrt{-1} \arctan(\tanh \frac{t}{2})).$$

Denote the Cartan decompositions of the corresponding symmetric spaces as $\mathfrak{g}_n = \mathfrak{k} \oplus \mathfrak{m}_n$ for M^n and $\mathfrak{g}_c = \mathfrak{k} \oplus \sqrt{-1}\mathfrak{m}_n = \mathfrak{k} \oplus \mathfrak{m}_c$ for M^c . Let $\mathfrak{a}_n \subset \mathfrak{m}_n, \mathfrak{a}_c = \sqrt{-1}\mathfrak{a}_n \subset \sqrt{-1}\mathfrak{m}_n = \mathfrak{m}_c$ be the maximal abelian subspaces of $\mathfrak{m}_n, \mathfrak{m}_c$ respectively. We will use X to represent vector in $\mathfrak{m}_n,$ and $\sqrt{-1}X$ to represent the corresponding vector in $\mathfrak{m}_c.$

First we construct an injective map from \mathfrak{a}_n to $\mathfrak{a}_c.$ As before, for the compact symmetric space $M^c,$ the unit lattice $\Gamma(G^c, K)$ is a lattice in \mathfrak{a}_c which is generated by $\{\sqrt{-1}A_1, \dots, \sqrt{-1}A_r\},$ where $r = \text{rank}(M).$ Since M^c is a symmetric R-space, we have $A_i \perp A_j$ and $|A_i| = |A_j|$ for any $i \neq j \in \{1, 2, \dots, r\}.$ Note that for any $\sqrt{-1}X = (\sqrt{-1}x_1A_1, \dots, \sqrt{-1}x_rA_r)$ in $\mathfrak{a}_c,$ $\sqrt{-1}X$ is a tangent cut point if and only if $\max |x_i| = 1/2.$

Define:

$$h : \mathfrak{a}_n \rightarrow \mathfrak{a}_c,$$

$$(x_1A_1, \dots, x_rA_r) \mapsto -\left(\frac{\arctan(\tanh \pi x_1)}{\pi} \sqrt{-1}A_1, \dots, \frac{\arctan(\tanh \pi x_r)}{\pi} \sqrt{-1}A_r\right),$$

note here the function $a(x) := \arctan(\tanh \pi x)/\pi$ is a monotone increasing odd function. Since $\arctan(\tanh \pi x)/\pi \in (-\frac{1}{4}, \frac{1}{4}),$ we have $\text{Im}(h) = \frac{\mathbb{R}}{2} \cap \mathfrak{a}_c,$ where $\frac{\mathbb{R}}{2} = \{\sqrt{-1}tX \in \mathfrak{m}_c \mid |X| = 1, t < \frac{\bar{t}_0(\sqrt{-1}X)}{2}\}.$

Now we want to extend h to the whole tangent space. For any tangent vector $X \in \mathfrak{m}_n,$ there exists $k \in K,$ such that $X = \text{Ad}k(H)$ for a unique $H \in \mathfrak{a}_n.$ Then we define $h'(X) = \text{Ad}k(h(\text{Ad}k^{-1}X)).$

It is easy to see that such a h' does not depend on the choice of the maximal abelian subspace \mathfrak{a}_n and $k \in K.$ In fact, $\text{Ad}k \cdot \mathfrak{a}_n = \mathfrak{a}'_n$ is also a maximal abelian subspace in \mathfrak{m}_n and $X \in \mathfrak{a}'_n.$ Consider their corresponding compact duals \mathfrak{a}_c and $\mathfrak{a}'_c = \text{Ad}k \cdot \mathfrak{a}_c,$ if $\{\sqrt{-1}A_1, \dots, \sqrt{-1}A_r\}$ are generators of $\Gamma(G^c, K),$ then $\{\sqrt{-1}A'_1 = \text{Ad}k \cdot \sqrt{-1}A_1, \dots, \sqrt{-1}A'_r = \text{Ad}k \cdot \sqrt{-1}A_r\}$ are generators of the lattice $\Gamma(G^c, K)'. Write $X = \sum x_i A_i,$ then under the map h' defined above:$

