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CLUSTER POINTS OF JUMPING COEFFICIENTS AND EQUISINGULARTIES OF PLURISUBHARMONIC FUNCTIONS*

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Abstract. In this article, we will construct a plurisubharmonic function whose jumping coefficients have a cluster point. We also give a class of plurisubharmonic functions which cannot be "strongly equisingular" to any plurisubharmonic function with generalized analytic singularities and present global equisingular approximations of quasi-plurisubharmonic functions with stable analytic pluripolar sets on compact complex manifolds.

Key words. Plurisubharmonic function, Multiplier ideal sheaf, Lelong number, Equisingularity.

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1. Introduction.

1.1. Cluster points of jumping coefficients. Let X be a complex manifold of dimension n and φ a quasi-plurisubharmonic (abbr. quasi-psh) function on X. The *multiplier ideal sheaf* $\mathscr{I}(\varphi)$ is defined to be the sheaf of germs of holomorphic functions f such that $|f|^2e^{-2\varphi}$ is locally integrable, which is a coherent sheaf of ideals (see [3]).

Let $x \in X$ be a point. By the solution to Demailly's strong openness conjecture [10], i.e.,

$$\mathscr{I}(\varphi) = \mathscr{I}_+(\varphi) := \bigcup_{\varepsilon>0} \mathscr{I}((1+\varepsilon)\varphi),$$

which was posed by Demailly in [2], there is an increasing sequence

$$0 = \xi_0(\varphi; x) < \xi_1(\varphi; x) < \xi_2(\varphi; x) < \cdots$$

of real numbers $\xi_k = \xi_k(\varphi; x)$ such that $\mathscr{I}(c\cdot\varphi)_x = \mathscr{I}(\xi_k\cdot\varphi)_x$ for $c \in [\xi_k, \xi_{k+1})$ and

$$\mathscr{I}(\xi_{k+1}\cdot\varphi)_x \subsetneqq \mathscr{I}(\xi_k\cdot\varphi)_x \quad \text{for every } k.$$

If the Lelong number $\nu(\varphi, x) = 0$, we put $\xi_1 = +\infty$.

DEFINITION 1.1 ([8], [5]). The real numbers $\xi_k(\varphi; x)$ is called the *jumping coefficients* or *jumping numbers* of φ at x. We say that ξ is a jumping coefficient of φ on an analytic set $A \subset X$ if it is a jumping coefficient of φ at some point $x \in A$. The collection of all jumping coefficients of φ at x is denoted by $\mathrm{Jump}(\varphi; x)$.

By the strong openness property of multiplier ideal sheaves, we know that $\mathrm{Jump}(\varphi; x)$ satisfies the descending chain condition (DCC): any decreasing sequence of jumping coefficients stabilizes. Moreover, it is easy to see that $\mathrm{Jump}(\varphi; x)$ does not

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satisfy ACC. By a discussion on Hironaka's log resolution [5], the sequence (ξ_k) will go to infinity when φ has analytic singularities, i.e., φ can be written locally as

$$\varphi = \frac{c}{2}\log\big(\sum_{k=1}^{m}|f_k|^2\big) + O(1),$$

where $c \in \mathbb{R}^+$ and f_k are holomorphic functions. However, the sequence (ξ_k) will not be a discrete subset of $\mathbb{R}^+$ for general plurisubharmonic functions as we will present, which is very different from the algebraic case. One can refer to [8] and [5] for more properties of jumping coefficients.

In the present subsection, we will construct a plurisubharmonic function φ such that $\mathrm{Jump}(\varphi;x)$ has a cluster point, i.e., there exists a jumping coefficient ξ_{k_0} such that for every small enough $\varepsilon > 0$ we have a jumping coefficient $\xi_k \in (\xi_{k_0} - \varepsilon, \xi_{k_0})$. Indeed, we will prove the following:

Theorem 1.1. *There exists a plurisubharmonic function φ on $\mathbb{C}^n$ $(n \geq 2)$ such that* 1 *is a cluster point of* $\mathrm{Jump}(\varphi;o)$.

By Theorem 1.1, one can obtain that the plurisubharmonic function φ will not be strongly equisingular to any plurisubharmonic function with analytic singularities (here, "strongly equisingular" for φ_1 and φ_2 means that $e^{-2\varphi_1} - e^{-2\varphi_2}$ is locally integrable). We will consider general case in the next subsection.

1.2. Equisingular problem on plurisubharmonic functions.

1.2.1. Strongly equisingular approximations with generalized analytic singularities. In [7], the following "equisingular" regularization process was established.

Theorem 1.2 ([7], see also [3, 4]). *Let $T = \alpha + i\partial\bar{\partial}\varphi$ be a closed $(1,1)$-current on a compact Hermitian manifold (X,ω), where α is a smooth closed $(1,1)$-form and φ a quasi-psh function. Let γ be a continuous real $(1,1)$-form such that $T \geq \gamma$. Then one can write $\varphi = \lim_{m\to+\infty}\varphi_m$, where*

(a) φ_m is smooth in the complement $X\backslash Z_m$ of an analytic set $Z_m \subset X$;
(b) (φ_m) is a decreasing sequence, and $Z_m \subset Z_{m+1}$ for all m;
(c) $\int_X(e^{-2\varphi} - e^{-2\varphi_m})dV_\omega$ is finite for every m and converges to 0 as $m \to +\infty$;
(d) ("equisingularity") $\mathscr{I}(\varphi_m) = \mathscr{I}(\varphi)$ for all m;
(e) $T_m = \alpha + i\partial\bar{\partial}\varphi_m$ satisfies $T_m \geq \gamma - \varepsilon_m\omega$, where $\varepsilon_m \to 0$ as $m \to +\infty$.

