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# LEHMANN-SUWA RESIDUES OF CODIMENSION ONE HOLOMORPHIC FOLIATIONS AND APPLICATIONS*

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**Abstract.** Let $\mathscr{F}$ be a singular codimension one holomorphic foliation on a compact complex manifold $X$ of dimension at least three such that its singular set has codimension at least two. In this paper, we determine *Lehmann-Suwa residues* of $\mathscr{F}$ as multiples of complex numbers by integration currents along irreducible complex subvarieties of $X$. We then prove a formula that determines the Baum-Bott residue of *simple almost Liouvillian foliations of codimension one*, in terms of Lehmann-Suwa residues, generalizing a result of Marco Brunella. As an application, we give sufficient conditions for the existence of dicritical singularities of a singular real-analytic Levi-flat hypersurface $M \subset X$ tangent to $\mathscr{F}$.

**Key words.** Residues formula, holomorphic foliations, Levi-flat hypersurfaces.

**Mathematics Subject Classification.** Primary 32V40, 32S65.

**1. Introduction.** In 1999, D. Lehmann and T. Suwa [22] gave a generalization to the case of arbitrary dimension, of the *variational index*, defined by Khanedani and Suwa [20] for singular holomorphic foliations on complex surfaces. More precisely, Lehmann and Suwa proved the following result.

THEOREM 1.1 (Lehmann-Suwa [22]). *Let $V$ be a complex subvariety of dimension $m \geq 1$ in a complex manifold $X$ and let $\mathscr{F}$ be a singular holomorphic foliation of dimension $k \geq 1$ on $X$ which leaves $V$ invariant. Denote by $\mathcal{N}_{\mathscr{F}}$ the normal sheaf of $\mathscr{F}$. Let $\varphi$ be a homogeneous symmetric polynomial of degree $d > m - k$.*

(1) *For each compact connected component $Z$ of the singular set $\mathsf{Sing}(\mathscr{F}|_V)$, there exists a homology class*

$$Res_{\varphi}(\mathscr{F}, \mathcal{N}_{\mathscr{F}}|_V; Z) \in H_{2m-2d}(Z; \mathbb{C}),$$

*which is determined by the local behavior of $\mathscr{F}$ near $Z$.*

(2) *If $V$ is compact,*

$$\sum_{Z} (i_Z)_* Res_{\varphi}(\mathscr{F}, \mathcal{N}_{\mathscr{F}}|_V; Z) = \varphi(\mathcal{N}_{\mathscr{F}}) \frown [V] \quad in \quad H_{2m-2d}(V; \mathbb{C}),$$

*where $i_Z$ denotes the embedding $Z \hookrightarrow V$ and the sum is taken over all the components $Z$ of $\mathsf{Sing}(\mathscr{F}|_V)$.*

When $\varphi = c_1$, the expression $\mathrm{Res}_{\varphi}(\mathscr{F}, \mathcal{N}_{\mathscr{F}}|_V; Z)$ is called the *variation of $\mathscr{F}$ with respect to $V$ at $Z$*. In general, the computation and determination of these residues is a difficult problem and few results are known. For example, if the foliation $\mathscr{F}$ is singular at $p \in \mathbb{C}^2$ and $V$ is a reduced complex curve through $p$ invariant by $\mathscr{F}$. Then the variation of $\mathscr{F}$ relative to $V$ at $p$ is given by

$$\mathrm{Res}_{c_1}(\mathscr{F}, \mathcal{N}_{\mathscr{F}}|_V; p) = \mathrm{Var}(\mathscr{F}, V, p)[p],$$

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where $\mathrm{Var}(\mathscr{F}, V, p)$ is the *variational index* defined by Khanedani and Suwa in [20]. When $\mathscr{F}$ is a one-dimensional holomorphic foliation on a complex manifold, that is, locally defined by holomorphic vector fields, there is an explicit formula in terms of Grothendieck residues for $\mathrm{Res}_{\varphi}(\mathscr{F}, \mathcal{N}_{\mathscr{F}}|_V; p)$, see for instance [22] and [29].

This paper aims to study of residues of codimension one holomorphic foliations on complex manifolds of dimension at least three. First, we will restrict our attention to *Lehmann-Suwa residues* (or *variations*) of a codimension one holomorphic foliation $\mathscr{F}$ on a compact complex manifold $X$ of dimension at least three. In Section 3, it is shown that Lehmann-Suwa residues localized at codimension two irreducible components of the singular set of $\mathscr{F}$ can be determined as multiples of complex numbers by integration currents along of these irreducible components.

In [23], Lins Neto introduced the Camacho-Sad index [8] for a codimension one holomorphic foliation concerning a *codimension one complex submanifold*, and Gmira [16] obtained a generalization of some results due to Lins Neto [23]. Recently, Corrêa and Machado [12] defined the *GSV-index* for holomorphic Pfaff systems on complex manifolds generalizing the *GSV-index* of Gómez-Mont–Seade–Verjovsky [17]. In Section 4, combining the Corrêa-Machado index with the Lehmann-Suwa residues, we recover the Camacho-Sad index for a codimension one holomorphic foliation $\mathscr{F}$ with respect to a codimension one complex subvariety $V$ (possibly with singularities).

In [7], Brunella and Perrone determine the Baum-Bott residue [3] of a codimension one holomorphic foliation concerning a singular component of codimension two via integration over a 3-sphere of a certain 3-form (see for instance Section 5). In general, the determination of Baum-Bott residues (in terms of the Grothendieck residues) of singular holomorphic foliations of arbitrary codimension have been obtained by Corrêa and Lourenço [13]. In Section 6, we will prove (see Theorem 6.2) that the Baum-Bott and Lehmann-Suwa residues are related when the codimension one foliation $\mathscr{F}$ is a *simple almost Liouvillian foliation* (see Definition 6.1).

In the last part of the paper, we apply our residual formulas to prove, under certain conditions, the existence of *dicritical singularities* of a real-analytic Levi-flat hypersurface tangent to a codimension one holomorphic foliation on a compact complex manifold of complex dimension at least three.

It is important pointing out that a general construction of residue theorems for holomorphic foliations and Pfaff systems of any dimension can be found in [1], [29] and [27]. In the special case of singular codimension one holomorphic distributions, we refer the reader to [18] and the references given there.

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**2. Holomorphic foliations.** Let $X$ be a complex manifold and $TX$ the holomorphic tangent bundle of $M$. Let $\Theta_X = \mathcal{O}(TM)$ be the sheaf of holomorphic vector fields on $X$. A *singular holomorphic foliation* $\mathscr{F}$ of dimension $r$ on $X$ is determined by a coherent subsheaf $\Theta_{\mathscr{F}} \subset \Theta_X$ of rank $r$, which is involutive (or integrable), i.e., such that

$$[\Theta_{\mathscr{F},p}, \Theta_{\mathscr{F},p}] \subset \Theta_{\mathscr{F},p} \quad \text{for all } p \in X.$$

We set $\mathcal{N}_{\mathscr{F}} = \Theta_X/\Theta_{\mathscr{F}}$ and define $S(\mathscr{F})$ by

$$S(\mathscr{F}) = \mathsf{Sing}(\mathcal{N}_{\mathscr{F}}).$$

Note that $S(\mathscr{F})$ is an analytic subset and, from [22], $S(\mathscr{F})$ is describe as follows: let $U$ be a sufficiently small coordinate neighborhood with coordinates $(z_1, \ldots, z_n)$ and let $v_1, \ldots, v_r$ be generators of $\mathscr{F}$ on $U$. We write $v_i = \displaystyle\sum_{j=1}^{n} f_{ij}(z)\frac{\partial}{\partial z_j}$. Then

$$S(\mathscr{F}) \cap U = \{z \in U : rank\,(f_{ij}(z)) < r\}.$$

Furthermore, the foliation $\mathscr{F}$ induce an exact sequence

$$0 \longrightarrow \Theta_{\mathscr{F}} \longrightarrow \Theta_X \longrightarrow \mathcal{N}_{\mathscr{F}} \longrightarrow 0.$$

