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RIEMANNIAN AND KÄHLERIAN NORMAL COORDINATES*

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Abstract. In every point of a Kähler manifold there exist special holomorphic coordinates well adapted to the underlying geometry. Comparing these Kähler normal coordinates with the Riemannian normal coordinates defined via the exponential map we prove that their difference is a universal power series in the curvature tensor and its iterated covariant derivatives and devise an algorithm to calculate this power series to arbitrary order. As a byproduct we generalize Kähler normal coordinates to the class of complex affine manifolds with $(1,1)$–curvature tensor. Moreover we describe the Spencer connection on the infinite order Taylor series of the Kähler normal potential and obtain explicit formulas for the Taylor series of all relevant geometric objects on symmetric spaces.

Key words. Kähler potential, Spencer connection, hermitean symmetric spaces.

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1. Introduction. Situated at the crossroads of differential and algebraic geometry the geometry of Kähler manifolds is a very attractive topic of research, in particular both analytical and algebraic tools and ideas have been brought to bear on the topic. The main motivation of the article at hand is to understand the interrelationship between the Taylor series of infinite order of several objects relevant for Kähler geometry. Philosophically our study of Kähler manifolds is thus based in the jet calculus of differential geometry, a calculus which in contrast to tensor or exterior calculus tries to avoid taking actual derivatives at all cost. Instead of taking derivatives jet calculus focuses on the algebraic constraints satisfied by the jets of all relevant objects, ideally then these algebraic constraints are sufficient to determine all the jets by multiplication, which at higher orders is much simpler than differentiation.

Not to the least the success of this strategy depends on our ability to isolate the objects to which it can be applied in the first place. In affine geometry for example the relevant object turns out to be the backward parallel transport Φ^{-1}, which encodes the differential of the exponential map and describes the Taylor series of all covariantly parallel tensors. The parallel transport equation proved in Lemma 3.1 of this article is the algebraic constraint

$$N(N+1)\Phi^{-1} = \mathscr{R}\,\Phi^{-1} + N(\mathscr{T}\Phi^{-1})$$

on the Taylor series of the backward parallel transport, whose solution Φ^{-1} is easily found by multiplication. In passing we remark that in the torsion free case the Taylor series of Φ^{-1} was calculated by several authors before using either implicitly or explicitly properties of iterated covariant derivatives. The novelty in our argument is that the parallel transport equation appears directly as an integrability condition for the Jacobi equation.

In order to find the relevant objects for the study of Kähler geometry we recall an observation originally due to Bochner [Bo]: Riemannian normal coordinates on Kähler

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manifolds are not complex coordinates unless the manifold is flat. Bochner found remedy to this nuisance in singling out complex coordinates centered in an arbitrary point p of a Kähler manifold M, which are unique up to unitary transformations. Their lack of uniqueness is easily fixed by thinking of Kähler normal coordinates as anchored coordinates, local diffeomorphisms

$$\mathrm{knc}_p : T_pM \longrightarrow M, \qquad X \longmapsto \mathrm{knc}_p X$$

mapping the origin to p with differential $(\mathrm{knc}_p)_{*,0} = \mathrm{id}_{T_pM}$ under the usual identification $T_0(T_pM) \cong T_pM$. The difference element $K := \exp_p^{-1} \circ \mathrm{knc}_p$ and its inverse K^{-1} are "hidden" relevant objects in Kähler geometry, because their Taylor series relate to commonly considered objects like the Kähler potential. Inevitably then a pivotal role in this article is played by the proof of the following statement about the Taylor series of K and K^{-1}:

Theorem 4.6 (Universality of Kähler Normal Coordinates). *Every term in the Taylor series of the difference element $K = \exp_p^{-1} \circ \mathrm{knc}_p$ and its inverse K^{-1} in the origin of Kähler normal coordinates is a universal polynomial, independent of the manifold and its dimension, in the complex structure I_p, the curvature tensor R_p and its iterated covariant derivatives $(\nabla R)_p$, $(\nabla^2 R)_p$, ... evaluated at $p \in M$.*

Of course the theorem in itself can hardly be called surprising, its validity for example is implicitly taken for granted without even a fleeting comment in [Hi]. The rigorous proof in Section 4 however allows us to draw several conclusions about the nature of Kähler normal coordinates, for example Kähler normal coordinates are inherited by totally geodesic complex submanifolds according to Corollary 4.9. Moreover the proof of Theorem 4.6 provides us with a simple recursion formula for the Taylor series of the difference element K^{-1}.

Perhaps the most striking insight however to be taken from the proof of Theorem 4.6 is that Kähler normal coordinates, very much like their Riemannian counterpart, do only depend on the complex affine geometry and *not* on the metric structure, despite the fact that their characterization apparently involves the metric in form of the potential. In fact Kähler normal coordinates generalize to the much more general class of balanced complex affine manifolds: Complex manifolds M endowed with a torsion free connection ∇ on their real tangent bundle TM such that the associated almost complex structure is parallel $\nabla I = 0$ and the curvature tensor is a $(1,1)$–form in the sense $R_{IX,\,IY} = R_{X,\,Y}$ for all X, Y.

Using the recursion formula for K^{-1} formulated in Remark 4.10 in analogy to the parallel transport equation we calculate the initial terms of the Taylor series of K^{-1}, of the Kähler normal potential and the Riemannian distance to the origin. Actually the recursion formula is simple enough to be readily programmed in a computer algebra system, the problem is to standardize the numerous terms cropping up in the process. An interesting conclusion from these calculations is in any case that the total holomorphic sectional curvature

$$S_{\mathrm{total}}(X) := \sum_{k \geq 4} \frac{1}{(k-4)!}\, g((\nabla^{k-4}_{X,\,\ldots,\,X} R)_{X,\,IX} IX,\ X) \ \in\ \Gamma(\overline{\mathrm{Sym}}^{\geq(2,2)} T^*M)$$

written as a sum of its bihomogeneous components $S_{\kappa,\,\overline{\kappa}}$ of degrees $\kappa,\,\overline{\kappa}\ \geq\ 2$ is congruent

$$\theta(X) \equiv g(X,\,X) - \sum_{\kappa,\,\overline{\kappa}\,\geq\,2} \frac{1}{2\kappa\,(\kappa-1)\,\overline{\kappa}\,(\overline{\kappa}-1)}\; S_{\kappa,\,\overline{\kappa}}(X) \tag{1}$$

$$\mathrm{dist}^2_g(p, \mathrm{knc}_p X) \equiv g(X,\,X) - \sum_{\kappa,\,\overline{\kappa}\,\geq\,2} \frac{1}{2(\kappa+\overline{\kappa}-1)\,(\kappa-1)\,(\overline{\kappa}-1)}\; S_{\kappa,\,\overline{\kappa}}(X) \tag{2}$$

to the Kähler normal potential θ and the Riemannian distance to the origin modulo terms at least *quadratic* in the curvature tensor and all its iterated covariant derivatives. A simple induction based on these congruences implies that the total holomorphic sectional curvature, the Kähler normal potential and the Riemannian distance to the origin all parametrize the underlying Kähler geometry uniquely up to covering.

For the time being the Kähler normal potential is certainly the most convenient of these three parameters, because the covariant derivative of the Kähler normal potential considered as a section $\theta \in \Gamma(\overline{\mathrm{Sym}}\,T^*M)$ can be calculated using the concept of holomorphically extended vector fields. More precisely we find in Lemma 4.11 an explicit formula of the form

$$\nabla_Z\theta = \mathrm{pr}_{\geq(2,2)}\,(Z \lrcorner\, \theta^{\mathrm{free}}) - \mathrm{pr}_{[\,1]}(Z \lrcorner\, \theta^{\mathrm{crit}})\; \bullet\; \theta^{\mathrm{free}}$$

where $\bullet$ is a simple bilinear operation, while θ^{free} and θ^{crit} denote the sums of all bihomogeneous components of θ at least or exactly quadratic respectively in the holomorphic or antiholomorphic coordinates. Note that this result is best thought of as describing the so called Spencer connection on the parameter vector bundle $\overline{\mathrm{Sym}}^{\geq(2,2)}T^*M$.

Specializing from general Kähler manifolds to concrete examples we study the locally symmetric Kähler manifolds, usually called hermitean locally symmetric spaces, in the second part of this article. In general the Lie theoretic setup [He] [Ber] makes many calculations like the determination of the Taylor series of the difference element K and its inverse K^{-1} feasible for symmetric spaces, making good use this simplification we obtain the formula:

Theorem 5.3 (Difference Elements of Hermitean Symmetric Spaces). *For every hermitean locally symmetric space the difference element* $K := \exp_p^{-1} \circ \mathrm{knc}_p$ *measuring the deviation between the Riemannian and Kählerian normal coordinates reads:*

$$KX = \frac{\mathrm{artanh}(\frac{1}{2}\,\mathrm{ad}\,IX)}{\frac{1}{2}\,\mathrm{ad}\,IX}\; X \qquad\qquad K^{-1}X = \frac{\tanh(\frac{1}{2}\,\mathrm{ad}\,IX)}{\frac{1}{2}\,\mathrm{ad}\,IX}\; X.$$

Although the arguments and statements in this second part are formulated for hermitean locally symmetric spaces, all results except of course Corollary 5.4 hold true for the locally symmetric among the balanced complex affine spaces, for which Kähler normal coordinates are defined. Unfortunately complex symmetric spaces are never balanced unless they are flat, nevertheless there certainly exist balanced complex affine manifolds, which are symmetric, but not even pseudo–hermitean symmetric spaces.

In Section 2 we briefly recall several important definitions for Kähler manifolds with a view towards the parametrization of Kähler geometry by a single power series in the parameter bundle $\mathrm{Sym}^{\geq(2,2)}T^*M$. In Section 3 we prove the parallel transport equation and discuss the notion of exponentially extended vector fields. Difference elements and their relation with other objects relevant for Kähler geometry are the topic of Section 4, in which we prove Theorem 4.6 and describe the Spencer connection for the potential in Lemma 4.11. In the very rigid context of locally symmetric spaces the characteristic power series for Kähler normal coordinates are calculated in the final Section 5. Appendix A details explicit calculations for the four families of compact hermitean symmetric spaces undertaken by the authors to vindicate the formula of Theorem 5.3 and verify Corollaries 4.9 and 5.5.

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2. Kähler manifolds and Kähler normal coordinates. Kähler geometry is a classical topic of Differential Geometry and blends the metric structure characteristic for Riemannian geometry with the complex structure making complex analytical tools available. Every Kähler manifold is real analytic, hence the infinite order Taylor series of the metric and the complex structure in an arbitrary point determine the manifold completely up to coverings. In this section we briefly recall the definition of Kähler manifolds and then establish a parametrization of these two Taylor series by a single power series in the parameter vector bundle $\overline{\mathrm{Sym}}^{\geq(2,2)}T^*M$ using two independent arguments. Excellent introductory texts to Kähler geometry are the recent textbooks [Ba] and [M].

A Kähler manifold is a smooth manifold M endowed with a Riemannian metric g and an orthogonal almost complex structure $I \in \Gamma(\mathrm{End}\,TM)$ satisfying the rather strong integrability condition that I is parallel with respect to the Levi–Civita connection ∇ for g:

$$\nabla_X I = 0 \tag{3}$$

Pseudo–Kähler manifolds generalize Kähler manifolds by weakening the positive–definiteness of the Riemannian metric g in the definition to non–degeneracy. Every Kähler manifold M is actually a complex manifold, because the integrability condition implies the vanishing of the Nijenhuis tensor and so the Theorem of Newlander–Nierenberg applies. A powerful tool to study the topological properties of (pseudo) Kähler manifolds is the Hodge decomposition

$$\Lambda\, T^*M = \bigoplus_{k\,\geq\,0} \Lambda^k T^*M \,\otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{\kappa,\,\overline{\kappa}\,\geq\,0} \Lambda^{\kappa,\,\overline{\kappa}} T^*M$$

of complex–valued differential forms on M into the eigenspaces of the derivation

$$(\mathrm{Der}_I\eta)(X_1, X_2, \ldots, X_k) := \eta(IX_1, X_2, \ldots, X_k) + \cdots + \eta(X_1, X_2, \ldots, IX_k) \tag{4}$$

extending the complex structure I, more precisely

$$\Lambda^{\kappa,\overline{\kappa}}T^* := \{\eta \in \Lambda^{\kappa+\overline{\kappa}}T^* \otimes_{\mathbb{R}} \mathbb{C} \mid \mathrm{Der}_I\eta = i\,(\kappa - \overline{\kappa})\,\eta\}$$

is the eigenspace of Der_I for the eigenvalue $i(\kappa - \overline{\kappa})$. In passing we remark that $\mathrm{Der}_I = -I\star$ agrees up to sign with the representation of the Lie algebra bundle

End TM on forms. Replacing alternating by symmetric forms on T we obtain the analoguous decomposition

$$\operatorname{Sym} T^*M \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{k\,\geq\,0} \operatorname{Sym}^k T^*M \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{\kappa,\,\overline{\kappa}\,\geq\,0} \operatorname{Sym}^{\kappa,\,\overline{\kappa}} T^*M \tag{5}$$

into the eigenspaces of Der_I for the eigenvalues $i\,(\kappa - \overline{\kappa}) \in \mathbb{C}$ on $\operatorname{Sym}^{\kappa+\overline{\kappa}} T^*M \otimes_{\mathbb{R}} \mathbb{C}$. The conjugation of the value of a complex valued symmetric multilinear form clearly commutes with Der_I and thus induces isomorphisms $\operatorname{Sym}^{\kappa,\,\overline{\kappa}} T^*M \longrightarrow \operatorname{Sym}^{\overline{\kappa},\,\kappa} T^*M$, $\eta \longmapsto \overline{\eta}$, for all $\kappa,\,\overline{\kappa} \geq 0$. In particular a real valued symmetric form $\eta = \overline{\eta}$ has conjugated bihomogeneous components in $\operatorname{Sym}^{\kappa,\,\overline{\kappa}} T^*M$ and $\operatorname{Sym}^{\overline{\kappa},\,\kappa} T^*M$ respectively. Bihomogeneous real valued symmetric forms $\eta = \overline{\eta}$ exist only for $\kappa = \overline{\kappa}$ and are characterized by $\operatorname{Der}_I \eta = 0$.

With the complex structure I being parallel on a Kähler manifold M the curvature tensor R of the Levi–Civita connection ∇ associated to the Riemannian metric g commutes with I

$$R_{X,\,Y} IZ = I\, R_{X,\,Y} Z$$

in addition to the standard symmetries of a Riemannian curvature tensor, namely

$$g(R_{X,\,Y}Z,\,W) = -g(R_{Y,\,X}Z,\,W) = -g(R_{X,\,Y}W,\,Z) \stackrel{!}{=} +g(R_{Z,\,W}X,\,Y)$$

and of course the first Bianchi identity $R_{X,\,Y}Z + R_{Y,\,Z}X + R_{Z,\,X}Y = 0$. Combining these standard identities with the characteristic commutativity of Kähler geometry we obtain

$$g(R_{X,\,Y}IU,\,IV) = g(R_{X,\,Y}U,\,V) = g(R_{IX,\,IY}U,\,V) \tag{6}$$

and conclude that $R_{X,\,IY} = R_{Y,\,IX}$ is symmetric in $X,\,Y$. Via polarization the curvature tensor R of a Kähler manifold M is thus completely determined by the biquadratic polynomial $T_pM \times T_pM \longrightarrow \mathbb{R}$, $(X, Y) \longmapsto -g(R_{X,IX}Y,\,IY)$, called the biholomorphic sectional curvature of M in order to distinguish it from the holomorphic sectional curvature:

Definition 2.1 (Holomorphic Sectional Curvature Tensor). *The holomorphic sectional curvature tensor of a Kähler manifold M with Riemannian metric g and orthogonal complex structure I is the section $S \in \Gamma(\operatorname{Sym}^4 T^*M)$ defined by:*

$$S(X,\,Y,\,U,\,V) := 8\Big(g(R_{X,\,IY}IU,\,V) + g(R_{X,\,IU}IV,\,Y) + g(R_{X,\,IV}IY,\,U)\Big)$$

The holomorphic sectional curvature is the associated quartic polynomial on TM defined by:

$$S : TM \longrightarrow \mathbb{R}, \quad X \longmapsto \frac{1}{4!}\, S(X,\,X,\,X,\,X) = g(R_{X,\,IX}IX,\,X)$$

In passing we observe that the symmetries (6) of the curvature tensor R of a Kähler manifold M imply that S is actually symmetric in all its four arguments, in particular it is completely determined by the associated quartic polynomial on TM, this is the holomorphic sectional curvature. In a similar vein we may deduce $\operatorname{Der}_I S = 0$ via polarization from the identity:

$$(\operatorname{Der}_I S)(X, X, X, X) = 4S(IX, X, X, X) = 96 g(R_{IX,\,IX}IX,\,X) = 0$$

Geometrically $\mathrm{Der}_I S = 0$ is equivalent to the statement that the holomorphic sectional curvature S is constant along the fibers of the Hopf fibration from the unit sphere $S(T_pM)$ to the set $\mathbb{P}(T_pM)$ of complex lines in T_pM with respect to the complex structure I_p.