As what Remark 2.4 pointed out in [7] (see also Remark 15.4 in [3]), it is interesting to know whether the φ_m can be taken to have analytic singularities along Z_m. In [9], the first author showed that the above result does not hold if φ_m are supposed to have analytic singularities.

We notice that, in the proof of Theorem 1.2, the equisingular part (d) directly follows from the truth of part (c). Hence, it is interesting to know whether the part (c) holds by supposing φ_m to have analytic singularities in Theorem 1.2.

A plurisubharmonic function φ will be said to have *generalized analytic singularities* if for any point $x_0 \in X$ there exists a ball $B_r(x_0)$ such that for every complex line $L \subset B_r(x_0)$, $\varphi|_L$ has analytic singularities. The set of plurisubharmonic functions on X with generalized analytic singularities is denoted by $\mathcal{E}(X)$. In particular, $\mathcal{E}(X)$ contains any $\varphi \in Psh(X)$, which can be written locally as

$$\varphi = \log\big(\sum_{j=1}^{N}\prod_{k=1}^{m_j}|f_{jk}|^{\lambda_{jk}}\big) + O(1),$$

where $\lambda_{jk} \in \mathbb{R}^+$ and f_{jk} are holomorphic functions.

Now, it is natural to put forward such a question:

QUESTION. *For any given $\varphi \in Psh(X)$, can one choose $\varphi_A \in \mathcal{E}(X)$ such that φ_A is strongly equisingular to φ, i.e., $e^{-2\varphi} - e^{-2\varphi_A}$ is locally integrable?*

In this subsection, we establish the following result on the above question.

THEOREM 1.3. *Let χ be a convex increasing function on $\mathbb{R}$ with $\lim_{t\to-\infty} \chi'(t) = C_0 > 0$. Let φ be a plurisubharmonic function near $o \in \mathbb{C}^n$ with $C_0 \cdot \nu(\varphi, o) \geq n$ and $\varphi_A \in \mathcal{E}(\mathbb{C}^n)$ has generalized analytic singularities. If $e^{-2\chi\circ\varphi} - e^{-2\varphi_A}$ is locally integrable near o, then $\lim_{t\to-\infty} |\chi(t) - C_0 t| < \infty$.*

REMARK 1.1. (1) By Theorem 1.3, for any plurisubharmonic function φ near $o \in \mathbb{C}^n$ with $\nu(\varphi, o) > 0$, there exists a convex increasing function χ on $\mathbb{R}$ such that $e^{-2\chi\circ\varphi} - e^{-2\varphi_A}$ is not locally integrable near o.

(2) By a similar discussion in [9], we have that, for any complex manifold X (compact or noncompact) with $\dim X \geq 2$ and $z_0 \in X$, there exists a quasi-psh function φ on X such that for any plurisubharmonic function φ_A with generalized analytic singularities near z_0, $e^{-2\varphi} - e^{-2\varphi_A}$ is not locally integrable near z_0.

Note that in Theorem 1.3, the Lelong number $\nu(\chi \circ \varphi, o) \geq n$ is necessary. For Lelong number near 1 case, following from the plurisubharmonic function φ as in Theorem 1.1 (M large enough), we obtain

THEOREM 1.4. *Let $\varphi \in Psh(\mathbb{C}^n)$ be the plurisubharmonic function as in Theorem 1.1 and φ_A a plurisubharmonic function with generalized analytic singularities near the origin o. Then, $e^{-2\varphi} - e^{-2\varphi_A}$ is not locally integrable near o.*

1.2.2. Global equisingular approximations with stable analytic pluripolar sets. In [1] (see also [3, 4]), Demailly proved the basic result on the approximation of psh functions by psh functions with analytic singularities via Bergman kernels, which provides less equisingularity than Theorem 1.2 in the sense that the multiplier ideal sheaves are preserved.

In this note, by combining the strong openness property of multiplier ideal sheaves, we will present the following equisingular approximations quasi-psh functions with stable analytic pluripolar set $A := N(\mathscr{I}(\varphi))$, the zero set of multiplier ideal sheaf $\mathscr{I}(\varphi)$ associated to φ.

