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RIGIDITY OF GRADIENT SHRINKING RICCI SOLITONS*

FEI YANG† AND LIANGDI ZHANG‡

Abstract. We prove that an n-dimensional ($n \geq 4$) gradient shrinking Ricci soliton with fourth-order divergence free Riemannian curvature tensor (i.e. $div^4 Rm = 0$) is rigid. In particular, such a soliton in dimension 4 is either Einstein, or a finite quotient of $\mathbb{R}^4$, $\mathbb{R}^2 \times \mathbb{S}^2$ or $\mathbb{R} \times \mathbb{S}^3$. Under the condition of $div^3 W(\nabla f) = 0$, we have the same results.

Key words. Rigidity, Gradient shrinking Ricci soliton, Riemannian curvature tensor, Weyl curvature tensor.

Mathematics Subject Classification. 53C24, 53C25.

1. Introduction. A complete Riemannian manifold (M^n, g, f) is called a gradient Ricci soliton if there exists a smooth function f on M^n such that the Ricci tensor Ric of the metric g satisfies the equation

$$Ric + Hess(f) = \lambda g \tag{1.1}$$

for some constant λ. For $\lambda > 0$ the Ricci soliton is shrinking, for $\lambda = 0$ it is steady and for $\lambda < 0$ expanding.

An Einstein manifold with constant potential function is called a trivial gradient Ricci soliton. $(\mathbb{R}^n, g_0, \frac{|x|^2}{4})$, where g_0 is the flat Euclidean metric, is called the Gaussian shrinking Ricci soliton. We may refer to an excellent survey by H. D. Cao [1] for more examples of Ricci solitons.

Taking a product $N \times \mathbb{R}^k$ with N being Einstein with Einstein constant λ and $f = \frac{\lambda}{2}|x|^2$ on $\mathbb{R}^k$ yields a mixed gradient soliton. A gradient soliton is rigid if it is of the type $N \times_\Gamma \mathbb{R}^k$, where Γ acts freely on N and by orthogonal transformations on $\mathbb{R}^k$ (no translational components). This concept was first introduced by P. Peterson and W. Wylie [13]. They also showed that a gradient Ricci soliton is rigid if and only if it has constant scalar curvature and is radially flat i.e., $sec(E, \nabla f) = 0$.

M. Fernández-López and E. Garcia-Río [8] proved that a compact Ricci soliton is rigid if and only if it has harmonic Weyl tensor. For the complete non-compact case, O. Munteanu and N. Sesum [11] showed that a gradient shrinking Ricci soliton with harmonic Weyl tensor is rigid. G. Catino, P. Mastrolia and D. D. Monticelli [4] proved that an n-dimensional ($n \geq 4$) gradient shrinking Ricci soliton with fourth-order divergence free Riemannian curvature tensor (i.e. $div^4 W = 0$) is rigid.

The classification of gradient Ricci solitons has been a subject of interest for many people in recent years. In the special case of dimension 4, A. Naber [12] showed that a non-compact shrinking Ricci soliton with bounded nonnegative Riemannian curvature is a finite quotient of $\mathbb{R}^4$, $\mathbb{R}^2 \times \mathbb{S}^2$ or $\mathbb{R} \times \mathbb{S}^3$. X. Chen and Y. Wang [6] classified four-dimensional anti-self dual gradient steady and shrinking Ricci solitons. H. D. Cao and Q. Chen [2] proved that a 4-dimensional Bach-flat gradient shrinking

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†Corresponding author. School of Mathematics and Physics, China University of Geosciences, Wuhan 430074, China (yangfei810712@163.com).

‡Center of Mathematical Sciences, Zhejiang University, Hangzhou 310027, China (zhangliangdi@zju.edu.cn).

Ricci soliton is either Einstein or a finite quotient of $\mathbb{R}^4$ or $\mathbb{R} \times \mathbb{S}^3$. More recently, J. Y. Wu, P. Wu and W. Wylie [15] proved that a 4-dimensional gradient shrinking Ricci soliton with half harmonic Weyl tensor (i.e. $divW^{\pm} = 0$) is either Einstein or a finite quotient of $\mathbb{R}^4$, $\mathbb{R}^2 \times \mathbb{S}^2$ or $\mathbb{R} \times \mathbb{S}^3$.

For general dimensions, M. Eminenti, G. La Nave and C. Mantegazza [7] proved that an n-dimensional compact shrinking Ricci soliton with vanishing Weyl tensor is a finite quotient of $\mathbb{S}^n$. More generally, P. Peterson and W. Wylie [14] showed that a gradient shrinking Ricci soliton with vanishing Weyl tensor is a finite quotient of $\mathbb{R}^n$, $\mathbb{S}^{n-1} \times \mathbb{R}$, or $\mathbb{S}^n$ by assuming $\int_M |Ric|^2 e^{-f} < \infty$. The integral assumption was proven to be true for gradient shrinking Ricci solitions (see Theorem 1.1 of [11]). Without additional assumptions, Z. H. Zhang [16] obtained the same classification of gradient shrinking Ricci solitons with vanishing Weyl tensor. More generally, H. D. Cao and Q. Chen [2] proved that an n-dimensional ($n \geq 5$) Bach-flat gradient shrinking Ricci soliton is either Einstein or a finite quotient of $\mathbb{R}^n$ or $\mathbb{R} \times N^{n-1}$, where N is an $(n-1)$-dimensional Einstein manifold.

In order to state our results precisely, we introduce the following definitions for the Riemannian curvature tensor.

$$(divRm)_{ijk} := \nabla_l R_{ijkl},$$

$$(div^2 Rm)_{ik} := \nabla_j \nabla_l R_{ijkl},$$

$$(div^3 Rm)_i := \nabla_k \nabla_j \nabla_l R_{ijkl},$$

$$div^4 Rm := \nabla_i \nabla_k \nabla_j \nabla_l R_{ijkl}.$$

For the Weyl curvature tensor, we define

$$(divW)_{ijk} := \nabla_l W_{ijkl},$$

$$(div^2 W)_{ik} := \nabla_j \nabla_l W_{ijkl},$$

$$(div^3 W)_i := \nabla_k \nabla_j \nabla_l W_{ijkl},$$

$$div^4 W := \nabla_i \nabla_k \nabla_j \nabla_l W_{ijkl}.$$

Our main theorems are following.

