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JOHN-NIRENBERG RADIUS AND COLLAPSING IN CONFORMAL GEOMETRY*

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Abstract. Given a positive function $u \in W^{1,n}$, we define its John-Nirenberg radius at point x to be the supreme of the radius such that $\int_{B_t(x)} |\nabla \log u|^n < \epsilon_0^n$ when $n > 2$, and $\int_{B_t(x)} |\nabla u|^2 < \epsilon_0^2$ when $n = 2$. We will show that for a collapsing sequence of metrics in a fixed conformal class under some curvature conditions, the radius is bounded below by a positive constant. As applications, we will study the convergence of a conformal metric sequence on a 4-manifold with bounded $\|K\|_{W^{1,2}}$, and prove a generalized Hélein's Convergence Theorem.

Key words. John-Nirenberg Radius, scalar curvature equation, Blow up analysis.

Mathematics Subject Classification. 53C21, 58J05.

1. Introduction. We say that a Riemannian manifold sequence collapses, if it converges to a low dimensional space in the Gromov-Hausdorff distance. When (M_k, g_k) collapses, a reasonable attempt is to blow up the sequence, i.e., to find $c_k \to +\infty$, such that $(M_k, c_k g_k)$ converges to a manifold of the same dimension. This usually needs some monotone properties, such as volume comparison. Then some sectional or Ricci curvature conditions are usually assumed for a collapsing sequence.

Recently, in [11] the first author and the third author of this paper considered collapsing sequences in a fixed conformal class with bounded L^p-norm of scalar curvature, where $p > \frac{n}{2}$. Let B_1 be the unit ball of $\mathbb{R}^n$ centered at the origin and g be a smooth metric over $\bar{B}_1$, where $n > 3$. Consider a sequence of metric $g_k = u_k^{\frac{4}{n-2}} g$ which satisfies

$$\int_{B_1} |R(g_k)|^p dV_{g_k} < \Lambda,$$

where $R(g_k)$ is the scalar curvature of g_k. Our conclusion is the following: "when $\mathrm{vol}(g_k) \to 0$, there exists a sequence $\{c_k\}$ which tends to $+\infty$, such that $c_k u_k$ converges to a positive function in $W^{2,p}$ weakly". The proof of the conclusion is rather analytic and the John-Nirenberg inequality plays an essential role in the procedure.

Recall that the John-Nirenberg inequality says that, given $u \in W^{1,q}(B_1)$, where $q \in [1, n]$ and B_1 is the unit ball of $\mathbb{R}^n$, if

$$\int_{B_r(x)} |\nabla u|^q < r^{n-q}, \quad \forall\, B_r(x) \subset B_1,$$

then there exists α and β, such that

$$\int_{B_1} e^{\alpha u} \int_{B_1} e^{-\alpha u} < \beta.$$

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Inspired by the John-Nirenberg inequality we define the John-Nirenberg radius of u_k at x in [11] as follows:

$$\rho(x, u_k, \Omega, \epsilon_0) = \sup\left\{ r : t^{2-n} \int_{B_t(x)\cap\Omega} |\nabla \log u_k|^2 < \epsilon_0^2, \quad \forall\, t < r \right\}, \tag{1.1}$$

where $u_k \in W^{1,2}(\Omega)$. The key ingredient of the arguments in [11] is that, when $\mathrm{vol}(g_k)$ converges to 0, there must exist an $a > 0$ which is independent of u_k, such that $\inf_{B_{\frac{1}{2}}} \rho_k(x) > a$. This means that

$$t^{2-n} \int_{B_t(x)} |\nabla \log u_k|^2 < \epsilon_0^2$$

for any $t < a$ and $x \in B_{\frac{1}{2}}$, hence the John-Nirenberg inequality holds for $\frac{\log c_k u_k}{\epsilon_0}$ on $B_a(x)$, where $\int_{B_{\frac{1}{2}}} \log c_k u_k = 0$. Then it follows the estimates of $L^{\frac{\alpha}{\epsilon_0}}$-norms of $\frac{1}{c_k u_k}$ and $c_k u_k$.

The arguments and calculations of the first half of [11] were so complicated that it is not easy for one to pay attention to the John-Nirenberg radius, which was introduced and discussed in the last section of [11]. While we think this new technique is very interesting and believe that it might be applied to some other nonlinear equations, we write this paper to highlight on the John-Nirenberg radius and give a simple explanation of how the John-Nirenberg inequality works.

In general, we can replace $t^{2-n}\int_{B_t(x)} |\nabla u|^2 < \epsilon_0^2$ in (1.1) by $t^{q-n}\int_{B_t(x)} |\nabla u|^q < \epsilon_0^q$, that is to say, define

$$\rho^q(x, u_k, \Omega, \epsilon_0) = \sup\left\{ r : t^{q-n} \int_{B_t(x)\cap\Omega} |\nabla \log u_k|^q < \epsilon_0^q, \quad \forall\, t < r \right\},$$

where $q \in [1, n]$. It is easy to check that the arguments in [11] still work. We discover that it is much more convenient to use $q = n$ to define the John-Nirenberg radius. For this situation, the John-Nirenberg inequality can be deduced from Moser-Trudinger inequality, which also gives the optimal constant in the John-Nirenberg inequality in the case of $q = n$. So we start our discussion from Moser-Trudinger inequality in Section 2, and define the John-Nirenberg radius to be the supreme of the radius such that

$$\int_{B_t(x)} |\nabla \log u|^n dx < \epsilon_0^n.$$

Then we prove Theorem 2.7 which tells us when the John-Nirenberg radius is positive.

Some applications of the John-Nirenberg radius will be given. In Section 3, we will use the John-Nirenberg radius to prove a well-known result (for example, c.f. [13]): a positive harmonic function defined in a domain of a manifold with a point removed is either a Green function or smooth across the removed point.

In Section 4 and 5, we will apply John-Nirenberg radius to study a collapsing sequence of metrics in conformal geometry, i.e., we will show that, if $g_k = u_k^{\frac{4}{n-2}} g$ collapses, then there exists c_k such that $c_k u_k$ converges to a positive function. Furthermore, we show that the ϵ-regularity in [11] can be also deduced by employing John-Nirenberg radius. In Section 5, we will use the John-Nirenberg radius to prove that a sequence of metrics on a 4-dimensional manifold in a fixed conformal class with $\|K\|_{W^{1,2}} < C$ and fixed volume is compact in $C^{1,\alpha}$. The idea is, if the sequence blows up, then the neck domains can be considered as collapsing sequences. Then, by multiplying a suitable constant, one of the neck sequences converges to a complete flat manifolds with at least two ends collared topologically by $S^3 \times \mathbb{R}$. Yet, this is impossible. Employing the same argument one can also give a new proof of the $C^{0,\alpha}$-compactness of a metric sequence, which is in a fixed conformal class and satisfies

$$\mathrm{vol}(g_k) + \|K(g_k)\|_{L^p} < C,$$

where $p > \frac{n}{2}$. It is well-known that such a problem has been deeply studied by Chang-Yang [2, 3], and solved by Gursky [5].

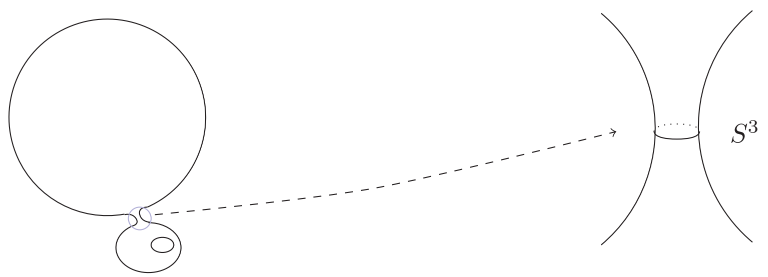

FIG. 1. *After an appropriate rescaling, one of the neck sequences converges to a complete flat manifold, which has at least 2 ends collared topologically by $S^3 \times \mathbb{R}$.*

In Section 6, we try to extend the definition of John-Nirenberg radius to the case of two dimensional manifolds. We will apply the John-Nirenberg radius to give a generalized Hélein's Convergence Theorem. However, it is worthy to point out that Lemma 4.1 does not hold true for the case of two dimensional manifolds.

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2. John-Nirenberg radius. First, we need to recall the following Moser's inequality on the ball B^n for functions with mean value zero, which was established in [8].

THEOREM 2.1 ([8]). *Let B_1 be the unit ball of $\mathbb{R}^n$, and $\alpha_n = n(\frac{\omega_{n-1}}{2})^{\frac{1}{n-1}}$, where ω_{n-1} is the measure of unit sphere in $\mathbb{R}^n$. Then*

$$\sup_{u\in W^{1,n}(B_1),\ \int_B u dx=0,\ \|\nabla u\|_{L^n(B_1)}\leq 1} \int_{B_1} e^{\alpha_n |u|^{\frac{n}{n-1}}} dx < +\infty.$$

From the Theorem above and the following inequality

$$|u| = \frac{|u|}{\|\nabla_g u\|_{L^n}} \|\nabla_g u\|_{L^n} \leq \frac{n-1}{n} \left(\frac{|u|}{\|\nabla u\|_{L^n}} \right)^{\frac{n}{n-1}} + \frac{1}{n} \|\nabla u\|^n_{L^n},$$

we derive the following:

COROLLARY 2.2. *Let B_1 be the unit ball of $\mathbb{R}^n$, and $u \in W^{1,n}(B_1)$ and $\int_{B_1} u dx = 0$. Then*

$$\int_{B_1} e^{\beta_n |u|} dx < C e^{\frac{\alpha_n}{n-1} \int_{B_1} |\nabla u|^n dx},$$

where $\beta_n = \frac{n}{n-1}\alpha_n$.