THEOREM 1.5. *Let φ be a quasi-psh function on a compact Hermitian manifold (X, ω) of dimension n such that $\frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\varphi \geq \gamma$ for some continuous real $(1,1)$-form γ. Then, there exists a sequence $(\hat{\psi}_k)$ of quasi-psh functions with analytic singularities on X and a decreasing sequence $\varepsilon_k > 0$ converging to 0 such that*

(1) $(\hat{\psi}_k)$ converges to φ almost everywhere;

(2) ("equisingularity") $\mathscr{I}(\hat{\psi}_k) = \mathscr{I}(\varphi)$ for every k;

(3) ("stability") the polar sets of $\hat{\psi}_k$ coincide with A for all k;

(4) $\nu(\varphi, x) - (1 + \frac{1}{k})\frac{n}{k} \leq \nu(\hat{\psi}_k, x) \leq (1 + \frac{1}{k})\nu(\varphi, x)$ for every $x \in A$;

(5) $\frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\hat{\psi}_k(\xi, \bar{\xi}) \geq \min\{\gamma(\xi, \bar{\xi}), p_k(\gamma - \lambda \cdot \omega)(\xi, \bar{\xi})\} - \varepsilon_k \omega(\xi, \bar{\xi})$ for any $\xi \in T_X$, where $\lambda > 0$ is some constant, $p_k < p_{k+1} \to +\infty$ as k tends to infinity.

REMARK 1.2. In [9], the first author proved that one cannot add the requirement to make $\hat{\psi}_k$ converge to φ decreasingly.

Whereas the pluripolar set Z_m of φ_m is larger and larger as m tends to infinity in Theorem 1.2. We infer from the strong openness property of multiplier ideal sheaves and Theorem 1.2 that

Theorem 1.6. *Let φ be a quasi-psh function on a compact Hermitian manifold (X,ω) of dimension n such that $\frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\varphi \geq \gamma$ for some continuous real $(1,1)$-form γ. Then, there exists a sequence $(\hat{\psi}_k)$ of quasi-psh functions on X such that*

(1) $(\hat{\psi}_k)$ is decreasing and convergent to φ;

(2) $\hat{\psi}_k$ is smooth on $X\backslash A$ for all k;

(3) $e^{-2\varphi} - e^{-2\hat{\psi}_k}$ is locally integrable for every k, which implies $\mathscr{I}(\hat{\psi}_k) = \mathscr{I}(\varphi)$;

(4) $\frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\hat{\psi}_k(\xi,\bar{\xi}) \geq \min\{(\gamma - \varepsilon_k\omega)(\xi,\bar{\xi}), p_k(\gamma - \lambda\cdot\omega)(\xi,\bar{\xi})\}$ for any $\xi \in T_X$, where $\lambda > 0$ is some constant, $p_k < p_{k+1} \to +\infty$ and $\varepsilon_k \to 0$ as k tends to infinity.

2. Some known results. To prove the main results, the following special case of Ohsawa-Takegoshi L^2 extension theorem and Siu's decomposition theorem are necessary.

Theorem 2.1 ([6], Theorem 2.1). *Let $\Omega \subset \mathbb{C}^n$ be a bounded pseudoconvex domain, and let H be an affine linear subspace of $\mathbb{C}^n$ of codimension $p \geq 1$ given by an orthogonal system s of affine linear equations $s_1 = \cdots = s_p = 0$. For every $\beta < p$, there exists a constant $C_{\beta,n,\Omega}$ depending only on β, n and the diameter of Ω, satisfying the following property. For every plurisubharmonic function $\varphi \in Psh(\Omega)$ and $f \in \mathcal{O}(\Omega\cap H)$ with $\int_{\Omega\cap H}|f|^2e^{-\varphi}d\lambda_H < +\infty$, there exists an extension $F \in \mathcal{O}(\Omega)$ of f such that*

$$\int_{\Omega}|F|^2|s|^{-2\beta}e^{-\varphi}d\lambda_n \leq C_{\beta,n,\Omega}\int_{\Omega\cap H}|f|^2e^{-\varphi}d\lambda_H,$$

where $d\lambda_n$ and $d\lambda_H$ are the Lebesgue volume elements in $\mathbb{C}^n$ and H respectively.

Theorem 2.2 (see [3], Theorem 2.18). *Let T be a closed positive current of bidimension (p,p). Then T can be written as a convergent series of closed positive currents*

$$T = \sum_{k=1}^{+\infty}\lambda_k[A_k] + R,$$

where $[A_k]$ is a current of integration over an irreducible analytic set of dimension p, and R is a residual current with the property that $\dim E_c(R) < p$ for every $c > 0$. This decomposition is locally and globally unique: the sets A_k are precisely the p-dimensional components occurring in the upperlevel sets $E_c(T)$, and $\lambda_k = \min_{x\in A_k}\nu(T,x)$ is the generic Lelong number of T along A_k.

The proof of Lemma 4.7 in [11] implies

Lemma 2.3. *Let $p : \Delta^2 \to \Delta$, $z = (z_1, z_2) \mapsto z_1$ and $\varphi \in Psh(\Delta^2)$. For almost all $z_1 \in \Delta$ (in the sense of the Lebesgue measure on Δ), the level set of Lelong numbers of $\varphi|_{p^{-1}(z_1)}$ satisfies*

$$\{z|\nu(\varphi|_{p^{-1}(z_1)}, z) = 0\} = (\{z|\nu(\varphi, z) = 0\} \cap p^{-1}(z_1)).$$

For the proof of Theorem 1.3, we also need the following two Lemmas.

Lemma 2.4. *Let φ_1, φ_2 be two subharmonic functions near $o \in \mathbb{C}$. If $e^{-2\varphi_1} - e^{-2\varphi_2}$ is locally integrable near o and the Lelong number $\nu(\varphi_1, o) \geq 1$, then $\nu(\varphi_1, o) = \nu(\varphi_2, o)$.*

Proof. As $e^{-2\varphi_1} - e^{-2\varphi_2}$ is locally integrable near o, it follows from $\nu(\varphi_1, o) \geq 1$ that $\nu(\varphi_2, o) \geq 1$.