$$X = (x'_1A'_1, \dots, x'_rA'_r) \xrightarrow{\text{Ad}k^{-1}} (x_1A_1, \dots, x_rA_r)$$

$$\begin{aligned} & \xrightarrow{h} -\left(\frac{\arctan(\tanh \pi x'_1)}{\pi} \sqrt{-1}A_1, \dots, \frac{\arctan(\tanh \pi x'_r)}{\pi} \sqrt{-1}A_r\right), \\ & \xrightarrow{\text{Ad}k} -\left(\frac{\arctan(\tanh \pi x'_1)}{\pi} \sqrt{-1}A'_1, \dots, \frac{\arctan(\tanh \pi x'_r)}{\pi} \sqrt{-1}A'_r\right), \end{aligned}$$

which has the same form with the map h on \mathfrak{a}_n .

From above, we have a globally defined map, also denoted by $h : \mathfrak{m}_n \rightarrow \mathfrak{m}_c$. We can now construct an injective map from M^n to M^c , $\mathbf{f} = \exp_o^c \circ h \circ (\exp_o^n)^{-1} : M^n \rightarrow \mathfrak{m}_n \rightarrow \mathfrak{m}_c \rightarrow M^c$.

Since $\text{Im}(h) = \frac{\mathbb{R}}{2} \subset \mathfrak{m}_c$, \mathbf{f} is injective and obviously

PROPOSITION 4.3. $\text{Im}(\mathbf{f}) = \exp_o^c(\frac{\mathbb{R}}{2})$.

Similar to the generalized embedding \mathbf{g} , the embedding \mathbf{f} is also K -equivariant.

LEMMA 4.3. *The embedding \mathbf{f} is K -equivariant.*

Proof. For any $x = e^X K \in M^n$ and $k \in K$,

$$h(\text{Ad}k \cdot X) = \text{Ad}k \cdot h(X),$$

$$\mathbf{f}(k \cdot x) = \mathbf{f}(e^{\text{Ad}k \cdot X} K) = e^{h(\text{Ad}k \cdot X)} K = e^{\text{Ad}k \cdot h(X)} K = k \cdot e^{h(X)} K = k \cdot \mathbf{f}(x).$$

□

The following proposition is the key argument for the rest of the paper.

PROPOSITION 4.4. *For any symmetric R-space M , if N is a reflective submanifold of M which is also a symmetric R-space, then $\mathbf{f}_N = \mathbf{f}_M|_N$.*

Proof. Let $M = G/K$ and $\rho : M \rightarrow M$ be the isometric involution whose fixed point set is N , then $N = H/I$, where $H \subset G^\rho$, $I = H \cap K$, $\tilde{\rho} : G \rightarrow G$, $g \mapsto \rho g \rho^{-1}$. For the corresponding Cartan decomposition, $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$, $\mathfrak{h} = \mathfrak{i} \oplus \mathfrak{n}$, $\mathfrak{m} \cong T_o M$, $\mathfrak{n} = \mathfrak{m}^{d\tilde{\rho}} \cong T_o N$.

Now take $\mathfrak{a}' \subset \mathfrak{n}$ to be a maximal abelian subspace in \mathfrak{n} , extend \mathfrak{a}' to \mathfrak{a} , which is maximal abelian in \mathfrak{m} . Define $\Gamma(G, K) := \{A \in \mathfrak{a} : \exp A \in K\}$ and $\Gamma(H, I) := \{A' \in \mathfrak{a}' : \exp A' \in I\}$. Since $I \subset K$ and $\mathfrak{a}' \subset \mathfrak{a}$, $\Gamma(H, I) \subset \Gamma(G, K) \cap \mathfrak{a}'$; since $I = H \cap K$, $\Gamma(H, I) \supset \Gamma(G, K) \cap \mathfrak{a}'$. So we have $\Gamma(H, I) = \Gamma(G, K) \cap \mathfrak{a}'$.

