ASIAN J. MATH.
Vol. 24, No. 3, pp. 483–500, June 2020

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THE STAR MEAN CURVATURE FLOW ON 3-SPHERE AND HYPERBOLIC 3-SPACE*

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Abstract. The Hodge star mean curvature flow on a 3-dimensional Riemannian or pseudo-Riemannian manifold is one of nonlinear dispersive curve flows in geometric analysis. Such a curve flow is integrable as its local differential invariants of a solution to the curve flow evolve according to a soliton equation. In this paper, we show that these flows on a 3-sphere and 3-dimensional hyperbolic space are integrable, and describe algebraically explicit solutions to such curve flows. Solutions to the (periodic) Cauchy problems of such curve flows on a 3-sphere and 3-dimensional hyperbolic space and its Bäcklund transformations follow from this construction.

Key words. moving frames, Hodge star MCF, Gross-Pitaevskii equation, periodic Cauchy problems, Bäcklund transformation.

Mathematics Subject Classification. 14H70, 37K10, 53C44, 70E40.

1. Introduction. Suppose g is a Riemannian or Lorentzian metric on a 3-dimensional manifold N^3. The hodge star mean curvature flow ($*$-MCF) on (N^3, g) is the following curve evolution on the space of immersed curves in N^3,

$$\gamma_t = *_\gamma(H(\gamma(\cdot, t))), \tag{1.1}$$

where $*_{\gamma(x)}$ is the Hodge star operator on the normal plane $\nu(\gamma)_x$ and $H(\gamma(\cdot, t))$ is the mean curvature vector for $\gamma(\cdot, t)$. It can be checked that $*$-MCF preserves arc length parameter. As shown in [15], the $*$-MCF on $\mathbb{R}^3$ parametrized by arc length is the *vortex filament equation* (VFE), first modeled by Da Rios [3] for a self-induced motion of vortex lines in an incompressible fluid,

$$\gamma_t = \gamma_x \times \gamma_{xx}, \tag{1.2}$$

which is directly linked to the famous nonlinear Schrödinger equation (NLS)

$$q_t = \frac{i}{2}(q_{xx} + 2|q|^2 q). \tag{1.3}$$

Hasimoto transform [4] shows that the correspondence between the VFE and the NLS is given as follows. If γ is a solution of the VFE, then there exists a function $\theta : \mathbb{R} \to \mathbb{R}$ such that

$$q(x,t) = k(x,t)e^{i(\theta(t)+\int_0^x \tau(s,t)ds)} \tag{1.4}$$

is a solution of the NLS, where $\tau(\cdot, t)$ is the torsion for $\gamma(\cdot, t)$ and x is the arc-length parameter. Due to this transform, the VFE is regarded as a completely integrable curve flow and has been studied widely (see [5, 16]). In [11, 12], Terng and Uhlenbeck provided a systematic method to construct such a correspondence and they further gave a way to derive explicit Bäcklund transformations for curve flows.

A large literature has been developed about a more generalized NLS or the *Gross-Pitaevskii equation* in [7] given by

$$i\psi_t + \mu\psi = -\psi_{xx} + u(x)\psi + \alpha\psi|\psi|^2 \tag{1.5}$$

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with a trapping potential $u(x)$ and a chemical potential constant μ. ψ and $|\psi|^2$ represent a condensated wave-function and its local density of matter, respectively. $\alpha = +1$ or -1 is repulsive or attractive interactions between atoms. The Gross-Pitaevskii equation (1.5) is a fundamental model in nonlinear optics and low temperature physics, such as Bose-Einstein condensation and Superfluids [2, 6, 9, 10].

In the present article, we will show that the $*$-MCF on 3-sphere and hyperbolic 3-space is respectively related to

$$q_t = \frac{i}{2}(q_{xx} \pm q + 2|q|^2 q), \tag{1.6}$$

the simplest case of Gross-Pitaevskii equation, which is more natural in the physical context [8]. This is a Schrödinger-type equation, however, the Gross-Pitaevskii equation is not integrable in general because of the external potential. For certain potentials, the Gross-Pitaevskii equation (1.5) admits special solutions. Fortunately, the equation (1.6) is completely integrable, since there exists a transform $q \mapsto e^{\pm it}q$ between solutions of (1.6) and that of NLS. From now on, we refer (1.6) to (GP$^{\pm}$), where the superscript $\pm$ indicates the sign in front of the external potential q in (1.6).

We aim to write down the explicit solutions of such a curve evolution on $\mathbb{S}^3$ and $\mathbb{H}^3$ using the relation with (GP$^{\pm}$). This paper is organized as follows. In Section 2, we review and modify moving frames along a curve in $\mathbb{R}^4$ from that in [15] and find periodic frames along a closed curve in order to investigate periodic Cauchy problems for $*$-MCF. We then show how the $*$-MCF is related to the nonlinear Schrödinger flow in Section 3 and a Lax pair is given for (1.6) in Section 4. Section 5 is devoted to find solutions of (periodic) Cauchy problems of $*$-MCF on certain spaces. We give the Bäcklund transformations (BT) in the last section.

2. Moving Frames along a Curve. Let $\gamma(x) : \mathbb{R} \to \mathbb{R}^4$ be a curve parametrized by its arc-length parameter x, then there exists an orthonormal frame $g \in SO(4)$ such that $g^{-1}g_x$ is a $\mathfrak{so}(4)$-valued connection 1-form consisting of 6 local invariants. Since two orthonormal frames are differed by an element in $SO(4)$, one may choose a *suitable* frame that contains the least number of local invariants. Let

$$R(c) = \begin{pmatrix} \cos c & \sin c \\ -\sin c & \cos c \end{pmatrix}$$

denote the rotation of $\mathbb{R}^2$ by angle c, we consider the following types of frames.

2.1. Parallel Frame along curves on $\mathbb{S}^3$. Given a curve $\gamma : \mathbb{R} \to \mathbb{S}^3 \subset \mathbb{R}^4$ parametrized by arc length x. Since the position vector γ is perpendicular to the tangent vector γ_x, we choose $e_0 = \gamma, e_1 = \gamma_x$, n_2 and n_3 normal to e_0 and e_1 such that $\{e_0, e_1, n_2, n_3\}$ is an orthonormal basis in $\mathbb{R}^4$. Then

$$(e_0, e_1, n_2, n_3)_x = (e_0, e_1, n_2, n_3) \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -\xi_1 & -\xi_2 \\ 0 & \xi_1 & 0 & -\omega \\ 0 & \xi_2 & \omega & 0 \end{pmatrix}. \tag{2.1}$$

Rotating n_2, n_3 by the angle θ for $\theta_x = -\omega$, i.e.,

$$(e_0, e_1, e_2, e_3) = (e_0, e_1, n_2, n_3) \begin{pmatrix} I_2 & 0 \\ 0 & R(\theta) \end{pmatrix}, \tag{2.2}$$

one obtains an orthonormal frame $g = (e_0, e_1, e_2, e_3)$ satisfying

$$g^{-1}g_x = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -k_1 & -k_2 \\ 0 & k_1 & 0 & 0 \\ 0 & k_2 & 0 & 0 \end{pmatrix}, \tag{2.3}$$

where $k_1 = \xi_1 \cos\theta + \xi_2 \sin\theta$, $k_2 = -\xi_1 \sin\theta + \xi_2 \cos\theta$. The frame g is called a *parallel* frame and k_{i-1} the principal curvature along e_i for $i = 2, 3$ for γ. Moreover, $g^{-1}g_x$ is $\mathfrak{o}(4)$-valued.