Suppose that $\nu(\varphi_1, o) = \nu(\varphi_2, o)$ does not hold. Without loss of generality, we put $\nu(\varphi_1, o) > \nu(\varphi_2, o)$. Let $a \in (\nu(\varphi_2, o), \min\{\nu(\varphi_1, o), \nu(\varphi_2, o) + 1\})$. Then, $\varphi_i - (a-1)\log|z|$ are subharmonic near o and

$$\nu(\varphi_1 - (a-1)\log|z|, o) > 1,\ \nu(\varphi_2 - (a-1)\log|z|, o) < 1.$$

That is to say $e^{-2(\varphi_1-(a-1)\log|z|)} - e^{-2(\varphi_1-(a-1)\log|z|)}$ is not locally integrable near o. However, the local integrability of $e^{-2\varphi_1} - e^{-2\varphi_2}$ near o implies that

$$|z|^{2(a-1)}(e^{-2\varphi_1} - e^{-2\varphi_2}) = e^{-2(\varphi_1-(a-1)\log|z|)} - e^{-2(\varphi_1-(a-1)\log|z|)}$$

is locally integrable near o, which is a contradiction. □

Lemma 2.5. *Let $\varphi_A = C_1 \log|z| + O(1)$ near $o \in \mathbb{C}$ and φ a subharmonic function near $o \in \mathbb{C}$ with $\nu(\varphi, o) = C_2$, where C_1, C_2 are positive constants. Let χ be a convex increasing function on $\mathbb{R}$ such that $\lim_{t\to-\infty} \chi'(t) = C_0 > 0$ with $C_0C_2 \geq k$ for some $k \in \mathbb{N}^+$. If $|z|^{2(k-1)}(e^{-2\chi\circ\varphi} - e^{-2\varphi_A})$ is locally integrable near o, then $\lim_{t\to-\infty} |\chi(t) - C_0t| < \infty$.*

Proof. As $\chi'' \geq 0$ and $\lim_{t\to-\infty} \chi'(t) = C_0 > 0$, we have $\chi'(t) \geq C_0$ for any $t \in \mathbb{R}$. Then, $\chi(t) - C_0t$ is a convex increasing function on $\mathbb{R}$. Suppose that $\lim_{t\to-\infty} |\chi(t) - C_0t| < \infty$ does not hold. Then $\lim_{t\to-\infty}(\chi(t) - C_0t) = -\infty$.

Since $\nu(\chi\circ\varphi, o) = C_0C_2 \geq k$, then $\nu(\chi\circ\varphi - (k-1)\log|z|, o) \geq 1$. It follows from the integrability of $|z|^{2(k-1)}(e^{-2\chi\circ\varphi} - e^{-2\varphi_A})$ near o that $|z|^{-2(k-1)}e^{-2\varphi_A}$ is not locally integrable near o, which implies that $\varphi_A - (k-1)\log|z|$ is subharmonic near $o \in \mathbb{C}$ and $\nu(\varphi_A - (k-1)\log|z|, o) \geq 1$. By Lemma 2.4, we have

$$\nu(\chi\circ\varphi - (k-1)\log|z|, o) = \nu(\varphi_A - (k-1)\log|z|, o),$$

which implies $\nu(\chi\circ\varphi, o) = \nu(\varphi_A, o) = C_1 \geq k$.

By the assumption, we have $\lim_{z\to0}(\chi\circ\varphi - C_0\varphi) = -\infty$. Then, we infer from $\varphi \leq C_2\log|z| + O(1)$ that $\lim_{z\to0}(\chi\circ\varphi - C_0C_2\log|z|) = -\infty$. Hence,

$$\lim_{z\to0}(e^{-2(\chi\circ\varphi - C_1\log|z|)} - e^{-2(\varphi_A - C_1\log|z|)}) = +\infty.$$

It follows from $\nu(\chi\circ\varphi, o) = \nu(\varphi_A, o) = C_1 \geq k$ that $|z|^{2(k-1)}(e^{-2\chi\circ\varphi} - e^{-2\varphi_A})$ is not locally integrable near o, which is a contradiction. □

3. Proof of the main results. We are now in a position to prove our main results.

Proof of Theorem 1.1. Let

$$\varphi(z) = \log|z_1| + \sum_{k=1}^{\infty} \alpha_k \log(|z_1| + |\frac{z_2}{k}|^{\beta_k}),$$

where $\alpha_k = M^{-k}$, $\beta_k = M^{2k}$ and $M \geq 2$. Then $\varphi \in Psh(\mathbb{C}^n)$, $\mathscr{I}(\varphi)_o = (z_1) \cdot \mathcal{O}_n$ and $\nu(\varphi, (0, z_2)) = 1$, $\forall z_2 \neq 0$.