THEOREM 1.1. *Let* (M^n, g, f) $(n \geq 4)$ *be a gradient shrinking Ricci soliton. If*
(i) $div^4 Rm = 0$, *or*
(ii) $div^3 Rm(\nabla f) = 0$,
then (M^n, g, f) *is rigid.*

G. Catino, P. Mastrolia and D. D. Monticelli [4] proved that a gradient shrinking Ricci soliton with $div^4 W = 0$ is rigid. We will give a different proof in Section 6 Appendix. Moreover, we have another rigid result.

THEOREM 1.2. *Let* (M^n, g, f) $(n \geq 4)$ *be a gradient shrinking Ricci soliton with* $div^3 W(\nabla f) = 0$, *then it is rigid.*

In dimension 4, we have a classification result.

COROLLARY 1.3. *Let (M^4, g, f) be a 4-dimensional gradient shrinking Ricci soliton. Under either of the following condition,*
(i) $div^4 Rm = 0$, *or*
(ii) $div^3 Rm(\nabla f) = 0$, *or*
(iii) $div^3 W(\nabla f) = 0$,
(M^4, g, f) is either Einstein, or a finite quotient of the Gaussian shrinking soliton $\mathbb{R}^4$, $\mathbb{R}^2 \times \mathbb{S}^2$ or the round cylinder $\mathbb{R} \times \mathbb{S}^3$.

REMARK 1.4. *As it will be clear from the proof, the vanishing assumptions on $div^4 Rm$, $div^3 Rm(\nabla f)$ and $div^3 W(\nabla f)$ in all the above theorems can be trivially relaxed to $div^4 Rm \geq 0$, $div^3 Rm(\nabla f) \geq 0$ and $div^3 W(\nabla f) \geq 0$, respectively.*

The rest of this paper is organized as follows. In Section 2, we recall some background material and prove some formulas which will be needed in the proof of the main theorems. In Section 3, we prove that a compact gradient Ricci soliton with $div^4 Rm = 0$ is Einstein. In Section 4, we finish the proof of Theorem 1.1. In Section 5, we give a direct proof of Theorem 1.2.

2. Preliminaries. First of all, we present some basic facts for gradient shrinking Ricci solitons.

PROPOSITION 2.1. *([7],[10],[11],[14]) Let (M^n, g, f) $(n \geq 3)$ be a gradient shrinking Ricci soliton, then the following identities hold.*

$$\nabla_l R_{ijkl} = \nabla_j R_{ik} - \nabla_i R_{jk}, \tag{2.1}$$

$$\nabla R = 2 div Ric, \tag{2.2}$$

$$R_{ijkl} \nabla_l f = \nabla_l R_{ijkl}, \tag{2.3}$$

$$\nabla_l (R_{ijkl} e^{-f}) = 0, \tag{2.4}$$

$$R_{jl} \nabla_l f = \nabla_l R_{jl}, \tag{2.5}$$

$$\nabla_l (R_{jl} e^{-f}) = 0, \tag{2.6}$$

$$\nabla R = 2 Ric(\nabla f, \cdot), \tag{2.7}$$

$$\Delta_f R_{ik} = 2\lambda R_{ik} - 2 R_{ijkl} R_{jl}, \tag{2.8}$$

$$\Delta_f R = 2\lambda R - 2|Ric|^2, \tag{2.9}$$

where $\Delta_f := \Delta - \nabla_{\nabla f}$,

$$\Delta_f |Ric|^2 = 4\lambda |Ric|^2 - 4 Rm(Ric, Ric) + 2|\nabla Ric|^2, \tag{2.10}$$

where $Rm(Ric, Ric) = R_{ijkl}R_{ik}R_{jl}$, *and*

$$R + |\nabla f|^2 - 2\lambda f = Const. \tag{2.11}$$

Next, we prove the following formulas.

PROPOSITION 2.2. *Let* (M^n, g, f) $(n \geq 3)$ *be a gradient shrinking Ricci soliton, then we have the following identities.*

$$(div^2 Rm)_{ik} = 2\lambda R_{ik} + \nabla_l R_{ik}\nabla_l f - \frac{1}{2}\nabla_i\nabla_k R - R^2_{ik} - R_{ijkl}R_{jl}, \tag{2.12}$$

$$(div^3 Rm)_i = -R_{ijkl}\nabla_k R_{jl}, \tag{2.13}$$

and

$$div^4 Rm = \nabla_l R_{jk}\nabla_k R_{jl} - |\nabla Ric|^2 - R_{ijkl}\nabla_i\nabla_k R_{jl}. \tag{2.14}$$

Proof. By direct computation,

$$\begin{aligned}
(div^2 Rm)_{ik} &= \nabla_j\nabla_l R_{ijkl} \\
&= \Delta R_{ik} - \nabla_j\nabla_i R_{jk} \\
&= \Delta_f R_{ik} + \nabla_l R_{ik}\nabla_l f - \nabla_i\nabla_j R_{jk} + R_{ijkl}R_{jl} - R^2_{ik} \\
&= 2\lambda R_{ik} - 2R_{ijkl}R_{jl} + \nabla_l R_{ik}\nabla_l f - \frac{1}{2}\nabla_i\nabla_k R + R_{ijkl}R_{jl} - R^2_{ik} \\
&= 2\lambda R_{ik} - R_{ijkl}R_{jl} + \nabla_l R_{ik}\nabla_l f - \frac{1}{2}\nabla_i\nabla_k R - R^2_{ik},
\end{aligned}$$

where we used (2.1) in the second equality. Moreover, we used (2.2) and (2.8) in the fourth equality.