In this paper, we study foliations of *codimension one* in $X$, i.e., foliations of dimension $\dim(X) - 1$. As is common, codimension one foliations can be described dually utilizing differential 1-forms: a *codimension one singular holomorphic foliation* $\mathscr{F}$ on $X$ is determined by a saturated locally free subsheaf

$$N^*_{\mathscr{F}} \subset \Omega^1_X$$

of rank one, which satisfies the *Frobenius integrability condition.* Locally, $N^*_{\mathscr{F}}$ is generated by holomorphic 1-forms $\omega_k \in \Omega^1_X(U_k)$, where $\{U_k\}_{k\in I}$ is an open covering of $X$, such that

$$\omega_k \wedge d\omega_k = 0$$

and

$$\omega_k = g_{k\ell}\omega_\ell \qquad \text{on } U_k \cap U_\ell.$$

The functions $g_{k\ell}$ are nowhere vanishing, and the multiplicative cocycle $\{g_{k\ell}\}$ defines a line bundle $N_{\mathscr{F}}$, called the *normal bundle* of $\mathscr{F}$. The *singular set* $\mathsf{Sing}(\mathscr{F})$ of $\mathscr{F}$ is the analytic subset of $X$ defined by

$$\mathsf{Sing}(\mathscr{F}) \cap U_k = \text{zeros of } \omega_k, \qquad \forall\, k \in I.$$

The saturated condition means that the zero set of every $\omega_k$ has codimension at least two. Therefore, by definition, $\mathsf{Sing}(\mathscr{F})$ has codimension at least two. We will denote $\mathsf{Sing}_2(\mathscr{F})$ the union of all irreducible components of $\mathsf{Sing}(\mathscr{F})$ of codimension two.

Throughout this paper, we will always work with $N_{\mathscr{F}}$ and $\mathsf{Sing}(\mathscr{F})$. The relation between $S(\mathscr{F})$ and $\mathsf{Sing}(\mathscr{F})$ can be found in [30] and [29] as well. Moreover, we will assume by hypotheses that $S(\mathscr{F}) = \mathsf{Sing}(\mathscr{F})$ and $\mathsf{Sing}_2(\mathscr{F}) \neq \emptyset$.

We remark that in general the characteristic classes of $\mathcal{N}_{\mathscr{F}}$ (as in Theorem 1.1) and $N_{\mathscr{F}}$ are not the same. However, $c_1(\mathcal{N}_{\mathscr{F}})$ and $c_1(N_{\mathscr{F}})$ are equal in the $K$-group $K(X)$. In fact, we have the exact sequence

$$0 \longrightarrow N^*_{\mathscr{F}} \longrightarrow \Omega^1_X \longrightarrow \Omega^1_X/N^*_{\mathscr{F}} \longrightarrow 0.$$

Taking the duals in the last sequence, we obtain an exact sequence

$$0 \longrightarrow (\Omega^1_X/N^*_{\mathscr{F}})^* \longrightarrow \Theta_X \longrightarrow N_{\mathscr{F}} \longrightarrow \mathcal{E}xt^1(\Omega^1_X/N^*_{\mathscr{F}}, \mathcal{O}_X) \longrightarrow 0.$$

We have $(\Omega^1_X/N^*_{\mathscr{F}})^* = \Theta_{\mathscr{F}}$ and an exact sequence

$$0 \longrightarrow \mathcal{N}_{\mathscr{F}} \longrightarrow N_{\mathscr{F}} \longrightarrow \mathcal{E}xt^1(\Omega^1_X/N^*_{\mathscr{F}}, \mathcal{O}_X) \longrightarrow 0.$$

The characteristic classes of a coherent sheaf are defined by taking a resolution of the sheaf by vector bundles and regarding it as an element in the $K$-group $K(X)$. Since $c_1$ is additive on $K(X)$, we have

$$c_1(\mathcal{N}_{\mathscr{F}}) = c_1(\Theta_X) - c_1(\Theta_{\mathscr{F}}) = -c_1(\Omega^1_X) + c_1(\Omega^1_X) - c_1(N^*_{\mathscr{F}}) = c_1(N_{\mathscr{F}}).$$

**3. Lehmann-Suwa formula.** Let $\mathscr{F}$ be a codimension one singular holomorphic foliation on a compact complex manifold $X$ of dimension at least three and let $V \subset X$ be a complex hypervariety invariant by $\mathscr{F}$. Here, *complex hypervariety* means codimension one complex subvariety and *invariant* means that if a point of $V$ belongs to the regular part of $\mathscr{F}$, then the whole leaf through this point is included in $V$. We shall assume furthermore that $V$ is *reduced*, that is, the divisor $V$ does not contain multiple irreducible components.

Let us denote by $\mathsf{Sing}(V)$ the singular set of $V$ and set

$$\mathsf{Sing}_2(\mathscr{F}, V) = \mathsf{Sing}(V) \cup (\mathsf{Sing}_2(\mathscr{F}) \cap V).$$

Let $Z$ be an irreducible component of $\mathsf{Sing}_2(\mathscr{F}, V)$ such that $Z$ has pure codimension two. Take a generic point $p \in Z$, that is, a point where $Z$ is smooth and disjoint from the other singular components. Take $B_p \subset X$ a small ball centered at $p$ such that $Z \cap B_p$ is the unique irreducible component of $\mathsf{Sing}_2(\mathscr{F}, V) \cap B_p$ and suppose that $\omega \in \Omega^1_X(B_p)$ represents $\mathscr{F}$ in $B_p$. Working with smooth sections of $N^*_{\mathscr{F}}$, instead of holomorphic ones, the corresponding cohomology group is trivial, and so we can certainly find a smooth (1,0)-form $\beta \in A^{1,0}(B^*_p)$ such that

$$d\omega = \beta \wedge \omega, \tag{1}$$

where $B^*_p = B_p \setminus (\mathsf{Sing}_2(\mathscr{F}, V) \cap B_p)$. Since $p$ is a generic point of $Z$ and the codimension of $Z$ with respect to $V$ is one, we may take a one-dimensional small transverse section $\sum_p$ to $Z$ at $p$ such that $\sum_p \subset V$. Then we define

$$\mathrm{Var}(\mathscr{F}, V, Z) := \frac{1}{2\pi i}\int_{\Gamma} \beta, \tag{2}$$

where $\Gamma$ is a generator of $H_1(\sum_p \setminus \{p\}, \mathbb{Z})$. We call this complex number the *Variational index of $\mathscr{F}$ concerning $V$ along $Z$*. By a connectedness argument, it does not depend on the choice of the generic point $p \in Z$. It is the natural extension of the variational index of Khanedani-Suwa [20].

The following result is a particular case of Theorem 1.1, the novelty will be to obtain a proof using the Variational index given in (2).

THEOREM 3.1. *Let $\mathscr{F}$ be a codimension one holomorphic foliation on a compact complex manifold $X$ of dimension at least three and let $V \subset X$ be a reduced complex hypervariety invariant by $\mathscr{F}$ such that $\mathsf{Sing}_2(\mathscr{F}, V) \neq \emptyset$. Then*

$$\sum_Z \mathit{Var}(\mathscr{F}, V, Z)[Z] = c_1(N_{\mathscr{F}}|_V) \frown [V],$$

*where the sum is done over all irreducible components $Z$ of $\mathsf{Sing}_2(\mathscr{F}, V)$ and $[Z]$ denotes the integration current associated to $Z$.*

*Proof.* We cover $X$ by open subsets $U_k$ where the foliation $\mathscr{F}$ is defined by integrable holomorphic 1-forms $\omega_k$, with $\omega_k = g_{k\ell}\omega_\ell$, where $g_{k\ell} \in \mathcal{O}^*(U_k \cap U_\ell)$ whenever $U_k \cap U_\ell \neq \emptyset$. Assume that $V \cap U_k = \{f_k = 0\}$, where $f_k \in \mathcal{O}(U_k)$. On $U_k \cap U_\ell$, we have $f_k = \varphi_{k\ell} f_\ell$ with $\varphi_{k\ell} \in \mathcal{O}^*(U_k \cap U_\ell)$ and the cocycle $\{\varphi_{k\ell}\}$ defines the line bundle $[V]$ on $X$.