Given $\varepsilon \in (0, 1]$. As $(z_1, z_2) \cdot \mathcal{O}_n \subset \mathrm{rad}\mathscr{I}((1-\varepsilon)\varphi)_o$, there exists an integer $N > 0$ such that $z_2^N \in \mathscr{I}((1-\varepsilon)\varphi)_o$. Note that $\mathscr{I}((1-\varepsilon)\varphi)_o \subset \mathscr{I}((1-\varepsilon)\varphi_k)_o$ for every k, where $\varphi_k = \log|z_1| + \alpha_k \log(|z_1| + |\frac{z_2}{k}|^{\beta_k})$. To prove the desired result, it is sufficient to show that for any integer $m > 0$, there exists k_0 and $\varepsilon_0 > 0$ such that $z_2^m \notin \mathscr{I}((1-\varepsilon)\varphi_{k_0})_o$, $\forall \varepsilon \in (0, \varepsilon_0]$.

Considering the integration of $|z_2|^m e^{-(1-\varepsilon)\varphi_k}$ over unit bidisk $\Delta \times \Delta$, we have

$$\begin{aligned}
&\int_{\Delta\times\Delta} (|z_2|^m e^{-(1-\varepsilon)\varphi_k})^2 d\lambda_2 \\
=&\int_{\Delta^*} (\frac{1}{|z_1|^{(1-\varepsilon)(1+\alpha_k)}})^2 \frac{\sqrt{-1}}{2} dz_1 \wedge d\bar{z}_1 \int_{\Delta} (\frac{|z_2|^m}{(1+\frac{|z_2/k|^{\beta_k}}{|z_1|})^{(1-\varepsilon)\alpha_k}})^2 \frac{\sqrt{-1}}{2} dz_2 \wedge d\bar{z}_2 \\
=&\int_{\Delta^*} (\frac{1}{|z_1|^{(1-\varepsilon)(1+\alpha_k)}})^2 \frac{\sqrt{-1}}{2} dz_1 \wedge d\bar{z}_1 \int_{\tilde{\Delta}} (\frac{|kz_1^{1/\beta_k}|^{m+1} \cdot |w|^m}{(1+|w|^{\beta_k})^{(1-\varepsilon)\alpha_k}})^2 \frac{\sqrt{-1}}{2} dw \wedge d\bar{w} \\
\geq& C \cdot \int_{\Delta^*} (\frac{1}{|z_1|^{(1-\varepsilon)(1+\alpha_k)-\frac{m+1}{\beta_k}}})^2 \frac{\sqrt{-1}}{2} dz_1 \wedge d\bar{z}_1.
\end{aligned}$$

where $C > 0$ is some constant and $\tilde{\Delta}$ is the domain of integration in the new variable. Then, for any m, it follows from $\alpha_k = M^{-k}$ and $\beta_k = M^{2k}$ that there exists k_0 and $\varepsilon_0 > 0$ such that $(1-\varepsilon_0)(1+\alpha_{k_0}) - \frac{m+1}{\beta_{k_0}} \geq 1$, which implies $z_2^m \notin \mathscr{I}((1-\varepsilon)\varphi_{k_0})_o$, $\forall \varepsilon \in (0, \varepsilon_0]$. □

Proof of Theorem 1.3. Let

$$\xi : \mathbb{C}^n \to \mathbb{C}^n, \ z \mapsto (z_1, z_1 z_2, \ldots, z_1 z_n)$$

be a holomorphic mapping. Then ξ is an isomorphism outside $\{z_1 = 0\}$. Hence, for sufficiently small polydisc Δ_r^n, we have

$$\int_{\Delta_r^n} |z_1|^{2(n-1)} (e^{-2\chi(\varphi\circ\xi)} - e^{-2(\varphi_A\circ\xi)}) d\lambda_n = \int_{\xi(\Delta_r^n)\setminus\{z_1=0\}} (e^{-2\chi\circ\varphi} - e^{-2\varphi_A}) d\lambda_n < \infty.$$

That is to say, for almost every $(z_2, ..., z_n) \in \Delta_r^{n-1}$, we have

$$\int_{\Delta_r} |z_1|^{2(n-1)} (e^{-2\chi(\varphi\circ\xi)} - e^{-2(\varphi_A\circ\xi)}) d\lambda_1 < \infty.$$

It follows from Lemma 2.5 that $\lim_{t\to-\infty} |\chi(t) - C_0 t| < \infty$. □

Proof of Theorem 1.4. It is enough to prove the dimension two case. In fact, if $e^{-2\varphi} - e^{-2\varphi_A}$ is locally integrable near the origin $o = (o', o'')$ for general n ($n \geq 3$) case, then for almost all $(a_3, ..., a_n) \in \mathbb{C}^{n-2}$ near o'', $\left(e^{-2\varphi} - e^{-2\varphi_A}\right)|_{\{z_3=a_3,...,z_n=a_n\}}$ is locally integrable near o', which is a contradiction.

Assume that $e^{-2\varphi} - e^{-2\varphi_A}$ is integrable on a relatively compact neighborhood U_0 of o. Then, $e^{-2c\varphi} - e^{-2c\varphi_A}$ is integrable on U_0 for any $0 \leq c \leq 1$. As $\mathscr{I}(\varphi) = (z_1) \cdot \mathcal{O}_{\mathbb{C}^2}$ on U_0, it follows that $e^{-2\varphi_A}$ is not locally integrable near $\{z_1 = 0\}|_{U_0}$. By Theorem 2.2, on U_0, we have $\varphi_A = \lambda \log|z_1| + \psi$, where $\lambda \geq 1$ and ψ is a plurisubharmonic function near o such that $(\{z_1 = 0\}, o) \not\subset (\{\nu(\psi, z) \geq c\}, o)$, $\forall c \in \mathbb{Q}^+$. It follows from Lemma 2.3 that $\psi|_{z_1=0} \not\equiv -\infty$.