Using (2.12), we have

$$\begin{aligned}
(div^3 Rm)_i &= \nabla_k\nabla_j\nabla_l R_{ijkl} \\
&= \nabla_k(2\lambda R_{ik} - R_{ijkl}R_{jl} + \nabla_l R_{ik}\nabla_l f - \frac{1}{2}\nabla_i\nabla_k R - R^2_{ik}) \\
&= \lambda\nabla_i R - \nabla_k R_{ijkl}R_{jl} - R_{ijkl}\nabla_k R_{jl} + \nabla_l R_{ik}\nabla_k\nabla_l f + \nabla_k\nabla_l R_{ik}\nabla_l f \\
&\quad -\frac{1}{2}\nabla_k\nabla_i\nabla_k R - R_{ij}\nabla_k R_{kj} - R_{kj}\nabla_k R_{ij} \\
&= \lambda\nabla_i R + (\nabla_j R_{il} - \nabla_i R_{jl})R_{jl} - R_{ijkl}\nabla_k R_{jl} + \nabla_l R_{ik}(\lambda g_{kl} - R_{kl}) \\
&\quad +(\nabla_l\nabla_k R_{ik} + R_{lj}R_{ij} + R_{klij}R_{jk})\nabla_l f - \frac{1}{2}\nabla_i\Delta_f R - \frac{1}{2}\nabla_i(\nabla_k R\nabla_k f) \\
&\quad -\frac{1}{2}R_{ij}\nabla_j R - \frac{1}{2}R_{ij}\nabla_j R - R_{kj}\nabla_k R_{ij} \\
&= \lambda\nabla_i R + R_{jl}\nabla_j R_{il} - \frac{1}{2}\nabla_i|Ric|^2 - R_{ijkl}\nabla_k R_{jl} + \frac{\lambda}{2}\nabla_i R \\
&\quad -R_{kl}\nabla_l R_{ik} + \frac{1}{2}\nabla_l\nabla_i R\nabla_l f + \frac{1}{2}R_{ij}\nabla_j R + R_{jk}\nabla_l R_{ijkl} \\
&\quad -\lambda\nabla_i R + \nabla_i|Ric|^2 - \frac{1}{2}\nabla_i\nabla_l R\nabla_l f - \frac{1}{2}\nabla_l R\nabla_i\nabla_l f \\
&\quad -R_{ij}\nabla_j R - R_{kj}\nabla_k R_{ij}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\nabla_i|Ric|^2 - R_{ijkl}\nabla_k R_{jl} + \frac{\lambda}{2}\nabla_i R - \frac{1}{2}R_{ik}\nabla_k R + R_{jk}\nabla_j R_{ik} \\
&\quad -\frac{1}{2}\nabla_i|Ric|^2 - \frac{\lambda}{2}\nabla_i R + \frac{1}{2}R_{il}\nabla_l R - R_{kj}\nabla_k R_{ij} \\
&= -R_{ijkl}\nabla_k R_{jl},
\end{aligned}
$$

where we used (2.2) in the third equality, (2.1) and (1.1) in the fourth equality. Moreover, we used (2.3), (2.7) and (2.9) in the fifth equality. In the sixth equality, we used (1.1) and (2.1).

It follows from (2.13) that

$$
\begin{aligned}
div^4 Rm &= \nabla_i\nabla_k\nabla_j\nabla_l R_{ijkl} \\
&= -\nabla_i R_{ijkl}\nabla_k R_{jl} - R_{ijkl}\nabla_i\nabla_k R_{jl} \\
&= (\nabla_l R_{jk} - \nabla_k R_{jl})\nabla_k R_{jl} - R_{ijkl}\nabla_i\nabla_k R_{jl} \\
&= \nabla_l R_{jk}\nabla_k R_{jl} - |\nabla Ric|^2 - R_{ijkl}\nabla_i\nabla_k R_{jl},
\end{aligned}
$$

where we used (2.1) in the third equality. □

REMARK 2.3. *It is clear from* (2.12) *that* $div^2 Rm$ *is a symmetric 2-tensor. Therefore,*

$$
(div^2 Rm)_{ik} = \nabla_j\nabla_l R_{ijkl} = \nabla_l\nabla_j R_{ijkl},
$$

$$
(div^3 Rm)_i = \nabla_k\nabla_j\nabla_l R_{ijkl} = \nabla_k\nabla_j\nabla_l R_{kjil} = \nabla_k\nabla_l\nabla_j R_{ijkl} = \nabla_k\nabla_l\nabla_j R_{kjil},
$$

and

$$
div^4 Rm = \nabla_i\nabla_k\nabla_j\nabla_l R_{ijkl} = \nabla_i\nabla_k\nabla_l\nabla_j R_{ijkl} = \nabla_k\nabla_i\nabla_j\nabla_l R_{ijkl} = \nabla_k\nabla_i\nabla_l\nabla_j R_{ijkl}.
$$

3. Compact gradient shrinking Ricci soltions with $div^4 Rm = 0$. In this section, we prove that a compact gradient shrinking Ricci soliton with $div^4 Rm = 0$ must be Einstein. The first step is to derive the following integral equation.

LEMMA 3.1. *Let* (M^n, g, f) $(n \geq 3)$ *be a compact gradient shrinking Ricci soliton, then*

$$
\int_M \nabla_l R_{jk}\nabla_k R_{jl} e^{-f} = \frac{1}{2}\int_M |\nabla Ric|^2 e^{-f}. \tag{3.1}
$$

Proof. Calculating directly, we have

$$
\begin{aligned}
&\int_M \nabla_l R_{jk}\nabla_k R_{jl} e^{-f} \\
&= -\int_M R_{jk}\nabla_l\nabla_k R_{jl} e^{-f} + \int_M R_{jk}\nabla_k R_{jl}\nabla_l f e^{-f} \\
&= -\int_M R_{jk}(\nabla_k\nabla_l R_{jl} + R_{jp}R_{pk} + R_{lkji}R_{il})e^{-f} \\
&\quad + \int_M R_{jk}\nabla_k(R_{jl}\nabla_l f)e^{-f} - \int_M R_{jk}R_{jl}\nabla_k\nabla_l f e^{-f}
\end{aligned}
$$

$$
\begin{aligned}
&= -\int_M R_{jk}(R_{jp}R_{pk} + R_{lkji}R_{il})e^{-f} - \int_M R_{jk}R_{jl}(\lambda g_{kl} - R_{kl})e^{-f} \\
&= -\int_M trRic^3 e^{-f} + \int_M Rm(Ric, Ric)e^{-f} - \lambda \int_M |Ric|^2 e^{-f} + \int_M trRic^3 e^{-f} \\
&= \int_M Rm(Ric, Ric)e^{-f} - \lambda \int_M |Ric|^2 e^{-f}, \qquad (3.2)
\end{aligned}
$$

where we used (2.5) and (1.1) in the third equality.