We may find smooth $(1,0)$-forms $\gamma_k$ on $U_k^* = U_k \setminus (\mathsf{Sing}_2(\mathscr{F}, V) \cap U_k)$ such that $d\omega_k = \gamma_k \wedge \omega_k$. We fix a small neighborhood $U$ of $\mathsf{Sing}_2(\mathscr{F}, V)$ and we regularize each $\gamma_k$ on $U$, that is, we choose a smooth $(1,0)$-form $\tilde{\gamma}_k$ on $U_k$ coinciding with $\gamma_k$ outside of $U_k \cap U$. Then the smooth $(1,0)$-forms

$$\zeta_{k\ell} = \frac{dg_{k\ell}}{g_{k\ell}} - \tilde{\gamma}_k + \tilde{\gamma}_\ell$$

vanish on $\mathscr{F}$ outside $U$. This cocycle can be trivialized

$$\zeta_{k\ell} = \zeta_k - \zeta_\ell,$$

where $\zeta_k$ is a smooth $(1,0)$-form on $U_k$ vanishing on $\mathscr{F}$ outside of $U_k \cap U$. Hence, after setting $\hat{\gamma}_k = \tilde{\gamma}_k + \zeta_k$, we get

$$\frac{dg_{k\ell}}{g_{k\ell}} = \hat{\gamma}_k - \hat{\gamma}_\ell. \tag{3}$$

Note that we still have $d\omega_k = \hat{\gamma}_k \wedge \omega_k$ outside of $U_k \cap U$. The globally defined closed 2-form (of mixed type $(2,0) + (1,1)$)

$$\Omega = \frac{1}{2\pi i} d\hat{\gamma}_k$$

represents, in the De Rham cohomology, the first Chern class of $N_{\mathscr{F}}$. Moreover, outside $U$, $\Omega$ vanishes when restricted to leaves of $\mathscr{F}$ $(\Omega \wedge \omega_k = 0)$, and in particular, when restricted to $V$, except on small neighborhoods of $\mathsf{Sing}_2(\mathscr{F}, V)$ in $V$. This means that

$$\mathrm{Supp}(\Omega|_V) \subset \overline{U}.$$

Let $\psi$ be a closed smooth $(2n-4)$-form on $V$ and let $\langle\, , \rangle$ be a hermitian metric on $[V]$. Let $\sigma$ be the global section of $[V]$ defined by $\sigma|_{V_k} = f_k$, where $V_k := V \cap U_k = \{f_k = 0\}$. Set $S = \mathsf{Sing}_2(\mathscr{F}, V)$. We consider the tubular neighborhood of $S$ in $V$ for some small number $\epsilon > 0$ as follows

$$T_S(\epsilon) = \{p \in V : ||\sigma(p)||_p \leq \epsilon\}.$$

Note that $\sigma(p) = 0$ if, and only if, $p \in S$, moreover

$$\partial T_S(\epsilon) = \{p \in V : ||\sigma(p)||_p = \epsilon\}.$$

Assume that $S = \bigcup_{j=1}^{m} Z_j$. For each $Z_j$ choose a small neighborhood $V_j$ in $U$ such that $Z_j \subset V_j$, $\overline{V_j} \subset U$ and $\mathrm{Supp}(\Omega|_V) = \bigcup_{j=1}^{m} V_j$. Denoting $T_{Z_j}(\epsilon) = T_S(\epsilon) \cap \overline{V_j}$, we have

$Z_j \subset T_{Z_j}(\epsilon)$. Therefore,

$$\begin{aligned}\int_V \Omega \wedge \psi &= \sum_{j=1}^{m} \int_{V_j} \Omega \wedge \psi \\ &= \frac{1}{2\pi i} \sum_{j=1}^{m} \int_{V_j} d\hat{\gamma}_j \wedge \psi \\ &= \frac{1}{2\pi i} \sum_{j=1}^{m} \left[ \int_{T_{Z_j}(\epsilon)} d\hat{\gamma}_j \wedge \psi + \int_{V_j - T_{Z_j}(\epsilon)} d\hat{\gamma}_j \wedge \psi \right].\end{aligned}$$

Since $\lim_{\epsilon \to 0} \int_{V_j - T_{Z_j}(\epsilon)} d\hat{\gamma}_j \wedge \psi = 0$, we get

$$\begin{aligned}\int_V \Omega \wedge \psi &= \frac{1}{2\pi i} \sum_{j=1}^{m} \lim_{\epsilon \to 0} \int_{T_{Z_j}(\epsilon)} d\hat{\gamma}_j \wedge \psi \\ &= \frac{1}{2\pi i} \sum_{j=1}^{m} \lim_{\epsilon \to 0} \int_{T_{Z_j}(\epsilon)} d(\hat{\gamma}_j \wedge \psi) \\ &= \frac{1}{2\pi i} \sum_{j=1}^{m} \lim_{\epsilon \to 0} \int_{\partial T_{Z_j}(\epsilon)} \hat{\gamma}_j \wedge \psi. \qquad (4)\end{aligned}$$

Now, take a smooth point $p \in Z_j - \bigcup_{\ell \neq j} Z_\ell$, then there exists a neighborhood $W_p \subset V_j$ of $p$ and a coordinate system $(z_1, z_2, \ldots, z_{n-1})$ centered at $p$ such that $Z_j \cap W_p = \{z_1 = 0\}$ and $\partial T_{Z_j}(\epsilon) \cap W_p = \{|z_1| = \epsilon, z' \in \Delta_\epsilon\}$, where $z' = (z_2, \ldots, z_{n-1})$ and $\Delta_\epsilon = \{z' \in \mathbb{C}^{n-2} : |z'| \leq \epsilon\}$. Note that

$$\begin{aligned}\lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{\partial T_{Z_j}(\epsilon) \cap W_p} \hat{\gamma}_j \wedge \psi &= \lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{\Delta_\epsilon} \int_{|z_1| = \epsilon} \hat{\gamma}_j \wedge \psi \\ &= \lim_{\epsilon \to 0} \int_{\Delta_\epsilon} \left[ \frac{1}{2\pi i} \int_{|z_1| = \epsilon} \hat{\gamma}_j \right] \psi \\ &= \lim_{\epsilon \to 0} \int_{\Delta_\epsilon} \operatorname{Var}(\mathscr{F}, V, Z_j) \psi \\ &= \operatorname{Var}(\mathscr{F}, V, Z_j) \lim_{\epsilon \to 0} \int_{\Delta_\epsilon} \psi \\ &= \operatorname{Var}(\mathscr{F}, V, Z_j) \int_{Z_j \cap W_p} \psi.\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{\epsilon \to 0} \frac{1}{2\pi i} \int_{\partial T_{Z_j}(\epsilon)} \hat{\gamma}_j \wedge \psi &= \operatorname{Var}(\mathscr{F}, V, Z_j) \int_{Z_j} \psi \\ &= \operatorname{Var}(\mathscr{F}, V, Z_j)[Z_j](\psi). \qquad (5)\end{aligned}$$

Hence, from (4) and (5) we get

$$\int_V \Omega \wedge \psi = \sum_{j=1}^{m} \operatorname{Var}(\mathscr{F}, V, Z_j)[Z_j](\psi),$$

for any closed smooth $(2n-4)$-form $\psi$ on $V$. Using Poincaré duality and the fact that $\Omega|_V$ represents, in the De Rham cohomology, the Chern class of $N_{\mathscr{F}}|_V$, we obtain

$$\sum_{j=1}^{m} \mathrm{Var}(\mathscr{F}, V, Z_j)[Z_j] = c_1(N_{\mathscr{F}}|_V) \frown [V].$$

□

Remark 3.1. The proof above gives more, namely, the Lehmann-Suwa residues are determined as follows:

$$\mathrm{Res}_{c_1}(\mathscr{F}, N_{\mathscr{F}}|_V; Z) = \mathrm{Var}(\mathscr{F}, V, Z)[Z].$$

## 4. GSV and Camacho-Sad indices for codimension one holomorphic foliations.

**4.1. Saito's decomposition.** The following lemma can be found in Saito [26, Section 1]. When $\mathscr{F}$ is a germ of holomorphic foliation at $0 \in \mathbb{C}^2$, we refer to reader to [24], [29].