By assumption, M is a symmetric R-space, its unit lattice is orthonormal, i.e. $\Gamma(G, K) = \text{span}_{\mathbb{Z}}(e_1, \dots, e_r)$ with $\{e_1, \dots, e_r\}$ orthonormal basis of \mathfrak{a} and $r = \text{rank}(M)$. We want to compute $\Gamma(H, I)$. For any $X \in \Gamma(G, K)$, we have $X \in \Gamma(H, I)$ if and only if $d\tilde{\rho}(X) = X$. Consider the map $d\tilde{\rho} : \mathfrak{g} \rightarrow \mathfrak{g}$, by Observation 4 in [19], \mathfrak{a} is invariant under $d\tilde{\rho}$. Since $d\tilde{\rho}$ is the differential of an isometric involution that leaves \mathfrak{a} invariant, $d\tilde{\rho}|_{\mathfrak{a}}$ is an orthogonal transformation of \mathfrak{a} that squares identity and hence preserves the unit lattice $\Gamma(G, K) \in \mathfrak{a}$. By Proposition 5 in [19], we can choose $\{e_1, \dots, e_r\}$ such that

- (i) $d\tilde{\rho}(e_{2j}) = e_{2j-1}$ for $1 \leq j \leq p$
- (ii) $d\tilde{\rho}(e_j) = e_j$ for $2p + 1 \leq j \leq q$
- (iii) $d\tilde{\rho}(e_j) = -e_j$ for $q + 1 \leq j \leq r$,

where $0 \leq 2p \leq q \leq r$.

Then $\Gamma(H, I) = \text{span}_{\mathbb{Z}}(e_1 + e_2, \dots, e_{2p-1} + e_{2p}, e_{2p+1}, \dots, e_q)$.

With the assumption that N is also a symmetric R-space, $\Gamma(H, L)$ must be orthonormal. That means $\Gamma(H, I)$ must either be $\text{span}_{\mathbb{Z}}(e_1 + e_2, \dots, e_{2p-1} + e_{2p})$ or $\text{span}_{\mathbb{Z}}(e_1, \dots, e_q)$, where $2p \leq r, q \leq r$.

When $\Gamma(H, I) = \text{span}_{\mathbb{Z}}(e_1 + e_2, \dots, e_{2p-1} + e_{2p})$, the map \mathbf{f}_N is induced by h_N (denote $\mathbf{a}'_n, \mathbf{a}_n$ as $\sqrt{-1}\mathbf{a}'_n = \mathbf{a}', \sqrt{-1}\mathbf{a}_n = \mathbf{a}$, and $\tilde{e}_i \in \sqrt{-1}\mathbf{a}_n$ such that $\sqrt{-1}\tilde{e}_i = e_i$)

$$h_N : \mathbf{a}'_n \rightarrow \mathbf{a}' ,$$

$$(x_1(\tilde{e}_1 + \tilde{e}_2), \dots, x_p(\tilde{e}_{2p-1} + \tilde{e}_{2p})) \mapsto -(a(x_1)(e_1 + e_2), \dots, a(x_p)(e_{2p-1} + e_{2p})),$$

which is just the restriction of h_M :

$$h_M : \mathbf{a}_n \rightarrow \mathbf{a} ,$$

$$(x_1\tilde{e}_1, \dots, x_r\tilde{e}_r) \mapsto -(a(x_1)e_1, \dots, a(x_r)e_r),$$

hence $\mathbf{f}_N = \mathbf{f}_M|_N$.

When $\Gamma(H, I) = \text{span}_{\mathbb{Z}}(e_1, \dots, e_q)$, it is obviously that $\mathbf{f}_N = \mathbf{f}_M|_N$. \square

5. Comparing these embeddings. We will first compare \mathbf{f} defined above with the space-like embedding \mathbf{p} , then compare \mathbf{f} with the Borel embedding \mathbf{b} , and finally compare \mathbf{f} with the generalized embedding \mathbf{g} . The images of these embeddings are described using cut loci.