Applying (2.10) to (3.2), we obtain

$$
\begin{aligned}
&\int_M \nabla_l R_{jk} \nabla_k R_{jl} e^{-f} \\
&= -\frac{1}{4}\int_M \Delta_f |Ric|^2 e^{-f} + \frac{1}{2}\int_M |\nabla Ric|^2 e^{-f} \\
&= -\frac{1}{4}\int_M (\Delta |Ric|^2 - \nabla_{\nabla f} |Ric|^2) e^{-f} + \frac{1}{2}\int_M |\nabla Ric|^2 e^{-f} \\
&= -\frac{1}{4}\int_M \nabla_{\nabla f} |Ric|^2 e^{-f} + \frac{1}{4}\int_M \nabla_{\nabla f}|Ric|^2 e^{-f} + \frac{1}{2}\int_M |\nabla Ric|^2 e^{-f} \\
&= \frac{1}{2}\int_M |\nabla Ric|^2 e^{-f}.
\end{aligned}
$$

□

THEOREM 3.2. *Let (M^n, g, f) $(n \geq 4)$ be a compact gradient shrinking Ricci soliton with $div^4 Rm = 0$, then it is Einstein.*

Proof. Note that $div^4 Rm = 0$. Integrating (2.14), we obtain

$$
\begin{aligned}
0 &= \int_M div^4 Rm e^{-f} \\
&= \int_M \nabla_l R_{jk} \nabla_k R_{jl} e^{-f} - \int_M |\nabla Ric|^2 e^{-f} - \int_M R_{ijkl} \nabla_i \nabla_k R_{jl} e^{-f} \\
&= \frac{1}{2}\int_M |\nabla Ric|^2 e^{-f} - \int_M |\nabla Ric|^2 e^{-f} \\
&= -\frac{1}{2}\int_M |\nabla Ric|^2 e^{-f}, \qquad (3.3)
\end{aligned}
$$

where we used Lemma 3.1 and (2.4) in the third equality.

It follows that $|\nabla Ric| = 0$. By (2.2), we know that

$$\nabla R = 0,$$

i.e., R is a constant on M^n.

Tracing (1.1), we know that $\Delta f = n\lambda - R = Const$. Since (M^n, g, f) is compact, we can easily conclude that the potential function f must be harmonic hence a constant function. It is clear that (M^n, g, f) is Einstein. □

4. The proof of Theorem 1.1. In this section, we finish the proof of Theorem 1.1. First of all, we derive a integral inequality for complete non-compact gradient shrinking Ricci solitons.

LEMMA 4.1. *Let (M^n, g, f) $(n \geq 3)$ be a complete non-compact gradient shrinking Ricci soliton. For every C^2 function $\phi : \mathbb{R}_+ \to \mathbb{R}$ with $\phi(f)$ having compact support in M and some constant $c > 0$, we have*

$$\int_M \nabla_l R_{jk} \nabla_k R_{jl} \phi^2(f) e^{-f} \leq c \int_M |Ric|^2 |\nabla f|^2 (\phi')^2 e^{-f} + \frac{3}{4} \int_M |\nabla Ric|^2 \phi^2(f) e^{-f}. \tag{4.1}$$

Proof. By direct computation, we have

$$\begin{aligned}
&\int_M \nabla_l R_{jk} \nabla_k R_{jl} \phi^2(f) e^{-f} \\
&= -\int_M R_{jk} \nabla_l \nabla_k R_{jl} \phi^2(f) e^{-f} - \int_M R_{jk} \nabla_k R_{jl} \nabla_l \phi^2(f) e^{-f} + \int_M R_{jk} \nabla_k R_{jl} \nabla_l f \phi^2(f) e^{-f} \\
&= -\int_M R_{jk} (\nabla_k \nabla_l R_{jl} + R_{jp} R_{pk} + R_{lkji} R_{il}) \phi^2(f) e^{-f} - 2\int_M R_{jk} \nabla_k R_{jl} \nabla_l f \phi\phi' e^{-f} \\
&\quad + \int_M R_{jk} \nabla_k (R_{jl} \nabla_l f) \phi^2(f) e^{-f} - \int_M R_{jk} R_{jl} \nabla_k \nabla_l f \phi^2(f) e^{-f} \\
&= -\int_M R_{jk} (R_{jp} R_{pk} + R_{lkji} R_{il}) \phi^2(f) e^{-f} - 2\int_M R_{jk} \nabla_k R_{jl} \nabla_l f \phi\phi' e^{-f} \\
&\quad - \int_M R_{jk} R_{jl} (\lambda g_{kl} - R_{kl}) \phi^2(f) e^{-f} \\
&= -\int_M trRic^3 \phi^2(f) e^{-f} + \int_M Rm(Ric, Ric) \phi^2(f) e^{-f} - 2\int_M R_{jk} \nabla_k R_{jl} \nabla_l f \phi\phi' e^{-f} \\
&\quad -\lambda \int_M |Ric|^2 \phi^2(f) e^{-f} + \int_M trRic^3 \phi^2(f) e^{-f} \\
&= \int_M Rm(Ric, Ric) \phi^2(f) e^{-f} - 2\int_M R_{jk} \nabla_k R_{jl} \nabla_l f \phi\phi' e^{-f} - \lambda \int_M |Ric|^2 \phi^2(f) e^{-f},
\end{aligned} \tag{4.2}$$

where we used (2.6) and (1.1) in the third equality.

Applying (2.10) to (4.2), we obtain

$$\begin{aligned}
&\int_M \nabla_l R_{jk} \nabla_k R_{jl} \phi^2(f) e^{-f} \\
&= -2\int_M R_{jk} \nabla_k R_{jl} \nabla_l f \phi\phi' e^{-f} - \frac{1}{4} \int_M \Delta_f |Ric|^2 \phi^2(f) e^{-f} + \frac{1}{2} \int_M |\nabla Ric|^2 \phi^2(f) e^{-f} \\
&= -2\int_M R_{jk} \nabla_k R_{jl} \nabla_l f \phi\phi' e^{-f} - \frac{1}{4} \int_M \Delta |Ric|^2 \phi^2(f) e^{-f} \\
&\quad + \frac{1}{4} \int_M \nabla_{\nabla f} |Ric|^2 \phi^2(f) e^{-f} + \frac{1}{2} \int_M |\nabla Ric|^2 \phi^2(f) e^{-f} \\
&= -2\int_M R_{jk} \nabla_k R_{jl} \nabla_l f \phi\phi' e^{-f} + \frac{1}{4} \int_M \langle \nabla |Ric|^2, \nabla \phi^2(f) \rangle e^{-f} \\
&\quad - \frac{1}{4} \int_M \nabla_{\nabla f} |Ric|^2 \phi^2(f) e^{-f} + \frac{1}{4} \int_M \nabla_{\nabla f} |Ric|^2 \phi^2(f) e^{-f} + \frac{1}{2} \int_M |\nabla Ric|^2 \phi^2(f) e^{-f} \\
&= -2\int_M R_{jk} \nabla_k R_{jl} \nabla_l f \phi\phi' e^{-f} + \int_M R_{ik} \nabla_l R_{ik} \nabla_l f \phi\phi' e^{-f} \\
&\quad + \frac{1}{2} \int_M |\nabla Ric|^2 \phi^2(f) e^{-f}
\end{aligned}$$

$$\leq c\int_M |Ric||\nabla f||\nabla Ric||\phi||\phi'|e^{-f} + \frac{1}{2}\int_M |\nabla Ric|^2\phi^2(f)e^{-f}$$
$$\leq c\int_M |Ric|^2|\nabla f|^2(\phi')^2 e^{-f} + \frac{3}{4}\int_M |\nabla Ric|^2\phi^2(f)e^{-f}$$

for some constant $c > 0$. □

LEMMA 4.2. *Let (M^n, g, f) $(n \geq 3)$ be a complete non-compact gradient shrinking Ricci soliton. For every C^2 function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ with $\varphi(f)$ having compact support in M, we have*

$$-\int_M R_{ijkl}\nabla_i\nabla_k R_{jl}\varphi(f)e^{-f} = \int_M (|\nabla Ric|^2 - \nabla_l R_{kj}\nabla_k R_{jl})\varphi' e^{-f}. \tag{4.3}$$