LEMMA 2.2 (Description of Curvature Tensor). *The curvature tensor R of a Kähler manifold M is determined by the holomorphic sectional curvature tensor $S \in \Gamma(\mathrm{Sym}^{2,2}T^*M)$ and can be reconstructed from S by means of:*

$$g(R_{X,\,Y}U,\,V) = \frac{1}{32}\Big(S(X,IY,IU,V) - S(X,IY,U,IV)\Big)$$

The proof of this lemma is completely straightforward: Expanding the definition of S

$$\begin{aligned} +S(X,IY,IU,V) &= 8g(R_{X,Y}U,\,V) - 8g(R_{X,U}IV,\,IY) - 8g(R_{X,IV}Y,\,IU) \\ -S(X,IY,U,IV) &= 8g(R_{X,Y}IU,\,IV) + 8g(R_{X,IU}V,\,IY) - 8g(R_{X,V}Y,\,U) \end{aligned}$$

adding and using the first Bianchi identity twice we obtain $4{\cdot}8\,g(R_{X,Y}U,V)$ as claimed. In particular we can calculate the sectional curvature of a Kähler manifold M for an arbitrary plane $\mathrm{span}\{X,Y\} \subset T_pM$ from the holomorphic sectional curvatures:

$$\begin{aligned} g(R_{X,\,Y}Y,\,X) &= \frac{1}{32}\Big(S(X,X,IY,IY) - S(IX,X,IY,Y)\Big) \\ &= \frac{1}{64}\Big(3S(X,X,IY,IY) - S(X,X,Y,Y)\Big) \end{aligned}$$

In the second equality we have replaced $S(IX,X,IY,Y)$ using the argument

$$-4S(X,X,Y,Y) + 4S(X,X,IY,IY) + 8S(IX,X,IY,Y) = 0$$

which follows directly from $\mathrm{Der}_I S = 0$ by expanding $(\mathrm{Der}_I^2 S)(X,X,Y,Y) = 0$. Generalizing the holomorphic curvature tensor to higher orders to incorporate information about the iterated covariant derivatives ∇R, $\nabla^2 R$, $\ldots$ of the curvature tensor R as well we arrive at:

DEFINITION 2.3 (Higher Holomorphic Sectional Curvature Tensors). *For all $k \geq 4$ the higher holomorphic sectional curvature $S_k \in \Gamma(\mathrm{Sym}^k T^*M)$ is defined by:*

$$S_k(X) \,\widehat{=}\, \frac{1}{k!}\,S_k(X,\,\ldots,\,X) := \frac{1}{(k-4)!}\,g((\nabla^{k-4}_{X,\,\ldots,\,X}R)_{X,\,IX}IX,\;X)$$

Calculating $\mathrm{Der}_I S_k$ we observe that the last four summands expected from the definition of Der_I all vanish by the symmetries (6) of curvature tensors of Kähler type. In consequence the bihomogeneous components $S_{\kappa,\,\overline{\kappa}} \in \Gamma(\mathrm{Sym}^{\kappa,\,\overline{\kappa}}T^*M \otimes_{\mathbb{R}} \mathbb{C})$ of the higher order holomorphic sectional curvature tensor S_k vanish unless $-k+4 \leq \kappa - \overline{\kappa} \leq k-4$, this is to say:

$$S_{\mathrm{total}} := \bigoplus_{k\,\geq\,4} S_k = \bigoplus_{\kappa,\,\overline{\kappa}\,\geq\,2} S_{\kappa,\,\overline{\kappa}} \;\in\; \Gamma(\overline{\mathrm{Sym}}^{\geq(2,2)}T^*M) \tag{7}$$

Of course the bihomogeneous components $S_{\kappa,\,\overline{\kappa}}$ and $S_{\overline{\kappa},\,\kappa}$ are conjugated, because the higher holomorphic sectional curvature tensor $S_k \in \Gamma(\mathrm{Sym}^k T^*M)$ is a real valued polynomial by definition. In light of Lemma 2.2 we may replace the curvature tensor

R with $S = S_4$, similarly we may identify S_5 with the covariant derivative ∇R in the following way:

Corollary 2.4 (Covariant Derivative of Curvature). *The first higher holomorphic sectional curvature $S_5 \in \Gamma(\mathrm{Sym}^5 T^*M)$ of a Kähler manifold M describes the covariant derivative ∇R of the curvature tensor by means of the formula:*

$$g((\nabla_X R)_{Y,\,Z}U,\,V) = \frac{1}{192}\Big(+S_5(X,\,Y,\,IZ,\,IU,\,V) - S_5(X,\,Y,\,IZ,\,U,\,IV) \\ +S_5(X,\,IY,\,Z,\,U,\,IV) - S_5(X,\,IY,\,Z,\,IU,\,V)\Big).$$

This formula for ∇R is a direct corollary of the description of R in Lemma 2.2, because

$$\frac{1}{4!}\,S_5(Y,\,X,\,X,\,X,\,X) = g((\nabla_Y R)_{X,\,IX}IX,\,X) + 4g((\nabla_X R)_{Y,\,IX}IX,\,X)$$

implies via the second Bianchi identity:

$$\begin{aligned} &\frac{1}{4!}\,S_5(Y,\,X,\,X,\,X,\,X) + \frac{1}{4!}\,S_5(Y,\,IX,\,IX,\,IX,\,IX) \\ &= 2g((\nabla_Y R)_{X,\,IX}IX,\,X) + 4g((\nabla_X R)_{Y,\,IX}IX,\,X) + 4g((\nabla_{IX} R)_{X,\,Y}IX,\,X) \\ &= 6g((\nabla_Y R)_{X,\,IX}IX,\,X) \\ &= \frac{6}{4!}\,(\nabla_Y S_4)(X,\,X,\,X,\,X). \end{aligned}$$

In consequence of Lemma 2.2 and Corollary 2.4 every Kähler geometry can be constructed up to covering from the total holomorphic sectional curvature S_{total} defined in (7), in particular all iterated covariant derivatives ∇R, $\nabla^2 R$, ... and thus the infinite order Taylor series of both g and I in exponential coordinates are determined by S_{total}. For the time being however this reconstruction is a theoretical possibility based on counting the free parameters in the sequence R, ∇R, ... using the formal theory of partial differential equations [BCG]. En nuce the problem addressed in this article is that we are lacking the explicit formulas needed for this reconstruction to work in practice. Moreover S_{total} is not the only power series with the correct number of parameters, another interesting candidate parametrizing Kähler geometry by a section of the parameter bundle $\overline{\mathrm{Sym}}^{\geq(2,2)}T^*M$ arises from a suitable potential:

Definition 2.5 (Local Potential Functions). *A local potential function for a Kähler manifold M with Riemannian metric g and complex structure I is a smooth function $\theta^{\mathrm{loc}} : U \longrightarrow \mathbb{R}$ defined on an open subset $U \subset M$, which is a preimage of the Kähler form $\omega(X,Y) := g(IX,Y)$ in the sense of the $\partial\overline{\partial}$–Lemma*

$$\omega = \frac{i}{2}\,\partial\overline{\partial}\,\theta^{\mathrm{loc}}$$

where ∂ and $\overline{\partial}$ are the $(1,0)$ and $(0,1)$–components of the exterior derivative $d := \partial + \overline{\partial}$. Note that $\omega \in \Gamma(\Lambda^2 T^*M)$ is parallel under the Levi–Civita connection and so is closed.

A local potential function is never unique as it can always be modified $\theta^{\mathrm{loc}} \rightsquigarrow \theta^{\mathrm{loc}} + \mathrm{Re}\,f$ by adding the real part of a holomorphic function $f : U \longrightarrow \mathbb{C}$. Instead of the Kähler form ω we may employ the hermitean form $h := g + i\,\omega$ associated to a Kähler geometry on a manifold M in the equation characterizing a local potential function θ^{loc}

$$h(X,\,Y) := g(X,\,Y) + i\,\omega(X,\,Y) = (\mathrm{Hess}\;\theta^{\mathrm{loc}})(\mathrm{pr}^{0,1}X,\;\mathrm{pr}^{1,0}Y) \tag{8}$$

where $\mathrm{pr}^{1,0} := \frac{1}{2}(\mathrm{id} - iI)$ and $\mathrm{pr}^{0,1} := \frac{1}{2}(\mathrm{id} + iI)$ are the I–eigenprojections, or alternatively:

$$g(X,\,Y) = \frac{1}{4}\Big((\mathrm{Hess}\;\theta^{\mathrm{loc}})(X,\,Y) + (\mathrm{Hess}\;\theta^{\mathrm{loc}})(IX,\,IY)\Big) \tag{9}$$

In both equations the Hessian $D \circ d$ can actually be taken with respect to an arbitrary torsion free connection D on the tangent bundle making I parallel, it is not necessary to use the Levi–Civita connection ∇ for this purpose. In fact the difference $A_XY := D_XY - \hat{D}_XY$ of two such connections is symmetric $A_XY = A_YX$ with $A_XIY = IA_XY$ and thus automatically satisfies $A_XY + A_{IX}IY = 0 = A_{\mathrm{pr}^{0,1}X}\mathrm{pr}^{1,0}Y$. The liberty granted by this observation is very useful in verifying equations (8) and (9) directly in local holomorphic coordinates $(z^1,\,\ldots,\,z^n)$, because we may simply choose D to be the trivial connection arising from the local trivialization of the complexified tangent bundle by the Wirtinger vector fields

$$\frac{\partial}{\partial z^\mu} := \frac{1}{2}\Big(\frac{\partial}{\partial x^\mu} - i\frac{\partial}{\partial y^\mu}\Big) \qquad\qquad \frac{\partial}{\partial \overline{z}^\mu} := \frac{1}{2}\Big(\frac{\partial}{\partial x^\mu} + i\frac{\partial}{\partial y^\mu}\Big)$$

where $(x^1,\,y^1,\,\ldots,\,x^n,\,y^n)$ are the real and imaginary parts of $(z^1,\,\ldots,\,z^n)$. The Riemannian metric g is completely determined by its mixed components $g_{\mu\,\overline{\nu}} := g(\frac{\partial}{\partial z^\mu},\,\frac{\partial}{\partial \overline{z}^\nu})$ in holomorphic coordinates, because the eigensubbundles $T^{1,0}M$ and $T^{0,1}M$ of the complexified tangent bundle $TM \otimes_{\mathbb{R}} \mathbb{C}$ under the skew symmetric endomorphism I are isotropic, so that all pure components $g_{\mu\,\nu} = 0 = g_{\overline{\mu}\,\overline{\nu}}$ vanish. In holomorphic coordinates we thus find the following local expansions of the metric, the Kähler and the hermitean form:

$$g = \sum_{\mu\,\nu} g_{\mu\,\overline{\nu}}\,dz^\mu \cdot d\overline{z}^\nu \qquad \omega = i\sum_{\mu\,\nu} g_{\mu\,\overline{\nu}}\,dz^\mu \wedge d\overline{z}^\nu \qquad h = 2\sum_{\mu\,\nu} g_{\mu\,\overline{\nu}}\,d\overline{z}^\nu \otimes dz^\mu$$

Comparing the expansion for ω with the analoguous local expansion of $\frac{i}{2}\,\partial\overline{\partial}\,\theta^{\mathrm{loc}}$ we find

$$\partial\,\overline{\partial}\,\theta^{\mathrm{loc}} = \sum_{\mu\,\nu} \frac{\partial^2\;\theta^{\mathrm{loc}}}{\partial z^\mu \partial \overline{z}^\nu}\,dz^\mu\,\wedge\,d\overline{z}^\nu \qquad \Longrightarrow \qquad g_{\mu\,\overline{\nu}} = \frac{1}{2}\frac{\partial^2}{\partial z^\mu \partial \overline{z}^\nu}\,\theta^{\mathrm{loc}}$$

and reinserting this relation between θ^{loc} and the mixed components of g into the expansions of the hermitean form h we verify equation (8) and in turn equation (9) for the Hessian $D \circ d$ associated to the local trivial connection $D\frac{\partial}{\partial z^\mu} = 0 = D\frac{\partial}{\partial \overline{z}^\nu}$.

The concept of local potentials for Kähler geometries allows us to single out special holomorphic coordinates charts $(z^1,\,\ldots,\,z^n)$ on a Kähler manifold by requiring that the local potential, after adding the real part of a holomorphic function, takes as simple a form as possible. Uniqueness of these holomorphic coordinates is usually stipulated modulo unitary transformations only, in order to remove this ambiguity we borrow from the Riemannian exponential map the idea of anchored coordinates centered in a point $p \in M$. An anchored local coordinate chart is a smooth map

$\varphi : T_pM \longrightarrow M$, which is a diffeomorphism of some neighborhood of $0 \in T_pM$ to a neighborhood of $\varphi(0) = p$ such that the differential

$$\varphi_{*,0} : T_pM \cong T_0(T_pM) \longrightarrow T_pM, \qquad Z \longmapsto Z$$

equals id_{T_pM} under the natural identification $T_0(T_pM) \cong T_pM$. Actually we do not insist on T_pM to be the domain of φ, hence it may not be defined outside a neighborhood of 0.

Theorem 2.6 (Kähler Normal Coordinates and Normal Potentials [Bo]). *In every point $p \in M$ of a Kähler manifold M there exist unique anchored local holomorphic coordinates* $\mathrm{knc}_p : T_pM \longrightarrow M$ *centered in p and a unique local potential function θ_p^{loc} such that the infinite order Taylor series θ_p of the pull back of θ_p^{loc} along* knc_p *is congruent*

$$\theta_p^{\mathrm{loc}}(\mathrm{knc}_p X) \underset{X\to 0}{\sim} \theta_p(X) \equiv g_p(X, X) \;\; mod \;\; \overline{\mathrm{Sym}}^{\geq(2,2)} T_p^*M$$

to the norm square function on T_pM modulo a power series, which is at least quadratic in both holomorphic and antiholomorphic coordinates. Due to their uniqueness the holomorphic coordinates knc_p *are called Kähler normal coordinates with normal potential $\theta_p^{\mathrm{loc}} \mathrel{\widehat{=}} \theta_p$.*

The double uniqueness statement in this theorem implies in particular that for given anchored local holomorphic coordinates $\varphi : T_pM \longrightarrow M$ the existence of a power series potential $\theta_p \in \overline{\mathrm{Sym}}\, T_p^*M$ satisfying the normalization congruence is sufficient to conclude that θ_p is the Taylor series of the normal potential θ_p^{loc} and that $\varphi = \mathrm{knc}_p$ are the Kähler normal coordinates, this argument will be used in explicit examples in Appendix A. The original account [Bo] of Bochner contains a very readable, elementary proof of Theorem 2.6, for this reason we will not discuss Theorem 2.6 independently of Theorem 4.6 below.

3. Affine Exponential Coordinates. In the study of the geometry of Riemannian or more generally affine manifolds the exponential map provides an indispensable tool in both explicit calculations and theoretic considerations. Classically Jacobi vector fields are used to describe the differential of the exponential map, alternatively the forward and backward parallel transport defined in this section can be used for this purpose. In particular we will use Jacobi vector fields to give an apriori proof of the parallel transport equation, which describes the Taylor series of parallel tensors, without calculating its solution first. Moreover we will introduce the concept of exponentially extended vector fields for general affine manifolds, which generalize the left invariant vector fields on Lie groups and the transvection Killing vector fields on symmetric spaces.

Affine linear spaces are characterized by the presence of a distinguished family of curves, the family of straight lines. The choice of a connection ∇ on the tangent bundle TM of a manifold M replaces this distinguished family by the family of geodesics, the solutions $\gamma : \mathbb{R} \longrightarrow M$ to the geodesic equation associated to the choice of connection. According to the Theorem of Picard–Lindelöf a unique solution γ to the geodesic equation $\frac{\nabla}{dt}\dot{\gamma} = 0$ exists for all initial values so that we may define the exponential map centered in a point

$$\exp_p : T_pM \longrightarrow M, \qquad X \longmapsto \gamma^X(1)$$

by sending a tangent vector X to the value $\gamma^X(1)$ of the unique geodesic γ^X with initial values $\gamma^X(0) = p$ and $\dot\gamma^X(0) = X$. Constant reparametrizations of geodesics are geodesics $\gamma^{\lambda X}(t) = \gamma^X(\lambda t)$, hence the image of the ray $t \longmapsto tX$ under exponential coordinates is the geodesic $t \longmapsto \gamma^{tX}(1) = \gamma^X(t)$ we started with. The parallel transport along this ray geodesic $\mathbf{PT}^\nabla(X) := \mathbf{PT}^\nabla_{\gamma^X}(1)$ is by construction a linear isomorphism of tangent spaces:

$$\mathbf{PT}^\nabla(X) :\ T_pM \stackrel{\cong}{\longrightarrow} T_{\exp_p X}M$$

This isomorphism allows us to describe the differential of exponential coordinates $\exp_p$ centered in $p \in M$, either by using the forward parallel transport defined as the composition

$$\Phi(X) :\ T_pM \stackrel{\mathbf{PT}^\nabla(X)}{\longrightarrow} T_{\exp_p X}M \stackrel{(\exp_p)^{-1}_{*,X}}{\longrightarrow} T_XT_pM \stackrel{\cong}{\longrightarrow} T_pM$$

wherever the exponential map $\exp_p$ is a local diffeomorphism, or the backward transport

$$\Phi^{-1}(X) :\ T_pM \stackrel{\cong}{\longrightarrow} T_XTpM \stackrel{(\exp_p)_{*,X}}{\longrightarrow} T_{\exp_p X}M \stackrel{\mathbf{PT}^\nabla(X)^{-1}}{\longrightarrow} T_pM$$

defined for all $X \in T_pM$ under the rather mild assumption that M is a complete affine manifold. The infinite order Taylor series of Φ^{-1} in the origin $0 \in T_pM$ is an explicitly known power series in the curvature tensor R of M and its iterated covariant derivatives ∇R, $\nabla^2 R$, ... evaluated at p. More precisely the Taylor series of Φ^{-1} in the origin equals the unique power series solution $\Phi^{-1} \in \overline{\mathrm{Sym}}\, T^*_pM \otimes \mathrm{End}\, T_pM$ to a formal differential equation with initial value $\Phi(0) = \mathrm{id}$ involving the number operator N on power series:

Lemma 3.1 (Parallel Transport Equation). *The infinite order Taylor series of the backward parallel transport $\Phi^{-1} : T_pM \longrightarrow \mathrm{End}\, T_pM$ in the center $0 \in T_pM$ of exponential coordinates of an affine manifold M is characterized as a formal power series $\Phi^{-1} \in \overline{\mathrm{Sym}}\, T^*_pM \otimes \mathrm{End}\, T_pM$ by the formal differential equation*

$$N\,(N+1)\,\Phi^{-1} = \mathscr{R}\,\Phi^{-1} + N(\mathscr{T}\,\Phi^{-1})$$

in which N denotes the number operator on power series and the power series $\mathscr{R}$ and $\mathscr{T}$ reflect the infinite order Taylor series of the curvature and the torsion respectively:

$$\mathscr{R}(X)\,Y := \sum_{k\geq 0} \frac{1}{k!}\,(\nabla^k_{X,\ldots,X}R)_{X,\,Y}X \qquad \mathscr{T}(X)\,Y := \sum_{k\geq 0} \frac{1}{k!}\,(\nabla^k_{X,\ldots,X}T)(X,\,Y)$$

Proof. In order to prove the lemma we want to study the standard Jacobi equation for a vector field $J \in \Gamma(\gamma^*TM)$ along a geodesic γ in a manifold M with respect to an affine, not necessarily torsion free connection ∇ on the tangent bundle with curvature R and torsion T:

$$\mathrm{Jac}_\gamma J := \frac{\nabla^2}{dt^2}\,J + \frac{\nabla}{dt}\,T(J,\,\dot\gamma) + R_{J,\,\dot\gamma}\dot\gamma \ \stackrel{?}{=}\ 0$$

Multiplying by t^2 we obtain for every solution J to this equation the equality:

$$t\,\frac{\nabla}{dt}\Big(t\,\frac{\nabla}{dt} - 1\Big)\,J = R_{t\,\dot\gamma,\,J}(t\,\dot\gamma) + \Big(t\,\frac{\nabla}{dt} - 1\Big)\,T(t\,\dot\gamma,\ J) \tag{10}$$

In particular we are interested in the family of solutions to the Jacobi equation, which arise naturally in exponential coordinates by varying the geodesic rays. More precisely we consider for a point $p \in M$ and an endomorphism $A \in \mathrm{End}\, T_pM$ of its tangent space the geodesic variation $(s, t) \longmapsto \exp_p(t\, e^{sA} X)$ for some tangent vector $X \in T_pM$. This geodesic variation induces a Jacobi field $J_A(t)$ along the geodesic ray $\gamma : t \longmapsto \exp_p(tX)$ determined by X:

$$J_A(t) := \frac{\partial}{\partial s}\Big|_0 \exp_p(t\, e^{sA}\, X) = (\exp_p)_{*,\, tX}(t\, AX) = t\, \mathbf{PT}^\nabla(tX)\; \Phi^{-1}(tX)\, AX$$