Lemma 4.1 (Saito [26]). *Let $\mathscr{F}$ be a germ of codimension one singular holomorphic foliation at $0 \in \mathbb{C}^n$, $n \geq 2$, defined by a germ of an integrable holomorphic 1-form $\omega$. Suppose $V = \{f = 0\}$ is a germ at $0 \in \mathbb{C}^n$ of reduced complex hypervariety invariant by $\mathscr{F}$. Then there exist germs of holomorphic functions $g$, $h$ and a germ of holomorphic 1-form $\eta$ at $0 \in \mathbb{C}^n$ such that*

$$g\omega = hdf + f\eta, \tag{6}$$

*where $h$ and $f$ have no common factors. Moreover, $g$ and $f$ also have no common factors.*

Using the Saito's decomposition, we can now state a similar result to Brunella [6, Proposition 5].

Proposition 4.2. *Let $\mathscr{F}$, $V$ and $Z$ be as in Lemma 4.1. Then*

$$Var(\mathscr{F}, V, Z) = \frac{1}{2\pi i} \int_{\Gamma} \left( \frac{g}{h} d\left(\frac{h}{g}\right) - \frac{\eta}{h} \right),$$

*where $\Gamma$ is as in equation (2).*

*Proof.* By Lemma 4.1 we have

$$\omega = \frac{h}{g} df + f \frac{\eta}{g}.$$

Therefore,

$$d\omega = d\left(\frac{h}{g}\right) \wedge df + df \wedge \frac{\eta}{g} + f d\left(\frac{\eta}{g}\right).$$

Restringing to $V$, we get

$$d\omega = d\left(\frac{h}{g}\right) \wedge df + df \wedge \frac{\eta}{g} \tag{7}$$

and

$$\omega = \frac{h}{g} df. \tag{8}$$

From (7) and (8) it follows that

$$\begin{aligned} d\omega &= \left( d\left(\frac{h}{g}\right) - \frac{\eta}{g} \right) \wedge df \\ &= \left( \frac{g}{h} d\left(\frac{h}{g}\right) - \frac{\eta}{h} \right) \wedge \omega. \end{aligned}$$

Hence

$$\mathrm{Var}(\mathscr{F}, V, Z) = \frac{1}{2\pi i} \int_\Gamma \left( \frac{g}{h} d\left(\frac{h}{g}\right) - \frac{\eta}{h} \right),$$

where $\Gamma$ is a curve as in (2). □

**4.2. GSV-index for codimension one holomorphic foliations.** A. G. Aleksandrov in [2] introduced the concept of *multiple residues* of logarithmic differentials forms and generalize the Saito's decomposition theorem [26]. Using Aleksandrov's decomposition theorem, Corrêa and Machado defined in [12] the GSV-index for holomorphic Pfaff systems. In this subsection, we particularizing this definition for codimension one holomorphic foliations.

Let $\mathscr{F}$ be a germ of codimension one singular holomorphic foliation at $0 \in \mathbb{C}^n$, $n \geq 3$, defined by a germ of an integrable holomorphic 1-form $\omega$. Suppose $V = \{f = 0\}$ is a germ at $0 \in \mathbb{C}^n$ of reduced complex hypervariety invariant by $\mathscr{F}$. Then by Lemma 4.1 we have

$$g\omega = hdf + f\eta.$$

For each irreducible component $Z$ of $\mathsf{Sing}_2(\mathscr{F}, V)$, Corrêa and Machado [12] defined the *GSV-index* as follows:

$$\mathrm{GSV}(\mathscr{F}, V, Z) := \mathsf{ord}_Z \left( \frac{h}{g}\Big|_V \right). \tag{9}$$

According to Corrêa-Machado [12, Theorem 3.2] we can formulate:

THEOREM 4.3 (Corrêa-Machado [12]). *Let $\mathscr{F}$ be a codimension one holomorphic foliation on a compact complex manifold $X$ of dimension at least three and let $V \subset X$ be a reduced complex hypervariety invariant by $\mathscr{F}$ such that $\mathsf{Sing}_2(\mathscr{F}, V) \neq \emptyset$. Denote by $N_{V/X}$ the normal bundle of $V$ in $X$. Then*

$$\sum_Z GSV(\mathscr{F}, V, Z)[Z] = c_1(N_{\mathscr{F}}|_V \otimes (N_{V/X})^{-1}) \frown [V], \tag{10}$$

*where the sum is done over all irreducible components $Z$ of $\mathsf{Sing}_2(\mathscr{F}, V)$ and $[Z]$ denotes the integration current associated to $Z$.*

**4.3. Camacho-Sad index for codimension one holomorphic foliations.** Define the *Camacho-Sad index* as follows:

$$\mathrm{CS}(\mathscr{F}, V, Z) := \mathrm{Var}(\mathscr{F}, V, Z) - \mathrm{GSV}(\mathscr{F}, V, Z). \tag{11}$$

When $V$ is smooth, $CS(\mathscr{F}, V, Z)$ coincide with the index defined by Lins Neto [23], see also [16]. Note that Theorem 3.1 and Theorem 4.3 implies the following result of global nature.

THEOREM 4.4. *Let $\mathscr{F}$ be a codimension one holomorphic foliation on a compact complex manifold $X$ of dimension at least three and let $V \subset X$ be a reduced complex hypervariety invariant by $\mathscr{F}$ such that $\mathsf{Sing}_2(\mathscr{F}, V) \neq \emptyset$. Then*

$$\sum_Z CS(\mathscr{F}, V, Z)[Z] = c_1(N_{V/X}) \frown [V]$$

*where the sum is done over all irreducible components $Z$ of $\mathsf{Sing}_2(\mathscr{F}, V)$ and $[Z]$ denotes the integration current associated to $Z$.*

**5. Baum-Bott index.** In this section, we define the Baum-Bott index, following [7]. Similarly to above section, we work with smooth sections of $N^*_{\mathscr{F}}$, instead of holomorphic ones, then there exists a smooth (1,0)-form $\beta \in A^{1,0}(B^*_p)$ such that $d\omega = \beta \wedge \omega$, where $\omega$ is a local generator of $N^*_{\mathscr{F}}$. The smooth 3-form (of mixed type $(3,0) + (2,1)$)

$$\frac{1}{(2\pi i)^2} \beta \wedge d\beta \tag{12}$$

is closed, and it has a De Rham cohomology class in $H^3(B^*_p, \mathbb{C})$, which does not depend on the choice of $\omega$ and $\beta$.

Let $Z$ be an irreducible component of $\mathsf{Sing}_2(\mathscr{F})$. Take a generic point $p \in Z$ and pick $B_p$ sufficiently small ball, so that $S(B_p) := \mathsf{Sing}_2(\mathscr{F}) \cap B_p$ is a codimension two subball of $B_p$. Then the above De Rham class can be integrated over an oriented 3-sphere $L_p \subset B^*_p$ positively linked with $S(B_p)$:

$$\mathrm{BB}(\mathscr{F}, Z) = \frac{1}{(2\pi i)^2} \int_{L_p} \beta \wedge d\beta.$$

This complex number is called *Baum-Bott residue of $\mathscr{F}$ along $Z$*. Again by a connectedness argument, it does not depend on the choice of the generic point $p \in Z$.

Let us recall that every irreducible component $Z$ of $\mathsf{Sing}_2(\mathscr{F})$ has a class $[Z] \in H^4(X, \mathbb{C})$ (conveniently defined via the integration current over $Z$). Therefore, we have the following result.