Proof. By direct computation, we have

$$\begin{aligned} -\int_M R_{ijkl}\nabla_i\nabla_k R_{jl}\varphi(f)e^{-f} &= \int_M R_{ijkl}\nabla_k R_{jl}\varphi'\nabla_i f e^{-f} \\ &= \int_M \nabla_i R_{ijkl}\nabla_k R_{jl}\varphi' e^{-f} \\ &= \int_M (\nabla_k R_{jl} - \nabla_l R_{kj})\nabla_k R_{jl}\varphi' e^{-f} \\ &= \int_M (|\nabla Ric|^2 - \nabla_l R_{kj}\nabla_k R_{jl})\varphi' e^{-f}, \end{aligned}$$

where we used (2.4), (2.3) and (2.1) in the first, second and third equality, respectively. □

The following result by Munteanu-Sesum [11] is needed.

LEMMA 4.3 (Munteanu-Sesum [11]). *Let (M, g) be a gradient shrinking Ricci soliton. If for some $\beta < 1$ we have $\int_M |Rm|^2 e^{-\beta f} < +\infty$, then the following identity holds.*

$$\int_M |divRm|^2 e^{-f} = \int_M |\nabla Ric|^2 e^{-f} < +\infty. \tag{4.4}$$

REMARK 4.4. *It is clear from their proof that a gradient shrinking Ricci soliton with $\int_M |divRm|^2 e^{-f} < +\infty$ or $\int_M |\nabla Ric|^2 e^{-f} < +\infty$ still has* (4.4).

Now we are ready to prove the result that a complete non-compact gradient shrinking Ricci soliton with $div^4 Rm = 0$ is rigid.

THEOREM 4.5. *Let (M^n, g, f) $(n \geq 4)$ be a complete non-compact gradient shrinking Ricci soliton with $div^4 Rm = 0$, then it is rigid.*

Proof. Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be a C^2 function with $\phi = 1$ on $(0, s]$, $\phi = 0$ on $[2s, \infty)$ and $-\frac{c}{t} \leq \phi'(t) \leq 0$ on $(s, 2s)$ for some constant $c > 0$. Define $D(r) := \{x \in M | f(x) \leq r\}$. By Lemma 4.2, we have

$$\begin{aligned} -\int_M R_{ijkl}\nabla_i\nabla_k R_{jl}\phi^2(f)e^{-f} &= \int_M (|\nabla Ric|^2 - \nabla_l R_{kj}\nabla_k R_{jl})(\phi^2)' e^{-f} \\ &= 2\int_M (|\nabla Ric|^2 - \nabla_l R_{kj}\nabla_k R_{jl})\phi\phi' e^{-f} \end{aligned}$$

$$\leq 0. \tag{4.5}$$

Integrating (2.14) and using Lemma 4.1 and (4.5), we have

$$\begin{aligned}
&\int_M div^4 Rm\phi^2(f)e^{-f} \\
&= \int_M \nabla_l R_{jk}\nabla_k R_{jl}\phi^2(f)e^{-f} - \int_M |\nabla Ric|^2\phi^2(f)e^{-f} - \int_M R_{ijkl}\nabla_i\nabla_k R_{jl}\phi^2(f)e^{-f} \\
&\leq c\int_M |Ric|^2|\nabla f|^2(\phi')^2 e^{-f} + \frac{3}{4}\int_M |\nabla Ric|^2\phi^2(f)e^{-f} - \int_M |\nabla Ric|^2\phi^2(f)e^{-f} \\
&\leq \frac{c}{s^2}\int_{D(2s)\setminus D(s)} |Ric|^2|\nabla f|^2 e^{-f} - \frac{1}{4}\int_M |\nabla Ric|^2\phi^2(f)e^{-f}.
\end{aligned} \tag{4.6}$$

Since $R \geq 0$ (see B. L. Chen [5]), it follows from (2.11) that $|\nabla f|^2 \leq c(f+1)$. Note that f is of quadratic growth (see Cao-Zhou [3]) and $\int_M |Ric|^2 e^{-\alpha f} < +\infty$ (see Munteanu-Sesum [11]), we can derive that

$$\int_M |Ric|^2|\nabla f|^2 e^{-f} \leq \int_M |Ric|^2 e^{-\gamma f} < +\infty \tag{4.7}$$

for some $\gamma \in (0,1]$. Therefore,

$$\frac{c}{s^2}\int_{D(2s)\setminus D(s)} |Ric|^2|\nabla f|^2 e^{-f} \to 0$$

as $s \to +\infty$.

By taking $r \to +\infty$ in (4.6), we obtain $\int_M |\nabla Ric|^2 e^{-f} = 0$. Since $\int_M |\nabla Ric|^2 e^{-f} < +\infty$, it follows from (4.4) that

$$\int_M |divRm|^2 e^{-f} = \int_M |\nabla Ric|^2 e^{-f} = 0.$$

Hence, $|divRm| = |\nabla Ric| = 0$.

It is clear $divRm = 0$ implies that M^n is radially flat. Moreover, by (2.2), we know that

$$\nabla R = 0,$$

i.e., R is a constant on M^n.

Since M^n is radially flat and has constant scalar curvature, it follows from Theorem 1.2 of Peterson-Wylie [13] that (M^n, g, f) is rigid. □

Next, we show that a gradient shrinking Ricci soliton with $div^3 Rm(\nabla f) = 0$ is rigid.