If $\lambda > 1$, then for every $z_2 \neq 0$, there exists $\varepsilon_0 > 0$ such that $\varphi_A \leq (1+\varepsilon_0)\log|z_1|$ near $(0, z_2)$. Hence, $e^{-2\frac{\varphi_A}{1+\varepsilon_0}} \geq e^{-2\log|z_1|}$ is not locally integrable near $(0, z_2)$, which is a contradiction to the local integrability of $e^{-2\frac{\varphi}{1+\varepsilon_0}}$ near $(0, z_2)$. Hence, we have $\varphi_A = \log|z_1| + \psi$ on U_0.

Since $\psi|_{z_1=0} \not\equiv -\infty$, then the Lelong number $\nu(\psi|_{z_1=0}, o') < \infty$. Hence, we have an integer number $N > 0$ such that

$$z_2^N \in \mathscr{I}(\psi|_{z_1=0})_o \subset \mathscr{I}(c\psi|_{z_1=0})_o,\ \forall c \in (0,1).$$

By Theorem 2.1, there exists a holomorphic function F_c on a Stein neighborhood $V_0 \subset\subset U_0$ of o such that

$$F_c(z)|_{V_0\cap\{z_1=0\}} = z_2^N|_{V_0\cap\{z_1=0\}} \text{ and } F_c(z)_o \in \mathscr{I}(c\psi + c\log|z_1|)_o = \mathscr{I}(c\varphi_A)_o.$$

Let $F_c(z) = z_2^N + z_1 \cdot G_c(z)$ for some holomorphic function $G_c(z)$ near o. By the assumption, we have

$$(z_1)\cdot \mathcal{O}_2 = \mathscr{I}(\varphi)_o = \mathscr{I}(\varphi_A)_o \subset \mathscr{I}(c\varphi_A)_o.$$

It follows that $z_2^N = F_c(z) - z_1 \cdot G_c(z) \in \mathscr{I}(c\varphi_A)_o$ for any $c \in (0,1)$. However, by the proof of Theorem 1.1, we know that there exists $\varepsilon_0 > 0$ such that

$$z_2^N \notin \mathscr{I}((1-\varepsilon)\varphi)_o,\ \forall \varepsilon \in (0, \varepsilon_0],$$

which is a contradiction to the assumption, i.e., $e^{-2\varphi} - e^{-2\varphi_A}$ is not locally integrable near o. □

Proof of Theorem 1.5. Step 1. By Theorem 13.12 in [3], there exists a finite open covering (U_k) of X by coordinate balls and a sequence of quasi-psh functions φ_m on X such that $\frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\varphi_1 \geq \gamma - \lambda\cdot\omega$ for some constant $\lambda > 0$ and on every U_k, we have

$$\varphi_m|_{U_k} = \frac{1}{2m}\log\sum_j |\sigma_{j,m}^{(k)}|^2 + O(1),$$

where $(\sigma_{j,m}^{(k)})$ is an orthonormal basis of $\mathcal{H}_{U_k}(m\varphi)$, the Hilbert space of holomorphic functions f on U_k satisfying $\int_{U_k}|f|^2 e^{-2m\varphi}d\lambda_n < \infty$.

Step 2 (see Proposition 2.1 in [11]). There exists a quasi-psh function $\tilde{\varphi}$ on X such that

(*i*) $e^{-2\varphi} - e^{-2\tilde{\varphi}}$ is locally integrable, which implies $\mathscr{I}(\varphi) = \mathscr{I}(\tilde{\varphi})$ on X;

(*ii*) $\tilde{\varphi} \in L^\infty_{loc}(X\backslash A)$ and $\nu(\tilde{\varphi}, x) = \nu(\varphi, x)$ for every $x \in A$.

Let $x_0 \in X$ be any point. Without loss of generality, we assume $x_0 \in U_k$. By Proposition 5.7 in [3], there exists $j_0 > 0$ and a neighborhood $V_0 \subset\subset U_k$ of x_0 such that $\sigma_{1,1}^{(k)}, ..., \sigma_{j_0,1}^{(k)}$ generate $\mathscr{I}(\varphi)$ on V_0, and

$$\log\sum_j |\sigma_{j,1}^{(k)}|^2 = \log\sum_{j=1}^{j_0}|\sigma_{j,1}^{(k)}|^2 + O(1).$$

on U_k. It follows from the strong openness property of multiplier ideal sheaves that there exists a real number $p_0 > 0$ satisfying

$$\int_{V_0}|\sigma_{j,1}^{(k)}|^2 e^{-2(1+\frac{1}{p})\varphi}d\lambda_n < \infty$$

for any $1 \le j \le j_0$ and any $p \ge p_0$. Then, we obtain that on V_0,

$$\begin{aligned}&\int_{V_0}(e^{-2\varphi}-e^{-2\max\{\varphi,p\varphi_1\}})d\lambda_n \le \int_{\{\varphi<p\varphi_1\}\cap V_0} e^{2\cdot\frac{1}{p}\varphi-2(1+\frac{1}{p})\varphi}d\lambda_n\\ \le&\int_{\{\varphi<p\varphi_1\}\cap V_0} e^{2\varphi_1-2(1+\frac{1}{p})\varphi}d\lambda_n \le \int_{V_0} e^{2\varphi_1-2(1+\frac{1}{p})\varphi}d\lambda_n\\ \le& C\cdot\int_{V_0}\sum_{j=1}^{j_0}|\sigma_{j,m}^{(k)}|^2e^{-2(1+\frac{1}{p})\varphi}d\lambda_n<\infty\end{aligned}$$

for some constant $C>0$.