Multiplying the Jacobi equation (10) for this Jacobi field by $\mathbf{PT}^\nabla(tX)^{-1}$ and using the characteristic property of the parallel transport $\mathbf{PT}^\nabla(tX)^{-1} \circ \frac{\nabla}{dt} = \frac{d}{dt} \circ \mathbf{PT}^\nabla(tX)^{-1}$ we obtain

$$\begin{aligned} t\frac{d}{dt}(t\frac{d}{dt} - 1)\Big(t\Phi^{-1}(tX)AX\Big) &= \Big(\mathbf{PT}^\nabla(tX)^{-1}R\Big)_{tX,\, t\,\Phi^{-1}(tX)AX}(tX) \\ &\quad + \Big(t\frac{d}{dt} - 1\Big)\Big(\mathbf{PT}^\nabla(tX)^{-1}T\Big)(tX,\, t\,\Phi^{-1}(tX)AX) \end{aligned} \tag{11}$$

where $\mathbf{PT}^\nabla(tX)^{-1}R$ for example denotes the parallel transport for the curvature tensor

$$\Big(\mathbf{PT}^\nabla(tX)^{-1}R\Big)_{U,\,V} W := \mathbf{PT}^\nabla(tX)^{-1}\Big(R_{\mathbf{PT}^\nabla(tX)\,U,\, \mathbf{PT}^\nabla(tX)\,V}\mathbf{PT}^\nabla(tX)\, W\Big)$$

the analoguous definition of the parallel transport $\mathbf{PT}^\nabla(tX)^{-1}T$ for the torsion tensor is omitted. In general the derivative of a geodesic like $t \longmapsto \exp_p(tX)$ is given by the parallel transport of the initial tangent vector $\dot\gamma(t) = (\exp_p)_{*,\, tX}X = \mathbf{PT}^\nabla(tX)\, X$, incidentally this argument directly implies the so called Gauß Lemma valid for all $X \in T_pM$ and $t \in \mathbb{R}$:

$$\Phi^{-1}(tX)\, X = X = \Phi(tX)\, X \tag{12}$$

Of course the point in defining the parallel transport for the curvature and torsion tensor is that $\mathbf{PT}^\nabla(tX)^{-1}R$ and $\mathbf{PT}^\nabla(tX)^{-1}T$ are trilinear and bilinear maps on T_pM respectively with values in T_pM independent of the argument $X \in T_pM$. Hence it makes sense to talk about their infinite order Taylor series as X approaches $0 \in T_pM$, which are given by

$$\mathbf{PT}^\nabla(X)^{-1}R \underset{X\,\to\,0}{\sim} \sum_{k\,\geq\,0} \frac{1}{k!}\nabla^k_{X,\,\ldots,\,X}R \qquad \mathbf{PT}^\nabla(X)^{-1}T \underset{X\,\to\,0}{\sim} \sum_{k\,\geq\,0} \frac{1}{k!}\nabla^k_{X,\,\ldots,\,X}T \tag{13}$$

due to the definition of iterated covariant derivatives, a more detailed derivation of these asymptotic expansions can be found for example in [W2]. Replacing $\mathbf{PT}^\nabla(tX)^{-1}R$ and $\mathbf{PT}^\nabla(tX)^{-1}T$ as well as the backward parallel transport $\Phi^{-1}(tX)$ by their respective infinite order Taylor series in equation (11) we obtain the following identity of power series

$$\begin{aligned} &t\frac{d}{dt}\,(t\frac{d}{dt} - 1)\Big(\Phi^{-1}(tX)\, A(tX)\Big) \\ &= \mathscr{R}(tX)\, \Phi^{-1}(tX)\, A(tX) + (t\frac{d}{dt} - 1)\Big(\mathscr{T}(tX)\, \Phi^{-1}(tX)\, A(tX)\Big) \end{aligned}$$

in which every occurrence of the argument $X \in T_pM$ comes along with a factor t and vice versa. In turn we may replace the differential operator $t\frac{d}{dt}$ by the number operator N on power series in X and evaluate at $t = 1$ to reduce this power series identity to:

$$N\,(N-1)\,(\Phi^{-1}(X)\,AX) = \mathscr{R}(X)\,\Phi^{-1}(X)\,AX + (N-1)\,(\mathscr{T}(X)\,\Phi^{-1}(X)\,AX)$$

Recall now that the endomorphism $A \in \mathrm{End}\,T_pM$ of the tangent space T_pM can be chosen arbitrarily in this identity. A fortiori the infinite order Taylor series Φ^{-1} of the backward parallel transport satisfies the differential equation $(N+1)N\Phi^{-1} = \mathscr{R}\,\Phi^{-1} + N(\mathscr{T}\,\Phi^{-1})$ in light of the observation that the following homogeneous linear map of degree $+1$

$$\iota:\ \mathrm{Sym}^{\bullet}T^* \otimes \mathrm{End}\,T \longrightarrow \mathrm{Hom}\,(\mathrm{End}\,T,\,\mathrm{Sym}^{\bullet+1}T^* \otimes T), \qquad \Psi \longmapsto \Big(A \longmapsto \Psi(\cdot)\,A(\cdot)\Big)$$

is injective for every finite dimensional vector space T; the degree $+1$ homogeneity of ι evidently accounts for the shift $N \rightsquigarrow N+1$ in the differential equation for Φ^{-1}. To justify our observation about the injectivity of ι we specify a linear map ι^* in the opposite direction

$$\iota^*:\ \mathrm{Hom}\,(\mathrm{End}\,T,\,\mathrm{Sym}^{\bullet+1}T^* \otimes T) \longrightarrow \mathrm{Sym}^{\bullet}T^* \otimes \mathrm{End}\,T, \qquad F \longmapsto \iota^*F$$

by summing over a pair $\{\,E_\mu\,\}$ and $\{\,dE_\mu\,\}$ of dual bases for T and T^* respectively:

$$[\iota^*F](X)\,Y := \sum_\mu (E_\mu \lrcorner\, F(dE_\mu \otimes Y))(X)$$

The injectivity of ι is then a direct consequence of the following identity for $\iota^*\iota$:

$$[\,\iota^*\iota\Psi](X)\,Y = \sum_\mu \Big(E_\mu \lrcorner (dE_\mu(\cdot)\,\Psi(\cdot)\,Y)\Big)(X) = [(N + \dim T)\,\Psi](X)\,Y.$$

□

Perhaps it is a good idea to remind the reader that the parallel transport equation formulated in Lemma 3.1 is only a formal differential equation for the infinite order Taylor series Φ^{-1} of the backward parallel transport. Its formal solution is easily expanded to arbitrary order in X, but it is much easier to recall the formal differential equation than its explicit solution. Expanding the solution to order 4 in X for example we obtain in the torsion free case:

$$\begin{aligned}\Phi^{-1}(X)\,Y = Y &+ \frac{1}{6}\,R_{X,\,Y}X + \frac{1}{12}\,(\nabla_X R)_{X,\,Y}X\\ &+ \frac{1}{40}\,(\nabla^2_{X,\,X}R)_{X,\,Y}X + \frac{1}{120}\,R_{X,\,R_{X,\,Y}X}X + O(X^5).\end{aligned}$$

Similarly the parallel transport equation can be solved explicitly for Lie groups. Consider for example the flat connection ∇^L on the tangent bundle TG of a Lie group G, which makes all left invariant vector fields parallel. By construction this connection is flat $R^L = 0$ and its torsion $T^L(X,Y) = -[X,\,Y]_{\mathrm{algebraic}}$ is parallel with respect to ∇^L so that $\mathscr{T}^L(X) = -\mathrm{ad}\,X$. Under the ansatz $\Phi^{-1}(X) = \varphi^{-1}(\mathrm{ad}\,X)$ the parallel transport equation becomes

$$(x\frac{d}{dx} + 1)\,x\frac{d}{dx}\,\varphi^{-1}(x) = x\frac{d}{dx}\,(-\,x\,\varphi^{-1}(x)) \qquad\qquad \varphi^{-1}(0) = 1$$

with unique solution $\varphi^{-1}(x) = \frac{e^{-x}-1}{-x}$, in turn we find the well–known explicit solution:

$$\Phi^{-1}(X) = \frac{e^{-\operatorname{ad} X} - \operatorname{id}}{-\operatorname{ad} X} \qquad\qquad \Phi(X) = \frac{-\operatorname{ad} X}{e^{-\operatorname{ad} X} - \operatorname{id}}.$$

The parallel transport equation for symmetric spaces can be solved using a similar ansatz.

Corollary 3.2 (Taylor Series of Parallel Tensors). *The unique power series solution $\Phi^{-1} \in \overline{\operatorname{Sym}}\, T_p^*M \otimes \operatorname{End} T_pM$ to the parallel transport equation $N(N+1)\,\Phi^{-1} = \mathscr{R}\,\Phi^{-1} + N\,(\mathscr{T}\,\Phi^{-1})$ determines the Taylor series of every tensor parallel with respect to ∇ in exponential coordinates. For a purely covariant parallel tensor η say the resulting Taylor series reads for $X \in T_pM$ and $A_1, \ldots, A_r \in T_X(T_pM) \cong T_pM$:*

$$(\operatorname{taylor}_0(\exp_p^* \eta))_X(A_1, \ldots, A_r) = \eta_p(\Phi^{-1}(X)\, A_1, \ \ldots, \ \Phi^{-1}(X)\, A_r).$$

Of course the restriction of a tensor η parallel with respect to ∇ on all of M is parallel along every radial geodesic $t \longmapsto \exp_p(tX)$, with this in mind we find the relation

$$\begin{aligned}(\exp_p^* \eta)_X(A_1, \ldots, A_r) &= \eta_{\exp_p X}((\exp_p)_{*,\,X} A_1, \ldots, (\exp_p)_{*,\,X} A_r) \\ &= \eta_p(\mathbf{PT}^{\nabla}(X)^{-1}(\exp_p)_{*,\,X} A_1, \ldots, \mathbf{PT}^{\nabla}(X)^{-1}(\exp_p)_{*,\,X} A_r)\end{aligned}$$

and so the definition of Φ^{-1} implies the Corollary. For mixed co– and contravariant tensors the formulas become slightly more complicated, but the argument in itself remains valid.

A notion closely related to the construction of the forward and backward parallel transport is the notion of exponentially extended vector fields, which in a sense describe the covariant derivative of the exponential map. The exponentially extended vector field associated to a tangent vector $Z \in T_pM$ is the smooth vector field defined on the domain of the exponential map $\exp_p$ in T_pM as the derivative of the following family of diffeomorphisms of T_pM

$$Z^{\exp} := \frac{d}{dt}\bigg|_0 \Big(T_pM \overset{\mathbf{PT}^{\nabla}_{\gamma}(t)}{\longrightarrow} T_{\gamma(t)}M \overset{\exp_{\gamma(t)}}{\longrightarrow} M \overset{(\exp_p)^{-1}}{\longrightarrow} T_pM\Big) \ \in \ \Gamma(T\,(T_pM))$$

where $\mathbf{PT}^{\nabla}_{\gamma}(t)$ denotes the parallel transport along a curve γ representing $Z = \frac{d}{dt}\big|_0 \gamma$. The exponentially extended vector field $Z^{\exp}$ does not depend on the curve γ used in its definition, moreover its value $Z^{\exp}(0) = Z$ in the origin equals the tangent vector we started with under the identification $T_0(T_pM) \cong T_pM$. Somewhat more general the infinite order Taylor series of $Z^{\exp}$ in $0 \in T_pM$ is determined by a power series Θ on T_pM with values in $\operatorname{End} T_pM$ via

$$Z^{\exp}(X) \underset{X \to 0}{\sim} \Theta(X)\, Z \qquad\qquad \Theta \ \in \ \overline{\operatorname{Sym}}\, T_p^*M \otimes \operatorname{End} T_pM$$

which has a universal expansion in terms of the curvature and the torsion of the connection ∇ together with all their covariant derivatives. In the torsion free case $T = 0$ for example

$$\begin{aligned}&Z^{\exp}(X) \\ &= Z + \frac{1}{3}\, R_{X,Z}X + \frac{1}{12}\,(\nabla_X R)_{X,Z}X + \Big(\frac{1}{60}\,(\nabla^2_{X,X} R)_{X,Z}X - \frac{1}{45}\, R_{X,R_{X,Z}X}X\Big) \\ &\quad + \Big(\frac{1}{360}\,(\nabla^3_{X,X,X} R)_{X,Z}X - \frac{1}{120}\, R_{X,(\nabla_X R)_{X,Z}X}X - \frac{1}{120}\,(\nabla_X R)_{X,R_{X,Z}X}X\Big) + O(X^6)\end{aligned}$$

reflects the terms of the power series Θ up to order 5. A simple algorithm to calculate the asymptotic expansion of Θ to arbitrary order is based on the asymptotic expansion

$$f(\exp_p X) \underset{X\to 0}{\sim} \sum_{k\geq 0} \frac{1}{k!}\,(\nabla^k_{X,\,\ldots,\,X} f)(p) \tag{14}$$

of the pull back of functions via the exponential map, which is the analogue for sections $f \in C^\infty M$ of the trivial line bundle of the expansions (13) used in the proof of Lemma 3.1. According to the definition of the exponentially extended vector field $Z^{\exp}$ the equality

$$\frac{d}{dt}\Big|_0 \exp_{\gamma(t)}\Big(\mathbf{PT}^\nabla(t)\,X\Big) = \frac{d}{dt}\Big|_0 \exp_p\Big(X + t\,Z^{\exp}(X)\Big) \in T_{\exp_p X}M$$

of tangent vectors holds true for all $X \in T_pM$ and all curves γ representing $Z = \frac{d}{dt}\big|_0 \gamma$, in turn the asymptotic expansion (14) implies an equality of asymptotic expansions of the form

$$\begin{aligned}\frac{d}{dt}\Big|_0 f(\exp_{\gamma(t)} \mathbf{PT}^\nabla(t)X) &\underset{X\to 0}{\sim} \frac{d}{dt}\Big|_0 \sum_{k\geq 0} \frac{1}{k!}(\nabla^k_{\mathbf{PT}^\nabla(t)X,\,\ldots,\,\mathbf{PT}^\nabla(t)X} f)(\gamma(t)) \\ &= \sum_{k\geq 0} \frac{1}{k!}(\nabla^{k+1}_{Z,\,X,\,\ldots,\,X} f)(p) \\ &\underset{X\to 0}{\sim} \frac{d}{dt}\Big|_0 \sum_{k\geq 0} \frac{1}{k!}(\nabla^k_{X+tZ^{\exp}(X),\,\ldots,\,X+tZ^{\exp}(X)} f)(p).\end{aligned}$$

The algorithm to calculate the Taylor series of the exponentially extended vector field $Z^{\exp}$ in $0 \in T_pM$ is thus based on calculating the unique power series Θ^{-1} satisfying the identity

$$\frac{d}{dt}\Big|_0 \sum_{k\geq 0} \frac{1}{k!}\,\nabla^k_{X+tZ,\,\ldots,\,X+tZ} f = \sum_{k\geq 0} \frac{1}{k!}\,\nabla^{k+1}_{\Theta^{-1}(X)Z,\,X,\,\ldots,\,X} f$$

of power series in $X \in T_pM$ for every function $f \in C^\infty M$, after formally inverting this power series we obtain the power series Θ describing the Taylor series of exponentially extended vector fields via $Z^{\exp}(X) \sim \Theta(X)\,Z$. The details of this algorithm can be worked out in analogy to the calculation of the backward parallel transport Φ^{-1} in [W1], somewhat hidden in equation (4.8) of the same reference the reader may find the relatively explicit formula

$$N\,(N-1)\,\Theta = N\Big((\mathrm{id} - \Phi)\,\Phi^*\Big) + (N\,\Phi^{-1})\,\Phi\,\Phi^* \tag{15}$$

valid in the torsion free case. The power series Φ^* occurring in this formula is obtained by expanding the power series Φ in terms of the homogeneous components of the power series $\mathscr{R}$ considered as indeterminates and reverse the order of the factors in all monomials. Alternatively Φ^* is the inverse of the unique power series solution Φ^{-*} of the wrong sided parallel transport equation $N(N+1)\,\Phi^{-*} = \Phi^{-*}\,\mathscr{R}$ with initial value $\Phi^{-*}(0) = \mathrm{id}$.

4. The Taylor Series of the Difference Element. In jet calculus the key idea is to find the appropriate difference elements and study the algebraic constraints they satisfy. Following this strategy we devise in this section an algorithm to calculate the infinite order Taylor series of the difference element $K := \exp^{-1} \circ \mathrm{knc}_p$ and its inverse in terms of the curvature tensor and its iterated covariant derivatives proving universality of the resulting expression on the way. Using this algorithm we calculate the lowest order terms of the Taylor series of the Kähler normal potential and the Riemannian distance function. Last but not least we define holomorphically extended vector fields and use them to determine the Spencer connection for the Kähler normal potential.

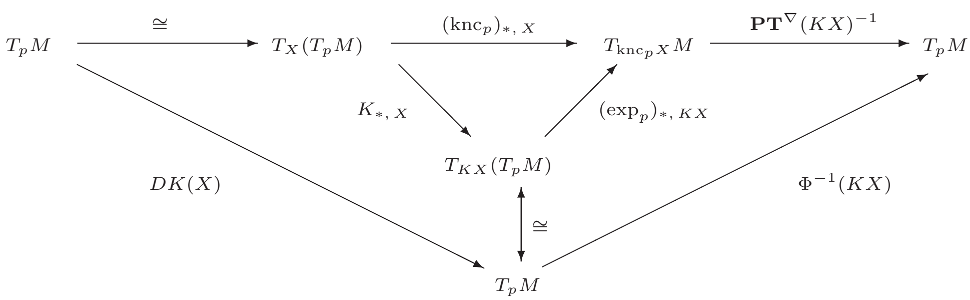
Definition 4.1 (Difference Element for Kähler Normal Coordinates). *In order to compare the unique Kähler normal coordinates* $\mathrm{knc}_p : T_pM \longrightarrow M$ *centered in a point* $p \in M$ *of a Kähler manifold* M *with the exponential map* $\exp_p : T_pM \longrightarrow M$ *associated to the Levi–Civita connection* ∇ *we consider their difference as a smooth map:*

$$K : T_pM \longrightarrow T_pM, \qquad X \longmapsto \exp_p^{-1}\Big(\mathrm{knc}_p X\Big).$$

The analogue of the backward parallel transport Φ^{-1} *in Riemannian normal coordinates is the Kähler backward parallel transport* $\Psi^{-1} \in C^\infty(T_pM, \mathrm{End}\, T_pM)$ *defined by means of:*

$$\Psi^{-1}(X)\, Y := \Phi^{-1}(KX)\; DK(X)\, Y := \Phi^{-1}(KX)\; \frac{d}{dt}\Big|_0 K(X + t\,Y).$$

Recall that the main role of the backward parallel transport Φ^{-1} in affine exponential coordinates is to describe the differential of the exponential map in covariant terms. In complete analogy the Kähler backward parallel transport Ψ^{-1} describes the differential of Kähler normal coordinates $\mathrm{knc}_p : T_pM \longrightarrow M$ as a power series of covariant tensors. More precisely the implicit form $\mathrm{knc}_p = \exp_p \circ K$ of the definition of the difference element K implies $(\mathrm{knc}_p)_{*,\,X} = (\exp_p)_{*,\,KX} \circ K_{*,\,X}$ on differentials leading to the commutative diagram



which allows us to identify the power series Ψ^{-1} from Definition 4.1 with the composition:

$$\Psi^{-1}(X) :\; T_pM \;\cong\; T_X(T_pM) \;\overset{(\mathrm{knc}_p)_{*,X}}{\longrightarrow}\; T_{\mathrm{knc}_pX}M \;\overset{\mathbf{PT}^\nabla(KX)^{-1}}{\longrightarrow}\; T_pM \tag{16}$$

In particular the pull back of the Riemannian metric g to Kähler normal coordinates reads:

$$(\mathrm{knc}_p^*g)_X(A,B) = (\exp_p^*g)_{KX}(K_{*,X}A, K_{*,X}B) = g_p(\Psi^{-1}(X)A,\; \Psi^{-1}(X)B) \tag{17}$$

Combined with equation (9) this description of the Riemannian metric g in Kähler normal coordinates can be used to calculate the Taylor series θ_p of the pull back $\mathrm{knc}_p^*\theta_p^{\mathrm{loc}}$ of the Kähler normal potential θ_p^{loc} to the tangent space T_pM via knc_p. According to our discussion of equation (9) we are free to choose the connection D among the torsion free connections making I parallel in order to evaluate Hess $= D \circ d$. With $\mathrm{knc}_p : T_pM \longrightarrow M$ being holomorphic with respect to the constant complex structure I_p on T_pM we may simply use the trivial connection D for this purpose and so we are lead to consider the consequence

$$\begin{aligned} 4\,(\mathrm{knc}_p^* g)_X(X,\ X) &= 4\, g_p(\Psi^{-1}(X)\,X,\ \Psi^{-1}(X)\,X) \\ &= (\mathrm{Hess}\ \theta_p)_X(X,\ X) + (\mathrm{Hess}\ \theta_p)_X(IX,\ IX) \end{aligned}$$

of equations (9) and (17), which implies that the formal differential equation

$$[(N^2 + \mathrm{Der}_I^2)\,\theta_p](X) = 4\, g(\Psi^{-1}(X)\,X,\ \Psi^{-1}(X)\,X) \tag{18}$$

is obeyed by the Taylor series θ_p of the pull back $\mathrm{knc}_p^*\theta_p^{\mathrm{loc}}$ of the normal potential to the tangent space T_pM; in fact the Hessian with respect to the trivial connection D satisfies

$$\begin{aligned} (\mathrm{Hess}\ \psi)_X(\ X\ ,\ \ X) &= [N\,(N-1)\,\psi](X) \\ (\mathrm{Hess}\ \psi)_X(IX,\ IX) &= [(\mathrm{Der}_I^2 + N)\,\psi](X) \end{aligned}$$

for every polynomial ψ on T_pM. Besides determining the pull backs of the Riemannian metric g and the Kähler normal potential θ^{loc} respectively to Kähler normal coordinates via equations (17) and (18) the difference element K offers a very direct description of the square of the Riemannian distance function in Kähler normal coordinates as well:

Remark 4.2 (Distance Function in Kähler Normal Coordinates). *In Riemannian geometry the exponential map is a radial isometry in the sense that the Riemannian distance between* $p \in M$ *and* $\exp_p X$ *equals* $\mathrm{dist}_g^2(p,\ \exp_p X) = g_p(X,\ X)$ *for* X *sufficiently small. In Kähler normal coordinates this property of the distance becomes:*

$$\mathrm{dist}_g^2(p,\ \mathrm{knc}_p X) = \mathrm{dist}_g^2(p,\ \exp_p(KX)) = g_p(KX,\ KX).$$

Let us now come to the more difficult task of finding an efficient method to calculate the Taylor series of the difference elements K and K^{-1}. For that purpose we recall the fact that a vector field $X \in \Gamma(TM)$ on a manifold M with integrable almost complex structure I is the real part of a holomorphic vector field in the sense that the map $X - iIX : M \longrightarrow T^{1,0}M$ between complex manifolds is actually holomorphic, if and only if

$$\mathfrak{Lie}_X I = 0$$

compare for example [Ba] and [M]. In order to construct a resolution for the symbolic differential operator $\mathscr{L}$ corresponding to this complex Killing equation in the formal theory of partial differential equations [BCG], [W3] we need the concept of special alternating forms:

Definition 4.3 (Special Alternating Forms). *Consider the graded vector space* $\Lambda^\circ T^* \otimes T$ *of alternating forms on a real vector space* T *with values in* T. *The choice of* *a complex structure* $I \in \mathrm{End}\, T$ *on* T *singles out the graded subspace* $\Sigma^\circ \subset \Lambda^\circ T^* \otimes T$ *of special alternating forms defined in all degrees* $k \in \mathbb{N}_0$ *by:*

$$\Sigma^k := \{\ F \in \Lambda^k T^* \otimes T \ \mid\ (\mathrm{Der}_I \otimes I)\, F \ =\ k\, F\ \}.$$

Alternatively a special alternating k*–form* $F : T \times \ldots \times T \longrightarrow T$ *is characterized by*

$$I\, F(X_1,\, \ldots,\, X_k) = -\, F(I X_1,\, X_2,\, \ldots,\, X_k)$$

in particular $\Sigma^1 \subset T^* \otimes T$ *is simply the subspace of endomorphisms anticommuting with* I*:*

$$\mathrm{pr}_{\Sigma^1} : \quad T^* \otimes T \longrightarrow \Sigma^1, \qquad F \longmapsto \tfrac{1}{2}(F + I\, F\, I).$$

The link between the complex Killing equation and special alternating forms is established directly by rewriting the definition of the Lie derivative $\mathfrak{Lie}_X I$ in terms of a torsion free connection ∇ on the tangent bundle making I parallel $\nabla I = 0$. On a Kähler manifold M we take for example the Levi–Civita connection and find for the Lie derivative of the orthogonal complex structure I in the direction of a vector field $X \in \Gamma(TM)$:

$$\begin{aligned}(\mathfrak{Lie}_X I)\, Y &= \Big(\nabla_X (IY) - \nabla_{IY} X\Big) - I\, \Big(\nabla_X Y - \nabla_Y X\Big)\\ &= I\Big(\nabla_Y X + I\, \nabla_{IY} X\Big) = 2\, I\, \mathrm{pr}_{\Sigma^1}(\nabla X)\, Y.\end{aligned}$$

Hence the real parts of holomorphic vector fields are precisely the sections $X \in \Gamma(TM)$ in the kernel of the differential operator $\mathscr{L}_{\mathrm{diff}}$ defined as the composition of the covariant derivative ∇ with the projection pr_{Σ^1} to the subbundle $\Sigma^1 M \subset \mathrm{End}\, TM$

$$\mathscr{L}_{\mathrm{diff}} : \Gamma(TM) \stackrel{\nabla}{\longrightarrow} \Gamma(T^*M \otimes TM) \stackrel{\mathrm{pr}_{\Sigma^1}}{\longrightarrow} \Gamma(\Sigma^1 M), \qquad X \longmapsto -\frac{1}{2}\, I\, \mathfrak{Lie}_X I \quad (19)$$

of endomorphisms anticommuting with I. From the definition of $\mathscr{L}_{\mathrm{diff}}$ as the composition $\mathrm{pr}_{\Sigma^1} \circ \nabla$ we read off its principal symbol and in turn the associated symbol comodule [W3]:

$$\mathscr{H}^\bullet := \ker\Big(\mathscr{L} : \mathrm{Sym}^\bullet T^* \otimes \Sigma^0 \stackrel{\Delta}{\longrightarrow} \mathrm{Sym}^{\bullet-1} T^* \otimes (T^* \otimes T) \stackrel{\mathrm{id} \otimes \mathrm{pr}_{\Sigma^1}}{\longrightarrow} \mathrm{Sym}^{\bullet-1} T^* \otimes \Sigma^1\Big)$$

The symbolic differential operator $\mathscr{L}$ defining the comodule $\mathscr{H}$ extends to a complete resolution of the comodule $\mathscr{H}$ by free comodules using an adjoint pair of boundary operators:

Definition 4.4 (Symbolic Differential Operators). *Consider a real vector space* T *endowed with a complex structure* $I \in \mathrm{End}\, T$. *On the bigraded vector space* $\mathrm{Sym}^\bullet T^* \otimes \Sigma^\circ \subset \mathrm{Sym}^\bullet T^* \otimes \Lambda^\circ T^* \otimes T$ *of special alternating forms on* T *with polynomial coefficients we define two bigraded boundary operators* $\mathscr{L}^*$ *and* $\mathscr{L}$ *by:*

$$\begin{aligned}[\, \mathscr{L}^* F\,]_X (Z_2, \ldots, Z_r) &:= \quad F_X(X,\, Z_2,\, \ldots,\, Z_r)\\ [\ \mathscr{L}\, F\ ]_X (Z_0, \ldots, Z_r) &:= \frac{1}{2} \sum_{\mu=0}^{r} (-1)^\mu \Big((Z_\mu \lrcorner F)_X (Z_0, \ldots, Z_r) + I\, (I Z_\mu \lrcorner F)_X (Z_0, \ldots, Z_r)\Big)\end{aligned}$$

In order to clarify this definition we recall the convention adopted in this article concerning the identification of a symmetric k–multilinear form $F = \mathrm{Sym}^k T^*$ on T

with a homogeneous polynomial on T of degree $k \in \mathbb{N}_0$: The polynomial corresponding to F is defined by $F(X) := \frac{1}{k!}F(X, \ldots, X)$ so that the operation $Z \lrcorner$ of inserting the first argument agrees with the directional derivative $\frac{\partial}{\partial Z}$ of the polynomial in direction Z. With this convention in place we may alternatively define the boundary operators $\mathscr{L}^*$ and $\mathscr{L}$ in terms of the tensor product decomposition $\mathrm{Sym}^\bullet T^* \otimes \Sigma^\circ \subset \mathrm{Sym}^\bullet T^* \otimes \Lambda^\circ T^* \otimes T$ as sums

$$\mathscr{L}^* := \sum_\mu dE_\mu \cdot \otimes E_\mu \lrcorner \otimes \mathrm{id}$$
$$\mathscr{L} := \frac{1}{2}\sum_\mu \Big(E_\mu \lrcorner \otimes dE_\mu \wedge \otimes \mathrm{id} + IE_\mu \lrcorner \otimes dE_\mu \wedge \otimes I \Big)$$

over a dual pair of bases $\{ E_\mu \}$ and $\{ dE_\mu \}$ of T and T^*. Alternatively we could complexify the domain $\mathrm{Sym}^\bullet T^* \otimes \Lambda^\circ T^* \otimes T$ of the boundary operator $\mathscr{L}$ and choose a complex basis $\{ F_\alpha \}$ of the subspace $T^{1,0} \subset T \otimes_{\mathbb{R}} \mathbb{C}$ with dual basis $\{ dF_\alpha \}$ of $T^{1,0*} \subset T^* \otimes_{\mathbb{R}} \mathbb{C}$ to rewrite

$$\mathscr{L} := \sum_\alpha \Big(F_\alpha \lrcorner \otimes dF_\alpha \wedge \otimes \mathrm{pr}^{0,1} + \overline{F}_\alpha \lrcorner \otimes d\overline{F}_\alpha \wedge \otimes \mathrm{pr}^{1,0} \Big)$$

with the eigenprojections $\mathrm{pr}^{1,0} := \frac{1}{2}(\mathrm{id} - iI)$ and $\mathrm{pr}^{0,1} := \frac{1}{2}(\mathrm{id} + iI)$ to $T^{1,0}$ and $T^{0,1}$, in this alternative formulation the operator $\mathscr{L}$ evidently preserves the subspace of special alternating forms with polynomial coefficients. The reader is invited to use this description of the operator $\mathscr{L}$ in order to provide a more enlightening proof of Corollary 4.5 below. Staying in the real domain we calculate the anticommutator $\{ \mathscr{L}, \mathscr{L}^* \}$ of the boundary operators $\mathscr{L}$ and $\mathscr{L}^*$ using the canonical commutation and anticommutation relations

$$\begin{aligned}\{\mathscr{L}, \mathscr{L}^*\} &= \frac{1}{2}\sum_{\mu\,\nu} \Big([E_\mu \lrcorner, dE_\nu \cdot] \otimes dE_\mu \wedge E_\nu \lrcorner \otimes \mathrm{id} + dE_\nu \cdot E_\mu \lrcorner \otimes \{dE_\mu \wedge, E_\nu \lrcorner\} \otimes \mathrm{id} \\ &\qquad + [IE_\mu \lrcorner, dE_\nu \cdot] \otimes dE_\mu \wedge E_\nu \lrcorner \otimes I + dE_\nu \cdot IE_\mu \lrcorner \otimes \{dE_\mu \wedge, E_\nu \lrcorner\} \otimes I \Big) \\ &= \frac{1}{2}(\mathrm{id} \otimes N \otimes \mathrm{id} + N \otimes \mathrm{id} \otimes \mathrm{id} + \mathrm{id} \otimes \mathrm{Der}_I \otimes I + \mathrm{Der}_I \otimes \mathrm{id} \otimes I)\end{aligned}$$

where N denotes the number operator either on polynomials or on forms depending on context. The equality of linear maps $\mathrm{Der}_I \otimes I = N \otimes \mathrm{id}$ defining the subspace of special alternating forms $\Sigma^\circ \subset \Lambda^\circ T^* \otimes T$ allows us to write this identity in the simpler form:

$$\Delta := \{\mathscr{L}, \mathscr{L}^*\} = \mathrm{id} \otimes N \otimes \mathrm{id} + \frac{1}{2}(N \otimes \mathrm{id} \otimes \mathrm{id} + \mathrm{Der}_I \otimes \mathrm{id} \otimes I). \qquad (20)$$

A short inspection reveals that the formal Laplace operator Δ acts diagonalizable on the space $\mathrm{Sym}^\bullet T^* \otimes \Sigma^\circ$ of special alternating forms with polynomial coefficients with eigenspaces

$$(\mathrm{Sym}^{\kappa,\,\overline{\kappa}} T^* \otimes \Lambda^d T^{0,1*} \otimes T^{1,0}) \oplus (\mathrm{Sym}^{\overline{\kappa},\,\kappa} T^* \otimes \Lambda^d T^{1,0*} \otimes T^{0,1}) \subset (\mathrm{Sym}^\bullet T^* \otimes \Sigma^d) \otimes_{\mathbb{R}} \mathbb{C}$$

and eigenvalues $d + \overline{\kappa}$ parametrized by $\kappa, \overline{\kappa} \geq 0$ and $d \geq 0$ with $\bullet = \kappa + \overline{\kappa}$. With Δ being diagonalizable we conclude from Hodge theory that the homology of the $\mathscr{L}-$ complex equals the homology of the eigensubcomplex $\ker \Delta$, because $\frac{1}{\lambda}\mathscr{L}^*$ is a zero homotopy for the eigensubcomplexes $\ker(\Delta - \lambda\,\mathrm{id})$ for all eigenvalues $\lambda \neq 0$. On the other hand $\ker\Delta$ equals

$$(\mathrm{Sym}^{\bullet,\,0}T^* \otimes T^{1,0}) \;\oplus\; (\mathrm{Sym}^{0,\,\bullet}T^* \otimes T^{0,1}) = \mathscr{H}^\bullet \;\otimes_{\mathbb{R}} \mathbb{C} \;\;\subset\;\; (\mathrm{Sym}^\bullet T^* \otimes T) \otimes_{\mathbb{R}} \mathbb{C}$$

and so $\mathscr{L} = 0$ vanishes on $\ker\Delta$ for trivial reasons. Summarizing this argument we conclude:

Corollary 4.5 (Free Resolution of Holomorphic Vector Fields). *The total prolongation comodule $\mathscr{H}^\bullet$ associated to the principal symbol $\mathrm{pr}_{\Sigma^1} : T^* \otimes T \longrightarrow \Sigma^1$ of the differential operator $\mathscr{L}_{\mathrm{diff}}$ characterizing the real parts of holomorphic vector fields in complex dimension $n := \frac{1}{2}\dim T$ has a resolution by free comodules of the form:*

$$\begin{aligned} 0 \;\longrightarrow\; \mathscr{H}^\bullet \;\xrightarrow{\;\subset\;}\; \mathrm{Sym}^\bullet T^* \otimes \Sigma^0 \;\xrightarrow{\;\mathscr{L}\;}\; \mathrm{Sym}^{\bullet-1}T^* \otimes \Sigma^1 \;\xrightarrow{\;\mathscr{L}\;}\; \ldots \\ \ldots \;\xrightarrow{\;\mathscr{L}\;}\; \mathrm{Sym}^{\bullet-n}T^* \otimes \Sigma^n \;\longrightarrow\; 0. \end{aligned}$$

An important property of the real parts of holomorphic vector fields needed below is that they can be reconstructed up to a linear term from their closure, their image under the map

$$\mathrm{cl}: \quad \mathrm{Sym}^\bullet T^* \otimes T \longrightarrow \mathrm{Sym}^{\bullet+1}T^*, \qquad Z \longmapsto \mathrm{cl}\,Z \tag{21}$$

from vector fields to polynomials defined by $(\mathrm{cl}\,Z)(X) := g(Z(X), X)$. Since the eigenspaces $T^{1,0}$ and $T^{0,1}$ of I are isotropic subspaces of $T \otimes_{\mathbb{R}} \mathbb{C}$, the closure of the real part of a holomorphic vector field lies in the vector space $\mathrm{Sym}^\bullet_{[1]}T^* := \ker((N-2)^2 + \mathrm{Der}_I^2)$ of polynomials linear in the holomorphic or antiholomorphic coordinates with complexification:

$$\mathrm{Sym}^\bullet_{[1]}T^* \otimes_{\mathbb{R}} \mathbb{C} = \bigoplus_{\substack{\kappa,\,\overline{\kappa} \geq 0 \\ \kappa=1 \text{ or } \overline{\kappa}=1}} \mathrm{Sym}^{\kappa,\,\overline{\kappa}}T^*. \tag{22}$$

Restricted to the real parts of holomorphic vector fields cl induces in fact an exact sequence

$$0 \;\longrightarrow\; \delta_{\bullet=1}\,\mathfrak{u}(T,\,g,\,I) \;\xrightarrow{\;\subset\;}\; \mathscr{H}^\bullet \;\xrightarrow{\;\mathrm{cl}\;}\; \mathrm{Sym}^{\bullet+1}_{[1]}T^* \;\longrightarrow\; 0 \tag{23}$$

in which $\mathfrak{u}(T,\,g,\,I)$ denotes the vector space of linear vector fields on T corresponding to infinitesimal unitary transformations with respect to the hermitean form $h = g + i\omega$.

Coming back to our original aim to derive recursion formulas for the infinite order Taylor series of the difference elements K and K^{-1} we recall that multilinear maps on a vector space with values in this vector space can be composed to produce new multilinear forms

$$\begin{aligned} &(A \circ_\mu B)(X_1, \ldots, X_{a+b+1}) \\ := \;&A(X_1, \ldots,, X_{\mu-1},\, B(X_\mu, \ldots, X_{\mu+b}),\, X_{\mu+b+1}, \ldots, X_{a+b+1}) \end{aligned}$$

for some position $\mu = 1, \ldots, a$. We will call a multilinear form arising from a fixed set of basic multilinear forms by iterated compositions in arbitrary positions a composition polynomial in the selected set of basic multilinear forms. In particular we will consider composition polynomials in the complex structure I_p, the curvature tensor

R_p and its iterated covariant derivatives $(\nabla R)_p$, $(\nabla^2 R)_p$, ... on the tangent space T_pM of a Kähler manifold M in a point $p \in M$. In general multilinear forms on T_pM are bigraded by degree and weight: The degree of a multilinear form is the number of arguments it takes minus 1 to make the degree additive under composition, while its weight is the eigenvalue for the weight operator:

$$(\delta A)(X_1, \ldots, X_k) := I\, A(X_1, \ldots, X_k) - (\mathrm{Der}_I A)(X_1, \ldots, X_k). \tag{24}$$

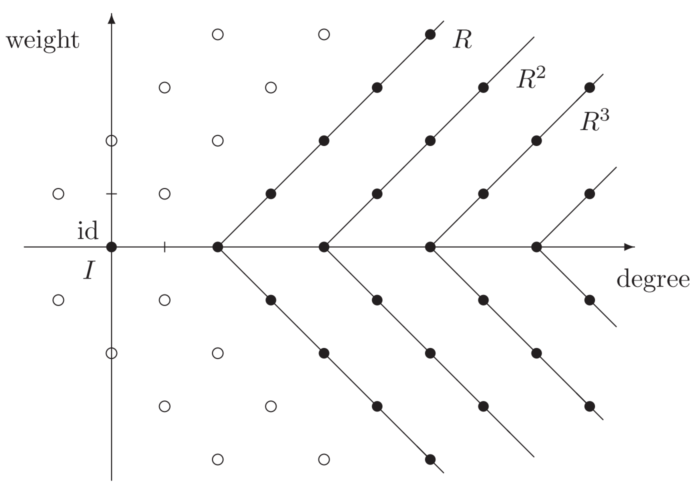
The weight qualifies as a grading, because the weight operator is a derivation for composition

$$\delta\,(A \circ_\mu B) = (\delta A) \circ_\mu B + A \circ_\mu (\delta B)$$

as the term $A(X_1, \ldots, X_{\mu-1}, IB(X_\mu, \ldots, X_{\mu+b}), X_{\mu+b+1}, \ldots, X_{a+b+1})$ appears twice with different sign in the expansion of the right hand side. The curvature tensor R of the Kähler manifold M for example is a degree 2 multilinear form on T_pM of weight 0

$$(\delta R)_{X,\,Y} Z = R_{X,\,Y} IZ - R_{I^2X,\,IY} Z - R_{X,\,IY} Z - R_{X,\,Y} IZ = 0$$

in consequence its iterated covariant derivatives $\nabla^r R$ decompose after complexification into a sum of multilinear forms of degree $r+2$ and weight $-ri, \ldots, +ri$. As both degree and weight are additive under composition a quadratic composition polynomial in the curvature tensor and its iterated covariant derivatives needs to have at least degree $r+4$ to decompose similarly into multilinear forms of weight $-ri, \ldots, +ri$ and so on. The possible combinations of degree and weight of multilinear forms in general and composition polynomials in the curvature tensor and its iterated covariant derivatives can be read off from the diagram:



(25)

These considerations play a crucial role in the proof of the following theorem:

Theorem 4.6 (Universality of Kähler Normal Coordinates). *Every term in the Taylor series of the difference elements* $K = \exp_p^{-1} \circ \mathrm{knc}_p$ *and* K^{-1} *in the origin* $0 \in T_pM$ *is a universal composition polynomial in the complex structure* I*, the curvature tensor* R *and its iterated covariant derivatives* $\nabla R, \nabla^2 R, \ldots$ *evaluated in* $p \in M$.