We say u is essentially positive, if there exists $\epsilon > 0$, such that $u > \epsilon$ almost everywhere. Given an essentially positive function $u \in W^{1,n}(\Omega)$, we define the John-Nirenberg radius as follows:

$$\rho(x, u, \Omega, \epsilon_0) = \sup \left\{ r : \int_{B_r(x)\cap\Omega} |\nabla \log u|^n dx < \epsilon_0^n \right\}.$$

Later, $\rho(x, u, \Omega, \epsilon_0)$ will be used to study convergence of a sequence of positive functions. Obviously, if $\Omega_1 \subset\subset \Omega$ and $\rho(x, u_k, \Omega, \epsilon_0) > a > 0$ for any $x \in \Omega_1$, then

$$\int_{\Omega_1} |\nabla \log u|^n dx < C(a, \Omega_1, \Omega, \epsilon_0).$$

LEMMA 2.3. *Let Ω be a domain of $\mathbb{R}^n$, $u_k \in W^{1,n}(\Omega)$ be essentially positive. Let domain $\Omega_0 \subset\subset \Omega_1 \subset\subset \Omega$, and $-\log c_k$ be the integral mean value of $\log u_k$ over Ω_0. Suppose $\rho(x, u_k, \Omega, \epsilon_0) > a > 0$ for any $x \in \Omega_1$. Then, $c_k u_k$ and $\frac{1}{c_k u_k}$ are bounded in $L^{\frac{\beta_n}{\epsilon_0}}(\Omega_1)$. Moreover, $\log c_k u_k$ converges weakly in $W^{1,n}(\Omega_1)$.*

Proof. Choose $a_1 < \frac{1}{2}\min\{d(\Omega_1, \partial\Omega), a\}$, and define $\Omega_1' = \{x : d(x, \Omega_1) < a_1\}$. By the assumptions, we have

$$\int_{\Omega_1'} |\nabla \log u_k|^n dx < C(\epsilon_0, \Omega_1').$$

The Poincaré inequality tells us $\log c_k u_k$ is bounded in $W^{1,n}(\Omega_1')$. Hence, we may assume that $\log c_k u_k$ converges in $L^q(\Omega_1')$ for any q.

Take $x_1, \cdots, x_m \in \overline{\Omega}_1$ such that $\{B_{a_1}(x_i) : i = 1, \cdots, m\}$ is an open cover of Ω_1. Without loss of generality, we may assume $\log u_k + \log c_k^i$ converges weakly in $W^{1,n}(B_{a_1}(x_i))$ and strongly in $L^1(B_{a_1}(x_i))$. Here $-\log c_k^i$ is the mean value of $\log u_k$ over $B_{a_1}(x_i)$. Since

$$(\log u_k + \log c_k^i) - (\log u_k + \log c_k)$$

converges in $L^1(B_{a_1}(x_i))$, we may assume $\log c_k^i - \log c_k$ converges.

By Corollary 2.2, we have

$$\int_{B_{a_1}(x_i)} e^{\frac{\beta_n}{\epsilon_0}|\log u_k + \log c_k^i|} < C(\epsilon_0, n),$$

and hence

$$\int_{B_{a_1}(x_i)} e^{\frac{\beta_n}{\epsilon_0}|\log u_k + \log c_k|} < C(\epsilon_0, n).$$

It turns out that both $\|c_k u_k\|_{L^{\frac{\beta_n}{\epsilon_0}}(\Omega_1)}$ and $\|\frac{1}{c_k u_k}\|_{L^{\frac{\beta_n}{\epsilon_0}}(\Omega_1)}$ are bounded. $\square$

Remark 2.4. *$-\log c_k$ in the above lemma can be chosen to be any constant which makes the Poincaré inequality hold. For example, we can set $-\log c_k$ to be the mean value of $\log u_k$ a compact $(n-1)$-dimensional submanifold (perhaps with boundary) embedded in Ω_1.*

We consider the operator

$$L(u) = a^{ij}u_{ij} + b^i u_i + cu,$$

where $a^{ij} = a^{ji}$ and

$$\|a^{ij}\|_{C^{0,\alpha}} + \|b^i\|_{C^{0,\alpha}} + \|c\|_{C^0} < A_1, \quad 0 < A_2 < a^{ij}\xi^i\xi^j < A_3 \ \ for \ \ all \ \ |\xi| = 1. \quad (2.1)$$

Later, we need to use the following:

Corollary 2.5. *Let $(\frac{1}{p} + \frac{2\epsilon_0}{\beta_n}) < \frac{1}{2} + \frac{1}{n}$. Let $u_k \in W^{2,p}(B_2)$ be a sequence of positive functions, each of which solves the equation $Lu_k = f_k u_k$ where $\|f_k\|_{L^p(B_2)} < \Lambda$. If*

$$\|u_k\|_{L^{\frac{\beta_n}{\epsilon_0}}(B_2)} + \|\frac{1}{u_k}\|_{L^{\frac{\beta_n}{\epsilon_0}}(B_2)} < \Lambda_2,$$

then, after passing to a subsequence, u_k converges weakly in $W^{2,q}(B_1)$ and $\log u_k$ converges weakly in $W^{2,q'}(B_1)$ for any

$$q \in \left(\frac{1}{\frac{n+2}{2n} - \frac{\epsilon_0}{\beta_n}}, \frac{p}{1 + \frac{\epsilon_0}{\beta_n}p}\right) \cap (1, n) \quad and \quad q' \in \left(1, \frac{1}{2(\frac{\epsilon_0}{\beta_n} + \frac{n-q}{nq})}\right) \cap (1, p).$$

Proof. Since $\frac{p}{1+\frac{\epsilon_0}{\beta_n}p} > q$, we have $\frac{pq}{p-q} < \frac{\beta_n}{\epsilon_0}$. Noting

$$\int_{B_2} |f_k u_k|^q dx \le \left(\int_{B_2} |f_k|^p dx\right)^{\frac{q}{p}} \left(\int_{B_2} |u_k|^{\frac{pq}{p-q}} dx\right)^{\frac{p-q}{p}},$$

by the standard elliptic theory we get the estimate of $\|u_k\|_{W^{2,q}(B_{\frac{3}{2}})}$. Then, it follows

$$\|\nabla u_k\|_{L^{\frac{nq}{n-q}}(B_{\frac{3}{2}})} < C.$$

It is easy to check that

$$2q' < \frac{nq}{n-q} \qquad \text{and} \qquad \frac{2nqq'}{nq - 2q'(n-q)} < \frac{\beta_n}{\epsilon_0}.$$

By Hölder inequality, we have

$$\begin{aligned}\int_{B_{\frac{3}{2}}} |\nabla \log u_k|^{2q'} dx &< \left(\int_{B_{\frac{3}{2}}} |\nabla u_k|^{\frac{nq}{n-q}} dx\right)^{\frac{2q'(n-q)}{nq}} \left(\int_{B_{\frac{3}{2}}} \left(\frac{1}{u_k}\right)^{\frac{2nqq'}{nq-2q'(n-q)}} dx\right)^{\frac{nq-2q'(n-q)}{nq}} \\ &< C_1.\end{aligned}$$

Define an operator $L' = L - c$. Obviously, $\log u_k$ satisfies the following equation

$$L'(\log u_k) = -a^{ij}(\log u_k)_i(\log u_k)_j + f_k - c.$$

By L^p estimate, we know $\|\log u_k\|_{W^{2,q'}} < C_2$. □

Remark 2.6. *In Corollary 2.5, in order to guarantee that*

$$\frac{p}{1+\frac{\epsilon_0}{\beta_n}p} > \frac{1}{\frac{n+2}{2n}-\frac{\epsilon_0}{\beta_n}},$$

we only need to choose p such that $\frac{1}{p}+\frac{2\epsilon_0}{\beta_n} < \frac{1}{2}+\frac{1}{n}$. *Hence, it follows that*

$$\left(\frac{1}{\frac{n+2}{2n}-\frac{\epsilon_0}{\beta_n}}, \frac{p}{1+\frac{\epsilon_0}{\beta_n}p}\right) \neq \emptyset \qquad \text{and} \qquad \left(1, \frac{1}{2(\frac{\epsilon_0}{\beta_n}+\frac{n-q}{nq})}\right) \neq \emptyset.$$

The following theorem is the key point of this paper:

Theorem 2.7. *Let* $p > \frac{n}{2}$, *and* $\frac{1}{p}+\frac{2\epsilon_0}{\beta_n} < \frac{2}{n}$. *Let* $\{u_k\} \in W^{2,p}(B_3)$ *be a sequence of positive functions which satisfy*

$$Lu_k = f_k u_k.$$

If for any $x_k \to x_0 \in \overline{B_2}$ *and* $r_k < 2\rho(x_k, u_k, B_3, \epsilon_0)$ *with* $r_k \to 0$, *a subsequence of* $r_k^2 f_k(r_k x + x_k)$ *is bounded in* $L^p(B_{\frac{1}{4}})$, *and converges to 0 in the sense of distribution on* $B_{\frac{1}{4}}$, *then there exists* $a > 0$, *such that*

$$\rho(x, u_k, B_3, \epsilon_0) > a, \quad \forall x \in B_1.$$

Proof. We argue by contradiction. Assume the conclusion is not true. Then we can find $x_k \in B_1$, s.t. $\rho(x_k, u_k, B_3, \epsilon_0) \to 0$. For simplicity, we denote $\rho(x, u_k, B_3, \epsilon_0)$ by $\rho_k(x)$.