THEOREM 4.6. *Let (M^n, g, f) $(n \geq 4)$ be a gradient shrinking Ricci soliton with $div^3 Rm(\nabla f) = 0$, then it is rigid.*

Proof. By (2.13), we have

$$\begin{aligned}
0 &= div^3 Rm(\nabla f) \\
&= \nabla_k\nabla_j\nabla_l R_{ijkl}\nabla_i f \\
&= -R_{ijkl}\nabla_k R_{jl}\nabla_i f
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}(\nabla_i R_{ijkl})(\nabla_l R_{jk} - \nabla_k R_{jl}) \\
&= -\frac{1}{2}|divRm|^2,
\end{aligned}
$$

where we used (2.3) in the third equality and (2.1) in the last. It follows that $divRm = 0$.

It is clear that $sec(E, \nabla f) = 0$, i.e., M^n is radially flat. Moreover, we have $\nabla_j R = 2\nabla_l R_{jl} = 2g^{ik}\nabla_l R_{ijkl} = 0$, i.e., R is a constant on M^n. □

To conclude, Theorem 1.1 follows immediately by Theorem 3.2, Theorem 4.5 and Theorem 4.6.

5. The proof of Theorem 1.2. In this section, we prove Theorems 1.2. We calculate the following formulas first.

PROPOSITION 5.1. *Let (M^n, g, f) $(n \geq 3)$ be a gradient shrinking Ricci soliton. We have the following identities.*

$$
(divW)_{ijk} = \frac{n-3}{n-2}(divRm)_{ijk} - \frac{n-3}{2(n-1)(n-2)}(g_{ik}\nabla_j R - g_{jk}\nabla_i R), \tag{5.1}
$$

$$
(div^2 W)_{ik} = \frac{n-3}{n-2}(div^2 Rm)_{ik} - \frac{n-3}{2(n-1)(n-2)}(g_{ik}\Delta R - \nabla_k\nabla_i R), \tag{5.2}
$$

$$
(div^3 W)_i = \frac{n-3}{n-2}(div^3 Rm)_i + \frac{n-3}{2(n-1)(n-2)}R_{ik}\nabla_k R, \tag{5.3}
$$

and

$$
div^4 W = \frac{n-3}{n-2}div^4 Rm + \frac{n-3}{2(n-1)(n-2)}(\frac{1}{2}|\nabla R|^2 + R_{ik}\nabla_i\nabla_k R). \tag{5.4}
$$

Proof. By direct computation,

$$
\begin{aligned}
(divW)_{ijk} &= \nabla_l W_{ijkl} \\
&= \nabla_l R_{ijkl} - \frac{1}{n-2}(g_{ik}\nabla_l R_{jl} - \nabla_i R_{jk} - g_{jk}\nabla_l R_{il} + \nabla_j R_{ik}) \\
&\quad + \frac{1}{(n-1)(n-2)}(g_{ik}\nabla_j R - g_{jk}\nabla_i R) \\
&= \nabla_l R_{ijkl} - \frac{1}{n-2}\nabla_l R_{ijkl} \\
&\quad - \frac{1}{2(n-2)}(g_{ik}\nabla_j R - g_{jk}\nabla_i R) + \frac{1}{(n-1)(n-2)}(g_{ik}\nabla_j R - g_{jk}\nabla_i R) \\
&= \frac{n-3}{n-2}\nabla_l R_{ijkl} - \frac{n-3}{2(n-1)(n-2)}(g_{ik}\nabla_j R - g_{jk}\nabla_i R),
\end{aligned}
$$

where we used (2.7) in the second equality.

It follows from (5.1) that

$$
(div^2 W)_{ik}
$$

$$
\begin{aligned}
&= \nabla_j \nabla_l W_{ijkl} \\
&= \frac{n-3}{n-2} \nabla_j \nabla_l R_{ijkl} - \frac{n-3}{2(n-1)(n-2)} (g_{ik} \Delta R - \nabla_k \nabla_i R),
\end{aligned}
$$

By (5.2), we have

$$
\begin{aligned}
(div^3 W)_i &= \nabla_k \nabla_j \nabla_l W_{ijkl} \\
&= \frac{n-3}{n-2} \nabla_k \nabla_j \nabla_l R_{ijkl} - \frac{n-3}{2(n-1)(n-2)} (\nabla_i \Delta R - \nabla_k \nabla_k \nabla_i R) \\
&= \frac{n-3}{n-2} \nabla_k \nabla_j \nabla_l R_{ijkl} + \frac{n-3}{2(n-1)(n-2)} R_{ik} \nabla_k R,
\end{aligned}
$$

From (5.3), we have

$$
\begin{aligned}
div^4 W &= \nabla_i \nabla_k \nabla_j \nabla_l W_{ijkl} \\
&= \frac{n-3}{n-2} \nabla_i \nabla_k \nabla_j \nabla_l R_{ijkl} + \frac{n-3}{2(n-1)(n-2)} \nabla_i (R_{ik} \nabla_k R) \\
&= \frac{n-3}{n-2} \nabla_i \nabla_k \nabla_j \nabla_l R_{ijkl} \\
&\quad + \frac{n-3}{2(n-1)(n-2)} (\frac{|\nabla R|^2}{2} + R_{ik} \nabla_i \nabla_k R),
\end{aligned}
$$

□

As a corollary of Proposition 5.1, we have

Corollary 5.2. *Let (M^n, g) $(n \geq 3)$ be a gradient shrinking Ricci soliton. We have the following identities.*

$$
(divW)_{ijk} = \frac{n-3}{n-2} (\nabla_j R_{ik} - \nabla_i R_{jk}) - \frac{n-3}{2(n-1)(n-2)} (g_{ik} \nabla_j R - g_{jk} \nabla_i R), \quad (5.5)
$$

$$
\begin{aligned}
(div^2 W)_{ik} &= \frac{n-3}{n-2} (2\lambda R_{ik} + \nabla_{\nabla f} R_{ik} - R^2_{ik} - R_{ijkl} R_{jl}) - \frac{n-3}{2(n-1)} \nabla_i \nabla_k R \\
&\quad - \frac{n-3}{2(n-1)(n-2)} (\nabla_{\nabla f} R + 2\lambda R - 2|Ric|^2) g_{ik}, \quad (5.6)
\end{aligned}
$$

$$
(div^3 W)_i = -\frac{n-3}{n-2} R_{ijkl} \nabla_k R_{jl} + \frac{n-3}{2(n-1)(n-2)} R_{ik} \nabla_k R, \quad (5.7)
$$

and

$$
\begin{aligned}
div^4 W &= \frac{n-3}{n-2} (\nabla_l R_{jk} \nabla_k R_{jl} - |\nabla Ric|^2 - R_{ijkl} \nabla_i \nabla_k R_{jl}) \\
&\quad + \frac{n-3}{2(n-1)(n-2)} (\frac{1}{2} |\nabla R|^2 + R_{ik} \nabla_i \nabla_k R). \quad (5.8)
\end{aligned}
$$

Proof. (5.5), (5.7) and (5.8) follows immediately by applying (2.1), (2.13) and (2.14) to (5.1), (5.3) and (5.4), respectively. Moreover, Plugging (2.9) and (2.12) into (5.2), we can get (5.6). □

Next, we prove that a gradient shrinking Ricci soliton with $div^3W(\nabla f) = 0$ is rigid.