In this context universality refers to the statement that these composition polynomials can be chosen without taking the Kähler manifold M or its dimension into account.

Universality does not imply uniqueness of course, and actually uniqueness wouldn't make sense anyhow:

Remark 4.7 (Non–Uniqueness of Composition Polynomials). *Composition polynomials in the complex structure, the curvature tensor and its covariant derivatives* I, R, ∇R, $\ldots$ *are in general not unique due to additional identities arising from Ricci type constraints. The following identity for example is valid on Kähler manifolds* M

$$(\nabla^2_{X,\,IX}R)_{X,\,IX} - (\nabla^2_{IX,\,X}R)_{X,\,IX} = [R_{X,\,IX},\,R_{X,\,IX}] - R_{R_{X,\,IX}X,\,IX} - R_{X,\,R_{X,\,IX}IX}$$
$$= -\,2\,R_{R_{X,\,IX}X,\,IX}$$

for all $X \in TM$ *and implies* $R_{X,\,R_{X,\,IX}IX} = 0$ *on hermitean locally symmetric spaces.*

Proof of Theorem 4.6. Both $\exp_p$ and knc_p are anchored coordinates for a Kähler manifold M, hence the differential of K in $0 \in T_pM$ equals the identity. In turn the decomposition of the infinite order Taylor series of K into homogeneous components reads

$$KX = X + K_2X + K_3X + \ldots \qquad K_n \;\in\; \mathrm{Sym}^nT^*_pM \otimes T_pM = \mathscr{H}^n \,\oplus\, \mathscr{H}^{n\perp}$$

where the complement $\mathscr{H}^{n\perp}$ to $\mathscr{H}^n = \ker\Delta$ is simply the direct sum of all eigenspaces of $2\Delta = (N\otimes\mathrm{id}+\mathrm{Der}_I\otimes I)$ with non–vanishing eigenvalue and thus the proper domain of:

$$(N \otimes \mathrm{id} + \mathrm{Der}_I \otimes I)^{-1} : \quad \mathscr{H}^{n\perp} \longrightarrow \mathscr{H}^{n\perp}. \tag{26}$$

Essentially the proof consists in verifying by induction on the degree n that the characteristic normalization constraint imposed on the Kähler normal potential θ is actually equivalent to the normalization constraint $K_n \in \mathscr{H}^{n\perp}$ for all $n \geq 2$.

In a first step we want to analyse the implications of the holomorphicity of Kähler normal coordinates knc_p on the difference element $K = \exp_p^{-1} \circ \mathrm{knc}_p$. On a Kähler manifold the complex structure I is a parallel tensor and so Corollary 3.2 implies for the infinite order Taylor series for its pull back $\exp^*_p I$ to the tangent space under the exponential map

$$(\exp^*_p I)_X \underset{X\to 0}{\sim} \Phi(X) \circ I_p \circ \Phi^{-1}(X)$$

whereas the holomorphicity of knc_p is equivalent to the statement $(\mathrm{knc}^*_p I)_X = I_p$ for all $X \in T_pM$. Hence the differential $K_{*,\,X} : T_X(T_pM) \longrightarrow T_{KX}(T_pM)$, which becomes $DK(X)$ under the identification of both domain and target with T_pM, intertwines I_p with $(\exp^*_p I)_{KX}$:

$$\Phi(KX) \circ I_p \circ \Phi(KX)^{-1} \circ DK(X) = DK(X) \circ I_p.$$

Dropping the explicit mention of the point $p \in M$ on $I = I_p$ we may write this equation

$$[I,\,\Phi^{-1}(KX) \circ DK(X)] = 0 = [I,\,DK^{-1}(X) \circ \Phi(X)] \tag{27}$$

where we replaced X by $K^{-1}X$ in the second equality. Note that this result is compatible with the interpretation of the Kähler backward parallel transport $\Psi^{-1}(X) := \Phi^{-1}(KX) \circ DK(X)$ as the differential of the Kähler normal coordinates knc_p.

For a moment let us forget that K is a diffeomorphism and think of it as a vector field with Taylor series $K \in \overline{\mathrm{Sym}}\, T_p^*M \otimes T_pM$. By construction the symbolic differential operator $\mathscr{L}$ equals $\mathscr{L}K = \mathrm{pr}_{\Sigma^1}(DK)$ on vector fields, whereas $\mathscr{L}^*K = 0$. Applying the anticommutator $\Delta = \{\, \mathscr{L}, \mathscr{L}^* \,\}$ to the vector field K we thus obtain the decisive formula

$$\Delta K = \mathscr{L}^* \,\mathrm{pr}_{\Sigma^1}\Big(DK\Big) = \mathscr{L}^* \,\mathrm{pr}_{\Sigma^1}\Big((\mathrm{id} - [\Phi^{-1} \circ K])\, DK\Big) \tag{28}$$

where $[\Phi^{-1} \circ K] \circ DK$ commutes with I and thus vanishes under pr_{Σ^1} according to (27). The resulting formula is actually a recursion formula, which allows us to calculate the term K_n in the Taylor series of K from the terms $K_2, \ldots, K_{n-2}$ up to addition by an element of $\ker \Delta$, because $\mathrm{id} - \Phi^{-1}(KX) = -\frac{1}{6}\, R_{X,\cdot}X + O(X^3)$ is at least quadratic in X.

In the ensuing induction on the degree $n \geq 2$ it is more convenient to begin with the inductive step. By induction hypothesis we may thus assume that for some $n \geq 3$ all the terms $K_2, \ldots, K_{n-1}$ are universal composition polynomials in the complex structure I, the curvature tensor and its iterated covariant derivatives $R, \nabla R, \ldots$. Under this assumption the homogeneous term of degree n in X on the right hand side of the recursion formula (28) is a universal composition polynomial in these generators and so then is the corresponding term on the left hand side. The formal Laplace operator $2\Delta = (N \otimes \mathrm{id} + \mathrm{Der}_I \otimes I)$ on the other hand has only finitely many non–zero eigenvalues for fixed degree $N = n$ so that its partial inverse (26) can be written as a polynomial in $\mathrm{Der}_I \otimes I$. In consequence the partial inverse maps composition polynomials in $I, R, \nabla R, \ldots$ to composition polynomials and so

$$K_n = H_n + \text{composition polynomial in } I,\ R,\ \nabla R,\ \nabla^2 R,\ \ldots$$

for some $H_n \in \mathscr{H}^n$. Incidentally the same conclusion is valid in the special case $n = 2$ forming the base of our induction, because the right hand side of the recursion formula (28) is at least cubic in X as $\mathscr{L}^*$ raises the degree by 1 and $\mathrm{id} - \Phi^{-1}(KX) = O(X^2)$.

In order to verify the base of induction and complete the induction step we need to show $H_n = 0$ for all $n \geq 2$. For this purpose we calculate the term homogeneous of degree n in X in the Kähler backward parallel transport observing $DK(X)\, X = (NK)\, X$:

$$\begin{aligned}[\Psi^{-1}(X)\, X]_n &= [\Phi^{-1}(KX)\, DK(X)\, X]_n \\ &= n\, H_n X + \text{composition polynomial in } I,\ R,\ \nabla R,\ \nabla^2 R,\ \ldots\end{aligned}$$

Inserting this expression into equation (18) we obtain the expansion

$$[(N^2 + \mathrm{Der}_I^2)\, \theta]_{n+1}(X) = 8\, n\, g(H_n X,\ X) + \ldots \tag{29}$$

for the homogeneous term of degree $n+1$ in X in the Kähler normal potential, where the ellipsis denotes a finite sum of homogeneous polynomials of degree $n+1$ in X of the form $g(\mathbb{A}, \mathbb{B})$ with composition polynomials $\mathbb{A}$ and $\mathbb{B}$ in the complex structure I, the curvature tensor R and its iterated covariant derivatives. For every polynomial of this form we find

$$\mathrm{Der}_I\, [g(\mathbb{A},\ \mathbb{B})] = g(\delta\, \mathbb{A},\ \mathbb{B}) + g(\mathbb{A},\ \delta \mathbb{B})$$

because the two additional terms $g(I\mathbb{A}, \mathbb{B}) + g(\mathbb{A}, I\mathbb{B})$ on the right hand side obtained upon expanding δ cancel out by the skew symmetry of I. Of course $\mathbb{A}$ and $\mathbb{B}$ can not be composition polynomials in I only, since we would not get a polynomial of degree $n+1 \geq 3$, hence at least one of $\mathbb{A}$ or $\mathbb{B}$ is at least linear in R, ∇R, $\nabla^2 R$, ... Diagram (25) thus tells us the possible combinations of degree and weight of the polynomial $g(\mathbb{A}, \mathbb{B})$, from which we deduce:

$$g(\mathbb{A},\, \mathbb{B}) \;\equiv\; 0 \qquad \mathrm{mod}\;\; \mathrm{Sym}^{\geq(2,2)} T_p^* M$$

In consequence equation (29) is a congruence modulo $\mathrm{Sym}^{\geq(2,2)} T_p^* M$, which we may write

$$[(N^2 + \mathrm{Der}_I^2)\, \theta]_{n+1} \equiv 8\, n\, (\mathrm{cl}\, H_n)$$

in light of the definition (21) of the closure map. The normalization constraint imposed on the Kähler normal potential implies on the other hand the congruence $[(N^2 + \mathrm{Der}_I^2)\, \theta]_{n+1} \;\equiv\; 0$ modulo $\mathrm{Sym}^{\geq(2,2)} T_p^* M$ for all $n+1 \neq 2$ so that $\mathrm{cl}\, H_n = 0$ and a fortiori $H_n = 0$ due to the exactness of the sequence (23). □

Although the preceeding proof of Theorem 4.6 takes the existence of Kähler normal coordinates stipulated in Theorem 2.6 for granted, it is easily rearranged to prove existence on the fly alongside the induction. In this setup we start with arbitrary anchored holomorphic coordinates $\mathrm{knc}_p^{(1)} : T_p M \longrightarrow M$ thought of as a first approximation to the Kähler normal coordinates we want to construct and use the flow of the real part H_2 of a quadratic holomorphic vector field to modify $\mathrm{knc}_p^{(1)}$ to the better approximation $\mathrm{knc}_p^{(2)}$ characterized by $H_2 = 0$. In turn we use the real part H_3 of a cubic holomorphic vector field to find an even better approximation $\mathrm{knc}_p^{(3)}$ characterized by $H_2 = 0 = H_3$ and so on. The advantage of this modified proof is that it applies verbatim to a wider class of complex affine manifolds:

Definition 4.8 (Balanced Complex Affine Manifolds). *A complex affine manifold is a smooth manifold M endowed with a torsion free connection ∇ on its tangent bundle and an almost complex structure I, which is parallel $\nabla I = 0$ and thus integrable by the Theorem of Newlander–Nierenberg. A complex affine manifold M is balanced provided the curvature tensor R of the connection ∇ is a $(1,1)$–form in the sense:*

$$R_{I\,\cdot\,,\,I\,\cdot} = R_{\,\cdot\,,\,\cdot} \qquad \Longleftrightarrow \qquad \delta R = 0.$$

In every point $p \in M$ of a balanced complex affine manifold M there thus exist unique anchored holomorphic coordinates $\mathrm{knc}_p : T_p M \longrightarrow M$ characterized by the congruence

$$K := \exp_p^{-1} \circ \mathrm{knc}_p \;\equiv\; \mathrm{id} \qquad \mathrm{mod}\;\; \mathscr{H}^{\perp}$$

imposed directly on the infinite order Taylor series of the difference element K. In consequence this Taylor series is given by the very same universal composition power series in I, R, ∇R, ... we found in the Kähler case. Although originally formulated in terms of the potential and so apparently depending on the metric structure of a Kähler manifold, the concept of Kähler normal coordinates turns out to arise from the underlying complex affine structure! The following corollary about totally geodesic submanifolds, whose proof is left to the reader, is a nice illustration of this dependence on the affine structure:

Corollary 4.9 (Totally Geodesic Complex Submanifolds). *For every totally geodesic complex submanifold $N \subset M$ of a Kähler manifold M the restriction of the*

Kähler normal coordinates $\mathrm{knc}_p : T_pM \longrightarrow M$ *in a point* $p \in N$ *to the vector subspace* $T_pN \subset T_pM$ *are the Kähler normal coordinates for the Kähler manifold* N.

Instead of using the difference element K we may consider using its inverse in the proof of Theorem 4.6. Lacking an analogue of equation (18) it is more difficult to relate the normalization constraint imposed on $K^{-1} = \mathrm{knc}_p^{-1} \circ \exp_p$ to the normalization constraint imposed on the potential θ, however the choice of K^{-1} allows us to use the Gauß Lemma to simplify the analogue of the recursion formula (28). Mimicking the argument we find

$$(N \otimes \mathrm{id} + \mathrm{Der}_I \otimes I)\, K^{-1} = 2\,\{\,\mathscr{L},\, \mathscr{L}^*\,\}\, K^{-1} = 2\,\mathscr{L}^*\,\mathrm{pr}_{\Sigma^1}\Big(DK^{-1} \circ (\mathrm{id} - \Phi)\Big)$$

since $DK^{-1} \circ \Phi$ still commutes with I according to (27). Inserting the definitions of the projection pr_{Σ^1} and the boundary operator $\mathscr{L}^*$ we find that the right hand side simplifies due to the Gauß Lemma (12) written in the form $(\mathrm{id} - \Phi(X))\, X = 0$, moreover we may replace the cumbersome notation involving DK^{-1} by the derivation extension of the endomorphism $F := (\mathrm{id} - \Phi(X)) \circ I$ using the identity $(\mathrm{Der}_F K^{-1})(X) = DK^{-1}(X)\, FX$:

Remark 4.10 (Explicit Version of Recursion Formula). *The Taylor series of the inverse difference element* K^{-1} *satisfies the recursion formula:*

$$\begin{aligned}\Big[(N \otimes \mathrm{id} + \mathrm{Der}_I \otimes I)\, K^{-1}\Big](X) &= I\Big[\mathrm{Der}_{(\mathrm{id}-\Phi(X))\, I} K^{-1}\Big](X)\\ &= +\sum_{k\geq 0} \frac{k+1}{(k+3)!}\, (\nabla^k_{X,\,\ldots,\,X} R)_{X,\,IX} IX \;+\; O(R^2)\end{aligned}$$

where $O(R^2)$ *signifies composition polynomials at least quadratic in* R, ∇R, $\nabla^2 R$, $\ldots$.

The relatively simple recursion formula for K^{-1} allows us to calculate the lowest order terms of the Taylor series for all relevant objects on a Kähler manifold like the difference element

$$\begin{aligned}K^{-1}X = X &+ \frac{1}{12}\, R_{X,\,IX} IX + \frac{1}{96}\Big(3\,(\nabla_X R)_{X,IX} IX - (\nabla_{IX} R)_{IX,X} X\Big)\\ &+ \frac{1}{960}\Big(7\,(\nabla^2_{X,X} R)_{X,IX} IX + 2\,(\nabla^2_{IX,X} R)_{X,IX} X + 2\,(\nabla^2_{X,IX} R)_{X,IX} X\\ &- (\nabla^2_{IX,IX} R)_{X,IX} IX\Big) - \frac{1}{120}\, R_{X,R_{X,IX}X} IX - \frac{1}{96}\, R_{X,R_{X,IX}IX} X + O(X^6)\end{aligned}$$

which provides the expansion of K by formal inversion. Equation (18) can be used to expand

$$\begin{aligned}\theta(X) = g(X,\, X) &- \frac{1}{8}\, g(R_{X,IX} IX, X) - \frac{1}{24}\, g((\nabla_X R)_{X,IX} IX, X)\\ &- \frac{1}{576}\Big(5\, g((\nabla^2_{X,X} R)_{X,IX} IX, X) - g((\nabla^2_{IX,IX} R)_{X,IX} IX, X)\Big)\\ &+ \frac{1}{48}\, g(R_{X,IX} X, R_{X,IX} X) + O(X^7)\end{aligned}$$

the Kähler normal potential and Remark 4.2 to expand the Riemannian distance function:

$$\begin{aligned}\operatorname{dist}_g^2(p,\,\mathrm{knc}_p X) = g(X,X) &- \frac{1}{6}\,g(R_{X,IX}IX,X) - \frac{1}{16}\,g((\nabla_X R)_{X,IX}IX,X)\\ &-\frac{1}{480}\Big(7\,g((\nabla^2_{X,X}R)_{X,IX}IX,X) - g((\nabla^2_{IX,IX}R)_{X,IX}IX,X)\Big)\\ &+\frac{23}{720}\,g(R_{X,IX}X,R_{X,IX}X) + O(X^7).\end{aligned}$$

With somewhat more effort one obtains the congruences (1), (2) given in the introduction.

Before closing this section we want to discuss the analogue of the exponentially extended vector fields $Z \rightsquigarrow Z^{\exp}$ defined in relation with the forward and backward parallel transport in Section 3. Replacing the affine exponential map $\exp_p : T_pM \longrightarrow M$ used in this construction with Kähler normal coordinates $\mathrm{knc}_p : T_pM \longrightarrow M$ we can define for every tangent vector $Z \in T_pM$ the holomorphically extended vector field Z^{knc} on T_pM by choosing a representative curve $\gamma :]-\varepsilon, +\varepsilon[\longrightarrow M$ for Z with associated parallel transport $\mathbf{PT}^\nabla_\gamma(t)$:

$$Z^{\mathrm{knc}} := \frac{d}{dt}\bigg|_0 \Big(T_pM \overset{\mathbf{PT}^\nabla_\gamma(t)}{\longrightarrow} T_{\gamma(t)}M \overset{\mathrm{knc}_{\gamma(t)}}{\longrightarrow} M \overset{(\mathrm{knc}_p)^{-1}}{\longrightarrow} T_pM\Big) \in \Gamma(T\,(T_pM)).$$

By construction Z^{knc} is actually the real part of a holomorphic vector field on T_pM, because the parallel transport $\mathbf{PT}^\nabla_\gamma(t) : T_pM \longrightarrow T_{\gamma(t)}M$ intertwines the complex structures I_p and $I_{\gamma(t)}$ and thus can be thought of as a biholomorphism between T_pM and $T_{\gamma(t)}M$ considered as complex manifolds, in consequence Z^{knc} is the derivative of a family of biholomorphisms. Expanding the composition defining this family of biholomorphisms in order to bring the properties of the difference element K to bear we end up with the alternative formulation

$$Z^{\mathrm{knc}} := \frac{d}{dt}\bigg|_0 K_p^{-1} \circ \Big(\exp_p^{-1} \circ \exp_{\gamma(t)} \circ \mathbf{PT}^\nabla_\gamma(t)\Big) \circ \Big(\mathbf{PT}^\nabla_\gamma(t)^{-1} \circ K_{\gamma(t)} \circ \mathbf{PT}^\nabla_\gamma(t)\Big)$$

in which the stipulated derivative in $t = 0$ leads to the rather simple formula:

$$Z^{\mathrm{knc}}(X) = K^{-1}_{*,\,KX}\Big(Z^{\exp}(KX) + (\nabla_Z K)\,X\Big). \tag{30}$$

The vector field Z^{knc} thus reflects the covariant derivative of the difference element K in the direction of $Z \in T_pM$ modified by adding the exponentially extended vector field $Z^{\exp}$. On the other hand Z^{knc} reflects the covariant derivative of the pull back $\theta_p := \theta_p^{\mathrm{loc}} \circ \mathrm{knc}_p$:

$$\begin{aligned}(Z^{\mathrm{knc}}\theta_p)(X) &= \frac{d}{dt}\bigg|_0 \theta_p\Big(\mathrm{knc}_p^{-1}(\mathrm{knc}_{\gamma(t)}\,\mathbf{PT}^\nabla_\gamma(t)X)\Big)\\ &= \frac{d}{dt}\bigg|_0 \Big(\theta_p^{\mathrm{loc}} - \theta_{\gamma(t)}^{\mathrm{loc}}\Big)\Big(\mathrm{knc}_{\gamma(t)}\,\mathbf{PT}^\nabla_\gamma(t)\,X\Big) + \frac{d}{dt}\bigg|_0 \theta_{\gamma(t)}\Big(\mathbf{PT}^\nabla_\gamma(t)\,X\Big)\\ &= \text{real part of holomorphic function} + (\nabla_Z\theta)_p(X).\end{aligned}$$

Recall at this point that the difference of the two local potentials θ_p^{loc} and $\theta_{\gamma(t)}^{\mathrm{loc}}$ is the real part of a holomorphic function and so then is the derivative $\frac{d}{dt}\big|_0\,(\theta_p^{\mathrm{loc}} - \theta_{\gamma(t)}^{\mathrm{loc}})$.