Since $u_k \in W^{1,n}(B_3)$, there exists $a_k > 0$, which depends on k, such that $\int_{B_{a_k}(x)} |\nabla u_k|^n < \epsilon_0^n$ for any $x \in \overline{B_2}$. Thus $\rho_k(x) > a_k$ for any $x \in \overline{B_2}$. Next, we show that ρ_k is lower semi-continuous on $x \in \overline{B}_2$. Let $x_m \to x$. Obviously,

$$\int_{B_{\rho_k(x)}(x)} |\nabla u_k|^n \leq \epsilon_0^n,$$

which yields that

$$\int_{B_{\rho_k(x)-|x_m-x|}(x_m)} |\nabla u_k|^n \leq \int_{B_{\rho_k(x)}(x)} |\nabla u_k|^n \leq \epsilon_0^n.$$

Then $\rho_k(x_m) \geq \rho_k(x) - |x_m - x|$, hence

$$\rho_k(x) \leq \varliminf_{m\to+\infty} \rho_k(x_m).$$

By $\rho_k(x) > a_k$ for any $x \in \overline{B}_2$,

$$\lim_{x\to\partial B_2} \frac{\rho_k(x)}{2-|x|} = +\infty.$$

Since $\rho_k(x)$ is lower semi-continuous, we can find $y_k \in B_2$, such that

$$\frac{\rho_k(y_k)}{2-|y_k|} = \inf_{x\in B_2} \frac{\rho_k(x)}{2-|x|} := \lambda_k.$$

Noting that

$$\lambda_k \le \frac{\rho_k(x_k)}{2-|x_k|} \le \rho_k(x_k) \to 0,$$

we have $\rho_k(y_k) \to 0$, and hence for any fixed R

$$\frac{\rho_k(y_k)}{2-|y_k|} \to 0, \qquad B_{R\rho_k(y_k)}(y_k) \subset B_{2-|y_k|}(y_k) \subset B_2,$$

when k is sufficiently large. Then, for any $y \in B_{R\rho_k(y_k)}(y_k)$ we have

$$\begin{aligned}\frac{\rho_k(y)}{\rho_k(y_k)} \ge \frac{2-|y|}{2-|y_k|} &\ge \frac{2-|y_k|-|y-y_k|}{2-|y_k|}\\ &\ge 1-\frac{R\rho_k(y_k)}{2-|y_k|}\\ &= 1-R\lambda_k.\end{aligned}$$

Hence, as k is large enough, there holds

$$\frac{\rho_k(y)}{\rho_k(y_k)} > \frac{1}{2}.$$

Assume $y_k \to y_0$. Let $v_k(x) = u_k(y_k + r_k x)$, where $r_k = \rho_k(y_k)$. Then, there holds

$$\rho(x, v_k, B_R, x) \ge 1/2 \quad \text{on} \quad B_{\frac{R}{2}}$$

and

$$\int_{B_1} |\nabla \log v_k|^n = \epsilon_0^n.$$

Moreover, v_k satisfies the following equation:

$$a^{ij}(y_k + r_k x)(v_k)_{ij} = -r_k b^i(y_k + r_k x)(v_k)_i - c r_k^2 v_k + r_k^2 f_k(y_k + r_k x) v_k.$$

By Lemma 2.3, we can find c_k, such that

$$\|c_k v_k\|_{L^{\frac{\beta_n}{\epsilon_0}}(B_{\frac{R}{2}})} + \|\frac{1}{c_k v_k}\|_{L^{\frac{\beta_n}{\epsilon_0}}(B_{\frac{R}{2}})} < C(R).$$

Noting that $\{r_k^2 f_k(y_k + r_k x_0' + r_k x)\}$ is bounded in $L^p(B_{\frac{1}{4}})$ for any $x_0' \in B_{\frac{R}{2}}$, by a covering argument we can see that the sequence $\{r_k^2 f_k(y_k + r_k x)\}$ is bounded in $L^p(B_{R/2})$ for any R. By the same arguments, we also know that in the sense of distribution on $\mathbb{R}^n$

$$r_k^2 f_k(y_k + r_k x) \to 0.$$

By the assumptions, we have

$$2\left(\frac{\epsilon_0}{\beta_n} + \frac{n-q}{nq}\right) < \frac{2}{n}$$

when $q = \frac{p}{1+\frac{\epsilon_0}{\beta_n}p}$. Noting that $\frac{2}{n} < \frac{1}{2} + \frac{1}{n}$, by Corollary 2.5, we can find q and $q' > \frac{n}{2}$, such that a subsequence of $c_k v_k$ converges to a function v weakly in $W^{2,q}(B_R)$, and $\log c_k v_k$ converges weakly to $\log v$ in $W^{2,q'}(B_R)$. Then $\log c_k v_k$ converges in $C^0(R)$, which implies that $v > 0$. Thus, by a standard diagonal argument, we obtain a positive function v which is defined on $\mathbb{R}^4$ and satisfies the equation $a^{ij}(y_0)v_{ij} = 0$. Then v is a positive constant. However, by the Sobolev embedding theorem, $|\nabla \log c_k v_k|$ converges in L^n. Then, it follows

$$\int_{B_1} |\nabla \log v|^n = \epsilon_0^n,$$

which is impossible since $\log v$ is a constant. Thus we complete the proof. □

COROLLARY 2.8. *Let p, ϵ_0 be as in Theorem 2.7. Let $u \in W^{2,p}(B_3)$ be a positive function which solves the equation*

$$Lu = fu.$$

Then there exist positive numbers ϵ and a which only depend on A_1, A_2, A_3, p and ϵ_0 such that, if

$$\sup_{x \in B_2, r < 2\rho(x,u,B_3,\epsilon_0)} r^{2p-n} \int_{B_r(x)} |f|^p < \epsilon, \tag{2.2}$$

then

$$\rho(x, u, B_3, \epsilon_0) > a, \quad \forall x \in B_1.$$

Proof. We argue by contradiction. Assume the above conclusion is not true. Then there exists a sequence of u_k satisfying

$$Lu_k = f_k,$$

such that

$$\inf_{B_1} \rho(x, u_k, B_3, \epsilon_0) \to 0$$

and

$$\sup_{x \in B_2, r < 2\rho(x,u_k,B_3,\epsilon_0)} r^{2p-n} \int_{B_r(x)} |f_k|^p \to 0.$$

It is easy to check from the above that $r_k^2 f_k(x_k + r_k x)$ converges to 0 in $L^p(B_{\frac{1}{4}})$. Thus we get the desired conclusion from Theorem 2.7. □

3. Positive harmonic function with isolated singularity. In this section, we will use the so-called John-Nirenberg radius or the John-Nirenberg inequality to study the positive harmonic functions with singularity on a manifold. We will give a proof of the following result, which is a special case of Theorem 1 of [13]:

LEMMA 3.1. *Let $g = dr^2 + g(r,\theta)d\mathbb{S}^{n-1}$ be a smooth metric over $B_1 \subset \mathbb{R}^n$, where $g(r,\theta) = r^2(1 + o(1))$. Assume u is a positive harmonic function on $B_1 \setminus \{0\}$. Then $u \in W^{1,q}$ for any $q \in (1, \frac{n}{n-1})$ and satisfies the weak equation*

$$-\Delta_g u = c\delta_0, \quad c \geq 0.$$

Proof. Let

$$c=-\int_{\partial B_r}\frac{\partial u}{\partial r}dS_r.$$

First, we prove that

$$\frac{u(rx)}{r^{2-n}}\to\frac{c}{(n-2)\omega_{n-1}}\qquad \textit{and}\qquad \frac{\nabla u}{r^{1-n}}(rx)\to-\frac{c}{\omega_{n-1}}\frac{\partial}{\partial r}$$

uniformly on S^{n-1}.

Assume this is not true. Then we can find $x_k\in S^{n-1}\subset\mathbb{R}^n$ and $r_k\to 0$, such that

$$\left|\frac{u(r_kx_k)}{r_k^{2-n}}-\frac{c}{(n-2)\omega_{n-1}}\right|>\epsilon\qquad\text{or}\qquad\left|\frac{\nabla u}{r_k^{1-n}}(r_kx_k)+\frac{c}{\omega_{n-1}}\frac{\partial}{\partial r}\right|>\epsilon.$$

Let $v_k=u(r_kx)$ and choose c_k such that

$$\int_{\partial B_1}\log c_kv_kd\mathcal{S}^{n-1}=0.$$

By the results in the above section, for any $r>0$ we can find $a(r)>0$ such that, for any $x\in B_{\frac{1}{r}}\setminus B_r$,

$$\rho(x,v_k,B_{\frac{2}{r}}\setminus B_{\frac{r}{2}},\epsilon_0)>a(r),$$

and hence both c_kv_k and $\frac{1}{c_kv_k}$ are bounded in $L^{\frac{\beta_n}{\epsilon_0}}(B_{\frac{1}{r}}\setminus B_r)$.

Since c_kv_k is harmonic, after passing to a subsequence, c_kv_k converges in $C^\infty_{loc}(\mathbb{R}^n)$ to a function v which is positive and harmonic on $\mathbb{R}^n\setminus\{0\}$. It is well-known that

$$v=ar^{2-n}+b,$$

where a and b are nonnegative real numbers with $a^2+b^2>0$ (c.f. [1, Corollary 3.14]).

Now, we need to discuss the following two cases.