Next, we consider the Minkowski spacetime, denoted by $\mathbb{R}^{3,1}$, with Lorentzian metric $-dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2$, and the hyperbolic 3-space $\mathbb{H}^3$ is the hyperquadric defined by

$$-x_0^2 + x_1^2 + x_2^2 + x_3^2 = -1.$$

2.2. Parallel Frame along curves on $\mathbb{H}^3$. Let $\gamma(x) \in \mathbb{H}^3 \subset \mathbb{R}^{3,1}$ be a curve parametrized by arc length x. A similar discussion to Subsection 2.1 shows that there exists a parallel frame $h = (e_0, e_1, e_2, e_3)$ with $e_0 = \gamma$, $e_1 = \gamma_x$ such that

$$h_x = h \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & -\mu_1 & -\mu_2 \\ 0 & \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 \end{pmatrix}. \tag{2.4}$$

Notice that $h^{-1}h_x \in \mathfrak{o}(1,3)$.

From (2.2), we see that there are other choices for orthonormal base. Hence, given a periodic curve $\gamma(x)$, there exists a periodic frame along the curve $\gamma(x)$.

2.3. Periodic moving frames on $\mathbb{S}^3$ and $\mathbb{H}^3$. (cf. [15]) The following idea is modified from the 3-dimensional Euclidean space in [15] to suit our case. Let $c_0 \in \mathbb{R}$ be a constant, and

$$M_{c_0} = \{\gamma : S^1 \to N \mid ||\gamma_x|| = 1, \text{ the normal holonomy of } \gamma \text{ is } R(-2\pi c_0)\},$$

where $N = \mathbb{S}^3$ and $\mathbb{H}^3$. If (e_0, e_1, e_2, e_3) is a parallel frame along $\gamma \in M_{c_0}$, then

$$(e_0, e_1, e_2, e_3)(2\pi) = (e_0, e_1, e_2, e_3)(0)\mathrm{diag}(I_2, R(-2\pi c_0)).$$

Let $(v_2(x), v_3(x))$ be the orthonormal normal frame obtained by rotating $(e_2(x), e_3(x))$ by $c_0 x$. Then the new frame

$$\tilde{g}(x) = (e_0, e_1, v_2, v_3)(x) = (e_0, e_1, e_2, e_3)(x) \begin{pmatrix} I_2 & 0 \\ 0 & R(c_0 x) \end{pmatrix}$$

is periodic in x (see Lemma 2.1). Moreover,

$$\tilde{g}^{-1}\tilde{g}_x = (e_{21} - \sigma e_{12}) + c_0(e_{43} - e_{34}) + \sum_{i=1}^{2} \tilde{k}_i (e_{i+2,2} - e_{2,i+2}), \tag{2.5}$$

where $\sigma = 1$ and -1 when $N = \mathbb{S}^3$ and $\mathbb{H}^3$, respectively, and e_{ij} denote a matrix with 1 in (i, j) entry and 0 elsewhere. Direct computations imply that

$$(\tilde{k}_1 + i\tilde{k}_2)(x, t) = e^{-ic_0 x}(k_1 + ik_2)(x, t)$$

is periodic, where $k_1(x,t)$ and $k_2(x,t)$ are principal curvatures along e_2 and e_3, respectively. We call $\tilde{g} = (e_0, e_1, v_2, v_3)$ a periodic h-frame along γ on N.

LEMMA 2.1. *Let $c_0 \in \mathbb{R}$ and $\gamma \in M_{c_0}$. If (e_0, e_1, e_2, e_3) is a parallel frame along γ and for any non-negative integer n, define*

$$(v_2^n, v_3^n)(x) = (e_2, e_3)(x)R((c_0 + n)x). \tag{2.6}$$

Then $(v_2^n, v_3^n)(x)$ is periodic in x for all n.

Proof. Since

$$\begin{aligned} v_2^n(2\pi) &= \cos((c_0+n)2\pi)e_2(2\pi) - \sin((c_0+n)2\pi)e_3(2\pi) \\ &= \cos(2\pi c_0)e_2(2\pi) - \sin(2\pi c_0)e_3(2\pi) \\ &= v_2^0(2\pi) \end{aligned} \tag{2.7}$$

and $v_2^0(0) = v_2^0(2\pi)$ followed from [15], $v_2^n(x)$ is periodic and so is $v_3^n(x)$ similarly. □

REMARK 2.2. One notes that if a curve $\gamma(x) \in \mathbb{R}^4$ parametrized by arc length is given to be periodic in x, it is obvious that $e_1 = \gamma_x$ is periodic as well. Therefore, $(e_0, e_1, v_2^n, v_3^n)(x)$ is a periodic h-frame in x, where $e_0 = \gamma$.

3. $*$-MCF on 3-Sphere $\mathbb{S}^3$ and Hyperbolic 3-space $\mathbb{H}^3$. In what follows, evolutions of invariants for the $*$-MCF on a 3-manifold $N = \mathbb{S}^3$ and $\mathbb{H}^3$ will be discussed. Denote $\sigma = 1$ and -1 for $N = \mathbb{S}^3$ and $\mathbb{H}^3$, respectively. Now, let us evolve $\gamma : \mathbb{R}^2 \to N$ with the $*$-MCF flow (1.1), where x is the arc-length parameter, $e_0 = \gamma$ and $e_1 = \gamma_x$.

Recall that the Hodge star operator on an oriented two dimensional inner product space is the rotation of $\frac{\pi}{2}$. So if (u_1, u_2) is an oriented orthonormal basis then

$$*(u_1) = u_2, *(u_2) = -u_1.$$

Under this orthonormal frame $\{e_0, e_1, n_2, n_3\}$ showed in Section 2, the mean curvature vector H is $\xi_1 n_2 + \xi_2 n_3$, and $*$-MCF (1.1) on N is written as

$$\gamma_t = \xi_1 n_3 - \xi_2 n_2. \tag{3.1}$$

On one hand, a direct computation implies the following properties.

PROPOSITION 3.1. *For any $\gamma(x,t) \in N$ satisfying the $*$-MCF (3.1) that is parametrized by arc length, there exists a moving frame $h = \{e_0, e_1, n_2, n_3\}$, where $e_0 = \gamma, e_1 = \gamma_x$, satisfying*

$$\begin{cases} h^{-1}h_x = \begin{pmatrix} 0 & -\sigma & 0 & 0 \\ 1 & 0 & -\xi_1 & -\xi_2 \\ 0 & \xi_1 & 0 & -\omega \\ 0 & \xi_2 & \omega & 0 \end{pmatrix}, \\ h^{-1}h_t = \begin{pmatrix} 0 & 0 & \sigma\xi_2 & -\sigma\xi_1 \\ 0 & 0 & (\xi_2)_x + \xi_1\omega & -(\xi_1)_x + \xi_2\omega \\ -\xi_2 & -(\xi_2)_x - \xi_1\omega & 0 & -u \\ \xi_1 & (\xi_1)_x - \xi_2\omega & u & 0 \end{pmatrix}. \end{cases} \tag{3.2}$$

where

$$\begin{cases} (\xi_1)_t = -(\xi_2)_{xx} - 2(\xi_1)_x\omega - \xi_1\omega_x + \xi_2(\omega^2 - \sigma + u), \\ (\xi_2)_t = (\xi_1)_{xx} - 2(\xi_2)_x\omega - \xi_2\omega_x - \xi_1(\omega^2 - \sigma + u), \\ \omega_t = u_x + \frac{1}{2}(\xi_1^2 + \xi_2^2)_x. \end{cases} \tag{3.3}$$

A similar result in [15] has been derived:

PROPOSITION 3.2. *Let $\gamma(x,t) : \mathbb{R}^2 \to N$ be a closed curve parametrized by arc length and $\gamma(0,t) = \gamma(2\pi,t)$ for all t. Then*

$$\int_0^{2\pi} \omega(x,t)\ dx \tag{3.4}$$

is independent of t.