A little consideration reveals that the only possible candidate for this real part is a multiple of $Z^\sharp$ and hence:

$$Z^{\mathrm{knc}}\,\theta_p = 2\,Z^\sharp + (\nabla_Z\theta)_p. \tag{31}$$

Decomposing this equation into its bihomogeneous components (5) it can be solved simultaneously for the vector field Z^{knc} on T_pM and the covariant derivative $(\nabla_Z\theta)_p$ of the Kähler normal potential. For this purpose let us decompose the normal potential into three parts

$$\theta_p = 2\,g_p + \theta_p^{\mathrm{free}} = 2\,g_p + (\theta_p^{\mathrm{crit}} + \theta_p^{\mathrm{rest}})$$

where $\theta_p^{\mathrm{rest}} \in \overline{\mathrm{Sym}}^{\geq(3,3)}T_p^*M$ and the critical part θ_p^{crit} comprises all bihomogeneous components of θ_p, which are exactly quadratic in the holomorphic or antiholomorphic coordinates. The irritating notation for the quadratic polynomial $(2g_p)(X) = g_p(X,X)$ is certainly a drawback of our convention relating symmetric forms with polynomials.

Lemma 4.11 (Spencer Connection on Kähler Potential). *The directional derivative $Z \lrcorner\, \theta^{\mathrm{crit}}$ of the critical part θ^{crit} of the Kähler normal potential θ in the direction of a tangent vector $Z \in T_pM$ in a point $p \in M$ decomposes into the sum*

$$Z \lrcorner\, \theta^{\mathrm{crit}} = \mathrm{pr}_{[1]}(Z \lrcorner\, \theta^{\mathrm{crit}}) + \mathrm{pr}_{\geq(2,2)}(Z \lrcorner\, \theta^{\mathrm{crit}})$$

where $\mathrm{pr}_{\geq(2,2)}(Z \lrcorner\, \theta^{\mathrm{crit}})$ is a power series at least quadratic in both the holomorphic and antiholomorphic coordinates and $\mathrm{pr}_{[1]}(Z \lrcorner\, \theta^{\mathrm{crit}}) \in \overline{\mathrm{Sym}}_{[1]}T_p^*M$ is the closure of the real part of a holomorphic vector field essentially equal to the holomorphically extended vector field:

$$Z^{\mathrm{knc}} = Z - \frac{1}{2}\,\mathrm{cl}^{-1}\Big(\mathrm{pr}_{[1]}(Z \lrcorner\, \theta^{\mathrm{crit}})\Big).$$

Moreover the covariant derivative of the Kähler normal potential is given by a projection

$$\nabla_Z\theta = \mathrm{pr}_{\geq(2,2)}\Big(Z \lrcorner\, \theta^{\mathrm{free}}\Big) - \frac{1}{2}\,\mathrm{cl}^{-1}\,\Big(\mathrm{pr}_{[1]}(Z \lrcorner\, \theta^{\mathrm{crit}})\Big)\,\theta^{\mathrm{free}}$$

of its formal derivative $Z \lrcorner\, \theta^{\mathrm{free}}$ and a term depending bilinearly on θ^{crit} and θ^{free}.

In order to justify Lemma 4.11 let us solve equation (31) with respect the holomorphically extended vector field Z^{knc} associated to a tangent vector $Z \in T_p$. Equation (30) together with the explicit expansions of Z^{exp} and K immediately imply that $Z^{\mathrm{knc}}(X) = Z + O(X^2)$ has no linear term and thus can be reconstructed without ambiguity from its closure, which implicitly appears on the left hand side of the equation considered. In fact the decomposition of power series into their bihomogeneous components (5) is parallel and hence the right hand side of equation (31) agrees with $\nabla_Z\theta = \nabla_Z\theta^{\mathrm{free}} \in \Gamma(\overline{\mathrm{Sym}}^{\geq(2,2)}T^*M)$ up to the linear term $2Z^\sharp$. Decomposing the left hand side $Z^{\mathrm{knc}}\theta$ analoguously into a linear term, a term in $\Gamma(\overline{\mathrm{Sym}}^{\geq(2,2)}T^*M)$ and a necessarily vanishing remainder term we find

$$\begin{aligned}(Z^{\mathrm{knc}}\theta)(X) = 2\,g(Z,X) &+ \Big(2\,g(Z^{\mathrm{rest}}(X),\,X) + \mathrm{pr}_{[1]}(Z \lrcorner\, \theta^{\mathrm{crit}})(X)\Big)\\ &+ \Big(\mathrm{pr}_{\mathrm{rest}}(Z \lrcorner\, \theta^{\mathrm{crit}})(X) + (Z \lrcorner\, \theta^{\mathrm{rest}})(X) + (Z^{\mathrm{rest}}\,\theta^{\mathrm{free}})\Big)\end{aligned}$$

where $Z^{\rm rest} = -\frac{1}{2}\,{\rm cl}^{-1}\,{\rm pr}_{[1]}(Z \lrcorner\, \theta^{\rm crit})$ denotes all non–constant components of the holomorphically extended vector field $Z^{\rm knc} = Z + Z^{\rm rest}$. Taking on the other hand a closer look at Definition 2.3 we observe that for all $k \geq 4$ the bihomogeneous component of the higher holomorphic sectional curvature tensor $S_k \in {\rm Sym}^k T_p^*M$ of bidegree $(k-2, 2)$ equals:

$$S_{k-2,\,2}(X) = \frac{1}{(k-4)!}\; g((\nabla_{{\rm pr}^{1,0}X,\,{\rm pr}^{1,0}X,\,\ldots,\,{\rm pr}^{1,0}X} R)_{X,\,IX} IX,\; X).$$

In turn the congruence (1) implies the following explicit formula for the critical part

$$\begin{aligned}\theta^{\rm crit}(X) &= -\frac{1}{8}\, g(R_{X,\,IX}IX,\; X) - \sum_{k>4} \frac{1}{4\,(k-2)\,(k-3)} \Big(S_{k-2,\,2}(X) + S_{2,\,k-2}(X) \Big) \\ &= -\frac{1}{8}\, g(R_{X,\,IX}IX,\; X) - \sum_{k>4} \frac{1}{2\,(k-2)!}\, {\rm Re}\, g\Big((\nabla_{{\rm pr}^{1,0}X,\,\ldots,\,{\rm pr}^{1,0}X} R)_{X,\,IX}IX,\; X \Big)\end{aligned}$$

of the Kähler normal potential, from which the closure ${\rm pr}_{[1]}(Z \lrcorner\, \theta^{\rm crit})$ of the holomorphically extended vector field $-2\,Z^{\rm knc}$ associated to a tangent vector $Z \in T_pM$ is easily calculated:

Corollary 4.12 (Holomorphically Extended Vector Fields). *The Taylor series of the holomorphically extended vector field $Z^{\rm knc}$ on T_pM associated to $Z \in T_pM$ depends linearly on the curvature tensor R_p and its iterated covariant derivatives:*

$$Z^{\rm knc}(X) = Z + \sum_{k \geq 4} \frac{1}{2^{k-3}\,(k-2)!}\, {\rm Re}\Big((\nabla^{k-4}_{X-iIX,\,\ldots,\,X-iIX} R)_{Z+iIZ,\,IX}(IX + iX) \Big).$$

5. Hermitean Symmetric Spaces. In differential geometry locally symmetric spaces form an important laboratory to test new concepts and ideas for plausibility, because the characteristic vanishing $\nabla R = 0$ of the covariant derivative of the curvature tensor R on a symmetric space sets up a Lie theoretic framework for doing calculations, which are infeasible or even impossible to do on general affine manifolds. In this section we use this framework to derive formulas for the difference elements, the normal potential and the holomorphically extended vector fields on hermitean symmetric spaces, although all results except Corollary 5.4 are valid for all symmetric among the balanced complex affine spaces. A standard reference for the Lie theoretic framework, albeit for Riemannian symmetric spaces only, is [He].

The starting point of our discussion of hermitean symmetric spaces is the affine Killing equation for a vector field on an affine manifold M endowed with a torsion free connection ∇ on its tangent bundle. In terms of the Lie derivative of connections this equation reads

$$0 \;\stackrel{?}{=}\; (\mathfrak{Lie}_X\nabla)_Y Z := [X,\, \nabla_Y Z] - \nabla_{[X,Y]}Z - \nabla_X[Y,\, Z] = R_{X,\,Y}Z + \nabla^2_{Y,\,Z}X$$

its solutions $X \in \Gamma(TM)$ are called affine Killing fields. The extended affine Killing field $X^{\rm ext} := \mathfrak{X} \oplus X$ with $\mathfrak{X} := \nabla X$ associated to a solution satisfies the extended equation

$$\nabla_Y X = \mathfrak{X}\,Y \qquad\qquad \nabla_Y \mathfrak{X} = -R_{X,\,Y} \tag{32}$$

the extended affine Killing field is thus a parallel section $\nabla^{\mathrm{Killing}}_Y(\mathfrak{X} \oplus X) = 0$ of the vector bundle End $TM \oplus TM$ with respect to the Killing connection defined by:

$$\nabla^{\mathrm{Killing}}_Y(\mathfrak{X} \oplus X) := (\nabla_Y \mathfrak{X} + R_{X,Y}) \oplus (\nabla_Y X - \mathfrak{X}\, Y). \tag{33}$$

A direct consequence of this construction is that the Lie subalgebra of affine Killing vector fields $\mathfrak{aff}(M, \nabla) \subset \Gamma(TM)$ is a vector space of dimension $\dim \mathfrak{aff}(M, \nabla) \leq m^2 + m$ on every connected manifold M of dimension m with equality only on affine spaces. Straightforward, but slightly tedious calculations result in an explicit formula for the curvature

$$R^{\mathrm{Killing}}_{Y,Z}(\mathfrak{X} \oplus X) = \Big((\nabla_X R)_{Y,Z} - (\mathfrak{X} \star R)_{Y,Z}\Big) \oplus 0 \ \stackrel{!}{=}\ (\mathfrak{Lie}_X R)_{Y,Z} \oplus 0$$

of the Killing connection, the relation with the Lie derivative of the curvature tensor R in the second equality requires $\mathfrak{X} = \nabla X$. Due to torsion freeness the Lie bracket of two affine Killing fields $X, Y \in \mathfrak{aff}(M, \nabla)$ can be calculated via $[X, Y] = \mathfrak{Y}X - \mathfrak{X}Y$ from their associated extended affine Killing fields $X^{\mathrm{ext}}, Y^{\mathrm{ext}} \in \Gamma(\mathrm{End}\, TM \oplus TM)$. Taking the covariant derivative and using (32) we find that the $C^\infty M$–bilinear “algebraic” bracket

$$\Big[\mathfrak{X} \oplus X,\, \mathfrak{Y} \oplus Y\Big]_{\mathrm{alg}} := ([\mathfrak{Y}, \mathfrak{X}] - R_{Y,X}) \oplus (\mathfrak{Y}\, X - \mathfrak{X}\, Y) \tag{34}$$

on the vector bundle End $TM \oplus TM$ reduces to $[X^{\mathrm{ext}}, Y^{\mathrm{ext}}]_{\mathrm{alg}} = [X, Y]^{\mathrm{ext}}$ for extended affine Killing fields. In general this algebraic bracket does not satisfy the Jacobi identity and thus does not define a fiberwise Lie algebra structure on the vector bundle End $TM \oplus TM$, the degree of failure of the Jacobi identity is however measured by the cyclic sum:

$$\begin{aligned}
&[\mathfrak{X} \oplus X,\, [\mathfrak{Y} \oplus Y,\, \mathfrak{Z} \oplus Z]_{\mathrm{alg}}]_{\mathrm{alg}} + \text{cyclic permutations} \\
&\qquad = \Big((\mathfrak{X} \star R)_{Y,Z} + (\mathfrak{Y} \star R)_{Z,X} + (\mathfrak{Z} \star R)_{X,Y}\Big) \oplus 0.
\end{aligned}$$

Last but not least we remark that the covariant derivative of $[,]_{\mathrm{alg}}$ under $\nabla^{\mathrm{Killing}}$ equals:

$$(\nabla^{\mathrm{Killing}}_Z [,]_{\mathrm{alg}})(\mathfrak{X} \oplus X, \mathfrak{Y} \oplus Y) = \Big((\nabla_Z R)_{X,Y} + (\mathfrak{X} \star R)_{Y,Z} - (\mathfrak{Y} \star R)_{X,Z}\Big) \oplus 0.$$

Let us now specify the preceeding equations to a hermitean locally symmetric space, this is a Kähler manifold M with covariantly parallel curvature tensor $\nabla R = 0$. In this case the joint stabilizer of the parallel Riemannian metric g, complex structure I and curvature tensor R defines a subbundle parallel with respect to the Levi–Civita connection ∇

$$\mathfrak{g}M := \mathfrak{stab}(g, I, R) \oplus TM \quad \subset \quad \mathrm{End}\, TM \oplus TM \tag{35}$$

which is moreover parallel under the Killing connection $\nabla^{\mathrm{Killing}}$ defined in equation (33), because the curvature tensor R takes values in the joint stabilizer $\mathfrak{stab}(g, I, R)$ of the parallel sections g, I and R. On the subbundle $\mathfrak{g}M$ the algebraic bracket $[,]_{\mathrm{alg}}$ satisfies the Jacobi identity and is parallel with respect to the Killing connection, thus $\mathfrak{g}M$ becomes a bundle of $\mathbb{Z}_2$–graded Lie algebras endowed with the flat algebra connection $\nabla^{\mathrm{Killing}}$, moreover the odd subbundle is exactly the tangent bundle TM.

In passing we remark that the Lie algebra structure (34) on $\mathfrak{g}_pM$ encodes the Jacobi operators of the Riemannian metric with

$$(\mathrm{ad}^2 X)\,A \;\widehat{=}\; [0 \oplus X,\; [0 \oplus X,\; 0 \oplus A]] = 0 \oplus R_{X,\,A}X \;\widehat{=}\; R_{X,\,A}X \qquad (36)$$

for all $X,\,A \in T_pM$. Under the ansatz $\Phi^{-1}(X) = \varphi^{-1}(\mathrm{ad}\,X)$ suggested by this identity the parallel transport equation of Lemma 3.1 becomes the ordinary differential equation

$$x\frac{d}{dx}\,(x\frac{d}{dx}+1)\,\varphi^{-1}(x) = x^2\,\varphi^{-1}(x) \qquad\qquad \varphi^{-1}(0) = 1$$

for the unknown power series $\varphi^{-1}(x) \in \mathbb{Q}[\![x]\!]$ with unique solution $\varphi^{-1}(x) = \frac{\sinh\,x}{x}$ or:

$$\Phi^{-1}(X) = \frac{\sinh\;\mathrm{ad}\,X}{\mathrm{ad}\,X} \qquad\qquad \Phi(X) = \frac{\mathrm{ad}\,X}{\sinh\;\mathrm{ad}\,X}\;. \qquad (37)$$

For hermitean locally symmetric spaces we will use in addition the following specific identity:

Lemma 5.1 (Fundamental Commutator Identity). *The Jacobi operator* (ad^2X) : $T_pM \longrightarrow T_pM$, $A \longmapsto R_{X,\,A}X$, *associated to a tangent vector* $X \in T_pM$ *commutes on every hermitean symmetric space* M *with the Jacobi operator* ad^2IX *associated to the tangent vector* $IX \in T_pM$. *For general Kähler manifolds the commutator of these Jacobi operators is a linear combination of iterated covariant derivatives of R:*

$$\begin{aligned} &3[R_{X,\,\cdot}X,\; R_{IX,\,\cdot}IX]A \\ &= \tfrac{1}{2}\,(\nabla^2_{X,\,IX}R - \nabla^2_{IX,\,X}R)_{X,\,IX}A + (\nabla^2_{X,\,IX}R - \nabla^2_{IX,\,X}R)_{A,\,X}IX \\ &\quad + (\nabla^2_{A,\,X}R - \nabla^2_{X,\,A}R)_{X,\,IX}IX + (\nabla^2_{A,\,IX}R - \nabla^2_{IX,\,A}R)_{X,\,IX}X. \end{aligned}$$

Proof. Although a Lie theoretic argument is significantly shorter we prefer to prove the general formula for arbitrary Kähler manifolds. It is rather difficult to believe though that such a complicated argument could be made up without knowledge of the Lie theoretic background. Using the first Bianchi identity three times we obtain the identity

$$\begin{aligned} &+R_{R_{A,\,U}V,\,W}Z - R_{R_{A,\,V}U,\,W}Z - R_{R_{A,\,W}U,\,V}Z + R_{R_{A,\,W}V,\,U}Z \\ &-R_{R_{A,\,Z}U,\,V}W + R_{R_{A,\,Z}V,\,U}W + R_{R_{A,\,Z}W,\,U}V - R_{R_{A,\,Z}W,\,V}U \\ = &-R_{R_{U,\,V}A,\,W}Z + (R_{A,\,W}R)_{U,\,V}Z - [R_{A,\,W},\;R_{U,\,V}]\,Z \\ &+(R_{A,\,Z}R)_{U,\,V}W - [R_{A,\,Z},\;R_{U,\,V}]\,W - R_{U,\,V}R_{A,\,Z}W \\ = &+(R_{U,\,V}R)_{A,\,W}Z + R_{A,\,R_{U,\,V}W}Z + (R_{A,\,W}R)_{U,\,V}Z \\ &+(R_{A,\,Z}R)_{U,\,V}W - R_{A,\,Z}R_{U,\,V}W \\ = &-R_{R_{U,\,V}W,\,Z}A + (R_{U,\,V}R)_{A,\,W}Z + (R_{A,\,W}R)_{U,\,V}Z + (R_{A,\,Z}R)_{U,\,V}W \end{aligned}$$

for arbitrary tangent vectors A and $U,\,V,\,W,\,Z$. Expressing the action of the curvature on curvature $(R_{U,V}R)_{A,W}Z = (\nabla^2_{U,V}R - \nabla^2_{V,U}R)_{A,W}Z$ by skew symmetrized iterated covariant derivatives and specifying $U = X = W$ and $V = IX = Z$ we conclude:

$$\begin{aligned} &3\,[R_{X,\,\cdot}X,\; R_{IX,\,\cdot}IX]\,A \\ &= -R_{R_{X,\,IX}X,\,IX}A + (\nabla^2_{X,\,IX}R - \nabla^2_{IX,\,X}R)_{A,\,X}IX \\ &\quad +(\nabla^2_{A,\,X}R - \nabla^2_{X,\,A}R)_{X,\,IX}IX + (\nabla^2_{A,\,IX}R - \nabla^2_{IX,\,A}R)_{X,\,IX}X. \end{aligned}$$