Case 1: $c\neq 0$. In this case, from

$$\frac{c_k}{r_k^{n-2}}\int_{\partial B_{r_k}}\frac{\partial u}{\partial r}dS_r\to\int_{\partial B_1}\frac{\partial v}{\partial r}d\mathcal{S}^{n-1}=a(2-n)\omega_{n-1},\tag{3.1}$$

it follows that

$$\frac{c_k}{r_k^{n-2}}\to a\frac{(n-2)\omega_{n-1}}{c}.\tag{3.2}$$

Then we have

$$\frac{u_k(r_kx)}{r_k^{2-n}}=\frac{c_kv_k(x)}{c_kr_k^{2-n}}\to\frac{(a+b)c}{a(n-2)\omega_{n-1}},$$

and

$$r_k^{n-1}\nabla u(r_kx)\to-\frac{c}{\omega_{n-1}}\frac{\partial}{\partial r}.$$

To get a contradiction, we need to prove $b = 0$. Let G be the Green function which satisfies $-\Delta_g G = \delta_0$ and $G|_{\partial B_\delta} = 0$. We have

$$\lim_{r\to 0} r^{n-2}G = \lim_{r\to 0}\frac{r^{n-1}}{2-n}\frac{\partial G}{\partial r} = \lambda \neq 0.$$

Then

$$\begin{aligned}
c_k\int_{\partial B_t}(u\frac{\partial G}{\partial r} - \frac{\partial u}{\partial r}G)dS_g &= c_k\int_{\partial B_{r_k}}(u\frac{\partial G}{\partial r} - \frac{\partial u}{\partial r}G)dS_g \\
&= \int_{\partial B_1}(\lambda c_k v_k(x)(2-n)r_k^{1-n}(1+o(1)) - \lambda\frac{\partial c_k v_k}{\partial r}r_k^{1-n}) \\
&\qquad \times r_k^{n-1}(1+o(1))dS \\
&\to \int_{\partial B_1}(v(1)(2-n)\lambda - v'(1)\lambda)\, dS \qquad (3.3)\\
&= -b(n-2)\omega_{n-1}\lambda.
\end{aligned}$$

Obviously (3.2) implies $c_k \to 0$, hence from the above equality (3.3) we derive that $b = 0$.

Case 2: $c = 0$. If $c_k r_k^{2-n} \to +\infty$, it is easy to check that

$$r_k^{n-2}u(r_k x) = \frac{c_k v_k(x)}{c_k r_k^{2-n}} \to 0 \qquad \text{and} \qquad r_k^{n-1}\nabla u(r_k x) \to 0.$$

On the other hand, if $c_k r_k^{2-n} < C$, it follows that $c_k \to 0$. From (3.1) we have $a = 0$. From (3.3) we can see that $b = 0$. Thus, we get a contradiction.

Therefore, we conclude that $u \in W^{1,q}(B)$ for any $q \in (1, \frac{n}{n-1})$. Given a smooth function φ whose support set is contained in B_1, we have

$$\begin{aligned}
\int_{B_1}\nabla\varphi\nabla u dV_g &= \lim_{r\to 0}\int_{B_1\setminus B_r}\nabla\varphi\nabla u dV_g \\
&= -\lim_{r\to 0}\int_{B_1\setminus B_r}\Delta u\varphi dV_g + \lim_{r\to 0}\int_{\partial B_r}\frac{\partial u}{\partial\nu}\varphi dV_g \\
&= c\varphi(0).
\end{aligned}$$

Thus, we get

$$-\Delta_g u = c\delta_0.$$

Thus we complete the proof of this lemma. □

Corollary 3.2. *Let (M, g) be a closed manifold with constant scalar curvature $R(g)$. Suppose $p_1, \cdots, p_m \in M$, and g' is a metric on $M \setminus \{p_1, \cdots, p_m\}$, which is conformal to g. If $R(g') = 0$, then (M, g') is complete near p_i or g' is smooth across p_i.*

Proof. We can find a metric g_0 which is conformal to g, such that $R(g_0) = 0$ in a neighborhood of p_i. Let $g' = u^{\frac{4}{n-2}}g_0$. Then u is harmonic in a neighborhood of p_i. Thus, either u can be extended smoothly to p_i, or $u \sim c_i r^{2-n}$ for a positive c_i, which implies that g' is complete near p_i. □

4. A collapsing sequence with bounded $\|R\|_{L^p}$. In the previous paper [11], the authors use the ϵ-regularity to study the bubble tree convergence of a metric sequence in a fixed conformal class with bounded volume and $L^p(M)$-norm of scalar curvature. Then, it has been shown that the John-Nirenberg radius is bounded below by a positive constant when the volume converges to 0. In this section, we will show that the ϵ-regularity is also a corollary of John-Nirenberg inequality, which was deduced directly from L^p estimate in [11].

First, we show the positivity of the John-Nirenberg radius for a collapsing sequence.

LEMMA 4.1. *Let $n > 3$ and $\{\hat{g}_k\}$ be a sequence of metrics over $B_2 \subset \mathbb{R}^n$ which converges to g. Let $g_k = u_k^{\frac{4}{n-2}} \hat{g}_k$, where u_k is smooth and positive. Suppose that $vol(B_2, g_k) \to 0$ and $\int_{B_2} |R(g_k)|^p d\mu_{g_k} < \Lambda$, where $p > \frac{n}{2}$. Then for any sufficiently small ϵ_0, there exists $a_0 > 0$, such that*

$$\rho(x, u_k, B_2, \epsilon_0) > a_0, \quad \forall\ x \in B_1.$$

Proof. We have the equation:

$$-\hat{g}_k^{ij} u_{k,ij} + \hat{g}_k^{ij} \Gamma_{ij}^m(\hat{g}_k) u_{k,m} + c(n) R(\hat{g}_k) u_k = f_k,$$

where

$$f_k = c(n) R(g_k) u_k^{\frac{4}{n-2}}.$$

Given $x_k \to x_0$, $r_k \to 0$, such that

$$r_k < 2\rho(x_k, u_k, B_2, \epsilon_0).$$

Note that $\hat{g}_k^{ij}$, $\Gamma_{ij}^m(\hat{g}_k)$, $R(\hat{g}_k)$ converges to g^{ij}, $\Gamma_{ij}^m(g)$, $R(g)$ respectively. We let $v_k(x) = r_k^{\frac{n-2}{2}} u_k(x_k + r_k x)$. Obviously, $\rho(0, v_k, B_2, \epsilon_0) \geq 1/2$ implies

$$\int_{B_{\frac{1}{3}}} |\nabla \log v_k|^n dx < \epsilon_0.$$

Then,

$$\rho(y, v_k, B_2(0), \epsilon_0) > \frac{1}{6}, \qquad \forall\ y \in B_{\frac{1}{3}}(0).$$

By Lemma 2.3,

$$\|c_k v_k\|_{L^{\frac{\beta_n}{\epsilon_0}}(B_{\frac{1}{3}})} + \|\frac{1}{c_k v_k}\|_{L^{\frac{\beta_n}{\epsilon_0}}(B_{\frac{1}{3}})} \leq C.$$

Since $\int |v_k|^{\frac{2n}{n-2}} \to 0$, we get $c_k \to +\infty$, then $\|v_k\|_{L^{\frac{\beta_n}{\epsilon_0}}(B_{\frac{1}{3}})} < C$. Fix a $p' \in (\frac{n}{2}, p)$, we can choose ϵ_0 to be sufficiently small such that

$$\|R(\hat{g}_k)(x_k + r_k x) v_k^{\frac{4}{n-2}}\|_{L^{p'}(B_{\frac{1}{3}})} < C(p', p, \Lambda), \quad and \quad \frac{p'}{1 + \frac{\epsilon_0}{\beta_n} p'} > \frac{n}{2}.$$

Applying Corollary 2.5 to $c_k v_k$, we get $c_k v_k$ converges to a positive function ϕ in $W^{2,q}(B_{\frac{1}{4}})$ for some $q > \frac{n}{2}$. Since $\text{vol}(B_2, g_k) \to 0$, $c_k \to \infty$, and $v_k \to 0$ in $B_{\frac{1}{4}}$ uniformly, we get

$$\begin{aligned}\int_{B_{\frac{1}{4}}(0)} r_k^{2p} |R(\hat{g}_k)|^p u_k(x_k + r_k x)^{\frac{4p}{n-2}} dx &= \int_{B_{\frac{1}{4}}(0)} |R(\hat{g}_k)|^p v_k^{\frac{4p}{n-2}} dx \\ &\leq ||v_k||_{L^\infty}^{\frac{4p-2n}{n-2}} \int_{B_{\frac{1}{4}}(0)} |R(\hat{g}_k)|^p v_k^{\frac{2n}{n-2}} dx \to 0\end{aligned}$$

Applying Theorem 2.7, we obtain the required result and complete the proof. □

Next, we prove the ϵ-regularity.