THEOREM 5.1 (Baum-Bott [3], Brunella-Perrone [7]).

$$\sum_Z BB(\mathscr{F}, Z)[Z] = c_1^2(N_{\mathscr{F}})$$

*where the sum is done over all irreducible components of $\mathsf{Sing}_2(\mathscr{F})$.*

**6. Almost Liouvillian foliations.** In this section, we consider germs at $0 \in \mathbb{C}^n$, $n \geq 3$, of singular holomorphic foliations of codimension one. Let $\mathscr{F}$ be a germ at $0 \in \mathbb{C}^n$ of a codimension one holomorphic foliation such that $0 \in \mathsf{Sing}(\mathscr{F})$ and suppose that $\mathscr{F}$ is defined by a germ of an integrable holomorphic 1-form $\omega \in \Omega^1(\mathbb{C}^n, 0)$. Let $\mathsf{Sing}_2(\mathscr{F})$ be the germ at $0 \in \mathbb{C}^n$ defined by the union of the germs at $0 \in \mathbb{C}^n$ of irreducible components of the singular set of $\mathscr{F}$ whose codimension is precisely two. In this section, we assume that $\mathsf{Sing}_2(\mathscr{F})$ is not empty.

Definition 6.1. We say that the germ $\mathscr{F}$ is an *almost Liouvillian foliation* at $0 \in \mathbb{C}^n$ if there exists a germ of closed meromorphic 1-form $\gamma_0$ and a germ of holomorphic 1-form $\gamma_1$ at $0 \in \mathbb{C}^n$ such that

$$d\omega = (\gamma_0 + \gamma_1) \wedge \omega. \tag{13}$$

We say that $\mathscr{F}$ is a *simple almost Liouvillian foliation* at $0 \in \mathbb{C}^n$ if we can choose $\gamma_0$ having only first-order poles.

The next lemma was proved by Brunella [6] in the two-dimensional case. We extend this fact for high dimension.

Lemma 6.1. *If $\mathscr{F}$ is almost Liouvillian foliation at $0 \in \mathbb{C}^n$ defined by $\omega \in \Omega^1(\mathbb{C}^n, 0)$, $n \geq 3$. Then the poles divisor of $\gamma = \gamma_0 + \gamma_1$ is invariant by $\mathscr{F}$.*

*Proof.* Let $V = (\gamma)_\infty$ be the poles divisor of $\gamma$. If $p$ is a smooth point of $V$ such that $p \notin \mathsf{Sing}(\mathscr{F})$, then there exists a coordinate system $(z_1, \ldots, z_n)$ at $p$ such that $z_1(p) = \ldots = z_n(p) = 0$ and $\omega = a(z_1, \ldots, z_n)dz_n$, where $a \in \mathcal{O}^*(\mathbb{C}^n, 0)$. Let $\gamma = b_1 dz_1 + \ldots + b_n dz_n$, where $b_1, \ldots, b_n$ are germs of meromorphic functions at $0 \in \mathbb{C}^n$. It follows from (13) that $b_1, \ldots, b_{n-1} \in \mathcal{O}(\mathbb{C}^n, 0)$ and therefore $\gamma_0 = b_n dz_n$. Since $\gamma_0$ is closed, we get $b_n = b_n(z_n)$ and $\gamma_0$ may be written as

$$\gamma_0 = \frac{h(z_n)}{z_n^k} dz_n,$$

where $h(z_n)$ is a holomorphic function and $k \geq 1$. Consequently, the germ of $V$ at $p$ is given by $\{z_n = 0\}$, which implies that $V$ is invariant by $\omega$. □

Remark 6.1. Let $\mathscr{F}$ be a germ at $0 \in \mathbb{C}^n$, $n \geq 3$, of a codimension one holomorphic foliation such that $Z$ is a germ at $0 \in \mathbb{C}^n$ of an irreducible component of $\mathsf{Sing}_2(\mathscr{F})$. Suppose that there exists a germ at $0 \in \mathbb{C}^n$ of a complex hypervariety $V$ invariant by $\mathscr{F}$ such that $V$ does not contain $Z$. Then it is not difficult to see that the definition of index variational $\mathrm{Var}(\mathscr{F}, V, Z)$ (see for instance (2)) may be extended to an irreducible component $Z$ of $\mathsf{Sing}_2(\mathscr{F})$ that is not contained in $V$. In this case, we have $\mathrm{Var}(\mathscr{F}, V, Z) = 0$.

The next theorem extends a result due to Brunella [6, Proposition 8]. This result provides an effective way of computing Baum-Bott residues of codimension one holomorphic foliations in high dimension. We remark that germs at $0 \in \mathbb{C}^3$ of codimension one holomorphic foliations with reduced singularities (see for instance Cano [9]), logarithmic foliations and some transversally affine foliations are examples of simple almost Liouvillian foliations.

Theorem 6.2. *Let $\mathscr{F}$ be a germ at $0 \in \mathbb{C}^n$, $n \geq 3$, of a simple almost Liouvillian foliation defined by $\omega \in \Omega^1(\mathbb{C}^n, 0)$ such that*

$$d\omega = (\gamma_0 + \gamma_1) \wedge \omega.$$

*Let $V$ be the divisor of poles of $\gamma = \gamma_0 + \gamma_1$ and $V_1, \ldots, V_\ell$ the irreducible components of $V$. Let $Z$ be an irreducible component of $\mathsf{Sing}_2(\mathscr{F})$. Then*

$$BB(\mathscr{F}, Z) = \sum_{j=1}^{k} Res(\gamma_0, V_j)\, Var(\mathscr{F}, V_j, Z),$$

*where $V_1, \ldots, V_k$ are the irreducible components of $V$ that contains $Z$.*

*Proof.* Take a generic point $p \in Z$ and pick $B_p$ a sufficiently small ball such that $S(B_p)$ is a codimension two subball of $B_p$ at $Z$ (as in Section 5). Let $S^3_\epsilon \subset B^*_p$ be an oriented 3-sphere positively linked with $S(B_p)$. Let $\partial V_j := S^3_\epsilon \cap V_j$ and let $W_j$ be a tubular neighborhood of $\partial V_j$ such that $W_j \cap Z = \emptyset$. Then there exists holomorphic 1-form $\gamma'_j$ in a neighborhood of $W_j$ such that

$$d\omega = \gamma'_j \wedge \omega. \tag{14}$$

Note that $W := \bigcup_{j=1}^{N} W_j$ is a tubular neighborhood of $\partial V = S^3_\epsilon \cap V$ and there is an partition of unity $\rho = \{\rho_j\}$ for $W$ subordinate to open cover $\{W_j\}$. With this we can define

$$\gamma' = \rho_1 \gamma'_1 + \ldots + \rho_N \gamma'_N.$$

It is easily seen that $\gamma'$ is a holomorphic 1-form in a neighborhood of $W$.

Let $\phi \in C^\infty_c(W)$ be equal to 1 on a smaller neighborhood of $\partial V$. Then $\beta = \phi\gamma' + (1-\phi)\gamma'$ is a smooth $(1,0)$-form on a neighborhood of $S^3_\epsilon$. Note that

$$d\omega = \beta \wedge \omega.$$

Denoting $\beta_j = \beta|_{S^3_\epsilon \cap W_j}$ and $\phi_j = \phi|_{S^3_\epsilon \cap W_j}$ for each $j = 1, \ldots, N$, we get

$$\beta_j = \phi_j \gamma'_j + (1-\phi_j)\gamma'_j, \qquad \text{on} \quad S^3_\epsilon \cap W_j.$$