In light of Remark 4.7 the latter identity implies the formula in question. □

The most important consequence of Lemma 5.1 is that we may evaluate a doubly even power series in two variables in the commuting Jacobi operators $\mathrm{ad}^2 X$ and $\mathrm{ad}^2 IX$ associated to a vector $X \in T_pM$ tangent to a hermitean symmetric space M. In particular the technical Lemma 5.2 is formulated in terms of the endomorphisms $F(\mathrm{ad}\, X,\, \mathrm{ad}\, IX) \in \mathrm{End}\, T_pM$ associated to tangent vectors $X \in T_pM$ and a power series $F \in \mathbb{Q}[\![x,\, \overline{x}]\!]$, which is doubly even in the sense $F(-x, \overline{x}) = F(x, \overline{x}) = F(x, -\overline{x})$. More specifically we are interested in doubly even power series $F^{\mathrm{ext}} \in \mathbb{Q}[\![x,\, \overline{x}]\!]$ arising from even power series $F \in \mathbb{Q}[\![x]\!]$ via:

$$F^{\mathrm{ext}}(x,\, \overline{x}) := \frac{1}{2\,x}\Big((x+\overline{x})\,F(x+\overline{x}) + (x-\overline{x})\,F(x-\overline{x})\Big). \tag{38}$$

With F being even the expression $(x+\overline{x})\,F(x+\overline{x}) + (x-\overline{x})\,F(x-\overline{x})$ vanishes at $x = 0$ so that $F^{\mathrm{ext}} \in \mathbb{Q}[\![x,\, \overline{x}]\!]$ is indeed a well–defined power series with doubly even expansion

$$F^{\mathrm{ext}}(x,\, \overline{x}) := \sum_{k\,\geq\,0}\sum_{\mu=0}^{k}\binom{2k+1}{2\mu}\,F_{2k}\,(x^2)^{k-\mu}(\overline{x}^2)^{\mu} \tag{39}$$

in terms of the coefficients $(F_{2k})_{k\geq 0}$ of F. Usually it is simpler to calculate F^{ext} directly from its definition (38), for the even power series $\frac{\tanh\frac{x}{2}}{\frac{x}{2}} \in \mathbb{Q}[\![x]\!]$ for example we find:

$$\begin{aligned}\Big(\frac{\tanh\frac{x}{2}}{\frac{x}{2}}\Big)^{\mathrm{ext}}(x,\, \overline{x}) &= \frac{1}{x}\Big(\frac{e^{x+\overline{x}}-1}{e^{x+\overline{x}}+1} + \frac{e^{x-\overline{x}}-1}{e^{x-\overline{x}}+1}\Big)\\ &= \frac{2}{x}\,\frac{(e^{2x}-1)\,e^{\overline{x}}}{(e^{x+\overline{x}}+1)\,(e^{x}+e^{\overline{x}})} = 4\,\frac{\sinh x}{x}\,\frac{e^{x}\,e^{\overline{x}}}{(e^{x+\overline{x}}+1)\,(e^{x}+e^{\overline{x}})}\;.\end{aligned}$$

In passing we remark that the power series $F^{\mathrm{ext}} \in \mathbb{Q}[\![x,\, \overline{x}]\!]$ satisfies the congruences

$$F^{\mathrm{ext}}(x,\overline{x}) \equiv F(x) \mod (\overline{x}^2) \qquad F^{\mathrm{ext}}(x,\overline{x}) \equiv \Big(\overline{x}\,\frac{d}{d\overline{x}}+1\Big)\,F(\overline{x}) \mod (x^2) \tag{40}$$

modulo the ideals generated respectively by $\overline{x}^2$ and x^2. Both congruences can be read off from expansion (39) or are readily derived from definition (38) using L'Hôpital's Rule.

Lemma 5.2 (Technical Lemma). *Consider an even formal power series $F \in \mathbb{Q}[\![x]\!]$ in one variable and its associated doubly even extension $F^{\mathrm{ext}} \in \mathbb{Q}[\![x,\, \overline{x}]\!]$ defined in equation (38). For every hermitean locally symmetric space M the evaluation of F^{ext} at the commuting Jacobi operators $\mathrm{ad}^2 X$ and $\mathrm{ad}^2 IX$ associated to $X \in T_pM$ results in an endomorphism $F^{\mathrm{ext}}(\mathrm{ad}\, X,\, \mathrm{ad}\, IX) \in \mathrm{End}\, T_pM$ of the tangent space in p, which makes prominent appearance in the directional derivative:*

$$D\Big[F(\mathrm{ad}\, IX)X\Big]\,A := \frac{d}{dt}\Big|_0\,F\Big(\mathrm{ad}\, I(X+tA)\Big)\,(X+tA) = F^{\mathrm{ext}}(\mathrm{ad}\, X,\, \mathrm{ad}\, IX)\,A.$$

Moreover the odd endomorphism $\mathrm{ad}\,[F(\mathrm{ad}\, IX)X]$ of the $\mathbb{Z}_2$–graded Lie algebra $\mathfrak{g}_pM$ defined in equation (35) as $\mathfrak{stab}_p(g, I, R) \oplus T_pM$ can be written in terms of

$F^{\mathrm{ext}}(\mathrm{ad}\,X,\,\mathrm{ad}\,IX)$*:*

$$\mathrm{ad}\Big(F(\mathrm{ad}\,IX)\,X\Big)\Big|_{T_pM} = (\mathrm{ad}\,X)\,\circ\,F^{\mathrm{ext}}(\mathrm{ad}\,X,\ \mathrm{ad}\,IX)$$
$$\mathrm{ad}\Big(F(\mathrm{ad}\,IX)\,X\Big)\Big|_{\mathfrak{stab}_p(g,I,R)} = F^{\mathrm{ext}}(\mathrm{ad}\,X,\ \mathrm{ad}\,IX)\,\circ\,(\mathrm{ad}\,X).$$

In consequence the square of this endomorphism restricted to the tangent space T_pM *reads:*

$$\mathrm{ad}^2\Big(F(\mathrm{ad}\,IX)\,X\Big) = (\mathrm{ad}^2\,X)\,F^{\mathrm{ext}}(\mathrm{ad}\,X,\ \mathrm{ad}\,IX)^2.$$

Proof. Recall that the even subalgebra $\mathfrak{stab}_p(g,I,R)$ of the $\mathbb{Z}_2$–graded Lie algebra $\mathfrak{g}_pM$ consists of endomorphisms of T_pM commuting with I. In consequence the symmetries of the curvature tensor R of Kähler type of M imply $[IX,\,I\hat{X}] = R_{IX,\,I\hat{X}} = [X,\,\hat{X}]$ and so

$$[IX,\,[I\hat{X},\,\mathfrak{Y}]] = [IX,\,I\,[\hat{X},\,\mathfrak{Y}]] = [X,\,[\hat{X},\,\mathfrak{Y}]]$$

for all $X,\,\hat{X}\in T_pM$ and all $\mathfrak{Y}\in\mathfrak{stab}_p(g,I,R)$. Generalizing this identity we conclude

$$\ldots[IX_{\mu+1},[IX_\mu,\ldots,[X_1,Y]\ldots]]\ldots = \ldots[X_{\mu+1},[X_\mu,\ldots,[X_1,Y]\ldots]]\ldots \tag{41}$$

for all $X_1,\,\ldots,\,X_\mu,\,X_{\mu+1},\,\ldots$ in T_pM provided μ is even and $Y\in T_pM$ or alternatively μ is odd and $Y\in\mathfrak{stab}_p(g,I,R)$. A straightforward induction on r or simply the binomial formula in the universal enveloping algebra generalizes the Jacobi identity for $\mathfrak{g}_pM$ to:

$$\mathrm{ad}\Big((\mathrm{ad}^rY)\,Z\Big) = \sum_{\mu=0}^{r}(-1)^\mu\binom{r}{\mu}\,(\mathrm{ad}^{r-\mu}Y)\,(\mathrm{ad}\,Z)\,(\mathrm{ad}^\mu Y). \tag{42}$$

Using this identity together with $\sum_{r=\mu}^{2k-1}\binom{r}{\mu} = \binom{2k}{\mu+1}$ we calculate for all $k\in\mathbb{N}_0$:

$$\begin{aligned}
&D\Big((\mathrm{ad}^{2k}IX)\,X\Big)\,A\\
&= \frac{d}{dt}\Big|_0\Big(\mathrm{ad}^{2k}\,I(X+tA)\Big)\,(X+tA)\\
&= (\mathrm{ad}^{2k}IX)\,A - \sum_{r=0}^{2k-1}(\mathrm{ad}^{2k-r-1}IX)\,[(\mathrm{ad}^rIX)\,X,\ IA]\\
&= (\mathrm{ad}^{2k}IX)\,A - \sum_{\mu=0}^{2k-1}(-1)^\mu\binom{2k}{\mu+1}\,(\mathrm{ad}^{2k-\mu-1}IX)\,(\mathrm{ad}\,X)\,(\mathrm{ad}^\mu IX)\,IA.
\end{aligned}$$

In order to evaluate this sum we treat the summands differently depending on the parity of the index μ: Identity (41) allows us to remove all I's to the right of $\mathrm{ad}\,X$ without changing the result provided the index μ is odd, in the opposite case with even index μ we employ identity (41) to replace $(\mathrm{ad}\,X)(\mathrm{ad}\,IX)$ by $-(\mathrm{ad}\,IX)(\mathrm{ad}\,X)$ and then remove all I's even further to the right without doing any harm. The net result of all these modifications reads

$$
\begin{aligned}
&D\Big((\mathrm{ad}^{2k}IX)\,X\Big)\,A\\
&= (\mathrm{ad}^{2k}IX)\,A + \sum_{\substack{\mu=0\\ \mu\text{ odd}}}^{2k}\binom{2k}{\mu+1}(\mathrm{ad}^2IX)^{k-\frac{\mu+1}{2}}\,(\mathrm{ad}^2X)^{\frac{\mu+1}{2}}\,A\\
&\qquad\qquad + \sum_{\substack{\mu=0\\ \mu\text{ even}}}^{2k}\binom{2k}{\mu+1}(\mathrm{ad}^2IX)^{k-\frac{\mu}{2}}\,(\mathrm{ad}^2X)^{\frac{\mu}{2}}\,A\\
&= (\mathrm{ad}^{2k}IX)\,A + \sum_{\nu=0}^{k}\left(\binom{2k}{2\nu+1}+\binom{2k}{2\nu}-\delta_{\nu=0}\right)(\mathrm{ad}^2IX)^{k-\nu}\,(\mathrm{ad}^2X)^{\nu}\,A\\
&= \sum_{\nu=0}^{k}\binom{2k+1}{2\nu+1}(\mathrm{ad}^2IX)^{k-\nu}\,(\mathrm{ad}^2X)^{\nu}\,A = (x^{2k})^{\mathrm{ext}}(\mathrm{ad}\,X,\ \mathrm{ad}\,IX)\,A
\end{aligned}
$$

where $\nu = \frac{\mu}{2}$ or $\nu = \frac{\mu+1}{2}$ depending on the parity of μ and the Kronecker delta $\delta_{\nu=0}$ is needed to cancel the non existent summand with $\mu = -1$ after switching to $\nu = 0$. With the directional derivative being linear we deduce the validity of the first statement of the lemma for all even power series $F \in \mathbb{Q}[\![x]\!]$ from its validity on the basis $(x^{2k})_{k\in\mathbb{N}_0}$.

The argument for the other two statements follows this line of reasoning very closely, the starting point for both statements is to use identity (42) to expand $\mathrm{ad}\,[(\mathrm{ad}^{2k}IX)\,X]$ into:

$$
\mathrm{ad}\,[(\mathrm{ad}^{2k}IX)\,X] = \sum_{\mu=0}^{2k}(-1)^{\mu}\binom{2k}{\mu}(\mathrm{ad}^{2k-\mu}IX)\,(\mathrm{ad}\,X)\,(\mathrm{ad}^{\mu}IX).
$$

In this situation making a case distinction on the parity of the index μ is not sufficient to employ identity (41) and so we have to restrict this endomorphism of $\mathfrak{g}_pM$ either to the tangent space T_pM or to $\mathfrak{stab}_p(g,I,R)$. Restriction to T_pM allows us to remove all I to the left of $\mathrm{ad}\,X$ without changing the result provided μ is even, for odd μ we have to replace $(\mathrm{ad}\,IX)(\mathrm{ad}\,X)$ by $-(\mathrm{ad}\,X)(\mathrm{ad}\,IX)$ first using (41) before removing all I further to the left. In analogy we remove all I to the right of $\mathrm{ad}\,X$ on $\mathfrak{stab}_p(g,I,R)$ provided μ is even, otherwise we replace $(\mathrm{ad}\,X)(\mathrm{ad}\,IX) = -(\mathrm{ad}\,IX)(\mathrm{ad}\,X)$ first and proceed as before. □

Theorem 5.3 (Difference Elements of Hermitean Symmetric Spaces). *For every hermitean locally symmetric space M the difference element $K := \exp_p^{-1}\circ\,\mathrm{knc}_p$ measuring the deviation between the Riemannian and Kählerian normal coordinates reads:*

$$
KX = \frac{\mathrm{artanh}\,(\frac{1}{2}\,\mathrm{ad}\,IX)}{\frac{1}{2}\,\mathrm{ad}\,IX}\,X \qquad\qquad K^{-1}X = \frac{\tanh(\frac{1}{2}\,\mathrm{ad}\,IX)}{\frac{1}{2}\,\mathrm{ad}\,IX}\,X.
$$

Proof. Before verifying the recursion formula (4.10) for K^{-1} let us first use the technical Lemma 5.2 to show that K and K^{-1} are actually composition inverses of each other. In order to simplify the exposition of this argument we consider the general case of two even power series $F,\,\hat{F}\in\mathbb{Q}[\![x]\!]$ parametrizing power series on T_pM with values in T_pM via

$$F(X) := F(\mathrm{ad}\,IX)\,X \;\overset{!}{=}\; -I\,F(\mathrm{ad}\,X)\,IX \tag{43}$$

the second equality rises from equation (41) in the form $[IX,[IX,Y]] = -I\,[X,[X,IY]]$ valid for all $X,\,Y\in T_pM$. The technical Lemma 5.2 implies for the composition:

$$\begin{aligned} F(\hat{F}(X)) &= -\,I\,F\left(\sqrt{\mathrm{ad}^2[\hat{F}(\mathrm{ad}\,IX)X]}\right)\,I\,\hat{F}(\mathrm{ad}\,IX)X \\ &= -\,I\,F\left(\sqrt{(\mathrm{ad}^2X)\,\hat{F}^{\mathrm{ext}}(\mathrm{ad}\,X,\mathrm{ad}\,IX)^2}\right)\,\hat{F}(\mathrm{ad}\,X)\,IX \\ &= -\,I\,F(\text{“}(\mathrm{ad}\,X)\,\hat{F}(\mathrm{ad}\,X)\text{”})\,\hat{F}(\mathrm{ad}\,X)\,IX = -\,I\,(F\circ\hat{F})(\mathrm{ad}\,X)\,IX \end{aligned}$$

where $(F\circ\hat{F})(x) := F(x\hat{F}(x))\,\hat{F}(x)$. For the series corresponding to K and K^{-1} we find

$$\frac{\tanh\frac{x}{2}}{\frac{x}{2}}\;\circ\;\frac{\mathrm{artanh}\,\frac{x}{2}}{\frac{x}{2}} = \frac{\tanh\frac{1}{2}\,(2\,\mathrm{artanh}\,\frac{x}{2})}{\frac{1}{2}\,(2\,\mathrm{artanh}\,\frac{x}{2})}\;\frac{\mathrm{artanh}\,\frac{x}{2}}{\frac{x}{2}} = 1$$

and conclude $K^{-1}(KX) = X$, mutatis mutandis we obtain $K(K^{-1}X) = X$ as well. Recall now that the difference element K^{-1} on a locally symmetric space is a composition polynomial in the curvature tensor alone as $\nabla R = 0$ and all iterated covariant derivatives vanish, thus it has weight 0 in the sense $\delta K^{-1} = 0$. Rewriting the definition (24) of the weight operator

$$\delta := \mathrm{id}\otimes I - \mathrm{Der}_I\otimes\mathrm{id} \;\overset{!}{=}\; (\mathrm{id}\otimes I)\,(\mathrm{Der}_I\otimes I + \mathrm{id})$$

we conclude $(\mathrm{Der}_I\otimes I)\,K^{-1} = -K^{-1}$ so that the recursion formula of Remark 4.10 becomes:

$$\Big[(N-1)\,K^{-1}\Big](X) = I\,DK^{-1}(X)\,(\mathrm{id} - \Phi(X))\,IX. \tag{44}$$

In order to verify that the stipulated power series $K^{-1}X = \frac{\tanh\frac{1}{2}\,\mathrm{ad}\,IX}{\frac{1}{2}\,\mathrm{ad}\,IX}\,X$ satisfies this formal differential equation characterizing the difference element let us consider the power series

$$F(x) := x\,\frac{d}{dx}\Big(\frac{\tanh\frac{x}{2}}{\frac{x}{2}}\Big) = \frac{1}{\cosh^2\frac{x}{2}} - \frac{\tanh\frac{x}{2}}{\frac{x}{2}} = -\,\frac{\tanh\frac{x}{2}}{\frac{x}{2}}\,(1 - \frac{x}{\sinh x})$$

where we have used the addition theorem $\frac{1}{\cosh^2\frac{x}{2}} = \frac{\tanh\frac{x}{2}}{(\cosh\frac{x}{2})(\sinh\frac{x}{2})} = \frac{\tanh\frac{x}{2}}{\frac{1}{2}\sinh x}$ for the hyperbolic sine in the second equality. Conveniently using the swap identity (43) we conclude

$$\begin{aligned}[(N-1)\,K^{-1}](X) &= F(\mathrm{ad}\,IX)\,X = -\,I\,F(\mathrm{ad}\,X)\,IX \\ &= I\,\frac{\tanh\frac{1}{2}\mathrm{ad}\,X}{\frac{1}{2}\mathrm{ad}\,X}\Big(\mathrm{id} - \frac{\mathrm{ad}\,X}{\sinh\,\mathrm{ad}\,X}\Big)\,IX\end{aligned}$$

where the shift from $N-1$ to $x\frac{d}{dx}$ reflects the fact that $x\frac{d}{dx}$ counts all arguments but one of the power series K^{-1}. On the other hand the technical Lemma 5.2 and equation

(37) for the forward parallel transport $\Phi(X) = \frac{\operatorname{ad} X}{\sinh \operatorname{ad} X}$ on symmetric spaces allow us to write the right hand side of the formal differential equation (44) for $K^{-1}X = \frac{\tanh \frac{1}{2}\operatorname{ad} IX}{\frac{1}{2}\operatorname{ad} IX} X$ in the form

$$\begin{aligned} I\,DK^{-1}(X)\,(\mathrm{id} - \Phi(X))\,IX &= I\Big(\frac{\tanh \frac{x}{2}}{\frac{x}{2}}\Big)^{\mathrm{ext}}(\operatorname{ad} X,\, \operatorname{ad} IX)\Big(\mathrm{id} - \frac{\operatorname{ad} X}{\sinh \operatorname{ad} X}\Big)\,IX \\ &= I\,\frac{\tanh \frac{1}{2}\operatorname{ad} X}{\frac{1}{2}\operatorname{ad} X}\Big(\mathrm{id} - \frac{\operatorname{ad} X}{\sinh \operatorname{ad} X}\Big)\,IX \end{aligned}$$

where we may reduce $(\frac{\tanh \frac{x}{2}}{\frac{x}{2}})^{\mathrm{ext}}$ modulo the ideal $(\overline{x}^2)$ according to the congruences (40), because $\operatorname{ad}^2 IX$ commutes with the power series $\frac{\operatorname{ad} X}{\sinh \operatorname{ad} X}$ in $\operatorname{ad}^2 X$ and kills its argument IX. Comparing the results for the left and right hand sides we conclude that the stipulated power series K^{-1} satisfies in fact the formal differential equation (44) uniquely characterizing the difference element $K^{-1} = \mathrm{knc}_p^{-1} \circ \exp_p$ as claimed. ∎