Lemma 4.2. *Let $B_r(x_0) \subset M$. We assume $\int_{B_r(x_0)} |R(g')|^p d\mu_{g'} < \Lambda$, where $p > \frac{n}{2}$. Then, there exists ϵ_0' which depends only on M, r and Λ, such that if vol$(B_r(x_0), g') < \epsilon_0'$, then*

$$\|u\|_{W^{2,p}(B_{\frac{r}{2}})(x_0)} < C vol(B_r(x_0), g')^{\frac{n-2}{2n}}.$$

Proof. Assume the result is not true. Then we can find $x_k \to x_0'$, $g_k = u_k^{\frac{4}{n-2}} g$, such that $\text{vol}(B_r(x_k), g_k) \to 0$, $\|R(g_k)\|_{L^p(B_r(x_k))} < \Lambda$ and

$$\|u_k\|_{W^{2,p}(B_{\frac{r}{2}}(x_0))} > k\text{vol}(B_r(x_0), g_k)^{\frac{n-2}{2n}}.$$

$g_k' = g|_{B_r(x_k)}$ can be regarded as a metric over $B_r \subset \mathbb{R}^n$ which converges smoothly. It follows that

$$-\Delta_{g_k'} u_k = (-c(n)R(g_k') + R(g_k) u_k^{\frac{4}{n-2}}) u_k.$$

By the above lemma, $\rho(x, u_k, B_r, \epsilon_0) > a > 0$ for any $x \in B_{\frac{7r}{8}}$. Choose ϵ_0 to be sufficiently small. By Corollary 2.5, $c_k u_k$ converges in $W^{2,q}(B_{\frac{3}{4}r})$ to a positive function for some $q > \frac{n}{2}$. Then, $\|c_k u_k\|_{L^\infty(B_{\frac{3}{4}r})}$ is bounded above. Since $\int_{B_r(x_0)} u_k^{\frac{2n}{n-2}} \to 0$, we get $c_k \to +\infty$. Then $u_k \to 0$ in $B_{\frac{r}{2}}(x_0)$ uniformly and

$$\|c_k u_k\|_{W^{2,p}(B_{\frac{r}{2}}(x_0))} \geq k\text{vol}(B_{\frac{3}{4}r}(x_0), c_k^{\frac{4}{n-2}} g_k)^{\frac{n-2}{2n}} \to +\infty.$$

On the other hand, since

$$\int_{B_{\frac{3}{4}r}} (|R(g_k)| u_k^{\frac{4}{n-2}} c_k u_k)^p \leq \int_{B_{\frac{3}{4}r}} |R(g_k)|^p u_k^{\frac{2n}{n-2}} \|(c_k u_k)^{\frac{p(n+2)-2n}{n-2}}\|_{L^\infty(B_{\frac{3}{4}r})} c_k^{p - \frac{p(n+2)-2n}{n-2}} \to 0,$$

we derive

$$\begin{aligned}\|c_k u_k\|_{W^{2,p}(B_{\frac{r}{2}})} &\leq C(\|R(g_k) u_k^{\frac{4}{n-2}} c_k u_k\|_{L^p(B_{\frac{3}{4}r})} + \|c_k u_k\|_{L^p(B_{\frac{3}{4}r})}) \\ &\leq C(\|R(g_k)\|_{L^p(B_r, g_k)} + 1) \\ &< C.\end{aligned}$$

We get a contradiction and finish the proof. □

5. 4-manifold in a conformal class with $\|K\|_{W^{1,2}} < \Lambda$. In this section, we let $\dim M = 4$, $u_k \in W^{3,2}(M,g)$ and $g_k = u_k^2 g$. Assume that for every k there holds

$$\mathrm{vol}(M,g_k) = 1 \quad \text{and} \quad \int (|\nabla_{g_k} K(g_k)|^2 + K^2(g_k))d\mu_{g_k} < \Lambda, \tag{5.1}$$

where $K(g_k)$ denotes the sectional curvature of g_k. We intend to study the convergence behavior of u_k.

First of all, we try to show that the John-Nirenberg inequality will imply the L^p-estimate of curvature. We want to prove the result under the assumption that

$$\|R(g_k)\|_{W^{1,2}(M,g_k)} < \Lambda.$$

LEMMA 5.1. *Let $g = g_{ij}dx^i \otimes dx^j$ be a smooth metric on $B_3 \subset \mathbb{R}^n$ with $\|g_{ij}\|_{C^{2,\alpha}(B_3)} < \gamma_1$. Suppose that $g' = u^2 g$ satisfies vol$(B_3, g') < \gamma_2$ and*

$$\int_{B_3} (|\nabla_{g'} R(g')|^2 + |R(g')|^2) dV_{g'} < \Lambda. \tag{5.2}$$

Then, for any $p < 4$, there exists $\hat{\epsilon}_0 = \hat{\epsilon}_0(p)$ such that, if $\epsilon_0 < \hat{\epsilon}_0$ and $\rho(x,u,B_3,\epsilon_0) \geq a > 0$, there holds true

$$\int_{B_1} |R(g')u^2|^p < C(p,\gamma_1,\gamma_2,\hat{\epsilon}_0,\Lambda,a).$$

Proof. For any $q \in (\frac{4}{3}, 2)$, we have

$$\begin{aligned}\int_{B_1} |\nabla_g (Ru^2)|^q dV_g &\leq C(q)\left(\int_{B_1} (|\nabla_g R|u)^q u^q dV_g + \int_{B_1} (|R|u^2)^q |\nabla_g \log u|^q dV_g\right)\\ &\leq C(q,\Lambda)\left(\int_{B_1} u^{\frac{2q}{2-q}} dV_g\right)^{\frac{2-q}{2}}\\ &\quad + C(q)\left(\int_{B_1} (|R|u^2)^{\frac{4q}{4-q}}\right)^{\frac{4-q}{4}} \left(\int_{B_1} |\nabla_g \log u|^4 dV_g\right)^{\frac{q}{4}}.\end{aligned}$$

Choose $\hat{\epsilon}_0$, such that $\frac{\beta_n}{\hat{\epsilon}_0} > \frac{2q}{2-q}$. Let $p = \frac{4q}{4-q}$ and $-\log c$ be the mean value of $\log u$ over B_1. By Corollary 2.2, we can find $C = C(\epsilon_0, q, \gamma_1, a)$, such that both $\|cu\|_{L^{\frac{2q}{2-q}}(B_1)}$ and $\|\frac{1}{cu}\|_{L^{\frac{2q}{2-q}}(B_1)}$ are bounded above by C. Then

$$|B_1|^2 \leq \int_{B_1} (cu)^4 dx \int_{B_1} (cu)^{-4} dx < C(\epsilon_0, q, \gamma_1, a) c^4 \int_{B_1} u^4,$$

which yields that c is bounded below by a positive constant $C = C(\epsilon_0, q, \gamma_1, \gamma_2, a)$. Then

$$\int_{B_1} |\nabla (Ru^2)|^q dV_g \leq C(q,\epsilon_0,\gamma_1,\gamma_2,\Lambda,a) + C(q,\gamma_1,a)\epsilon_0^q \left(\int_{B_1} (|R|u^2)^{\frac{4q}{4-q}} dV_g\right)^{\frac{4-q}{4}}.$$

Let $\epsilon = C(q,\gamma_1,a)^{\frac{1}{q}}\epsilon_0$. We get

$$\|\nabla(Ru^2)\|_{L^q(B_1,g)} < C(q,\epsilon_0,\gamma_1,\gamma_2,\Lambda,a) + \epsilon \|Ru^2\|_{L^{\frac{4q}{4-q}}(B_1,g)}.$$

By Sobolev inequality,

$$\|Ru^2\|_{L^{\frac{4q}{4-q}}(B_1,g)} \le C(q,\gamma_1)(\|\nabla(Ru^2)\|_{L^q(B_1,g)} + \|Ru^2\|_{L^q(B_1,g)}).$$

Put $\epsilon C(q,\gamma_1) < \frac{1}{2}$, we get

$$\|Ru^2\|_{L^{\frac{4q}{4-q}}(B_1)} \le C(q,\epsilon_0,\gamma_1,\gamma_2,\Lambda,a).$$

□

Next, we show that small $\|R\|_{L^2}$ implies the boundness of John-Nirenberg radius.

LEMMA 5.2. *Let g, u and g' be as in the above lemma. Then, there exist $\tau > 0$ and $a > 0$ such that, if $\int_{B_3} R^2(g')dV_{g'} < \tau$, then*

$$\inf_{x\in B_1} \rho(x,u,B_3,\epsilon_0) > a, \quad \forall\, x \in B_1.$$

Proof. We prove it by contradiction. Assume there exists $g_k = u_k^2 g$ such that

$$\inf_{x\in B_1} \rho(x,u_k,B_3,\epsilon_0) \to 0.$$

Given $y_k \to y_0 \in \overline{B_2}$, $r_k < 2\rho(y_k,u_k,B_3,\epsilon_0)$, we set $v_k(x) = r_k u_k(y_k + r_k x)$ and

$$\hat{g}_k = v_k^2 g_{ij}(y_k + r_k x)dx^i \otimes dx^j.$$

Then, it is easy to see that $\rho(0,v_k,B_{\frac{1}{2}},\epsilon_0) > \frac{1}{2}$ and $\rho(x,v_k,B_{\frac{1}{2}},\epsilon_0) > \frac{1}{4}, \forall x \in B_{\frac{1}{4}}$.

Obviously,

$$\|R(\hat{g}_k)\|_{W^{1,2}(B_3,\hat{g}_k)} = \|R(g_k)\|_{W^{1,2}(B_{3r_k}(y_k),g_k)}.$$

By Lemma 5.1, for some $p \in (2,\, 4)$ there holds

$$\int_{B_{\frac{1}{4}}} |R(\hat{g}_k)v_k^2|^p < C.$$

Since

$$\int_{B_{\frac{1}{4}}} |r_k^2 R(g_k)(r_k x + y_k)u_k^2(r_k x + y_k)|^p dx \le C\int_{B_{\frac{1}{4}}} |R(\hat{g}_k)v_k^2|^p dx,$$

and

$$\int_{B_{\frac{1}{4}}} |r_k^2 R(r_k)(r_k x + y_k)u_k^2(y_k + r_k x)|^2 = \int_{B_{\frac{1}{4}}} |R(\hat{g}_k)v_k^2|^2 = \int_{B_{\frac{1}{4}r_k}(y_k)} |R(g_k)|^2 u_k^4 \to 0.$$

From Lemma 2.7, it follows that $\rho(x,u_k,B_3,\epsilon_0) > a, \forall x \in B_1$. Then, we get a contradiction. □

For convenience, given a subset $A \subset \mathbb{S}^{n-1}$, we set

$$A_r = \bigcup_{t\in(0,r]} tA, \qquad C(A,r) = \bigcup_{t\in[\frac{r}{2},r]} tA.$$

We need to establish the following lemma:

LEMMA 5.3. *Let g be a smooth metric over $B_1 \subset \mathbb{R}^4$ and $g' = u^2 g$, where $u \in W^{3,2}(B_1)$ is a positive function. Assume $g = dr^2 + g(r,\theta)d\mathbb{S}^3$ with $g(r,\theta) = r^2(1+o(1))$. If*

$$vol(B_1, g') + \int_{B_1} (|K(g')|^2 + |\nabla_{g'} K(g')|^2) dV_{g'} < +\infty,$$

then, when r is small enough, there holds

$$vol(C(A, r/2), g') < \frac{1}{2^3} vol(C(A,r), g').$$

Proof. We claim that: there exists r_0, such that if $r < r_0$, then

$$\mathrm{vol}(C(A, r/4), g') < \frac{1}{2^3}\mathrm{vol}(C(A, r/2), g') < \frac{1}{2^6}\mathrm{vol}(C(A,r), g'), \tag{5.3}$$

or

$$\mathrm{vol}(C(A, r), g') < \frac{1}{2^3}\mathrm{vol}(C(A, r/2), g') < \frac{1}{2^6}\mathrm{vol}(C(A,r/4), g'). \tag{5.4}$$

Assume there exists $r_k \to 0$, such that none of the above holds. Put $u_k(x) = r_k u(r_k x)$ and $g_k = u_k^2 g(r_k x)$. For any fixed R, we have

$$\int_{B_R} (|K(g_k)|^2 + |\nabla_{g_k} K(g_k)|^2) dV_{g_k} = \int_{B_{Rr_k}} (|K(g')|^2 + |\nabla_{g'} K(g')|^2) dV_{g'} \to 0.$$

Then by Lemma 5.1-5.2, Lemma 2.5 and Lemma 2.3, we can find $\tilde{c}_k$ such that $\tilde{c}_k u_k$ converges to a positive function φ. Let $\tilde{g}_k = \tilde{c}_k^2 g_k$ and $\tilde{g} = \varphi^2 g_{\mathbb{R}^4}$.

Since $\mathrm{vol}(g_k, B_{Rr_k} \setminus \{0\}) \to 0$, we have $\tilde{c}_k \to \infty$. Then it is easy to check that

$$\int_{B_{\frac{1}{r}} \setminus B_r} (|\nabla_{\tilde{g}_k} K(\tilde{g}_k)|^2 + |K(\tilde{g}_k)|^2) dV_{\tilde{g}_k} \to 0.$$

By Lemma 5.2 again, $\tilde{c}_k u_k$ converges weakly in $W^{3,2}_{loc}(\mathbb{R}^4 \setminus \{0\})$ and $K(\varphi) = 0$, thus φ is a positive harmonic function.

Theorem 9.8 in [1] tells us that φ can be written as $\varphi = ar^{-2} + b$. Since $K(\varphi) = 0$, we get $a = 0$ or $b = 0$. When $a = 0$ and $b \neq 0$, we have

$$\frac{\mathrm{vol}(C(A,1),\tilde{g})}{\mathrm{vol}(C(A,1/2),\tilde{g})} = \frac{\mathrm{vol}(C(A,1/2),\tilde{g})}{\mathrm{vol}(C(A,1/4),\tilde{g})} = 2^4.$$

Since

$$\frac{\mathrm{vol}(C(A,r_k),g')}{\mathrm{vol}(C(A,r_k/2),g')} \to \frac{\mathrm{vol}(C(A,1),\tilde{g})}{\mathrm{vol}(C(A,1/2),\tilde{g})}$$

and

$$\frac{\mathrm{vol}(C(A,r_k/2),g')}{\mathrm{vol}(C(A,r_k/4),g')} \to \frac{\mathrm{vol}(C(A,1/2),\tilde{g})}{\mathrm{vol}(C(A,1/4),\tilde{g})},$$

we get (5.3) for $r = r_k$. A contradiction appears.

When $b=0$ and $a\neq 0$, we have

$$\frac{\mathrm{vol}(C(A,1),\tilde{g})}{\mathrm{vol}(C(A,1/2),\tilde{g})}=\frac{\mathrm{vol}(C(A,1/2),\tilde{g})}{\mathrm{vol}(C(A,1/4),\tilde{g})}=\frac{1}{2^4}.$$

We can get another contradiction by the same argument.

To prove the lemma, now we only need to show (5.4) does not hold. When (5.4) holds, we can pick r_0 such that

$$\mathrm{vol}(C(A,2^{-k}r_0),g')>2^3\mathrm{vol}(C(A,2^{-k+1}r_0),g'),$$

which contradicts $\mathrm{vol}(B_1,g')<+\infty$. □

Using the same method, or applying Klein transformation, we have the following:

LEMMA 5.4. *Let $u\in W^{3,2}_{loc}(\mathbb{R}^4\setminus B_R)$ and $g'=u^2g_{\mathbb{R}^4}$. If*

$$vol(\mathbb{R}^4\setminus B_R,g')+\int_{\mathbb{R}^4\setminus B_R}(|K(g')|^2+|\nabla_{g'}K(g')|^2)dV_{g'}<+\infty,$$

then, when r is large enough, there holds true

$$vol(C(A,r))<\frac{1}{2^3}vol(C(A,r/2)).$$

Now, we are in the position to prove the main theorem of this section:

THEOREM 5.5. *Let (M,g) be a closed 4-dimensional Riemannian manifold with constant scalar curvature. Let $u_k\in W^{3,2}(M,g)$ be a positive function and $g_k=u_k^2g$. Assume*

$$vol(M,g_k)=a_0 \quad \text{and} \quad \int_M(|\nabla_{g_k}K(g_k)|^2+|K(g_k)|^2)dV_{g_k}<\Lambda,$$

where $a_0>0$ and $\Lambda>0$. Then,

1) as (M,g) is not conformal to $\mathbb{S}^4$, u_k converges in $W^{3,2}(M,g)$ to a positive function weakly.

2) as $M=\mathbb{S}^4$, there exist Möbius transformation σ_k such that $\sigma_k^*(g_k)$ converges to $W^{3,2}$-metric weakly in $W^{3,2}$.

Proof. After passing to a subsequence, we find a finite set $\mathcal{S}$ such that

$$\lim_{r\to 0}\liminf_{k\to\infty}\int_{B_r(x)}R_k^2u_k^4>\frac{\tau}{2},\qquad x\in\mathcal{S}$$

and

$$\lim_{r\to 0}\limsup_{k\to\infty}\int_{B_r(x)}R_k^2u_k^4<\frac{\tau}{2},\qquad x\notin\mathcal{S}.$$

For more details we refer to Section 5 in [11].

By Lemma 5.1-5.2, and Corollary 2.5, we can find $c_k>0$ such that c_ku_k converges to a positive function ϕ weakly in $W^{3,2}_{loc}(M\setminus\mathcal{S})$. When $\mathcal{S}=\emptyset$, c_ku_k converges weakly in $W^{3,2}(M,g)$, then it follows from $\mathrm{vol}(M,g)=a_0$ that a subsequence of $\{c_k\}$ converges to a positive constant. Hence $\mathcal{S}=\emptyset$ implies that u_k converges weakly in $W^{3,2}(M,g)$.

Now, we assume $\mathcal{S} \neq \emptyset$. First, we consider the case M is not conformal to $\mathbb{S}^4$. For this case, we claim that

$$\int_{M\setminus\mathcal{S}} \phi^4 dV_g < \infty.$$

Assume this is not true. Since

$$\lim_{k\to\infty} \int_{M\setminus\mathcal{S}} (c_k u_k)^4 dV_g = \int_{M\setminus\mathcal{S}} \phi^4 dV_g,$$

it follows from $\int_M u_k^4 = a_0$ that $c_k \to \infty$.

Let $\hat{g}_k = c_k^2 g_k = c_k^2 u_k^2 g$. We have

$$\int_M |K(\hat{g}_k)|^2 dV_{\hat{g}_k} = \int_M |K(g_k)|^2 dV_{g_k} \leq \Lambda,$$

and

$$\int_M |\nabla_{\hat{g}_k} K(\hat{g}_k)|^2 dV_{\hat{g}_k} = \frac{1}{c_k^2} \int_M |\nabla_{g_k} K(g_k)|^2 dV_{g_k} \to 0.$$

Then, $K(\hat{g}_k)$ converges to a constant weakly in $W^{1,2}_{loc}(M \setminus \mathcal{S})$. Noting that

$$\int_M |K(\phi)|^2 \phi^4 dV_g \leq \Lambda,$$

we get $K(\phi) = 0$, which implies that $R(\phi) = 0$.

By Corollary 3.2, we know that $(M, \phi^2 g)$ is complete. On the other hand, each end of $M \setminus \mathcal{S}$ is collared topologically by $S^3 \times \mathbb{R}$. Therefore, we conclude that $(M \setminus \mathcal{S}, \phi^2 g)$ is just $\mathbb{R}^4$ (c.f. [4, Theorem 1]). This contradicts the assumption that M is not conformal to $\mathbb{S}^4$. Thus, we get the claim.