THEOREM 5.3. *Let (M^n, g, f) $(n \geq 4)$ be a gradient shrinking Ricci soliton with $div^3W(\nabla f) = 0$, then it is rigid.*

Proof. By (5.7), we have

$$\begin{aligned} & div^3W(\nabla f) \\ = & -\frac{n-3}{n-2} R_{ijkl}\nabla_k R_{jl}\nabla_i f + \frac{n-3}{2(n-1)(n-2)} R_{ik}\nabla_k R \nabla_i f \\ = & \frac{n-3}{2(n-2)}(\nabla_i R_{ijkl})(\nabla_l R_{jk} - \nabla_k R_{jl}) + \frac{n-3}{4(n-1)(n-2)}|\nabla R|^2 \\ = & -\frac{n-3}{2(n-2)}|divRm|^2 + \frac{n-3}{4(n-1)(n-2)}|\nabla R|^2, \end{aligned} \tag{5.9}$$

where we used (2.3) and (2.7) in the second equality and (2.1) in the last.

It follows from (2.7) that $|\nabla R|^2 \leq 4|Ric|^2|\nabla f|^2$. Using (4.7), we obtain that

$$\int_M |\nabla R|^2 e^{-f} < +\infty,$$

Integrating (5.9) and using the condition of $div^3W(\nabla f) = 0$, we obtain

$$\int_M |divRm|^2 e^{-f} = \frac{1}{2(n-1)} \int_M |\nabla R|^2 e^{-f} < +\infty.$$

It follows from (4.4) that

$$\begin{aligned} \int_M |\nabla Ric|^2 e^{-f} &= \int_M |divRm|^2 e^{-f} \\ &= \frac{1}{2(n-1)} \int_M |\nabla R|^2 e^{-f} \\ &\leq \frac{n}{2(n-1)} \int_M |\nabla Ric|^2 e^{-f}, \end{aligned} \tag{5.10}$$

where we used $|\nabla R|^2 \leq n|\nabla Ric|^2$.

Note that $\frac{n}{2(n-1)} < 1$, we conclude from (5.9) that

$$\int_M |divRm|^2 e^{-f} = \int_M |\nabla R|^2 e^{-f} = 0,$$

i.e., $|divRm| = |\nabla R| = 0$. It follows that M^n is radially flat, i.e., $sec(E, \nabla f) = 0$. Moreover, $|\nabla R| = 0$ on M, i.e., R is a constant on M. By Theorem 1.2 of Peterson-Wylie [13], (M^n, g, f) is rigid. □

6. Appendix. As it is mentioned in the introduction, G. Catino, P. Mastrolia and D. D. Monticelli [4] proved that an n-dimensional $(n \geq 4)$ gradient shrinking Ricci soliton with $div^4W = 0$ is rigid. In their paper, div^4W is defined as $\nabla_k\nabla_j\nabla_l\nabla_i W_{ikjl}$. They showed that $div^4W = 0$ if and only if $div^3C = 0$, where $div^3C = \nabla_i\nabla_j\nabla_k C_{ijk}$ and C_{ijk} is the Cotton tensor equals to $-\frac{n-2}{n-3}\nabla_l W_{ijkl}$ for $n \geq 4$ (see e.g. [2]). Then, they proved that $div^3C = 0$ implies $C = 0$. Therefore, the rigidity result follows (see [8] and [11]). Moreover, it is clear from their proof that this result holds for $\nabla_k\nabla_j\nabla_l\nabla_i W_{ikjl} \le 0$. We give a different proof in this section.

Remark 6.1. *The definition of $div^4 W$ in G. Catino, P. Mastrolia and D. D. Monticelli [4] differs from ours by a minus sign. To be more precise, we have*

$$\nabla_k\nabla_j\nabla_l\nabla_i W_{ikjl} = \nabla_j\nabla_k\nabla_l\nabla_i W_{ijkl} = -\nabla_j\nabla_k\nabla_l\nabla_i W_{jikl} = -\nabla_i\nabla_k\nabla_l\nabla_j W_{ijkl}. \tag{6.1}$$

It follows from (5.2) that $\nabla_j\nabla_l W_{ijkl}$ is symmetric on i and k, then it is also symmetric on j and l, i.e.,

$$\nabla_j\nabla_l W_{ijkl} = \nabla_l\nabla_j W_{ijkl}. \tag{6.2}$$

Combining (6.1) and (6.2), we have

$$\nabla_k\nabla_j\nabla_l\nabla_i W_{ikjl} = -\nabla_i\nabla_k\nabla_j\nabla_l W_{ijkl}.$$

It is clear from (5.4) that

$$div^4 W = \frac{n-3}{n-2} div^4 Rm + \frac{n-3}{2(n-1)(n-2)}\left(\frac{1}{2}|\nabla R|^2 + R_{ik}\nabla_i\nabla_k R\right).$$

Theorem 6.2. *Let (M^n, g, f) $(n \ge 4)$ be a compact gradient shrinking Ricci soliton with $div^4 W = 0$, then it is Einstein.*

Proof. Integrating (5.8), we have

$$\begin{aligned}
&\int_M div^4 W e^{-f} \\
&= \frac{n-3}{n-2}\int_M (\nabla_l R_{jk}\nabla_k R_{jl} - |\nabla Ric|^2 - R_{ijkl}\nabla_i\nabla_k R_{jl})e^{-f} \\
&\quad + \frac{n-3}{2(n-1)(n-2)}\int_M \left(\frac{1}{2}|\nabla R|^2 + R_{ik}\nabla_i\nabla_k R\right)e^{-f} \\
&= -\frac{n-3}{2(n-2)}\int_M |\nabla Ric|^2 e^{-f} + \frac{n-3}{4(n-1)(n-2)}\int_M |\nabla R|^2 e^{-f} \\
&\le -\frac{n-3}{4n(n-1)}\int_M |\nabla R|^2 e^{-f},
\end{aligned} \tag{6.3}$$

where we used Lemma 3.1, (2.4) and (2.6) in the second equality. Moreover, we used $|\nabla R|^2 \le n|\nabla Ric|^2$ in the inequality.