Note also that $S^3_\epsilon \cap W = \bigcup_{j=1}^{N} (S^3_\epsilon \cap W_j)$. To continue, we may choose holomorphic coordinates $(z_1, \ldots, z_n)$ near each $\partial V_j = S^3_\epsilon \cap V_j$, with $z_1$ varying on a neighborhood of the unitary circle and $(z_2, \ldots, z_n)$ on a neighborhood of the origin of $\mathbb{C}^{n-1}$ such that

$$\begin{aligned} V_j &= \{z_n = 0\}, \\ \partial V_j &= \{|z_1| = 1, z_2 = z_3 = \ldots = z_n = 0\}, \\ S^3_\epsilon \cap W_j &= \{|z_1| = 1, |z_n| \leq \epsilon, z_2 = z_3 = \ldots = z_{n-1} = 0\}, \\ \partial(S^3_\epsilon \cap W_j) &= \{|z_1| = 1, |z_n| = \epsilon, z_2 = z_3 = \ldots = z_{n-1} = 0\}. \end{aligned}$$

We claim $Supp(\beta \wedge d\beta) \subset S^3_\epsilon \cap W$. In fact, by construction we have

$$\beta \wedge d\beta = \beta_j \wedge d\beta_j \quad \text{in} \quad S^3_\epsilon \cap W_j. \tag{15}$$

On the other hand, since

$$\beta_j \wedge d\beta_j = \phi^2 \gamma'_j \wedge d\gamma'_j - \gamma'_j \wedge d\phi \wedge \gamma_j + \phi(1-\phi)\gamma'_j \wedge d\gamma_j + (1-\phi)\phi\gamma_j \wedge d\gamma'_j + (1-\phi)^2 \gamma_j \wedge d\gamma_j,$$

and $\gamma_j' \wedge d\gamma_j' = \gamma_j \wedge d\gamma_j' = \gamma_j' \wedge d\gamma_j = \gamma_j \wedge d\gamma_j = 0$ in $S^3_\epsilon \cap W_j$, we get

$$\beta \wedge d\beta = \beta_j \wedge d\beta_j = d\phi \wedge \gamma_j' \wedge \gamma_j.$$

Therefore, $Supp(\beta \wedge d\beta) \subset S^3_\epsilon \cap W$ and the assertion is proved.

Now

$$\int_{S^3_\epsilon} \beta \wedge d\beta = \int_{S^3_\epsilon \cap W} \beta \wedge d\beta = \sum_{j=1}^{N} \int_{S^3_\epsilon \cap W_j} \beta_j \wedge d\beta_j. \tag{16}$$

Since $\beta_j \wedge d\beta_j = d\phi \wedge \gamma_j' \wedge \gamma_j$ in $S^3_\epsilon \cap W_j$, we obtain $d((1-\phi)\gamma \wedge \gamma_j') = \beta_j \wedge d\beta_j$. Then

$$\begin{aligned}
\int_{S^3_\epsilon \cap W_j} \beta_j \wedge d\beta_j &= \int_{S^3_\epsilon \cap W_j} d((1-\phi)\gamma \wedge \gamma_j') \\
&= \int_{\partial(S^3_\epsilon \cap W_j)} (1-\phi)\gamma \wedge \gamma_j' \\
&= \int_{\partial(S^3_\epsilon \cap W_j)} \gamma \wedge \gamma_j' \\
&= \int_{\partial(S^3_\epsilon \cap W_j)} (\gamma_0 + \gamma_1) \wedge \gamma_j' \\
&= \int_{\partial(S^3_\epsilon \cap W_j)} \gamma_0 \wedge \gamma_j'.
\end{aligned} \tag{17}$$

In the coordinate system $(z_1, \ldots, z_n)$, we have

$$\gamma = b_1 dz_1 + \ldots + b_{n-1} dz_{n-1} + b_n dz_n$$

where $\gamma_1 = b_1 dz_1 + \ldots + b_{n-1} dz_{n-1}$ and $\gamma_0 = b_n dz_n$. Furthermore, since $\mathscr{F}$ is a simple almost Liouvillian foliation, we have

$$\gamma_0 = \lambda_j \frac{dz_n}{z_n} + \gamma_{0j},$$

where $\lambda_j = \mathrm{Res}(\gamma_0, V_j)$ and $\gamma_{0j}$ is a suitable holomorphic 1-form. On the other hand, $\gamma_j' = a_1 dz_1 + \ldots + a_n dz_n$, with $a_j \in \mathcal{O}(W_j)$ for all $i = 1, \ldots, n$, and in particular $\gamma_j'|_{\partial V_j} = a_1(z_1, 0, \ldots, 0) dz_1$, where $\partial V_j = \{|z_1| = 1, z_2 = \ldots = z_n = 0\}$. Then

$$\begin{aligned}
\int_{\partial(S^3_\epsilon \cap W_j)} \gamma_0 \wedge \gamma_j' &= \int_{|z_1|=1, |z_n|=\epsilon} \left( \lambda_j \frac{dz_n}{z_n} + \gamma_{0j} \right) \wedge (a_1 dz_1 + a_n dz_n) \\
&= \int_{|z_1|=1, |z_n|=\epsilon} \left( \lambda_j a_1 \frac{dz_n}{z_n} \wedge dz_1 \right) \\
&= (2\pi i)\lambda_j \int_{|z_1|=1} \left( \frac{1}{2\pi i} \int_{|z_n|=\epsilon} \frac{a_1(z_1, 0, \ldots, z_n)}{z_n} dz_n \right) dz_1 \\
&= (2\pi i)\lambda_j \int_{|z_1|=1} a_1(z_1, 0, \ldots, 0) dz_1 \\
&= (2\pi i)\lambda_j \int_{\partial V_j} \gamma_j'|_{\partial V_j}.
\end{aligned} \tag{18}$$

If $V_j$ contains $Z$ then it follows from (14) that

$$\frac{1}{2\pi i}\int_{\partial V_j}\gamma_j'|_{\partial V_j} = \mathrm{Var}(\mathscr{F}, V_j, Z).$$

Thus from (18) we get

$$\frac{1}{(2\pi i)^2}\int_{\partial(S_\epsilon^3\cap W_j)}\gamma_0\wedge\gamma_j' = \lambda_j\mathrm{Var}(\mathscr{F}, V_j, Z).$$

If $V_j$ does not contain $Z$ then $\mathrm{Var}(\mathscr{F}, V_j, Z) = 0$ by Remark 6.1. Finally, from (16) and (17), we obtain

$$\mathrm{BB}(\mathscr{F}, Z) = \sum_{j=1}^{k}\mathrm{Res}(\gamma_0, V_j)\mathrm{Var}(\mathscr{F}, V_j, Z),$$

where $V_1, \ldots, V_k$ are the irreducible components of $V$ that contains $Z$. □

To end this section we give an example where Theorem 6.2 applies.

EXAMPLE 6.1. Let $\mathscr{F}$ be the germ at $0 \in \mathbb{C}^3$ of a holomorphic foliation defined by

$$\omega = 2yzdx + 3xzdy + 4xydz.$$

We have

$$d\omega = \gamma_0\wedge\omega, \quad \text{where} \quad \gamma_0 = -\frac{dx}{x} - 2\frac{dy}{y} - 3\frac{dz}{z}.$$

In particular, $\mathscr{F}$ is a codimension one Liouvillian foliation at $0 \in \mathbb{C}^3$. Let $V = (\gamma_0)_\infty = \bigcup_{j=1}^{3} V_j$, where $V_1 = \{x = 0\}$, $V_2 = \{y = 0\}$ and $V_3 = \{z = 0\}$. Note that

$$\mathrm{Res}(\gamma_0, V_1) = -1, \qquad \mathrm{Res}(\gamma_0, V_2) = -2, \qquad \mathrm{Res}(\gamma_0, V_3) = -3.$$

Let $Z = \{y = z = 0\}$, it is evident that $Z \subset \mathsf{Sing}_2(\mathscr{F})$. Furthermore $Z \subset V_2$ and $Z \subset V_3$. To compute $\mathrm{Var}(\mathscr{F}, V_3, Z)$, we pick $p = (1, 0, 0) \in Z$ and the transverse section $\sum_p = \{|y| \le 1, x = 1, z = 0\}$ to $Z$ in $V_3$. By Proposition 4.2, we get

$$\mathrm{Var}(\mathscr{F}, V_3, Z) = \frac{1}{2\pi i}\int_{\Gamma_3}\left(\frac{d(4xy)}{4xy} - \frac{2ydx + 3xdy}{4xy}\right)|_{x=1} = \frac{1}{4},$$

where $\Gamma_3 \in H_1(\sum_p\backslash\{p\}, \mathbb{Z})$.