5.1. Comparing \mathbf{f} with \mathbf{p} .

EXAMPLE 5.1. *The Real Grassmannian $M = O(n + m)/O(n)O(m)$ ($m \geq n$), following the notations in Example 4.1, $\mathbf{a} = \text{span}_{\mathbb{R}}(R_{1,n+1}, \dots, R_{n,2n})$ is a Cartan subalgebra of (G, K) and $\Gamma(G, K) = \{A \in \mathfrak{a} \mid \exp A \in O(n) \times O(m)\} = \{\sum_{i=1}^n m_i \pi R_{i,n+i}, m_i \in \mathbb{Z}\}$. For any $X = \sum_{i=1}^n x_i R_{i,n+i} \in \mathfrak{a}$, $\sum_{i=1}^n x_i^2 = 1$, we have $\bar{t}_0(X) := \min_{\{A \in \Gamma(G, K) - \{0\}\}} \frac{\langle A, A \rangle}{2|\langle X, A \rangle|} = \frac{\pi}{2 \max |x_i|}$. Put*

$$o := \{(u_1, \dots, u_n, 0, \dots, 0)^t : u_i \in \mathbb{R}\} \cong \mathbb{R}^n$$

$$o^\perp := \{(0, \dots, 0, u_{n+1}, \dots, u_{n+m})^t : u_i \in \mathbb{R}\} \cong \mathbb{R}^m$$

$$W_l := \{L \in M^c : \dim(L \cap o^\perp) = l\}, (l = 0, 1, \dots, n).$$

Fact 1: *Cut loci of o in $M^c = O(n + m)/O(n)O(m)$ is given by $V = W_1 \cup \dots \cup W_n = \{L \in M^c : L \cap o^\perp \neq \{0\}\}$.*

Proof. For any $X = \sum_{i=1}^n x_i R_{i,n+i} \in \mathfrak{a}$, $\sum_{i=1}^n x_i^2 = 1$.

$\exp(tX) \cdot o = \{(u_1 \cos tx_1, \dots, u_n \cos tx_n, -u_1 \sin tx_1, \dots, -u_n \sin tx_n, 0, \dots, 0)^t : u_i \in \mathbb{R}\}$.

For any $0 \leq t < \bar{t}_0(X) = \frac{\pi}{2 \max |x_i|}$, $\cos tx_i \neq 0$ for any i , $\exp(tX) \cdot o \in W_0$.

Cut loci of o in M^c is given by all $\exp(\bar{t}_0 X) \cdot o$, when $\max |x_i| = |x_{i_1}| = \dots = |x_{i_k}| > |x_{i_{k+1}}| \geq \dots \geq |x_{i_n}|$, $\exp(\bar{t}_0 X) \cdot o \in W_k$. \square

Fact 2: $M^+ = \exp_o^c(\frac{R}{2})$.

Proof. Recall that $R := \{tX \in T_oM^c \mid |X| = 1, t < \bar{t}_0(X)\} \subset T_oM^c$, For any $X = \sum_{i=1}^n x_i R_{i,n+i} \in \mathfrak{a}$, we have

$$\begin{aligned} 0 \leq t < \frac{\bar{t}_0(X)}{2} = \frac{\pi}{4 \max |x_i|} &\iff |tx_i| < \frac{\pi}{4}, \quad \forall i \\ &\iff (\cos tx_i)^2 > (\sin tx_i)^2, \quad \forall i \\ &\iff \exp(tX) \cdot o \text{ is space-like.} \end{aligned}$$

Since the embedding is K -equivariant, we have $\exp_o^c(\frac{R}{2}) = M^+$. \square

That is, for the Real Grassmannian, the following is true:

$$M^n = \{L \subset V\}^+ = \exp_o^c(\frac{R}{2}) \subset M^c = \{L \subset V\} = \exp_o^c(R) \cup \text{Cut}(o).$$

The generalization of the results for real Grassmannians is as follows:

PROPOSITION 5.1. *For any compact generalized Grassmannian M^c and its non-compact dual M^n , the image of the space-like embedding \mathbf{p} is $\exp_o^c(\frac{R}{2})$.*

Proof. Any compact generalized Grassmannian M^c is the fixed point set of $O(m+n)/O(m)O(n)$ by some isometric involutions σ_{w_i} , $i = 1, \dots, s$, where σ'_{w_i} are the involutions on $O(m+n)/O(m)O(n)$ defined by some quadratic forms w_i on it. i.e. $M^c = \bigcap (O(m+n)/O(m)O(n))^{\sigma_{w_i}}$. For convenience, use R_o to denote R for $M^c = O(m+n)/O(m)O(n)$. By Lemma 7 in [19], we have

$$\begin{aligned} \exp_o^c(\frac{R}{2}) &= \exp_o^c(\frac{R_o}{2} \bigcap \mathfrak{m}_c) \\ &= O(m, n)/O(m)O(n) \bigcap M^c \\ &= \{\text{space-like subspace in } M^c\} \\ &= M^n. \end{aligned}$$

\square

Now for M^c a compact generalized Grassmannian with noncompact dual M^n , the two embeddings $\mathbf{p}, \mathbf{f} : M^n \hookrightarrow M^c$ are both K -equivariant and have the same image $\text{Im}(\mathbf{p}) = \text{Im}(\mathbf{f}) = \exp_o^c(\frac{R}{2})$. We want to show $\mathbf{p} = \mathbf{f}$ in fact.

EXAMPLE 5.2. $O(1, 1)/O(1)^2 \hookrightarrow O(2)/O(1)^2$.

For $O(1, 1)/O(1)^2$, we have $\mathfrak{a}_n = \left\{ \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix} : t \in \mathbb{R} \right\} = \mathfrak{m}_n$. After the exponential map \exp_o^n , we get $O(1, 1)/O(1)^2 = \left\{ \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \cdot K : t \in \mathbb{R} \right\}$. By viewing the manifold as a Grassmannian, we can use $\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \cdot K$ to represent a dimension one linear subspace in \mathbb{R}^2 with basis $\begin{bmatrix} \cosh t \\ \sinh t \end{bmatrix} = \begin{bmatrix} 1 \\ \tanh t \end{bmatrix}$.

For $O(2)/O(1)^2$, we have $\mathfrak{a}_c = \left\{ \begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} : \theta \in \mathbb{R} \right\} = \mathfrak{m}_c$. After the exponential map \exp_o^c , we get $O(2)/O(1)^2 = \left\{ \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \cdot K : \theta \in \mathbb{R} \right\}$. By viewing

the manifold as a Grassmannian, we can use $\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \cdot K$ to represent a dimension one linear subspace in \mathbb{R}^2 with basis $\begin{bmatrix} \cos \theta \\ -\sin \theta \end{bmatrix} = \begin{bmatrix} 1 \\ -\tan \theta \end{bmatrix}$.

Then $\begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \cdot K$ and $\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \cdot K$ represent the same linear subspace in \mathbb{R}^2 if and only if $\tanh t = -\tan \theta$. Comparing with the formula of \mathbf{f} , we see that the space-like embedding \mathbf{p} is the same with the embedding \mathbf{f} .

THEOREM 5.1. *For generalized Grassmannians, the space-like embedding \mathbf{p} and the embedding \mathbf{f} are the same.*

Proof. First we prove it for the real Grassmannians. Follow the notations in Example 5.1, we have for any $X = \sum_{i=1}^n x_i R_{i,n+i} \in \mathfrak{a}_c$,

$$\exp(X) \cdot o = \{(u_1 \cos x_1, \dots, u_n \cos x_n, u_1(-\sin x_1), \dots, u_n(-\sin x_n), 0, \dots, 0)^t : u_i \in \mathbb{R}\}.$$