Proof. It follows from Proposition 3.1 that there exists a moving frame h satisfying the system (3.2). The third equation of (3.3) implies

$$\frac{d}{dt}\int_0^{2\pi} \omega(x,t)\ dx = \int_0^{2\pi} \omega_t(x,t)\ dx = \int_0^{2\pi} u_x + \frac{1}{2}(\xi_1^2 + \xi_2^2)_x\ dx = 0.$$

∎

REMARK 3.3. The normal holonomy of γ (cf. [15]) is then defined as

$$\frac{1}{2\pi}\int_0^{2\pi} \omega(x,t)\ dx.$$

On the other hand, if one considers parallel frames for curves, we have the following consequence.

PROPOSITION 3.4. *For any $\gamma(x,t) \in N$ satisfying the $*$-MCF (3.1) that is parametrized by arc length, there exists a parallel frame $g = (e_0, e_1, e_2, e_3) \in G$ with $e_0 = \gamma$ and $e_1 = \gamma_x$ such that*

$$\begin{cases} g^{-1}g_x = \begin{pmatrix} 0 & -\sigma & 0 & 0 \\ 1 & 0 & -k_1 & -k_2 \\ 0 & k_1 & 0 & 0 \\ 0 & k_2 & 0 & 0 \end{pmatrix}, \\ g^{-1}g_t = \begin{pmatrix} 0 & 0 & \sigma k_2 & -\sigma k_1 \\ 0 & 0 & (k_2)_x & -(k_1)_x \\ -k_2 & -(k_2)_x & 0 & \frac{1}{2}(k_1^2 + k_2^2) \\ k_1 & (k_1)_x & -\frac{1}{2}(k_1^2 + k_2^2) & 0 \end{pmatrix}, \end{cases} \tag{3.5}$$

where k_1, k_2 are principal curvatures along e_2, e_3, and $(G,\sigma) = (SO(4),1)$ and $(O(1,3),-1)$ when $N = \mathbb{S}^3$ and $\mathbb{H}^3$, respectively.

Proof. We give computations for the 3-sphere case below, and omit the similar calculations for the case $\mathbb{H}^3$. Using the curve evolution (3.1), we get

$$\begin{aligned} (e_0)_t &= k_1 e_3 - k_2 e_2, \\ (e_1)_t &= (k_1)_x e_3 - (k_2)_x e_2. \end{aligned} \tag{3.6}$$

Therefore, we may assume

$$(e_0,e_1,e_2,e_3)_t=(e_0,e_1,e_2,e_3)\begin{pmatrix}0&0&k_2&-k_1\\0&0&(k_2)_x&-(k_1)_x\\-k_2&-(k_2)_x&0&-\nu\\k_1&(k_1)_x&\nu&0\end{pmatrix}.$$

Since $(e_2)_{tx}\cdot e_3=(e_2)_{xt}\cdot e_3$, it is easy to see

$$(e_2)_{xt}\cdot e_3=-k_1(k_1)_x \text{ and } (e_2)_{tx}\cdot e_3=k_2(k_2)_x+\nu_x,$$

which implies

$$\nu_x=-k_1(k_1)_x-k_2(k_2)_x=-\frac{1}{2}(k_1^2+k_2^2)_x.$$

So,

$$\nu=-\frac{1}{2}(k_1^2+k_2^2)+c(t).$$

Changing frames again to make $c(t)=0$, and then we have the equation (3.5) as desired. □

The compatibility condition of (3.5) leads to

$$\begin{cases}(k_1)_t=-(k_2)_{xx}-\sigma k_2-k_2\frac{k_1^2+k_2^2}{2}\\(k_2)_t=(k_1)_{xx}+\sigma k_1+k_1\frac{k_1^2+k_2^2}{2}\end{cases}, \tag{3.7}$$

therefore we have the following theorem.

THEOREM 3.5. *Let $\gamma(x,t)\in N$ be a solution to the curve evolution (3.1) parametrized by arc length with principal curvatures k_1,k_2 and a parallel frame $g\in G$ along γ with $G=SO(4)$ and $SO(1,3)$ when $N=\mathbb{S}^3$ and $\mathbb{H}^3$, respectively. Then $k(\cdot,t)=(k_1+ik_2)(\cdot,t)$ solves*

$$k_t=i(k_{xx}+\sigma k+\frac{1}{2}|k|^2k). \tag{3.8}$$

Proof. The assertion follows directly from (3.7). □

Let $q=\frac{k}{2}$. Then it is obvious that

$$q_t=i(q_{xx}+\sigma q+2|q|^2q). \tag{3.9}$$

We denoted the equation (3.9) by (GP$^\pm$), where $\pm$ indicates the sign of the external potential q, i.e., (3.8) is called (GP$^+$) or (GP$^-$) when $\sigma=1$ or $\sigma=-1$, respectively.

4. Lax Pair of GP$^\pm$. In the previous section, we see that the $*$-MCF is related to (3.9), which also can be transformed to the NLS. In this section, we will give Lax pairs of (GP$^\pm$), and the relation between (GP$^\pm$) and NLS, which implies that (GP$^\pm$) is integrable. We first review some known results of the NLS.

Suppose a is a 2-by-2 diagonal matrix with entries i and $-i$,

$$u=\begin{pmatrix}0&q\\-\bar{q}&0\end{pmatrix},\quad Q_{-1}(u)=\frac{i}{2}\begin{pmatrix}-|q|^2&q_x\\\bar{q}_x&|q|^2\end{pmatrix},$$

a Lax pair of the NLS

$$q_t = \frac{i}{2}(q_{xx} + 2|q|^2 q) \tag{4.1}$$

is known as

$$\theta = (a\lambda + u)dx + \big(a\lambda^2 + u\lambda + Q_{-1}(u)\big)\, dt. \tag{4.2}$$

The following statement is known and the readers can see more details in other articles or books.

Proposition 4.1 (cf. [13], [14]). *Given a smooth $q : \mathbb{R}^2 \to \mathbb{C}$, the following statements are equivalent:*

1) *q is a solution of the NLS,*
2) *θ is a flat connection one form,*
3) *the ODE system*

$$\begin{cases} E^{-1}E_x = a\lambda + u \\ E^{-1}E_t = a\lambda^2 + u\lambda + Q_{-1}(u) \end{cases}$$

is solvable for $E(x,t,\lambda) \in SL(2,\mathbb{C})$ satisfying $\overline{E(x,t,\bar{\lambda})} = E(x,t,\lambda)$. (A solution E that is holomorphic for $\lambda \in \mathbb{C}$ is called a frame).

Due to the connection between solutions of ($\mathrm{GP}^{\pm}$) and NLS, we are able to construct Lax pairs of ($\mathrm{GP}^{\pm}$).

Proposition 4.2. *Let $\sigma = \pm 1$, $a = \mathrm{diag}(i, -i)$,*

$$u = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix} \quad \textit{and} \quad Q_{-1} = \frac{i}{2}\begin{pmatrix} -|q|^2 & q_x \\ \bar{q}_x & |q|^2 \end{pmatrix}.$$

1) *If v is a solution of the focusing NLS (4.1), then a solution to ($GP^{\pm}$) is given via the transform*

$$q = e^{\sigma \frac{i}{2} t} v. \tag{4.3}$$

2) *The equation ($GP^{\pm}$)*

$$q_t = \frac{i}{2}(q_{xx} + \sigma q + 2|q|^2 q)$$

has a Lax pair

$$\tau = (a\lambda + u)dx + (a\lambda^2 + u\lambda + Q_{-1} - \frac{\sigma}{4}a)dt. \tag{4.4}$$

3) *If $E(x,t,\lambda)$ is a frame of the solution q of the NLS, then*

$$F(x,t,\lambda) = E(x,t,\lambda)e^{\pm \frac{at}{4}}$$

is a frame of the solution $q^{\pm} = qe^{\pm \frac{it}{2}}$ of the ($GP^{\pm}$).