Thus, we infer from the compactness of X that for sufficiently large p_0, by taking $\tilde{\varphi} := \max\{\varphi, p\varphi_1\}$ with $p \ge p_0$, we obtain that $e^{-2\varphi} - e^{-2\tilde{\varphi}}$ is locally integrable, $\tilde{\varphi} \in L^\infty_{loc}(X\backslash A)$ and $\frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\tilde{\varphi}(\xi,\bar{\xi}) \ge \min\{\gamma(\xi,\bar{\xi}), p(\gamma-\lambda\cdot\omega)(\xi,\bar{\xi})\}$ for any $\xi \in T_X$. In addition, since $\nu(\varphi,x)$ is bounded above on A, we have $\nu(\tilde{\varphi},x)=\nu(\varphi,x)$ for every $x\in A$.

Step 3. As the discussion of Remark 3 in [4], by taking $\hat{\varphi}_m = (1+\frac{1}{m})\tilde{\varphi}_m$, we have $\mathscr{I}(\hat{\varphi}_m)=\mathscr{I}(\tilde{\varphi})$ for large enough m, where $\tilde{\varphi}_m$ is the Bergman approximation sequence of $\tilde{\varphi}$. Moreover, $\hat{\varphi}_m$ is smooth outside the polar set Z_m of $\hat{\varphi}_m$.

Since $\hat{\varphi}_m|_{X\backslash A}$ is smooth, we have $Z_m \subset A$. Then, it follows from

$$\mathscr{I}(\hat{\varphi}_m)_x=\mathscr{I}(\tilde{\varphi})_x \ne \mathcal{O}_{X,x}$$

for any $x\in A$ that $Z_m = A$ for large enough m.

Step 4. Equisingular approximation of φ.

Without loss of generality, we assume that both φ and φ_1 are negative on X. Let $\psi_k := \max\{\varphi, p_k\varphi_1\}$, where $p_0 \le p_k < p_{k+1} \to \infty$ $(k\to\infty)$. Then ψ_k converges to φ and $\mathscr{I}(\psi_k)=\mathscr{I}(\varphi)$ for every k. By Remark 3 in [4], for every k, there exists a sequence of quasi-psh functions ψ_{jk} with analytic singularities such that ψ_{jk} is convergent to ψ_k, $\mathscr{I}(\psi_{jk})=\mathscr{I}(\psi_k)$ for each j,

$$\frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\psi_{jk}(\xi,\bar{\xi}) \ge (1+\frac{1}{j})(\min\{\gamma(\xi,\bar{\xi}), p_k(\gamma-\lambda\cdot\omega)(\xi,\bar{\xi})\}-\varepsilon_{jk}\omega(\xi,\bar{\xi})),\ \varepsilon_{jk}\to 0\ (j\to\infty),$$

for any $\xi\in T_X$ and

$$(1+\tfrac{1}{j})(\nu(\psi_k,x)-\tfrac{n}{j}) \le \nu(\psi_{jk},x) \le (1+\tfrac{1}{j})\nu(\psi_k,x) \qquad (*)$$

for every $x\in X$. It follows from $\psi_{jk}\to\psi_k$ $(j\to\infty)$ that for every k, there exists $N_k>0$ such that $m(E\{|\psi_{jk}-\psi_k|\ge\frac{1}{2^k}\})\le\frac{1}{2^k}$, for any $j\ge N_k$.

Take a subsequence ψ_{j_kk} of ψ_{jk} with $N_k \le j_k < j_{k+1}$ and

$$\frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\psi_{j_kk}(\xi,\bar{\xi}) \ge \min\{\gamma(\xi,\bar{\xi}), p_k(\gamma-\lambda\cdot\omega)(\xi,\bar{\xi})\}-\varepsilon_k\omega(\xi,\bar{\xi})$$

for any $\xi\in T_X$ and some $\varepsilon_k>0$ with $\varepsilon_k\downarrow 0$ $(k\to\infty)$. It follows from

$$|\psi_{j_kk}-\varphi| \le |\psi_{j_kk}-\psi_k|+|\psi_k-\varphi|$$

that for every $\varepsilon>0$ and all k with $\frac{1}{2^k}<\varepsilon$, we have

$$E\{|\psi_{j_kk}-\varphi|\ge\varepsilon\}\subset E\{|\psi_{j_kk}-\psi_k|\ge\frac{1}{2^k}\}\cup E\{|\psi_k-\varphi|\ge\varepsilon-\frac{1}{2^k}\},$$

which implies

$$m(E\{|\psi_{j_k k} - \varphi| \geq \varepsilon\}) \leq \frac{1}{2^k} + \frac{\varepsilon}{2} < \varepsilon$$

for large enough k, i.e., $\psi_{j_k k}$ is convergent to φ in Lebesgue measure. Let $\hat{\psi}_k$ be a subsequence of $\psi_{j_k k}$, which converges to φ almost everywhere. Then, it follows from Step 3 that $\hat{\psi}_k$ satisfies (1), (2), (3) and (5) as desired.