On the other hand, to compute $\mathrm{Var}(\mathscr{F}, V_2, Z)$, take again $p = (1, 0, 0) \in Z$ and the transverse section $\sum_p = \{|z| \le 1, x = 1, y = 0\}$ to $Z$ in $V_2$. Again by Proposition 4.2, we get

$$\mathrm{Var}(\mathscr{F}, V_2, Z) = \frac{1}{2\pi i}\int_{\Gamma_2}\left(\frac{d(3xz)}{3xz} - \frac{2zdx + 4xdz}{3xz}\right)|_{x=1} = -\frac{1}{3},$$

where $\Gamma_2 \in H_1(\sum_p\backslash\{p\}, \mathbb{Z})$. Hence, applying Theorem 6.2, we conclude

$$\mathrm{BB}(\mathscr{F}, Z) = (-2)\left(-\frac{1}{3}\right) + (-3)\left(\frac{1}{4}\right) = -\frac{1}{12}.$$

Using a recently result of Corrêa-Lourenço [13], we can verify that the above computations are correct. In fact, as in [13, Example 4.1], let us consider $p = (1, 0, 0) \in Z$, $D = \{|(y, z)| \le 1, x = 1\}$ and

$$\omega|_D = 3zdy + 4ydz.$$

The dual vector field of $\omega|_D$ is $X = 4y\frac{\partial}{\partial y} - 3z\frac{\partial}{\partial z}$. A straightforward calculation shows that

$$JX(0,0) = \begin{bmatrix} 4 & 0 \\ 0 & -3 \end{bmatrix}.$$

Thus

$$\mathrm{BB}(\mathscr{F}, Z) = \frac{\mathrm{Tr}(JX(0,0))^2}{\det(JX(0,0))} = -\frac{1}{12}.$$

**7. Singular holomorphic foliations tangent to singular Levi-flat hypersurfaces.** Motived by [4] and [15], we study singular codimension one holomorphic foliations tangent to singular real-analytic Levi-flat hypersurfaces in compact complex manifolds with emphasis on the type of singularities of them.

Let us clarify these terms. A closed set $M$ of a complex manifold $X$ is a *real-analytic subvariety* if it is defined, in some neighborhood of each point of $M$, by the vanishing of finitely many real-analytic functions with real values. We say that a real-analytic subvariety $M$ is *irreducible* if it cannot be written as the union of two real-analytic subvarieties properly contained in it. If $M$ is irreducible, it has a well-defined dimension $\dim_{\mathbb{R}} M$. A *hypervariety* is a subvariety of real codimension one.

If $M \subset X$ is a real-analytic submanifold of real codimension one. For each $p \in M$, there is a unique complex hyperplane $L_p$ contained in the tangent space $T_pM \subset T_pX$. This defines a real-analytic distribution $p \mapsto L_p$ of complex hyperplanes in $TM$. When this distribution is integrable in the sense of Frobenius, we say that $M$ is a *Levi-flat hypersurface*. In this case, $M$ is foliated by immersed complex manifolds of dimension $n-1$. This foliation, denoted by $\mathcal{L}$, is known as *Levi foliation*. A normal form for such an object was given by E. Cartan [10, Theorem IV]: at each $p \in M$, there are holomorphic coordinates $(z_1, \ldots, z_n)$ in a neighborhood $U$ of $p$ such that

$$M \cap U = \{\mathrm{Im}(z_n) = 0\}. \tag{19}$$

As a consequence, the leaves of $\mathcal{L}$ have local equations $z_n = c$, for $c \in \mathbb{R}$.

In the singular case, an irreducible real-analytic hypervariety $M \subset X$ is said to be *Levi-flat* if its *regular part* is a Levi-flat hypersurface. We denote by $M_{reg}$ its *regular part* — the points near which $M$ is a real-analytic manifold of dimension equal to $\dim_{\mathbb{R}} M$. Let $\mathsf{Sing}(M)$ be the singular points of $M$, points near which $M$ is not a real-analytic submanifold (of any dimension). Because we are working with real-analytic sets, the set $\mathsf{Sing}(M)$ is not in general equal to the complement of $M_{reg}$ as defined above, and is only a semi-analytic set (see for instance [21]). If $M \subset X$ is a real-analytic Levi-flat hypervariety, Cartan's local trivialization allows the extension of the Levi foliation to a non-singular holomorphic foliation in a neighborhood of $M_{reg}$ in $X$, which is unique as a germ around $M_{reg}$. In general, it is not possible to extend $\mathcal{L}$ to a singular holomorphic foliation in a neighborhood of $M$. There are examples of

Levi-flat hypervarieties whose Levi foliations extend to singular $k$-webs in the ambient space [14]. However, there is an extension in some "holomorphic lifting" of $M$ (see for instance [5]). If a singular holomorphic foliation $\mathscr{F}$ in the ambient space $X$ coincides with the Levi foliation on $M_{reg}$, we say either that $M$ is *invariant* by $\mathscr{F}$ or that $\mathscr{F}$ is *tangent* to $M$.

Definition 7.1. A singular point $p \in \mathsf{Sing}(M)$ is called *dicritical* if, for every neighborhood $U$ of $p$, infinitely many leaves of the Levi-foliation on $M^* \cap U$ have $p$ in their closure.

Recently dicritical singularities of singular real-analytic Levi-flat hypersurfaces have been characterized in terms of the *Segre varieties*, see for instance Pinchuk-Shafikov-Sukhov [25].

We recall the definition of meromorphic and holomorphic first integral for holomorphic foliations. Let $\mathscr{F}$ be a singular holomorphic foliation on $X$. Recall that $\mathscr{F}$ admits a *meromorphic* (*holomorphic*) first integral at $p \in X$, if there exists a neighborhood $U$ of $p$ and a *meromorphic* (*holomorphic*) function $h$ defined in $U$ such that its indeterminacy (zeros) set is contained in $\mathsf{Sing}(\mathscr{F}) \cap U$ and its level curves contain the leaves of $\mathscr{F}$ in $U$.

To prove the main result of this section, we need the following result.

Theorem 7.1 (Cerveau-Lins Neto [11]). *Let $\mathscr{F}$ be a germ of codimension one holomorphic foliation at $0 \in \mathbb{C}^n$, $n \geq 2$, tangent to a germ of an irreducible real-analytic hypersurface $M$. Then $\mathscr{F}$ has a non-constant meromorphic first integral. In the case of dimension two, we can precise more:*

(1) *If $\mathscr{F}$ is dicritical then it has a non-constant meromorphic first integral.*
(2) *If $\mathscr{F}$ is non-dicritical then it has a non-constant holomorphic first integral.*

We now prove a generalization of [4, Lemma 3.2]. To prove this we use Theorem 6.2 and Theorem 7.1.