Proof. (1) follows from direct calculations and the flatness of τ implies (2). For (3), we let $g = e^{\frac{\sigma}{4}at}$ and then

$$\begin{aligned} F^{-1}dF &= \tau \\ &= g\theta g^{-1} - dgg^{-1} \\ &= gE^{-1}(dEg^{-1} - Eg^{-1}dgg^{-1}) \\ &= gE^{-1}d(Eg^{-1}). \end{aligned}$$

□

We also notice that two frames of the solution for the ($\mathrm{GP}^{\pm}$) are differed by a constant element in $SU(2)$.

5. Solutions to $*$-MCF on $\mathbb{S}^3$ and $\mathbb{H}^3$. In this section, we construct solutions of the $*$-MCF on $\mathbb{S}^3$ and $\mathbb{H}^3$ under identifications of the ambient spaces $\mathbb{R}^4$ and $\mathbb{R}^{1,3}$. For later use, we write down the identifications in need as below.

$\mathbb{R}^4$ as the Quaternions. (cf. [1]) Now we identify $\mathbb{R}^4$ as the quaternion matrices $\mathcal{H}$, where

$$\mathcal{H} \equiv \{\begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix} \mid \alpha, \beta \in \mathbb{C} = \mathbb{R}^2\}.$$

As a real vector space, $\mathcal{H}$ has a standard basis consisting of the four elements

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, a = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, b = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \text{ and } c = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{5.1}$$

with multiplication rules

$$\begin{aligned} &a^2 = b^2 = c^2 = -I_2, \\ &ab = -ba = c, bc = -cb = a, \text{ and } ca = -ac = b. \end{aligned}$$

We identify $\mathcal{H}$ as the Euclidean $\mathbb{R}^4$ via

$$xI_2 + ya + zb + wc = \begin{pmatrix} x+iy & z+iw \\ -z+iw & x-iy \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix}. \tag{5.2}$$

Let $SU(2) \times SU(2)$ act on $\mathbb{R}^4$ by

$$(h_-, h_+) \cdot v = h_- v h_+^{-1},$$

where $(h_-, h_+) \in SU(2) \times SU(2)$ and $v \in \mathbb{R}^4$.

This gives an isomorphism $SO(4) \cong SU(2) \times SU(2)/ \pm I_2$. Let $\delta = (I_2, a, b, c)$ be an orthonormal basis of $\mathcal{H}$, where I_2, a, b and c are defined as in (5.1). Denote $(h_-, h_+) \cdot \delta$ by

$$(h_-, h_+) \cdot \delta = (h_- h_+^{-1}, h_- a h_+^{-1}, -h_- c h_+^{-1}, h_- b h_+^{-1}). \tag{5.3}$$

Then $(h_-, h_+) \cdot \delta$ is in $SO(4)$.

$\mathbb{R}^{1,3}$ **as the collection of Hermitian matrices.** (cf. [1]) Every point $\mathbf{x} \in \mathbb{R}^{1,3}$ can be identified as a 2×2 self-adjoint matrix as follows:

$$x = \begin{pmatrix} x_0 - x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 + x_3 \end{pmatrix} \text{ represents } \mathbf{x} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}. \tag{5.4}$$

Notice that

$$x^* = \bar{x}^t = x \text{ and } \det x = \|\mathbf{x}\|^2 = x_0^2 - x_1^2 - x_2^2 - x_3^2.$$

Let $\mathcal{M}$ be the collection of 2×2 self-adjoint matrices. Then $\mathcal{M}$ is a 4-dimensional real vector space, whose basis consists of the identity matrix I_2 and

$$a_M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, b_M = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \text{ and } c_M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{5.5}$$

We have identified the vector $\mathbf{x} = (x_0, x_1, x_2, x_3)^t$ with the matrix

$$x = x_0 I_2 + x_1 c_M + x_2 b_M + x_3 a_M.$$

Let $\mathcal{M}$ be equipped with an inner product

$$< x, y >= \frac{1}{2}\text{tr}(xy),$$

then $\delta_M = (I_2, a_M, b_M, c_M)$ is an orthonormal basis of $\mathcal{M}$. Let $SL(2,\mathbb{C})$ act on $\mathcal{M}$ by

$$h \cdot v = hv\bar{h}^t,$$

where $h \in SL(2,\mathbb{C})$ and $v \in \mathcal{M}$. This action preserves the determinant. Therefore, it maps to a subgroup of $O(1,3)$. Indeed, the image is the identity component of $O(1,3)$. Moreover, $h \cdot \delta_{\mathcal{M}}$ is in $SO(1,3)$. Note that

$$(a_M, b_M, c_M) = -i(a, b, c), \tag{5.6}$$

where (a, b, c) is defined as in (5.1).

5.1. Constructions of Solutions to $*$-MCF on $\mathbb{S}^3$. There are many solutions for the NLS constructed, and hence solutions to (GP$^\pm$) are obtained via $e^{it/2}$-transform. We have seen how the $*$-MCF relates to (GP$^\pm$) and their Lax pairs in previous sections. The natural question is that can we construct a $*$-MCF corresponding to a given solution of (GP$^\pm$)? The answer is positive. We will use such a correspondence to establish solutions to $*$-MCF. That is, solutions to $*$-MCF can be written down explicitly.

THEOREM 5.1. *Let $\lambda_1, \lambda_2 \in \mathbb{R} \setminus \{0\}$ and $\lambda_1 \neq \lambda_2$. Suppose E is a frame of a solution q to (GP^+) and define*

$$\eta(x,t) = E(x,t,\lambda_1)E(x,t,\lambda_2)^{-1}. \tag{5.7}$$

Then

$$\gamma(x,t) = \eta\left(\frac{1}{\lambda_1 - \lambda_2}x - \frac{\lambda_1 + \lambda_2}{\lambda_1 - \lambda_2}t, \frac{1}{\lambda_1 - \lambda_2}t\right) \tag{5.8}$$

is a solution of $*$-*MCF* (3.1) *on* $\mathbb{S}^3$.

Proof. Note that for any arbitrary λ, $E(x,t,\lambda)$ satisfies (4.4), that is,

$$E^{-1}E_x = a\lambda + u \quad \text{and} \quad E^{-1}E_t = a\lambda^2 + u\lambda + Q_{-1} - \frac{a}{4},$$

where $a = \mathrm{diag}(i,-i)$,

$$u = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix} \quad \text{and} \quad Q_{-1} = \frac{i}{2}\begin{pmatrix} -|q|^2 & q_x \\ \bar{q}_x & |q|^2 \end{pmatrix}.$$

We denote $E(x,t,\lambda_j)$ by E_j and $E_{j,z}$ indicates the partial derivative of E_j with respect to the variable z. Then it is easy to see

$$\begin{aligned} \eta_x &= E_{1,x}E_2^{-1} - E_1E_2^{-1}E_{2,x}E_2^{-1} \\ &= E_1(a\lambda_1 + u)E_2^{-1} - E_1(a\lambda_2 + u)E_2^{-1} \\ &= (\lambda_1 - \lambda_2)E_1aE_2^{-1}, \end{aligned} \tag{5.9}$$

and similarly,

$$\eta_t = (\lambda_1^2 - \lambda_2^2)E_1aE_2^{-1} + (\lambda_1 - \lambda_2)E_1uE_2^{-1}. \tag{5.10}$$