Choose a normal chart of a point $p \in \mathcal{S}$. By the definition of $\mathcal{S}$, we can get a sequence (x_k, r_k), such that $x_k \to 0$ and $r_k \to 0$ and

$$\int_{B_{r_k}(x_k)} |R(g_k)|^2 dV_{g_k} = \frac{\tau}{2},$$

$$\int_{B_r(y)} |R(g_k)|^2 dV_{g_k} \leq \frac{\tau}{2}, \qquad \forall y \in B_\delta(0), \quad r \leq r_k.$$

Let $v_k(x) = r_k u_k(x_k + r_k x)$ and

$$g_k' = r_k^2 u_k^2 (x_k + r_k x) g(x_k + r_k x).$$

It is easy to check that

$$\|K(g_k')\|_{W^{1,2}(B_R, g_k')} < C(R), \qquad \forall R,$$

$$\int_{B_1} |R(g_k')|^2 dV_{g_k'} = \frac{\tau}{2}, \quad \text{and} \quad \int_{B_1(y)} |R(g_k')|^2 dV_{g_k'} \leq \frac{\tau}{2}, \qquad \forall y.$$

By Lemma 5.1-5.2, Lemma 2.5 and Lemma 2.3, there exists a sequence of positive numbers $\{c'_k\}$ such that $c'_k v_k$ converges weakly to a positive function ψ in $W^{3,2}_{loc}(R^4)$ weakly. Noting $\int_{B_R} v_k^4 < a_0$, we have $\inf_k c'_k > 0$.

We claim that

$$\int_{\mathbb{R}^4} \psi^4 dx = \mathrm{vol}(\mathbb{R}^4, \psi^2 g_{\mathbb{R}^4}) < +\infty.$$

Assume this is not true. By a similar argument with the proof of $\int_M \phi^4 < +\infty$, we can get $c_k \to +\infty$ and $K(\psi) = 0$. Noting that

$$\int_{B_1} |R(g'_k)|^2 dV_{g'_k} = \int_{B_1} |R({c'_k}^2 g'_k)|^2 dV_{c_k^2 g'_k} = \frac{\tau}{2}$$

we get

$$\int_{B_1} |R(\psi)|^2 dV_{\psi^2 g_{\mathbb{R}^n}} = \frac{\tau}{2},$$

which is impossible. Therefore, the claim is true.

Let A' be an open ball in $\mathbb{S}^{n-1}$ such that, after passing to a subsequence,

$$\int_{x_k + A'_\delta} |R(g_k)|^2 dV_{g_k} < \frac{\tau}{2}.$$

Let $A \subset A'$ be a closed ball in $\mathbb{S}^{n-1}$, and δ be sufficiently small. Take $t_k \in [\frac{r_k}{\delta}, \delta]$, such that

$$\mathrm{vol}(C(A, t_k) + x_k, g_k) = \inf_{t \in [\frac{r_k}{r}, r]} \mathrm{vol}(C(A, t) + x_k, g_k).$$

By Lemma 5.3, for any fixed sufficiently small r, we have

$$\frac{\mathrm{vol}(C(A, r) + x_k, g_k)}{\mathrm{vol}(C(A, r/2) + x_k, g_k)} \to \frac{\mathrm{vol}(C(A, r), \phi^2 g)}{\mathrm{vol}(C(A, r/2), \phi^2 g)} > 2^3.$$

Then, $t_k \to 0$. By the same argument, we deduced from Lemma 5.4 that $\frac{t_k}{r_k} \to +\infty$.

Set

$$\tilde{v}_k = t_k u_k(x_k + t_k x), \qquad \tilde{g}_k = \tilde{v}_k g(x_k + t_k x).$$

Using the same method as we get ϕ, we can find a finite set $\tilde{\mathcal{S}}$ and a number $\tilde{c}_k$, such that $\tilde{c}_k \tilde{v}_k$ converges to a positive function v weakly in $W^{3,2}_{loc}(\mathbb{R}^4 \setminus (\{0\} \cup \mathcal{S}))$. By the definition of A, we have

$$\mathcal{S} \cap \{tA : t > 0\} = \emptyset,$$

hence it follows

$$\mathrm{vol}(C(A, 1), v^2 g_{\mathbb{R}^n}) = \inf_{t > 0} \mathrm{vol}(C(A, t), v^2 g_{\mathbb{R}^n}). \tag{5.5}$$

Then, by the same arguments as we derive $\int_M \phi^4 < +\infty$, we also obtain that $\tilde{c}_k \to +\infty$ and $K(v) = 0$. Then v is a positive harmonic function defined on $\mathbb{R}^4 \setminus (\mathcal{S} \cup \{0\})$. Furthermore, by Theorem 9.8 in [1], for any $x_0 \in \mathcal{S} \cup \{0\}$ we have

$$v(x) \sim c(x_0)|x - x_0|^{-2},$$

where $c(x_0)$ is a nonnegative constant.

Let

$$\mathcal{S}' = \{x \in \mathcal{S} \cup \{0\} : c(x) > 0\}.$$

If $\mathcal{S}'$ is nonempty, then, $(\mathbb{R}^4 \setminus \mathcal{S}', v^2 g_{\mathbb{R}^4})$ is a complete flat manifold, whose ends are collared topologically by $S^3 \times \mathbb{R}$. It is impossible. This means that $\mathcal{S}' = \emptyset$, hence $v \in C^\infty(\mathbb{R}^4)$ which contradicts (5.5). Therefore, we finish the proof of 1).

Next, we consider the case (M, g) is conformal to $\mathbb{S}^4$. Let P be the stereographic projection from $\mathbb{S}^4$ to $\mathbb{R}^4$, which sends $x_0 \in \mathcal{S}$ to $0 \in \mathbb{R}^4$. Under the coordinate system defined by P, as before, we can find $x_k \to 0$, $r_k \to 0$, and c_k', such that $c_k' r_k u_k(x_k + r_k x)$ converges to a positive function ψ, which satisfies

$$\int_{\mathbb{R}^4} \psi^4 dx < +\infty.$$

Let $\sigma_k(y) = P^{-1}(r_k P(y) + x_k)$. It is well-known that σ_k defines a Möbius transformation of $\mathbb{S}^4$. It is easy to check that for the new sequence $g_k' = \sigma_k^*(g_k) = (u_k')^2 g_{\mathbb{S}^4}$, there exist c_k and a finite set $\mathcal{S}'$, such that $c_k u_k$ converges weakly in $W^{3,2}(M \setminus \mathcal{S}')$ to a positive function ϕ, which satisfies $\int \phi^4 < +\infty$. Then, following the arguments taken in 1), we complete the proof easily. □

6. Hélein's convergence Theorem. The arguments in the previous sections seem useless to the Gauss equation in 2 dimensional case under Gauss curvature condition. However, we can apply them to study the convergence of a $W^{2,2}$-conformal immersion with bounded $\|A\|_{L^2}$ to give a generalized Hélein's Convergence Theorem.

In [7], we defined the $W^{2,2}$-conformal immersion as follows:

Definition 6.1. *Let (Σ, g) be a Riemann surface. A map $f \in W^{2,2}(\Sigma, g, \mathbb{R}^n)$ is called a conformal immersion, if the induced metric $g_f = df \otimes df$ is given by*

$$g_f = e^{2u} g \qquad \text{where } u \in L^\infty(\Sigma).$$

For a Riemann surface Σ the set of all $W^{2,2}$-conformal immersions is denoted by $W^{2,2}_{conf}(\Sigma, g, \mathbb{R}^n)$. When $f \in W^{2,2}_{loc}(\Sigma, g, \mathbb{R}^n)$ and $u \in L^\infty_{loc}(\Sigma)$, we say $f \in W^{2,2}_{conf,loc}(\Sigma, g, \mathbb{R}^n)$.

Hélein's Convergence Theorem was first proved by Hélein [6]. An optimal version of the theorem was stated in [7] as follows:

Theorem 6.2. *Let $f_k \in W^{2,2}(D, \mathbb{R}^n)$ be a sequence of conformal immersions with induced metrics $(g_{f_k})_{ij} = e^{2u_k}\delta_{ij}$ and satisfy*

$$\int_D |A_{f_k}|^2 \, d\mu_{f_k} \le \gamma < \gamma_n = \begin{cases} 8\pi & \text{for } n = 3, \\ 4\pi & \text{for } n \ge 4. \end{cases} \tag{6.1}$$

If $\mu_{f_k}(D) \le C$ and $f_k(0) = 0$, where μ_{f_k} is the measure defined by f_k, then f_k is bounded in $W^{2,2}_{loc}(D, \mathbb{R}^n)$, and there is a subsequence such that one of the following two alternatives holds:

(a) *u_k is bounded and f_k converges weakly in $W^{2,2}_{loc}(D, \mathbb{R}^n)$ to a conformal immersion $f \in W^{2,2}_{loc}(D, \mathbb{R}^n)$.*

(b) *$u_k \to -\infty$ and $f_k \to 0$ locally uniformly on D.*

Note that in case of (a), $\|u_k\|_{W^{1,2}} < C$ follows from the boundness of $\|u_k\|_{L^\infty}$ and $\|f\|_{W^{2,2}}$.

Hélein's convergence Theorem is a very powerful tool to study variational problem concerning Willmore functional [7, 12]. However, Theorem 6.2 can not get rid of a collapsing sequence. For this case, generally it is not true that f_k converges to a non-trivial map after rescaling. For example, if $f_k = a_k e^{kz}$, which is a sequence of conformal maps from D to $\mathbb{C}$, where a_k is chosen such that $\mu_{f_k}(D) = 1$, then f_k converges to a point, and for any c_k, $c_k f_k$ does not converge. However, in [9] (also see [10]) Y. Li showed that, if $f_k(D)$ can be extended to a closed surface immersed in $\mathbb{R}^n$ with $\|A_k\|_{L^2} < C$, then we can find c_k, such that $c_k f_k$ converges weakly in $W^{2,2}(D_r)$ for any r to a conformal immersion. The proof provided in [9] is based on the conformal invariant of Willmore functional and Simon's monotonicity formula.

In this section, we will use the John-Nirenberg inequality to give a new sufficient condition to guarantee the above assertion is still valid.