Since $div^4 W = 0$, it follows from (6.3) that $\nabla R = 0$, i.e., R is a constant on M. Therefore, $Ric(\nabla f, \nabla f) = \frac{1}{2}\langle \nabla R, \nabla f\rangle = 0$. By Lemma 2.3, (M^n, g) is Einstein. □

Theorem 6.3. *Let (M^n, g, f) $(n \ge 4)$ be a complete non-compact gradient shrinking Ricci soliton with $div^4 W = 0$, then it is rigid.*

Proof. Let $\phi : \mathbb{R}_+ \to \mathbb{R}$ be a C^2 function with $\phi = 1$ on $(0, s]$, $\phi = 0$ on $[2s, \infty)$ and $-\frac{c}{t} \le \phi'(t) \le 0$ on $(s, 2s)$ for some constant $c > 0$. Define $D(r) := \{x \in M | f(x) \le r\}$.
From Lemma 4.2, we obtain

$$-\int_M R_{ijkl}\nabla_i\nabla_k R_{jl}\phi^2(f)e^{-f} = \int_M (|\nabla Ric|^2 - \nabla_l R_{kj}\nabla_k R_{jl})(\phi^2)'e^{-f}$$

$$
\begin{aligned}
&= 2\int_M (|\nabla Ric|^2 - \nabla_l R_{kj}\nabla_k R_{jl})\phi\phi' e^{-f} \\
&\le 0. \qquad (6.4)
\end{aligned}
$$

By direct computation, we have

$$
\begin{aligned}
&\int_M R_{ik}\nabla_i\nabla_k R\phi^2(f)e^{-f} \\
&= -\int_M \nabla_i(R_{ik}e^{-f})\nabla_k R\phi^2(f) - \int_M |\nabla R|^2\phi\phi' e^{-f} \\
&= -\int_M |\nabla R|^2\phi\phi' e^{-f} \\
&\le \frac{1}{12}\int_M |\nabla R|^2\phi^2 e^{-f} + c\int_M |\nabla R|^2(\phi')^2 e^{-f} \\
&\le \frac{1}{12}\int_M |\nabla R|^2\phi^2 e^{-f} + c\int_M |Ric|^2|\nabla f|^2(\phi')^2 e^{-f} \qquad (6.5)
\end{aligned}
$$

where we used (2.6) in the second equality and (2.7) to obtain the last inequality.

Integrating (5.9), we have

$$
\begin{aligned}
&\int_M div^4 W\phi^2(f)e^{-f} \\
&= \frac{n-3}{n-2}\int_M (\nabla_l R_{jk}\nabla_k R_{jl} - |\nabla Ric|^2 - R_{ijkl}\nabla_i\nabla_k R_{jl})\phi^2(f)e^{-f} \\
&\quad + \frac{n-3}{2(n-1)(n-2)}\int_M (\frac{1}{2}|\nabla R|^2 + R_{ik}\nabla_i\nabla_k R)\phi^2(f)e^{-f} \\
&\le c\int_M |Ric|^2|\nabla f|^2(\phi')^2 e^{-f} + \frac{3(n-3)}{4(n-2)}\int_M |\nabla Ric|^2\phi^2(f)e^{-f} \\
&\quad - \frac{n-3}{n-2}\int_M |\nabla Ric|^2\phi^2(f)e^{-f} + \frac{n-3}{4(n-1)(n-2)}\int_M |\nabla R|^2\phi^2 e^{-f} \\
&\quad + \frac{n-3}{12(n-1)(n-2)}\int_M |\nabla R|^2\phi^2 e^{-f} + c\int_M |Ric|^2|\nabla f|^2(\phi')^2 e^{-f} \\
&\le \frac{c}{s^2}\int_{D(2s)\setminus D(s)} |Ric|^2|\nabla f|^2 e^{-f} - \frac{n-3}{4(n-2)}\int_M |\nabla Ric|^2\phi^2(f)e^{-f} \\
&\quad + \frac{n-3}{3(n-1)(n-2)}\int_M |\nabla R|^2\phi^2(f)e^{-f}, \qquad (6.6)
\end{aligned}
$$

where we used Lemma 4.2, (6.6) and (6.5) in the first inequality.

Applying $div^4 W = 0$ and $|\nabla R|^2 \le n|\nabla Ric|^2$ to (6.4), we obtain

$$
0 \le \frac{c}{s^2}\int_{D(2s)\setminus D(s)} |Ric|^2|\nabla f|^2 e^{-f} - \frac{n^2-9}{12(n-1)(n-2)}\int_M |\nabla Ric|^2\phi^2(f)e^{-f} \qquad (6.7)
$$

It follows from (4.7) that $\frac{c}{s^2}\int_{D(2s)\setminus D(s)} |Ric|^2|\nabla f|^2 e^{-f} \to 0$ as $s \to +\infty$.

Note that $n \ge 4$. By taking $r \to +\infty$ in (6.7), we obtain $\int_M |\nabla Ric|^2 e^{-f} = 0$. Since $\int_M |\nabla Ric|^2 e^{-f} < +\infty$, it follows (2.20) that

$$
\int_M |divRm|^2 e^{-f} = \int_M |\nabla Ric|^2 e^{-f} = 0. \qquad (6.8)
$$

Hence, $|divRm| = 0$.

It is clear that $sec(E, \nabla f) = 0$, i.e., M^n is radially flat. Moreover, we have $\nabla_j R = 2\nabla_l R_{jl} = 2g^{ik}\nabla_l R_{ijkl} = 0$, i.e., R is a constant on M^n. By Theorem 1.2 of Peterson-Wylie [13], we conclude that (M^n, g, f) is rigid. □

From Theorem 6.2 and Theorem 6.3, we have a classification theorem for 4-dimensional gradient shrinking Ricci solitons with $div^4 W = 0$:

THEOREM 6.4. *Let (M^4, g, f) be a 4-dimensional gradient shrinking Ricci soliton with $div^4 W = 0$, then it is either*

(i) *Einstein, or*

(ii) *a finite quotient of the Gaussian shrinking soliton $\mathbb{R}^4$, $\mathbb{R}^2 \times \mathbb{S}^2$ or the round cylinder $\mathbb{R} \times \mathbb{S}^3$.*

REMARK 6.5. *It is obvious that Theorems 6.2 to 6.4 hold for $div^4 W \geq 0$. Moreover, it follows from* (6.1) *that Theorem 6.2 to 6.4 still hold if indices of $div^4 W$ permutate.*

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