Let

$$\begin{aligned} &e_0 = E_1E_2^{-1}(\tilde{x},\tilde{t}), \quad e_1 = E_1aE_2^{-1}(\tilde{x},\tilde{t}), \\ &e_2 = -E_1cE_2^{-1}(\tilde{x},\tilde{t}), \quad e_3 = E_1bE_2^{-1}(\tilde{x},\tilde{t}), \end{aligned}$$

where $\tilde{x} = \frac{1}{\lambda_1-\lambda_2}x - \frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}t$, $\tilde{t} = \frac{1}{\lambda_1-\lambda_2}t$ and a, b, c are the same as stated in (5.1). Since $E_1, E_2 \in SU(2)$ and $(I_2, a, -c, b)$ is an orthonormal basis for $\mathcal{H}$, we see $(e_0, e_1, e_2, e_3) \in SO(4)$. We write $q = q_1 + iq_2$ and hence $u = q_1b + q_2c$. Use (5.9) and (5.10) to obtain

$$\begin{aligned} \gamma_x &= \tfrac{1}{\lambda_1-\lambda_2}\eta_x = e_1, \\ \gamma_t &= -\tfrac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}\eta_x + \tfrac{1}{\lambda_1-\lambda_2}\eta_t \\ &= E_1uE_2^{-1} \\ &= q_1e_3 - q_2e_2. \end{aligned} \tag{5.11}$$

We compute the following items:

$$\begin{aligned} (e_0)_x &= e_1, \\ (e_1)_x &= \tfrac{1}{(\lambda_1-\lambda_2)^2}\eta_{xx} \\ &= \tfrac{1}{(\lambda_1-\lambda_2)^2}\left\{-(\lambda_1-\lambda_2)^2e_0 + (\lambda_1-\lambda_2)(2q_1e_2 + 2q_2e_3)\right\} \\ &= -e_0 + \tfrac{1}{\lambda_1-\lambda_2}(2q_1e_2 + 2q_2e_3), \\ (e_2)_x &= -\tfrac{2q_1}{\lambda_1-\lambda_2}e_1 + \tfrac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}e_3, \end{aligned} \tag{5.12}$$

namely, if $g = (e_0, e_1, e_2, e_3)$, then

$$g^{-1}g_x = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -\frac{2q_1}{\lambda_1-\lambda_2} & -\frac{2q_2}{\lambda_1-\lambda_2} \\ 0 & \frac{2q_1}{\lambda_1-\lambda_2} & 0 & -\frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2} \\ 0 & \frac{2q_2}{\lambda_1-\lambda_2} & \frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2} & 0 \end{pmatrix}. \tag{5.13}$$

Next, we compute the t-derivative of g. We also notice that (5.11) implies $(e_0)_t = \gamma_t = q_1e_3 - q_2e_2$ and $(e_1)_t = (\gamma_x)_t = (\gamma_t)_x = (q_1e_3 - q_2e_2)_x$. Together with (5.13), we get

$$(e_1)_t = (-\frac{1}{\lambda_1-\lambda_2}(q_2)_x - \frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}q_1)e_2 + (\frac{1}{\lambda_1-\lambda_2}(q_1)_x - \frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}q_2)e_3.$$

Since $(e_2)_t$ can be computed as

$$q_2e_0 + (\frac{1}{\lambda_1-\lambda_2}(q_2)_x + \frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}q_1)e_1 + (-\frac{1}{2(\lambda_1-\lambda_2)} - \frac{2\lambda_1\lambda_2+|q|^2}{\lambda_1-\lambda_2})e_3,$$

we obtain $g^{-1}g_t = A$, where A is equal to

$$\begin{pmatrix} 0 & 0 & q_2 & -q_1 \\ 0 & 0 & \frac{(q_2)_x}{\lambda_1-\lambda_2} + \frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}q_1 & -\frac{(q_1)_x}{\lambda_1-\lambda_2} + \frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}q_2 \\ -q_2 & -\frac{(q_2)_x}{\lambda_1-\lambda_2} - \frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}q_1 & 0 & \frac{1}{2(\lambda_1-\lambda_2)} + \frac{2\lambda_1\lambda_2+|q|^2}{\lambda_1-\lambda_2} \\ q_1 & \frac{(q_1)_x}{\lambda_1-\lambda_2} - \frac{\lambda_1+\lambda_2}{\lambda_1-\lambda_2}q_2 & -\frac{1}{2(\lambda_1-\lambda_2)} - \frac{2\lambda_1\lambda_2+|q|^2}{\lambda_1-\lambda_2} & 0 \end{pmatrix}. \quad (5.14)$$

□

In particular, by choosing $\lambda_1 = 1, \lambda_2 = 0$, one obtains a neater solution. We state as follows:

Corollary 5.2. *Let E be a frame of a solution q of (GP^+) and $\eta(x,t)$ is defined as in* (5.7). *Then $\gamma(x,t) = \eta(x-t,t)$ solves the $*$-MCF on $\mathbb{S}^3$* (3.1) *with principal curvatures $2q_1, 2q_2$. Moreover, let $\phi(x,t) = E(x,t,1), \psi(x,t) = E(x,t,0)$ and $g = (\phi,\psi)\cdot\delta(x-t,t)$. Then g satisfies*

$$\begin{cases} g^{-1}g_x = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -2q_1 & -2q_2 \\ 0 & 2q_1 & 0 & -1 \\ 0 & 2q_2 & 1 & 0 \end{pmatrix}, \\ g^{-1}g_t = \begin{pmatrix} 0 & 0 & q_2 & -q_1 \\ 0 & 0 & (q_2)_x + q_1 & -(q_1)_x + q_2 \\ -q_2 & -(q_2)_x - q_1 & 0 & \frac{1}{2} + |q|^2 \\ q_1 & (q_1)_x - q_2 & -\frac{1}{2} - |q|^2 & 0 \end{pmatrix}, \end{cases} \quad (5.15)$$

where $q = q_1 + iq_2$.

Notice that a frame E is not unique. In other words, the solution constructed in this method is unique (up to the conjugation).

Proposition 5.3. *Let q be a solution of (GP^+) and E its frame. Let $F = CE$ for some constant $C \in SU(2)$ and $\gamma(x,t)$ defined as in Corollary 5.2. Define*

$$\tilde\eta(x,t) = F(x,t,1)F(x,t,0)^{-1} \quad \textit{and} \quad \tilde\gamma(x,t) = \tilde\eta(x-t,t).$$

Then F is again a frame for q and $\tilde\gamma(x,t)$ is also a solution of $*$-MCF, (3.1), *on $\mathbb{S}^3$.*

Proof. Denote E_1, E_0 by $E(x,t,1), E(x,t,0)$ and F_1, F_0 by $F(x,t,1), F(x,t,0)$, respectively. One sees that $F^{-1}dF = E^{-1}dE$ and $F_1F_0^{-1} = CE_1E_0^{-1}C^{-1}$, that is, $\tilde\gamma = C\gamma C^{-1}$. Since γ solves (3.1), we have

$$\tilde\gamma_t = C\gamma_tC^{-1} = CE_1uE_0^{-1}C^{-1} = F_1uF_0^{-1}. \quad (5.16)$$

□

According to (4.3), $*$-MCF on $\mathbb{H}^3$ is closely related to (GP$^-$) as well. And using the transform (4.3) again, the (GP$^-$) can be linked to (GP$^+$). Therefore, we derive a solution by means of frames for a solution u to (GP$^+$) and (5.6). A similar construction of solutions to $*$-MCF on $\mathbb{H}^3$ can be derived using Theorem 5.1. We state it as follows.

THEOREM 5.4 (Solutions on $\mathbb{H}^3$). *Let $\lambda = \frac{1}{2}(1-i)$ and suppose E is a frame of a solution q to (GP$^+$) and define*

$$\eta(x,t) = E(x,t,\lambda)E(x,t,\bar{\lambda})^{-1}. \tag{5.17}$$

Then

$$\gamma(x,t) = \eta(x-t,t) \tag{5.18}$$

is a solution of $*$-MCF (3.1) on $\mathbb{H}^3$.