Similarly, for its noncompact dual, denote \tilde{R}_{ij} ($1 \leq i \leq n, n+1 \leq j \leq n+m$) to be the matrix with 1 at (i, j) -entry, 1 at (j, i) -entry, and 0 otherwise. We have for any $Y = \sum_{i=1}^n y_i \tilde{R}_{i,n+i} \in \mathfrak{a}_n$,

$$\exp(Y) \cdot o = \{(u_1 \cosh y_1, \dots, u_n \cosh y_n, u_1 \sinh y_1, \dots, u_n \sinh y_n, 0, \dots, 0)^t : u_i \in \mathbb{R}\}.$$

Then $\exp(X) \cdot o$ and $\exp(Y) \cdot o$ represent the same subspace if and only if $\tanh y_i = -\tan x_i$ for each i . This means the two maps \mathbf{p} and \mathbf{f} are the same on the maximal flat $T := \exp(\mathfrak{a}_n) \cdot o$. Since both \mathbf{p} and \mathbf{f} are K -equivariant, they are the same.

For the other generalized Grassmannians, both the embeddings \mathbf{p} and \mathbf{f} are just the restrictions of the embeddings in the corresponding real Grassmannian case (Proposition 3.1 for \mathbf{p} and Proposition 4.3 for \mathbf{f}). Since \mathbf{p} and \mathbf{f} coincide in the real Grassmannian case, they are the same for all the generalized Grassmannians. \square

5.2. Comparing \mathbf{f} with \mathbf{b} . Now we compare the embedding \mathbf{f} with the Borel embedding \mathbf{b} in the Hermitian symmetric space case. There are six types compact Hermitian symmetric spaces, namely, $U(n+m)/U(n)U(m)$, $SO(2n)/U(n)$, $Sp(n)/U(n)$, $SO(n+2)/SO(n)SO(2)$, $E_6/(Spin(10)U(1)/\mathbb{Z}_4)$ and $E_7/(E_6U(1)/\mathbb{Z}_3)$.

THEOREM 5.2. *For all the compact Hermitian symmetric spaces except $SO(n+2)/SO(n)SO(2)$, the Borel embedding \mathbf{b} and the embedding \mathbf{f} are the same.*

Proof. For $U(n+m)/U(n)U(m)$, $SO(2n)/U(n)$, $Sp(n)/U(n)$, they are all generalized Grassmannians, we have $\mathbf{b}=\mathbf{g}=\mathbf{p}$. And also $\mathbf{p}=\mathbf{f}$, hence $\mathbf{b}=\mathbf{f}$. (Though $SO(2n)/U(n)$ is not in the generalized Grassmannian list when n is odd, we can also define the space-like embedding \mathbf{p} from Remark 3.3, and this \mathbf{p} is also the same with \mathbf{g} , \mathbf{f} obviously.)

For $M = E_6/(Spin(10)U(1)/\mathbb{Z}_4)$ with rank = 2. It has a reflective submanifold $N = U(6)/U(2)U(4)$ with rank = 2 which is also a symmetric R-space. M and N have the same maximal flat torus T . Since $\mathbf{b}_N = \mathbf{f}_N$, $\mathbf{b}_N = \mathbf{b}_M|_N$, $\mathbf{f}_N = \mathbf{f}_M|_N$ and $T \subset N$, we have $\mathbf{b}_M|_T = \mathbf{f}_M|_T$. Since \mathbf{b} and \mathbf{f} are both K -equivariant, we have $\mathbf{b}_M = \mathbf{f}_M$.

For $M = E_7/(E_6U(1)/\mathbb{Z}_3)$ with rank = 3. It has a reflective submanifold $N = SO(12)/U(6)$ with rank = 3 which is also a symmetric R-space. The argument is the same as above. \square

For $SO(n+2)/SO(n)SO(2)$, we compute directly.