Since $\nu(\psi_k, x) = \nu(\varphi, x)$ for every $x \in A$ by Step 2, it follows from $(*)$ that

$$(1 + \frac{1}{j_k})(\nu(\varphi, x) - \frac{n}{j_k}) \leq \nu(\hat{\psi}_k, x) \leq (1 + \frac{1}{j_k})\nu(\varphi, x),$$

which implies $\nu(\varphi, x) - (1 + \frac{1}{k})\frac{n}{k} \leq \nu(\hat{\psi}_k, x) \leq (1 + \frac{1}{k})\nu(\varphi, x)$ for every $x \in A$, i.e., part (4) holds. □

Proof of Theorem 1.6. Let (ψ_k) be the equisingular approximation sequence of φ as in Theorem 1.2 and φ_1, p_0 as in Step 2 of Theorem 1.5. In addition, we can assume all ψ_k and φ_1 are negative. Then, we obtain that $e^{-2\psi_k} - e^{-2\varphi}$ and $e^{-2\varphi} - e^{-2\max\{\varphi, p\varphi_1\}}$ are locally integrable for any $p \geq p_0$, which implies that $e^{-2\psi_k} - e^{-2\max\{\varphi, p\varphi_1\}}$ is locally integrable. Since

$$e^{-2\psi_k} - e^{-2\max\{\psi_k, \max\{\varphi, p\varphi_1\}\}} = \max\{0, e^{-2\psi_k} - e^{-2\max\{\varphi, p\varphi_1\}}\} \in L^1_{loc}(X),$$

then $e^{-2\psi_k} - e^{-2\max\{\psi_k, p\varphi_1\}}$ is locally integrable.

Let

$$M_\eta(t_1, t_2) := \int_{\mathbb{R}^2} \max\{t_1 + x_1, t_2 + x_2\} \prod_{1 \leq j \leq 2} \eta_j^{-1}\theta(x_j/\eta_j) dx_1 dx_2$$

be the regularized max function, where $\eta = (\eta_1, \eta_2)$ with $\eta_j > 0$, and θ is a non-negative smooth function on $\mathbb{R}$ with support in $[-1, 1]$ such that $\int_{\mathbb{R}} \theta(x)dx = 1$ and $\int_{\mathbb{R}} x\theta(x)dx = 0$. By setting $\eta = (1, 1)$, we have $M_\eta(\max\{\psi_k, p_0\varphi_1\}, p\varphi_1) = \max\{\psi_k, p_0\varphi_1\}$ near A for any $p > p_0$. Then, it follows that

$$e^{-2\max\{\psi_k, p_0\varphi_1\}} - e^{-2M_\eta(\max\{\psi_k, p_0\varphi_1\}, p\varphi_1)}$$

is locally integrable.

Note that

$$\begin{aligned} 0 \leq & e^{-2\psi_k} - e^{-2M_\eta(\psi_k, p\varphi_1)} \\ \leq & e^{-2\psi_k} - e^{-2M_\eta(\max\{\psi_k, p_0\varphi_1\}, p\varphi_1)} \\ = & (e^{-2\psi_k} - e^{-2\max\{\psi_k, p_0\varphi_1\}}) + (e^{-2\max\{\psi_k, p_0\varphi_1\}} - e^{-2M_\eta(\max\{\psi_k, p_0\varphi_1\}, p\varphi_1)}) \end{aligned}$$

for any $p > p_0$. Thus, $e^{-2\psi_k} - e^{-2M_\eta(\psi_k, p\varphi_1)}$ is locally integrable for any $p > p_0$ and all k.

Let $\hat{\psi}_k := M_\eta(\psi_k, p_k\varphi_1)$, where $\eta = (1, 1)$ and $p_0 < p_k < p_{k+1} \to \infty$ $(k \to \infty)$. Then, $\hat{\psi}_k$ decreasingly converges to φ and smooth on $X \backslash A$. It follows from

$$e^{-2\varphi} - e^{-2\hat{\psi}_k} = (e^{-2\varphi} - e^{-2\psi_k}) + (e^{-2\psi_k} - e^{-2M_\eta(\psi_k, p_k\varphi_1)})$$

that $e^{-2\varphi} - e^{-2\hat{\psi}_k}$ is locally integrable.

Moreover,

$$\frac{\sqrt{-1}}{\pi}\partial\bar{\partial}\hat{\psi}_k(\xi, \bar{\xi}) \geq \min\{(\gamma - \varepsilon_k\omega)(\xi, \bar{\xi}), p_k(\gamma - \lambda \cdot \omega)(\xi, \bar{\xi})\}, \ \varepsilon_k \to 0 \ (k \to \infty),$$

for any $\xi \in T_X$, by Theorem 1.2 and Step 2 in Theorem 1.5. □

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