Proof. Since E satisfies the $SU(2)$-reality condition, it is clear that

$$\eta_x = -iE(x,t,\lambda)aE(x,t,\bar{\lambda})^{-1} = E(x,t,\lambda)a_M E(x,t,\lambda)^*,$$

where a and a_M are defined as in (5.1) and (5.5).

Using the fact that E is a frame of (GP$^+$) together with(5.2) and (5.5), it is easy to show

$$\begin{aligned} \eta_t &= -iE(x,t,\lambda)aE(x,t,\bar{\lambda})^{-1} + (-i)E(x,t,\lambda)uE(x,t,\bar{\lambda})^{-1} && (5.19)\\ &= \eta_x + q_1E(x,t,\lambda)b_ME(x,t,\bar{\lambda})^{-1} + q_2E(x,t,\lambda)c_ME(x,t,\bar{\lambda})^{-1}, && (5.20)\end{aligned}$$

and

$$\begin{aligned} \eta_{xx} &= -i\left(E(x,t,\lambda)(\bar{\lambda}-\lambda+[u,a])E(x,t,\bar{\lambda})^{-1}\right)\\ &= E(x,t,\lambda)E(x,t,\bar{\lambda})^{-1} - iE(x,t,\lambda)[q_1b+q_2c,a]E(x,t,\bar{\lambda})^{-1}\\ &= \eta - iE(x,t,\lambda)(2q_2b-2q_1c)E(x,t,\bar{\lambda})^{-1}\\ &= \eta + 2q_2E(x,t,\lambda)b_ME(x,t,\bar{\lambda})^{-1} - 2q_1E(x,t,\lambda)c_ME(x,t,\bar{\lambda})^{-1}.\end{aligned}$$

Therefore, $\gamma_x = \eta_x$ and $\gamma_t = -\eta_x + \eta_t$ as desired. □

Similarly, choose any arbitrary $\lambda \in \mathbb{C}\setminus\mathbb{R}$, solutions to $*$-MCF on $\mathbb{H}^3$ are constructed as follows.

THEOREM 5.5. *Suppose $\lambda = r - is$, where $r,s \in \mathbb{R}$ and $s > 0$. Let $E(x,t,\lambda)$ be a frame of a solution q to (GP$^+$). Define*

$$\eta(x,t) = E(x,t,\lambda)E(x,t,\bar{\lambda})^{-1} \text{ and } \gamma(x,t) = \eta\left(\frac{x}{2s} - \frac{r}{s}t, \frac{t}{2s}\right). \tag{5.21}$$

Then $\gamma(x,t)$ solves $*$-MCF on $\mathbb{H}^3$.

An immediate example follows from our discussion above and simple computations.

EXAMPLE 5.6. Note that $q = 0$ is the trivial solution of both NLS and (GP$^+$). Then we see that $E(x,t,\lambda) = \exp(a\lambda x + (a\lambda^2 - \frac{1}{4}a)t)$ is a frame of the $*$-MCF on $\mathbb{S}^3$, namely, $E^{-1}dE = \tau$ in (4.4). By Corollary 5.2 and having $\lambda = 0, 1$, we have $\eta(x,t) = e^{ax+at}$, which implies the solution of the $*$-MCF on $\mathbb{S}^3$ is the matrix

$$\begin{pmatrix} e^{ix} & 0 \\ 0 & e^{-ix}\end{pmatrix}.$$

Such a solution is identified as a vector $(\cos x, \sin x, 0, 0)^t \in \mathbb{R}^4$. On the other hand, Theorem 5.4 ($\lambda = \frac{1}{2}(1-i)$) shows that

$$\eta(x,t) = E(x,t,\lambda)E(x,t,\bar{\lambda})^{-1} = e^{-i(ax+at)},$$

and the solution of $*$-MCF on $\mathbb{H}^3$ is therefore $\gamma(x,t) = e^{-iax}$. Indeed, the solution on $\mathbb{H}^3$ is $(\cosh x, 0, 0, -\sinh x)^t$.

5.2. Cauchy Problems. Using the correspondence between Lax pairs of $(\mathrm{GP}^{\pm})$ and $*$-MCF, we are able to write down the explicit solutions to the curve evolution. In this section, we further investigate the Cauchy problem of the $*$-MCF with an arbitrary initial curve or a periodic one. Without loss of generality, we assume the period is 2π.

THEOREM 5.7 (Cauchy problem for $*$-MCF on $\mathbb{S}^3$). *Let $\gamma_0(x) : \mathbb{R} \to \mathbb{S}^3$ be an arc-length parametrized curve, $g_0(x)$ a parallel frame along γ_0, and q_1, q_2 the corresponding principal curvatures. Given $\phi_0, \psi_0 \in SU(2)$ such that $g_0(0) = (\phi_0, \psi_0) \cdot \delta$. Suppose $q = k_1 + ik_2$ is a solution of (GP^+) with $q(\cdot, 0) = q_1 + iq_2$. Let E, F be the frames of q satisfying $E(0,0,\lambda) = \phi_0, F(0,0,\lambda) = \psi_0$, $\eta(x,t) = E(x,t,1)F(x,t,0)^{-1}$ and $\alpha(x,t) = \eta(x-t,t)$. Then $\gamma(x,t) = \alpha(x,t) - \eta(0,0) + \gamma_0(0)$ is a solution of* (3.1) *with $\gamma(x,0) = \gamma_0(x)$.*

Proof. Note that $\gamma_t = \alpha_t$ and Theorem 5.1 shows that γ satisfies the $*$-MCF (3.1). In particular, $\alpha(x,0) = \eta(x,0)$. We claim that $\eta(x,0) = \gamma_0(x) + \eta(0,0) - \gamma_0(0)$. In this case, one obtains $\gamma(x,0) = \gamma_0(x)$. Note that

$$\eta_x(x,0) = E(x,0)aF(x,0)^{-1} = \phi a \psi^{-1} = \gamma_0'(x), \tag{5.22}$$

which implies

$$\eta(x,0) = \gamma_0(x) + c,$$

for some constant c. So $c = \eta(0,0) - \gamma_0(0)$. ◻

Next, we turn our attention to construct x-periodic solutions to $*$-MCF (3.1) on $\mathbb{S}^3$. By the construction of solutions in Theorem 5.1, the formula (5.7) implies that it suffices to find periodic frames E.

THEOREM 5.8. *Let $\gamma(x,t)$ be an arc-length parametrized solution of the $*$-MCF on $\mathbb{S}^3$ and periodic in x with period 2π. Suppose $(e_0, e_1, \vec{n}_2, \vec{n}_3)$ is orthonormal along γ such that $e_0 = \gamma$ and $e_1 = \gamma_x$. Let $\omega = (\vec{n}_2)_x \cdot \vec{n}_3$. Then $c_0 = \frac{1}{2\pi}\int_0^{2\pi} \omega(x,t)\, dx$ is constant for all t, and there exists $g = (u_0, u_1, u_2, u_3)(x,t)$ such that*

1) *$g(\cdot, t)$ is a periodic h-frame along $\gamma(\cdot, t)$,*

2) $$g^{-1}g_x = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & -\zeta_1 & -\zeta_2 \\ 0 & \zeta_1 & 0 & -2c_0 \\ 0 & \zeta_2 & 2c_0 & 0 \end{pmatrix},$$

3) *$q = \frac{1}{2}(\zeta_1 + i\zeta_2)$ is a solution of the* (GP^+).