If $n = 1$, it is a rank one symmetric space. In this case, $\mathrm{SO}(3)/\mathrm{SO}(2)$ is the sphere. Through the Borel embedding \mathbf{b} , its noncompact dual $\mathrm{SO}_0(1, 2)/\mathrm{SO}(2)$ in it consists of those points with distance less than $\frac{1}{4}$ of the diameter of the sphere from the south pole (directly from Remark 3.3). Hence $\mathrm{Im}(\mathbf{b}) = \exp_o^c(\frac{R}{4})$. Through the embedding \mathbf{f} , we have $\mathrm{Im}(\mathbf{f}) = \exp_o^c(\frac{R}{2})$. Of course, $\mathbf{b} \neq \mathbf{f}$.

If $n \geq 2$, it is a rank two symmetric space, follow the notations in example 4.1, the lattice $\Gamma(G, K)$ is generated by $\pi(R_{1,n+1} + R_{2,n+2})$ and $\pi(R_{1,n+1} - R_{2,n+2})$ which are orthogonal to each other and have the same length. By direct computations, we have $\mathbf{b} \neq \mathbf{f}$, but $\mathrm{Im}(\mathbf{b}) = \mathrm{Im}(\mathbf{f}) = \exp_o^c(\frac{R}{2})$.

5.3. Comparing \mathbf{f} with \mathbf{g} . Finally, we compare the embedding \mathbf{f} with the generalized embedding \mathbf{g} , they both coincide with the space-like embedding \mathbf{p} in the generalized Grassmannian case and the Borel embedding \mathbf{b} in the Hermitian case (except $\mathrm{SO}(n+2)/\mathrm{SO}(n)\mathrm{SO}(2)$), when are they equal to each other?

THEOREM 5.3. *For all the symmetric R-spaces except $\mathrm{SO}(n+2)/\mathrm{SO}(n)\mathrm{SO}(2)$ and $\mathrm{SO}(n+1) \times \mathrm{SO}(m+1)/S(\mathrm{O}(n) \times \mathrm{O}(m))$, $\mathbf{g}=\mathbf{f}$.*

Proof. For any symmetric R-space N , it is a compact Hermitian symmetric space or it is a real form of a compact Hermitian symmetric space M .

When it is a compact Hermitian symmetric space except $\mathrm{SO}(n+2)/\mathrm{SO}(n)\mathrm{SO}(2)$, we have $\mathbf{g}=\mathbf{b}=\mathbf{f}$.

When it is a real form of a compact Hermitian symmetric space M , that means it is a reflective submanifold of M under a antiholomorphic isometric involution. By Proposition 3.1 and Proposition 4.4, we have $\mathbf{g}_N = \mathbf{g}_M|_N$ and $\mathbf{f}_N = \mathbf{f}_M|_N$. If $\mathbf{g}_M = \mathbf{f}_M$, then we have $\mathbf{g}_N = \mathbf{f}_N$. From the list of symmetric R-spaces, $\mathrm{SO}(n+m+2)/\mathrm{SO}(n+m)\mathrm{SO}(2)$ has real forms $\mathrm{SO}(n+1) \times \mathrm{SO}(m+1)/S(\mathrm{O}(n) \times \mathrm{O}(m))$. Hence $\mathrm{SO}(n+1) \times \mathrm{SO}(m+1)/S(\mathrm{O}(n) \times \mathrm{O}(m))$ is the only exception. \square

Combining all the above results, we have

PROPOSITION 5.2. *For $\mathrm{SO}(3)/\mathrm{SO}(2)$, $\mathrm{Im}(\mathbf{f}) = \exp_o^c(\frac{R}{2})$ and $\mathrm{Im}(\mathbf{g}) = \exp_o^c(\frac{R}{4})$. For all the symmetric R-spaces except $\mathrm{SO}(3)/\mathrm{SO}(2)$, $\mathrm{Im}(\mathbf{f}) = \mathrm{Im}(\mathbf{g}) = \exp_o^c(\frac{R}{2})$.*

Since both the constructions of \mathbf{g} and \mathbf{f} use the intrinsic characterizations of symmetric R-spaces among compact symmetric spaces, they can't be generalized to all compact symmetric spaces.

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