We define

$$\rho(u_k, x) = \sup\left\{t : \int_{D_r(x)\cap D} |\nabla u_k|^2 < \epsilon_0^2\right\}.$$

We first prove the following:

Lemma 6.3. *Let $f \in W^{2,2}_{conf}(D, \mathbb{R}^n)$ and $df \otimes df = e^{2u}(dx^2 + dy^2)$. Suppose that there exists a positive number β such that, for any $y \in \mathbb{R}^n$ and $r > 0$,*

$$\frac{\mu_f(f^{-1}(B_r(y)))}{\pi r^2} < \beta. \tag{6.2}$$

Then there exists $\epsilon > 0$ and $a > 0$ such that, if $\int_D |A|^2 < \epsilon$, then

$$\inf_{x\in D_{\frac{1}{4}}} \rho(u, x) > a.$$

Proof. If this is not true, then, we can find a sequence of f_k, such that $\int_D |A_k|^2 \to 0$ and $\inf_{D_{\frac{1}{4}}} \rho(u_k, x) \to 0$. Take $x_k \in D_{\frac{1}{4}}$, such that $\rho(u_k, x_k) \to 0$ and $x_k \to x_0$.

Put $z_k \in D_{\frac{1}{2}}$ such that

$$\frac{\rho(u_k, z_k)}{1/2 - |z_k|} = \inf_{x\in D_{\frac{1}{2}}} \frac{\rho(u_k, x)}{1/2 - |x|} := \lambda_k.$$

As the proof of Corollary 2.7, we have $\rho_k := \rho(u_k, z_k) \to 0$, $D_{R\rho_k}(z_k) \subset D_{\frac{1}{2}}$ and

$$\frac{\rho(u_k, z)}{\rho(u_k, z_k)} > \frac{1}{2}, \qquad \forall z \in D_{R\rho_k}(z_k),$$

when k is sufficiently large.

Assume $z_k \to z_0$ and put $f'_k(x) = c_k(f_k(z_k + \rho_k z) - f(z_k))$, where c_k is chosen such that

$$\int_D u'_k = 0.$$

It is easy to see that f'_k also satisfies (6.2). Then, as the proof of Corollary 2.7, we have $\int_{D_R} e^{2u'_k} < C(R)$ for any R. Since $\int_D u'_k$ does not converge to $-\infty$, by Theorem

6.2, we know that f'_k converges weakly in $W^{2,2}_{loc}(\mathbb{C},\mathbb{R}^n)$ to an $f'\in W^{2,2}_{loc}(\mathbb{C},\mathbb{R}^n)$ with $A_{f'}=0$. Since f' is conformal, it is a holomorphic immersion from $\mathbb{C}$ to a plain L in $\mathbb{R}^n$.

Moreover, from

$$-\Delta u_k = K_{f_k}e^{2u_k}$$

we deduce that u' is a harmonic function on $\mathbb{R}^2$ and hence $\nabla u'$ is harmonic, since $K_{f_k}e^{2u_k}$ converges to 0 in L^1 and u'_k converges to u' weakly in $W^{1,2}_{loc}(\mathbb{C})$. Obviously, we also have that for any $x\in\mathbb{R}^2$

$$\int_{D_{\frac{1}{2}}(x)}|\nabla u'|^2dx\leq\epsilon_0^2,$$

which follows from that for any x

$$\int_{D_{\frac{1}{2}}(x)}|\nabla u'_k|^2dx\leq\epsilon_0^2.$$

By mean value theorem, $\nabla u'$ is bounded. Therefore, $\nabla u'$ is a constant vector. Choosing an appropriate coordinates of L, we may write f' as $f'=az$ or e^{az+b}, where $a\neq 0$.

When $f'=az$, u' is a constant. Note that i) of Theorem 6.2 implies that for any r

$$\|u'_k\|_{W^{1,2}(D_r)}<C(r).$$

Without loss of generality, we assume u'_k converges to u' weakly in $W^{1,2}_{loc}(\mathbb{C})$. Given an positive cut-off function η which is 1 on D_1, we have

$$\begin{aligned}\epsilon_0^2\leq\int_{\mathbb{C}}\eta|\nabla u'_k|^2&=\int_{\mathbb{C}}\nabla(\eta u'_k)\nabla u'_k-u'_k\nabla\eta\nabla u'_k\\&=\int_{\mathbb{C}}\eta u'_kKe^{2u'_k}-\int_{\mathbb{C}}(u'_k-u')\nabla\eta\nabla u'_k-u'\int_{\mathbb{C}}\nabla\eta\nabla u'_k\\&\to 0.\end{aligned}$$

This is a contradiction.

When $f'(z)=e^{az+b}$, there exists $P_0\in L$, such that ${f'}^{-1}(\{P_0\})$ contains infinity many points. Let $m>\beta+1$. Take $z_1,\cdots,z_m\in {f'}^{-1}(\{P_0\})$ and choose $r>0$ and $r'>0$ such that $B_{r'}(P_0)\cap L\subset f(D_r(z_i))$ and f is injective on $D_r(z_i)$. Then we get

$$\frac{\mu_{f'}\left({f'}^{-1}(B_{r'}(P_0))\right)}{\pi {r'}^2}=m,$$

and hence

$$\frac{\mu_{f'_k}\left({f'_k}^{-1}(B_{r'}(P_0))\right)}{\pi {r'}^2}>m-1>\beta,$$

when k is sufficiently large. This contradicts (6.2). □

THEOREM 6.4. *Let $f_k \in W^{2,2}_{conf}(D,\mathbb{R}^n)$ and satisfy (6.2). Then, there exists an $\epsilon > 0$ such that, if $\int_D |A_{f_k}|^2 < \epsilon$, then there exist c_k such that $c_k f_k$ converges weakly in $W^{2,2}_{loc}(D_r)$ to an $f \in W^{2,2}_{conf,loc}(D,\mathbb{R}^n)$ for any $r < 1$.*

Proof. We only need to prove that, there exists c_k, such that

$$\int_{D_r} e^{2|u_k+\log c_k|} < C(r)$$

for any r. The proof goes almost the same as in the proof of Lemma 2.3, we omit it. □

When f_k can be extended to a closed immersed surface with $\|A_k\|_{L^2} < C$, by (1.3) in [14], we know that (6.2) must hold true.

COROLLARY 6.5. *Let $f_k \in W^{2,2}_{conf}(D,\mathbb{R}^n)$, which satisfies (6.2). If f_k satisfies*

$$\int_D |A_{f_k}|^2 < \gamma_n - \tau,$$

then there exist c_k such that $\{c_k f_k\}$ converges weakly in $W^{2,2}_{loc}(D_r)$ to an $f \in W^{2,2}_{conf,loc}(D,\mathbb{R}^n)$ for any $r < 1$.

Proof. Let ϵ be the same as in the Theorem 6.4. Take m such that $\frac{8\pi}{m} \cdot 5 < \epsilon$. For convenience, we set $r \in (\frac{3}{4}, 1)$ and $l = \frac{1-r}{m}$. After passing to a subsequence, there exists $2 \le i \le m-2$ such that

$$\int_{D_{r+(i+2)l} \setminus D_{r+(i-2)l}} |A_{f_k}|^2 < \epsilon, \qquad \forall k.$$

By Theorem 6.2, Theorem 6.4 and a covering argument, we know there exists c'_k such that $c'_k f_k$ converges weakly in $W^{2,2}_{loc}$ to a function

$$f_0 \in W^{2,2}_{conf,loc}(D_{r+(i+1)l} \setminus D_{r+(i-1)l}, \mathbb{R}^n),$$

and

$$\|u_k + \log c'_k\|_{L^\infty(D_{r+(i+1)l} \setminus D_{r+(i-1)l})} < C.$$

In particular, there holds true

$$\|u_k + \log c'_k\|_{L^\infty(\partial D_{r+il})} < C.$$

Since

$$\int_D |A_{f_k}|^2 < \gamma_n - \tau,$$

by Corollary 2.4 in [7], we know there exists a function $v_k : \mathbb{C} \to \mathbb{R}$ solving the equation

$$-\Delta v_k = K_{c'_k f_k} e^{2(u_k + \log c'_k)}$$

in D and satisfying the following estimates:

$$\|v_k\|_{L^\infty(D)} \le C.$$

The maximal principle yields

$$\|u_k + \log c_k' - v_k\|_{L^\infty(D_{r+il})} < C,$$

hence it follows

$$\|u_k + \log c_k'\|_{L^\infty(D_{r+il})} < C.$$

By the fact f_k satisfies the equation

$$\Delta c_k' f_k = e^{2(u_k + \log c_k)} H_{c_k' f_k}$$

for every k, we obtain

$$\int_{D_{r+il}} |\Delta c_k' f_k|^2 dx \leq e^{2\|u_k + \log c_k'\|_{L^\infty(D_{r+il})}} \int_{D_{r+il}} |H_{c_k' f_k}|^2 d\mu_{c_k' f_k} \leq C.$$

This implies

$$\|c_k' f_k\|_{W^{2,2}(D_{r+il})} < C.$$

Thus, there exists a subsequence of $\{c_k' f_k\}$ converges weakly to a $W^{2,2}$ conformal immersion in D_r.

Applying Theorem 6.2 again, we get

$$\|\nabla(u_k + \log c_k')\|_{L^2(D_r)} + \|u_k + \log c_k'\|_{L^\infty(D_r)} < C(r).$$

Let

$$\log c_k = -\frac{1}{|D_{\frac{1}{2}}|} \int_{D_{\frac{1}{2}}} u_k.$$

By Poincaré inequality, we have

$$\|u_k + \log c_k\|_{L^2(D_r)} < C.$$

Hence, it follows that

$$|\log c_k - \log c_k'| < C.$$

Thus, after passing to a subsequence, $c_k f_k$ converges weakly in $W^{2,2}(D_r)$ to a conformal map. □

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