LEMMA 5.9. *Let q be a x-periodic solution of* (GP^+) with period 2π, $\lambda_0 \in \mathbb{R}$, and $E(x,t,\lambda)$ the extended frame of q. If $E(x,0,\lambda_0)$ is periodic in x with period 2π, then so is $E(x,t,\lambda_0)$ for all t.

Proof. Recall that E satisfies the following linear system

$$\begin{cases} E^{-1}E_x = a\lambda + u \\ E^{-1}E_t = a\lambda^2 + u\lambda + Q_{-1} - \frac{a}{4} \end{cases}. \tag{5.23}$$

Let $y(t) = E(2\pi, t, \lambda_0) - E(0, t, \lambda_0)$ and $A(x,t) = a\lambda_0^2 + u\lambda_0 + Q_{-1} - \frac{a}{4}$. Note that $A(2\pi, t) = A(0,t)$ because of periodicity of q. Take the derivative with respect to t to obtain

$$\begin{aligned} y'(t) &= E_t(2\pi, t, \lambda_0) - E_t(0, t, \lambda_0) \\ &= E(2\pi, t, \lambda_0)A(2\pi, t) - E(0, t, \lambda_0)A(0,t) \\ &= y(t)A(0,t). \end{aligned} \tag{5.24}$$

Since $y(0) = 0$ solves $y'(t) = y(t)A(0,t)$, the uniqueness theorem of ODE implies that $y(t)$ is identically zero. □

As a consequence of Theorem 5.1 and Lemma 5.9, we have the following.

THEOREM 5.10 (Periodic Cauchy problem for $*$-MCF on $\mathbb{S}^3$). *Given a closed curve $\gamma_0(x) : [0, 2\pi] \to \mathbb{S}^3$ parametrized by arc length and principal curvatures q_1^0, q_2^0. Let $(e_0^0, e_1^0, u_2^0, u_3^0)$ be a h-frame along γ_0 and $\phi, \psi \in SU(2)$ such that*

$$(e_0^0, e_1^0, u_2^0, u_3^0) = (\phi\psi^{-1}, \phi a\psi^{-1}, -\phi c\psi^{-1}, \phi b\psi^{-1}),$$

where a, b, c are defined as in (5.1). *In addition, let $q : \mathbb{R}^2 \to \mathbb{C}$ be a periodic solution of (GP^+) with initial data $q(x,0) = \frac{1}{2}(q_1^0 + iq_2^0)e^{-ic_0x}$, where c_0 is the normal holonomy of γ_0. Let E and F be frames with $E(0,0,c_0+1) = \phi$ and $F(0,0,c_0) = \psi$. Define*

$$\eta(x,t) = EF^{-1}(x,t) \quad \textit{and} \quad \alpha(x,t) = \eta(x - (2c_0+1)t, t).$$

Then $\gamma(x,t) = \alpha(x,t) - \eta(0,0) + \gamma_0(0)$ is a solution of periodic Cauchy problem of $*$-MCF on $\mathbb{S}^3$ with initial data $\gamma_0(x)$.

Proof. Theorems 5.7 and 5.1 imply that γ is a solution of $*$-MCF and the periodicity of γ follows from Lemma 2.1 and Lemma 5.9. □

6. Bäcklund Transformation. From a conceptual point of view, once a solution $q(x)$ of the NLS is given, and consequently, through formula (4.3), one obtains a solution $Q(x)$ of GP. By solving the standard Bäcklund Transformation (BT) for the NLS [11], a new solution $q(x)$ of NLS is derived and thus, using the transformation (4.3) again, one recovers a new solution of the GP equation. Of course, since we have established the correspondence between $*$-MCF and the GP equation, we shall try to construct a BT for $*$-MCF on a 3-sphere and $\mathbb{H}^3$. In this section, we first state the BT for NLS and give one-soliton solutions for both NLS and $*$-MCF.

Given $\alpha \in \mathbb{C} \setminus \mathbb{R}$, a Hermitian projection π of $\mathbb{C}^2$, and let

$$g_{\alpha,\pi}(\lambda) = I + \frac{\alpha - \bar{\alpha}}{\lambda - \alpha}\pi^{\perp}, \tag{6.1}$$

where $\pi^{\perp} = I - \pi$. Then $g_{\alpha,\pi}(\lambda)^{-1} = g_{\alpha,\pi}(\bar{\lambda})^*$.

THEOREM 6.1 (Algebraic BT for NLS). ([11]) *Let $E(x,t,\lambda)$ be a frame of a solution $u = \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}$ of the NLS, π the Hermitian projection of $\mathbb{C}^2$ onto $\mathbb{C}v$, and*

$\alpha \in \mathbb{C} \setminus \mathbb{R}$. Let $\tilde{v} = E(x,t,\alpha)^{-1}(v)$, and $\tilde{\pi}$ the Hermitian projection of $\mathbb{C}^2$ onto $\mathbb{C}\tilde{v}$. Then

$$\tilde{u} = u + (\bar{\alpha} - \alpha)[\tilde{\pi}, a]$$

is a solution of the NLS. Moreover, $\tilde{E}(x,t,\lambda) = g_{\alpha,\pi}(\lambda)E(x,t,\lambda)g^{-1}_{\alpha,\tilde{\pi}(x,t)}$ is a new frame for $\tilde{u}$.

Let

$$W = e^{-\frac{\sigma}{4}at},$$

where $\sigma = 1$ and -1 for $\mathbb{S}^3$ and $\mathbb{H}^3$, respectively. Since $\tilde{E}(x,t,\lambda)$ described in Theorem 6.1 is a new frame for a solution $\tilde{u}$ to NLS, the relation between frames the GP and NLS in Proposition 4.2 implies that $\tilde{F} := g_{\alpha,\pi}(\lambda)E(x,t,\lambda)g^{-1}_{\alpha,\tilde{\pi}(x,t)}(\lambda)W$ is a new frame of the (GP$^+$).

Lemma 6.2. *Suppose $\tilde{E}(x,t,\lambda) = g_{\alpha,\pi}(\lambda)E(x,t,\lambda)g^{-1}_{\alpha,\tilde{\pi}(x,t)}(\lambda)$ and $F(x,t,\lambda) = E(x,t,\lambda)W$ is a frame of the (GP$^+$), where $\alpha, E(x,t,\lambda), \tilde{\pi}$ and $g_{\alpha,\tilde{\pi}(x,t)}$ are defined as that in Theorem 6.1. Then*

$$\tilde{F} = g_{\alpha,\pi}FW^{-1}g^{-1}_{\alpha,\tilde{\pi}(x,t)}W \tag{6.2}$$

is a new frame of the (GP$^+$).

Theorem 6.3 (Algebraic BT for $*$-MCF on $\mathbb{S}^3$). *Let γ be a solution of $*$-MCF on $\mathbb{S}^3$ and F the frame of a solution q of (GP$^+$). Let π be the Hermitian projection of $\mathbb{C}^2$ onto $\mathbb{C}v$, and $\alpha \in \mathbb{C} \setminus \mathbb{R}$. Let $\tilde{v} = WF(x,t,\alpha)^{-1}(v)$, and $\tilde{\pi}$ the Hermitian projection of $\mathbb{C}^2$ onto $\mathbb{C}\tilde{v}$. Then*

$$\tilde{\eta}(x,t) = g_0\left(\frac{\alpha(1-\bar{\alpha})}{\bar{\alpha}(1-\alpha)}\gamma + \frac{\bar{\alpha}-\alpha}{\bar{\alpha}(1-\alpha)}\phi_0 W^{-1}\tilde{\pi}W\phi_1^{-1}\right)g_1^{-1} \tag{6.3}$$

and $\tilde{\gamma}(x,t) = \tilde{\eta}(t-x,-t)$ is a new solution of $*$-MCF, where $W = e^{-\frac{1}{4}at}, g_0 = g_{\alpha,\pi}(0), g_1 = g_{\alpha,\pi}(1), \phi_0 = F(x,t,0)$, and $\phi_1 = F(x,t,1)$.