Proposition 7.2. *Let $\mathscr{F}$ be a germ of a codimension one holomorphic foliation at $0 \in \mathbb{C}^n$, $n \geq 3$. Suppose that $\mathsf{Sing}_2(\mathscr{F}) \neq \emptyset$ and that $\mathscr{F}$ has a non-constant holomorphic first integral, then for every irreducible component $Z$ of $\mathsf{Sing}_2(\mathscr{F})$, we have*

$$BB(\mathscr{F}, Z) \leq 0.$$

*Proof. Let $g = g_1^{m_1} g_2^{m_2} \dots g_k^{m_k}$ be a germ at $0 \in \mathbb{C}^n$ of a holomorphic first integral for $\mathscr{F}$, where $g_1, \dots, g_k$ are irreducible germs at $0 \in \mathbb{C}^n$ and $m_1, \dots, m_k$ are non-negative integers. Then the germ $\omega = m_1 g_2 \dots g_k dg_1 + \dots + m_k g_1 \dots g_{k-1} dg_k$ at $0 \in \mathbb{C}^n$ defines $\mathscr{F}$. Since $dg = h\omega$ with $h = g_1^{m_1-1} g_2^{m_2-1} \dots g_k^{m_k-1}$ we get*

$$d\omega = -\frac{dh}{h} \wedge \omega, \tag{20}$$

*where $\frac{dh}{h} = (m_1 - 1)\frac{dg_1}{g_1} + \dots + (m_k - 1)\frac{dg_k}{g_k}$. In particular, $\mathscr{F}$ is a simple Liouvillian foliation at $0 \in \mathbb{C}^n$.*

*Let $Z$ be an irreducible component of $\mathsf{Sing}_2(\mathscr{F})$, and let $V_j = \{g_j = 0\}$. Note that $Z \subset V_j$ for all $1 \leq j \leq k$. To compute $BB(\mathscr{F}, Z)$, we need to compute $Var(\mathscr{F}, V_j, Z)$.*

*By Proposition 4.2 we get*

$$\begin{aligned} Var(\mathscr{F}, V_j, Z) &= \frac{1}{2\pi i}\int_{\partial V_j}\left[\sum_{\ell\neq j}^{k}\frac{dg_\ell}{g_\ell} - \sum_{\ell\neq j}^{k}\frac{m_\ell}{m_j}\frac{dg_\ell}{g_\ell}\right] \\ &= \frac{1}{2\pi i}\int_{\partial V_j}\sum_{\ell\neq j}^{k}\left(1-\frac{m_\ell}{m_j}\right)\frac{dg_\ell}{g_\ell} \\ &= \sum_{\ell\neq j}^{k}\left(1-\frac{m_\ell}{m_j}\right)\mathsf{ord}_Z(g_\ell|_{V_j}). \end{aligned}$$

*On the other hand,* $Res\left(-\frac{dh}{h}, V_j\right) = -(m_j-1) = 1-m_j$*. According to Theorem 6.2, we obtain*

$$\begin{aligned} BB(\mathscr{F}, Z) &= \sum_{j=1}^{k} Res\left(-\frac{dh}{h}, V_j\right) Var(\mathscr{F}, V_j, Z) \\ &= \sum_{j=1}^{k}(1-m_j)\left[\sum_{\ell\neq j}^{k}\left(1-\frac{m_\ell}{m_j}\right)\mathsf{ord}_Z(g_\ell|_{V_j})\right]. \end{aligned}$$

*Since* $\mathsf{ord}_Z(g_\ell|_{V_j}) = \mathsf{ord}_Z(g_j|_{V_\ell})$ *for* $\ell\neq j$*, we get*

$$BB(\mathscr{F}, Z) = -\sum_{1\leq \ell<j\leq k}\frac{(m_\ell-m_j)^2}{m_\ell m_j}\mathsf{ord}_Z(g_j|_{V_\ell}) \leq 0. \tag{21}$$

□

Now we state the main result of this section.

THEOREM 7.3. *Let* $\mathscr{F}$ *be a codimension one singular holomorphic foliation on a compact complex manifold* $X$ *of dimension at least three, tangent to an irreducible real-analytic Levi-flat hypervariety* $M\subset X$*. Suppose that:*

(1) $Sing_2(\mathscr{F})$ *is not empty and* $\mathsf{Sing}_2(\mathscr{F})\subset M$*,*
(2) $h^4(X,\mathbb{C}) = 1$ *and denote by* $\zeta$ *the generator of* $H^4(X,\mathbb{C})$*,*
(3) *for every fundamental class* $[W]\in H^4(X,\mathbb{C})$ *of an irreducible complex subvariety* $W\subset X$ *of codimension two, there exists* $\alpha>0$ *such that* $[W]=\alpha\zeta$*,*
(4) *for the Chern class* $c_1^2(N_{\mathscr{F}})\in H^4(X,\mathbb{C})$*, there exists* $\alpha_0>0$ *such that*

$$c_1^2(N_{\mathscr{F}}) = \alpha_0\zeta.$$

*Then there exists an irreducible component* $Z$ *of* $\mathsf{Sing}_2(\mathscr{F})$ *such that it contains some dicritical point* $p\in\mathsf{Sing}(M)$*. Moreover,* $\mathscr{F}$ *has a non-constant meromorphic first integral at* $p$*.*

*Proof.* Suppose by contradiction that $\mathsf{Sing}_2(\mathscr{F})$ consists of irreducible components with only non-dicritical singularities of $M$. Take an irreducible component $Z$ of $\mathsf{Sing}_2(\mathscr{F})$ and a generic point $q\in Z$. By hypothesis (1) we have $Z\subset M$. Let $U$ be a small neighborhood of $q$ in $X$ such that $\mathscr{F}$ is represented by a holomorphic 1-form $\omega$ on $U$ and $Z\cap U$ is the unique singular component of $\omega$. Then, since $\mathscr{F}$ and $M$ are tangent in $U$ we have $\mathscr{F}|_U$ admits a meromorphic first integral $g$ on $U$, by Theorem

7.1. But since $q \in U$ is a non-dicritical singularity, $g$ must be a holomorphic first integral.

Applying Proposition 7.2 to $\mathscr{F}|_U$, we get $\mathrm{BB}(\mathscr{F}, Z) \leq 0$, for any $Z \subset \mathsf{Sing}_2(\mathscr{F})$. Assume that $\mathsf{Sing}_2(\mathscr{F}) = \bigcup_{j=1}^{k} Z_j$. Then Baum-Bott's formula (cf. Theorem 5.1) implies that

$$\begin{aligned} c_1^2(N_{\mathcal{F}}) &= \sum_{j=1}^{k} \mathrm{BB}(\mathscr{F}, Z_j)[Z_j], \quad \text{in} \quad H^4(X, \mathbb{C}) \\ &= \left( \sum_{j=1}^{k} \mathrm{BB}(\mathscr{F}, Z_j)\alpha_j \right) \zeta, \quad \text{for some} \quad \alpha_j > 0 \end{aligned}$$

which is absurd with hypothesis (4), because $\alpha_0 = \sum_{j=1}^{k} \mathrm{BB}(\mathscr{F}, Z_j)\alpha_j \leq 0$. Therefore, there exists an irreducible component $Z$ of $\mathsf{Sing}_2(\mathscr{F})$ such that it contains some dicritical point $p \in M$. Applying again Theorem 7.1, we obtain a non-constant meromorphic first integral for $\mathscr{F}$ in a neighborhood of $p$. □

When $X = \mathbb{P}^n$, the complex projective space, $n \geq 3$, we recall the singular set of any codimension one holomorphic foliation on $\mathbb{P}^n$, $n \geq 3$, has an irreducible component of codimenion two, see for instance [19, Proposition 2.6, page 95].

Let $h$ be the hyperplane class in $\mathbb{P}^n$. Then $H^4(\mathbb{P}^n, \mathbb{C})$ is generated by $h^2$. Thus for every codimension two irreducible component $W$ in $\mathbb{P}^n$ we have $[W] = \deg(W)h^2$. Moreover, for a codimension one foliation $\mathscr{F}$ on $\mathbb{P}^n$, $n \geq 3$, of degree $d \geq 0$, we have

$$c_1^2(N_{\mathscr{F}}) = (d+2)^2 h^2.$$

This implies that the hypotheses (2), (3) and (4) of Theorem 7.3 are satisfied for codimension one foliations of $\mathbb{P}^n$. Hence, we can state the following corollary.

Corollary 7.4. *Let $\mathscr{F}$ be a codimension one singular holomorphic foliation on $\mathbb{P}^n$, $n \geq 3$, tangent to an irreducible real-analytic Levi-flat hypervariety $M \subset \mathbb{P}^n$. Suppose that $\mathsf{Sing}_2(\mathscr{F}) \subset M$. Then there exists an irreducible component $Z$ of $\mathsf{Sing}_2(\mathscr{F})$ such that it contains some dicritical point $p \in \mathsf{Sing}(M)$. Moreover, $\mathscr{F}$ has a non-constant meromorphic first integral at $p$.*

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