Proof. Since $\tilde{F}$ defined as in Lemma 6.2 is a new frame of the (GP$^+$), Theorem 5.1 implies that

$$\begin{aligned}\tilde{\gamma} &= \tilde{F}(x,t,0)\tilde{F}(x,t,1)^{-1} && (6.4)\\ &= g_{\alpha,\pi}(0)F(x,t,0)W^{-1}(\tilde{g}_{\alpha,\tilde{\pi}(x,t)}(0)^{-1}\tilde{g}_{\alpha,\tilde{\pi}(x,t)}(1))WF(x,t,1)^{-1}g_{\alpha,\pi}(1)^{-1} && (6.5)\\ &= g_{\alpha,\pi}(0)F(x,t,0)W^{-1}\left(\frac{\alpha(1-\bar{\alpha})}{\bar{\alpha}(1-\alpha)}\mathrm{I} + \frac{\bar{\alpha}-\alpha}{\bar{\alpha}(1-\alpha)}\tilde{\pi}\right)WF(x,t,1)^{-1}g_{\alpha,\pi}(1)^{-1}. && (6.6)\end{aligned}$$

The equality (6.6) is obtained by the definition (6.1) and properties of the Hermitian projection $\tilde{\pi}$. Multiplying it out to have the desired result. □

Based on the connection between constructions of solutions to $*$-MCF on $\mathbb{S}^3$ and $\mathbb{H}^3$ (see Theorems 5.1, 5.4), we omit the similar statement for BT for $*$-MCF on $\mathbb{H}^3$. One-soliton solutions are illustrated below for the $*$-MCF.

Example 6.4 (Soliton solutions for the $*$-MCF). Note that $\gamma(x,t) = e^{ax}$ is a solution of the star mean curvature flow on $\mathbb{S}^3$. Then $q = 0$ is the corresponding trivial solution of NLS and $E(x,t,\lambda)=\exp(a\lambda x+a\lambda^2 t)$ is a frame for $q=0$ with $E(x,t,0)=\mathrm{Id}$.

Let $\alpha=r+is\in\mathbb{C}\setminus\mathbb{R}$, $v=(1,i)^t$, and $\pi=\frac{1}{||v||^2}vv^*$ the Hermitian projection of $\mathbb{C}^2$ onto $\mathbb{C}v$. Then

$$\tilde{v}(x,t):=E(x,t,\alpha)^{-1}v=(e^{-(A+iB)},ie^{A+iB})^t,$$

where $A=-(sx+2rst), B=rx+(r^2-s^2)t$ and the Hermitian projection $\tilde{\pi}(x,t)$ onto $\mathbb{C}\tilde{v}(x,t)$ is

$$\tilde{\pi}(x,t)=\frac{1}{e^{-2A}+e^{2A}}\begin{pmatrix} e^{-2A} & -ie^{-2iB}\\ ie^{2iB} & e^{2A}\end{pmatrix}. \tag{6.7}$$

It follows from Theorem 6.3 that a new solution on $\mathbb{S}^3$ is given by

$$\tilde{\eta}=\left(\frac{\bar{\alpha}}{\alpha}\mathrm{Id}+\frac{\alpha-\bar{\alpha}}{\alpha}\pi\right)\left(\frac{\alpha(1-\bar{\alpha})}{\bar{\alpha}(1-\alpha)}\mathrm{Id}+\frac{\bar{\alpha}-\alpha}{\bar{\alpha}(1-\alpha)}\tilde{\pi}\right)e^{-(ax+at)}\left(\frac{1-\alpha}{1-\bar{\alpha}}\mathrm{Id}+\frac{\alpha-\bar{\alpha}}{1-\bar{\alpha}}\pi\right).$$

If we choose $\alpha=i$, then

$$\tilde{\eta}=\frac{1}{D}\begin{pmatrix} e^{i(t+x)}\left((1+i)+2e^{(2-2i)x}+(1-i)e^{4x}\right) & e^{i(x+t)}\left(-1-i+2e^{(2-2i)x}+(-1+i)e^{4x}\right)\\ e^{-i(x+t)}\left(1-i-2e^{(2+2i)x}+(1+i)e^{4x}\right) & e^{-i(t+x)}\left((1-i)+2e^{(2+2i)x}+(1+i)e^{4x}\right)\end{pmatrix}, \tag{6.8}$$

where $D=2(1+e^{4x})$. The vector form of $\tilde{\eta}(x,t)$ on $\mathbb{S}^3$ is

$$\frac{1}{2}\begin{pmatrix} \cos(t-x)\,\mathrm{sech}(2x)+\cos(t+x)+\sin(t+x)\tanh(2x)\\ \sin(t-x)\,\mathrm{sech}(2x)+\sin(t+x)-\cos(t+x)\tanh(2x)\\ \cos(t-x)\,\mathrm{sech}(2x)-\cos(t+x)-\sin(t+x)\tanh(2x)\\ \sin(t-x)\,\mathrm{sech}(2x)-\sin(t+x)+\cos(t+x)\tanh(2x)\end{pmatrix}.$$

A 1-soliton solution of the $*$-MCF on $\mathbb{S}^3$ is given by $\tilde{\gamma}(x,t)=\tilde{\eta}(t-x,-t)$.

Furthermore, Theorem 5.5 suggests that a 1-soliton solution of the $*$-MCF on $\mathbb{H}^3$ is given by a new frame $\tilde{F}$ in Theorem 6.3 with $\lambda_0=\frac{1-i}{2}$. With $\alpha=i$, we indeed have 1-soliton solution on $\mathbb{H}^3$

$$\tilde{\eta}(x,t)=\begin{pmatrix}\tilde{\eta}_{11} & \tilde{\eta}_{12}\\ \tilde{\eta}_{21} & \tilde{\eta}_{22}\end{pmatrix},$$

where

$$\begin{aligned}
\tilde{\eta}_{11} &= \frac{1}{10}\,\mathrm{sech}(2x)(15\cosh(t-x)+3\cosh(t+3x)-8(\cos(2t-x)+\sin(2t-x))\\
&\qquad -5\sinh(t-x)-\sinh(t+3x)),\\
\tilde{\eta}_{12} &= \frac{1}{5}\,\mathrm{sech}(2x)(6i\cos(2t-x)-5i\cosh(t-x)-i\cosh(t+3x)-2\sin(2t-x)\\
&\qquad +5\sinh(t-x)+\sinh(t+3x)),\\
\tilde{\eta}_{21} &= \frac{1}{5}\,\mathrm{sech}(2x)(-6i\cos(2t-x)+5i\cosh(t-x)+i\cosh(t+3x)-2\sin(2t-x)\\
&\qquad +5\sinh(t-x)+\sinh(t+3x)),\\
\tilde{\eta}_{22} &= \frac{1}{10}\,\mathrm{sech}(2x)(-8\cos(2t-x)+15\cosh(t-x)+3\cosh(t+3x)+8\sin(2t-x)\\
&\qquad +5\sinh(t-x)+\sinh(t+3x)).
\end{aligned}$$

Or, equivalently the following vector on $\mathbb{H}^3$

$$\frac{1}{10}\operatorname{sech}(2x)\begin{pmatrix} -8\cos(2t-x)+15\cosh(t-x)+3\cosh(t+3x) \\ -4\sin(2t-x)+10\sinh(t-x)+2\sinh(t+3x) \\ -12\cos(2t-x)+10\cosh(t-x)+2\cosh(t+3x) \\ 8\sin(2t-x)+5\sinh(t-x)+\sinh(t+3x) \end{pmatrix}.$$

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