Corollary 5.4 (Kähler Potential on Symmetric Spaces). *On every hermitean locally symmetric space M the normal potential takes the form:*

$$\theta_p(X) = g_p\Big(X,\ \frac{\log(\mathrm{id} - \frac{1}{4}\operatorname{ad}^2 IX)}{-\frac{1}{4}\operatorname{ad}^2 IX}\,X\Big).$$

Proof. Essentially the proof consists of a calculation of the expression $\Psi(X)X$ on hermitean locally symmetric spaces using again the key technical Lemma 5.2. In a prelude to the proof we remark that the power series identity $2\,\mathrm{artanh}\,x = \log\frac{1+x}{1-x}$ is a direct consequence of the differential equation $\mathrm{artanh}'x = \frac{1}{1-x^2}$. The second congruence in (40) thus becomes

$$\Big(\frac{\mathrm{artanh}\,\frac{x}{2}}{\frac{x}{2}}\Big)^{\mathrm{ext}}(x,\,\overline{x}) \equiv \Big(\overline{x}\,\frac{d}{d\overline{x}} + 1\Big)\,\frac{\mathrm{artanh}\,\frac{\overline{x}}{2}}{\frac{\overline{x}}{2}} = \frac{1}{1 - (\frac{\overline{x}}{2})^2}$$

modulo the ideal generated by x^2. In consequence the formula (37) for the backward parallel transport on symmetric spaces and the technical Lemma 5.2 allow us to expand the definition $\Psi^{-1}(X) = \Phi^{-1}(KX)\,K_{*,X}$ of the Kähler backward parallel transport to the effect that

$$\begin{aligned} \Psi^{-1}(X)\,X &= \frac{\sinh \operatorname{ad} KX}{\operatorname{ad} KX}\,\Big(\frac{\mathrm{artanh}\,\frac{x}{2}}{\frac{x}{2}}\Big)^{\mathrm{ext}}(\operatorname{ad} X,\, \operatorname{ad} IX)\,X \\ &= \Big(\sum_{k\,\geq\,0}\frac{1}{(2k+1)!}\operatorname{ad}^{2k} KX\Big)\Big(1 - \tfrac{1}{4}\operatorname{ad}^2 IX\Big)^{-1} X = \Big(1 - \tfrac{1}{4}\operatorname{ad}^2 IX\Big)^{-1} X \end{aligned}$$

because $\operatorname{ad}^2 X$ kills the eventual argument X in the first line and so then does

$$\operatorname{ad}^2 KX = (\operatorname{ad}^2 X)\ \circ\ \Big(\frac{\mathrm{artanh}\,\frac{x}{2}}{\frac{x}{2}}\Big)^{\mathrm{ext}}(\operatorname{ad} X,\, \operatorname{ad} IX)$$

in the second. On the other hand we know that the Kähler normal potential θ on a hermitean locally symmetric space is necessarily of weight 0 in the sense $\mathrm{Der}_I\theta = 0$, because it is essentially a composition polynomial in the curvature tensor alone. With the Jacobi operator $\mathrm{ad}^2 IX = R_{IX,\cdot} IX$ being a symmetric endomorphism we thus simplify equation (18) to:

$$[N^2\,\theta](X) = 4\,g(\Psi^{-1}(X)\,X,\ \Psi^{-1}(X)\,X) = 4\,g(X,\ (1 - \tfrac{1}{4}\,\mathrm{ad}^2 IX)^{-2}\,X)$$

The corollary now follows from the identity $(x\frac{d}{dx})^2 \log(1 - x^2) = -4x^2\,(1 - x^2)^{-2}$, no additional shift is needed in this argument between the number operator N and $x\frac{d}{dx}$. □

Before closing this section we want to verify equation (31) describing the Spencer connection for the Kähler normal potential on hermitean locally symmetric spaces. In light of the explicit formulas for the forward and backward parallel transport (37) it seems justified to make the ansatz $\Theta(X) = \theta(\mathrm{ad}\,X)$ for the power series Θ describing the exponentially extended vector fields on symmetric spaces with an unknown even power series $\theta \in \mathbb{Q}[\![x]\!]$ as parameter. Under this ansatz the equation (15) for Θ becomes the ordinary differential equation

$$\begin{aligned} x\frac{d}{dx}\Big(x\frac{d}{dx} - 1\Big)\,\theta(x) &= x\frac{d}{dx}\Big((1 - \frac{x}{\sinh x})\,\frac{x}{\sinh x}\Big) + \Big(x\frac{d}{dx}\,\frac{\sinh x}{x}\Big)\Big(\frac{x}{\sinh x}\Big)^2 \\ &= -\,x\frac{d}{dx}\Big(\frac{x}{\sinh x}\Big)^2 \end{aligned}$$

with initial value $\theta(0) = 1$, the simplication in the second line is due to the standard identity $(x\frac{d}{dx}f^{-1})\,f^2 = -x\frac{d}{dx}f$ valid for every invertible, commutative power series f. Hence

$$\Big(x\frac{d}{dx} - 1\Big)\,\theta(x) = -\Big(\frac{x}{\sinh x}\Big)^2 + \text{integration constant} \qquad\qquad \theta(0) = 1$$

where the initial value $\theta(0) = 1$ forces the integration constant to vanish. The unique even power series solving the latter ordinary differential equation equals $\theta(x) = \frac{x}{\tanh x}$ due to $\tanh' x = \frac{1}{\cosh^2 x}$, hence exponentially extended vector fields on symmetric spaces read:

$$Z^{\exp}(X) = \Theta(X)\,Z = \frac{\mathrm{ad}\,X}{\tanh\ \mathrm{ad}\,X}\,Z. \tag{45}$$

Incidentally this formula is exactly the formula describing the unique Killing vector field with vanishing covariant derivative and value $Z \in T_pM$ in a point $p \in M$ in exponential coordinates [He]. With the difference element K being parallel $\nabla K = 0$ on locally symmetric spaces formula (30) tells us that the holomorphically extended vector fields are simply these transvection Killing vector fields written alternatively in Kähler normal coordinates:

COROLLARY 5.5 (Holomorphically Extended Vector Fields). *In accordance with Corollary 4.12 the holomorphically extended vector field Z^{knc} associated to a tangent vector $Z \in T_pM$ of a hermitean symmetric space equals the quadratic vector field*

$$Z^{\mathrm{knc}}(X) = Z - \frac{1}{4}\Big(R_{Z,\,X}X - R_{Z,\,IX}IX\Big)$$

on the tangent space T_pM. In particular all Killing vector fields on a hermitean symmetric space become at most quadratic vector fields when written in Kähler normal coordinates.

Proof. Recall first of all that the curvature tensor of a locally symmetric space is covariantly constant $\nabla R = 0$ and so then is the difference element K. In consequence equation (30) describing the Taylor series of holomorphically extended vector fields becomes

$$Z^{\mathrm{knc}}(X) = K^{-1}_{*,\,KX}\, Z^{\exp}(KX) = DK^{-1}(KX)\,\frac{\mathrm{ad}\,KX}{\tanh\;\mathrm{ad}\,KX}\,Z \tag{46}$$

in light of the formula (45) for exponentially extended vector fields. Using the technical Lemma 5.2 and the swap identity (43) we may rewrite $\mathrm{ad}^2(KX)$ and $\mathrm{ad}^2(IKX)$ in the form

$$\begin{aligned} \mathrm{ad}^2(\,KX\,) &= (\mathrm{ad}\,X)^2\Big(\frac{\mathrm{artanh}\,\frac{x}{2}}{\frac{x}{2}}\Big)^{\mathrm{ext}}(\mathrm{ad}\,X,\;\mathrm{ad}\,IX)^2 &&= F^2(\mathrm{ad}\,X,\;\mathrm{ad}\,IX) \\ \mathrm{ad}^2(IKX) &= -\,I\circ(\mathrm{ad}^2KX)\circ I &&= F^2(\mathrm{ad}\,IX,\;\mathrm{ad}\,X) \end{aligned}$$

defining the power series $F \in \mathbb{Q}[\![x,\,\overline{x}]\!]$ in two variables arising in this way as:

$$F(x,\,\overline{x}) := x\Big(\frac{\mathrm{artanh}\,\frac{x}{2}}{\frac{x}{2}}\Big)^{\mathrm{ext}}(x,\,\overline{x}) = \frac{1}{2}\Big(\log\frac{2+x+\overline{x}}{2-x-\overline{x}} + \log\frac{2+x-\overline{x}}{2-x+\overline{x}}\Big).$$

The identity $2\,\mathrm{artanh}\,\frac{x}{2} = \log\frac{2+x}{2-x}$ can be used conveniently to arrive at the extension (38) needed in the second equality. In turn the description (46) of holomorphically extended vector fields expands with the help of the technical Lemma 5.2 into the power series

$$Z^{\mathrm{knc}}(X) = DK^{-1}(KX)\,\frac{\mathrm{ad}\,KX}{\tanh\;\mathrm{ad}\,KX}\,Z = H(\mathrm{ad}^2X,\;\mathrm{ad}^2IX)\,Z \tag{47}$$

in ad^2X and ad^2IX alone, where the doubly even power series $H \in \mathbb{Q}[\![x,\,\overline{x}]\!]$ reads

$$\begin{aligned} H(x,\,\overline{x}) &:= \Big(\frac{\tanh\,\frac{x}{2}}{\frac{x}{2}}\Big)^{\mathrm{ext}}(F(x,\,\overline{x}),\;F(\overline{x},\,x))\,\frac{F(x,\,\overline{x})}{\tanh\;F(x,\,\overline{x})} \\ &= 4\,\cosh\,F(x,\,\overline{x})\,\frac{e^{F(x,\,\overline{x})+F(\overline{x},\,x)}}{(e^{F(x,\,\overline{x})+F(\overline{x},\,x)}+1)\,(e^{F(x,\,\overline{x})}+e^{F(\overline{x},\,x)})} \end{aligned}$$

the exemplary calculation of $\big(\frac{\tanh\,\frac{x}{2}}{\frac{x}{2}}\big)^{\mathrm{ext}}$ following definition (38) has been used here in the second line. In order to simplify the power series $F(x,\overline{x})$ and $F(\overline{x},x)$ in this formula we change variables from $x,\,\overline{x}$ to the new variables $a := \frac{x+\overline{x}}{2}$ and $b := \frac{x-\overline{x}}{2}$ and obtain:

$$F(x,\,\overline{x}) = \frac{1}{2}\Big(\log\frac{1+a}{1-a} + \log\frac{1+b}{1-b}\Big) \qquad F(\overline{x},\,x) = \frac{1}{2}\Big(\log\frac{1+a}{1-a} + \log\frac{1-b}{1+b}\Big).$$

Inserting these expressions into the power series H simplifies it drastically, namely we find

$$\begin{aligned} H(x,\,\overline{x}) &= 2\,\frac{(\frac{1+a}{1-a})\Big(\sqrt{\frac{1+a}{1-a}}\sqrt{\frac{1+b}{1-b}} + \sqrt{\frac{1-a}{1+a}}\sqrt{\frac{1-b}{1+b}}\Big)}{(\frac{1+a}{1-a}+1)\Big(\sqrt{\frac{1+a}{1-a}}\sqrt{\frac{1+b}{1-b}} + \sqrt{\frac{1+a}{1-a}}\sqrt{\frac{1-b}{1+b}}\Big)} \\ &= \frac{(1+a)\sqrt{\frac{1+b}{1-b}} + (1-a)\sqrt{\frac{1-b}{1+b}}}{\sqrt{\frac{1+b}{1-b}} + \sqrt{\frac{1-b}{1+b}}} = 1 + ab = 1 + \frac{1}{4}\,(x^2 - \overline{x}^2) \end{aligned}$$

by using the identity $\frac{\sqrt{t}}{\sqrt{t}+\sqrt{t^{-1}}} = \frac{t}{t+1}$ twice. Reinserting this result into our description of holomorphically extended vector fields (47) on hermitean symmetric spaces we get eventually

$$Z^{\mathrm{knc}}(X) = H(\mathrm{ad}^2 X,\ \mathrm{ad}^2 IX)\, Z = Z + \frac{1}{4}\Big((\mathrm{ad}^2 X)\, Z - (\mathrm{ad}^2 IX)\, Z\Big)$$

which converts via equation (36) or $(\mathrm{ad}^2 X)Z = -R_{Z,X}X$ into the first statement of the Lemma. The second statement follows from the fact that the complementary Killing vector fields with vanishing value in p are linear vector fields in Kähler normal coordinates. □

Having established Corollary 5.5 about holomorphically extended vector fields on hermitean locally symmetric spaces we can eventually verify the important equation (31). According to Corollary 5.4 the Kähler normal potential on hermitean locally symmetric spaces can be written in the form $\theta(X) = g(X,\ F(\mathrm{ad}\, IX)\, X)$ with the even power series:

$$F(x) := \frac{\log(1 - (\frac{x}{2})^2)}{-(\frac{x}{2})^2} = \sum_{k\,\geq\,0} \frac{1}{k+1}\,(\frac{x}{2})^{2k}.$$

Recall now that the Jacobi operators $\mathrm{ad}^2 X = -R_{\cdot,X}X$ and $\mathrm{ad}^2 IX$ are commuting symmetric endomorphisms and so then is $F^{\mathrm{ext}}(\mathrm{ad}\, X,\, \mathrm{ad}\, IX)$. Using the technical Lemma (5.2) we find

$$\begin{aligned}\frac{d}{dt}\Big|_0\ \theta(X + t\, Z) &= g\Big(Z,\ \big[F(\mathrm{ad}\, IX) + F^{\mathrm{ext}}(\mathrm{ad}\, X,\, \mathrm{ad}\, IX)\ \big]\ X\Big)\\ &= 2\, g(Z,\ (\mathrm{id} - \tfrac{1}{4}\,\mathrm{ad}^2 IX)^{-1} X)\end{aligned}$$

for every fixed direction $Z \in T_pM$ in light of the congruence of power series

$$F(\overline{x}) + F^{\mathrm{ext}}(x,\, \overline{x})\ \equiv\ (\overline{x}\,\frac{d}{d\overline{x}} + 2)\ F(\overline{x}) = 2\Big(1 - (\frac{\overline{x}}{2})^2\Big)^{-1} \qquad \mathrm{mod}\ (x^2)$$

in the variables x, $\overline{x}$ modulo the ideal generated by x^2, the equality in this congruence is most easily deduced from the power series expansion of F. In consequence we obtain:

$$\begin{aligned}(Z^{\mathrm{knc}}\theta)(X) &= 2\, g(Z + \tfrac{1}{4}\,(\mathrm{ad}^2 X - \mathrm{ad}^2 IX)\, Z,\ (\mathrm{id} - \tfrac{1}{4}\,\mathrm{ad}^2 IX)^{-1} X)\\ &= 2\, g(Z,\ (\mathrm{id} - \tfrac{1}{4}\,\mathrm{ad}^2 IX)\ (\mathrm{id} - \tfrac{1}{4}\,\mathrm{ad}^2 IX)^{-1} X) = 2\, g(Z,\ X).\end{aligned}$$

Appendix A. Kähler Normal Coordinates in Examples. In order to illustrate the concept of Kähler normal coordinates we want to describe these coordinates and the associated normal potentials for a couple of examples in this appendix, namely the four infinite series of compact hermitean symmetric spaces: The complex Grassmannians, which include the complex projective spaces as a special case, the real Grassmannians of oriented planes and the real and quaternionic twistor spaces. A classical reference for formulas pertaining to Riemannian symmetric spaces like the relation between the curvature tensor and the Lie algebra structure on the Lie algebra of Killing vector fields exploited in Section 5 is certainly [He]. From among these formulas we will use in particular the description of the Riemannian exponential map in terms of matrix exponentials.

Probably the most prominent examples of Kähler manifolds besides flat $\mathbb{C}^n$ are the complex projective spaces, which we will discuss in the context of complex Grassmann manifolds in the first part of this appendix. The Kähler metric of choice on the complex Grassmannian $\mathrm{Gr}_k V$ of k–dimensional subspaces of a complex vector space V is a generalization of the Fubini–Study metric on the set $\mathbb{P}V = \mathrm{Gr}_1 V$ of lines defined in terms of a positive definite hermitean form $h : \overline{V} \times V \longrightarrow \mathbb{C}$ on V and the associated identification of the vector space

$$\mathrm{Hom}\,(P,\ P^\perp) \ \stackrel{\cong}{\longrightarrow}\ T_P\mathrm{Gr}_k V, \qquad X \ \longmapsto\ \frac{d}{dt}\Big|_0 \ \mathrm{im}\,(\mathrm{id} + t\,X : \quad P \longrightarrow V)$$

tangent to $\mathrm{Gr}_k V$ in a point $P \in \mathrm{Gr}_k V$ with h–orthogonal complement $P^\perp \subset V$. Under this identification the complex structure I becomes the multiplication by i in $\mathrm{Hom}\,(P, P^\perp)$ and

$$h_{\mathrm{FS}} : \quad \mathrm{Hom}\,(P,\ P^\perp) \ \times\ \mathrm{Hom}\,(P,\ P^\perp) \ \longrightarrow\ \mathbb{C}, \qquad (X,\ Y) \ \longmapsto\ \mathrm{tr}_P(X^*Y)$$

defines the hermitean Fubini–Study metric with Riemann metric $g_{\mathrm{FS}} := \mathrm{Re}\, h_{\mathrm{FS}}$ and Kähler form $\mathrm{Im}\, h_{\mathrm{FS}}$, where $X^* \in \mathrm{Hom}\,(P^\perp, P)$ denotes the adjoint of X with respect to h. In order to calculate the exponential map we apply the matrix exponential $\exp : \mathrm{End}\ V \longrightarrow \mathbf{GL}\, V$ to the Lie algebra element $X - X^* \in \mathfrak{su}\, V$ corresponding to the tangent vector X

$$\exp\begin{pmatrix} 0 & -X^* \\ X & 0 \end{pmatrix} = \begin{pmatrix} \cos\sqrt{X^*X} & * \\ X\,\frac{\sin\sqrt{X^*X}}{\sqrt{X^*X}} & * \end{pmatrix}$$

in which the ill–defined square root never materializes, because both $\cos x$ and $\frac{\sin x}{x}$ are even power series in x. The Riemannian exponential is now covered by the matrix exponential in the isometry group for symmetric spaces like $\mathrm{Gr}_k V$, see for example [He], in consequence the exponential map $\exp_P : T_P\mathrm{Gr}_k V \longrightarrow \mathrm{Gr}_k V$ for the Grassmannians can be written:

$$\exp_P :\ \mathrm{Hom}\,(P,\ P^\perp) \longrightarrow \mathrm{Gr}_k V, \quad X \longmapsto \mathrm{im}\Big(v \longmapsto \cos\sqrt{X^*X}\,v + X\frac{\sin\sqrt{X^*X}}{\sqrt{X^*X}}\,v\Big).$$

Based on this description we simply make the following guess for Kähler normal coordinates

$$\mathrm{knc}_P : \quad \mathrm{Hom}\,(P,\ P^\perp) \ \longrightarrow\ \mathrm{Gr}_k V, \quad X \ \longmapsto\ \mathrm{im}\,(v \ \longmapsto\ v + Xv) \tag{48}$$

which we will verify in due course by calculating the normal potential. In any case the image of these Kähler normal coordinates is exactly the big cell consisting of all subspaces transversal to $P^\perp$, since these big cells form the coordinate charts in the standard holomorphic atlas for $\mathrm{Gr}_k V$ the proposed map $\mathrm{knc}_P : T_P\mathrm{Gr}_k V \longrightarrow \mathrm{Gr}_k V$ is certainly holomorphic. Whenever the operator norm of the tangent vector $X \in \mathrm{Hom}\,(P, P^\perp)$ satisfies $\|\,X\,\| < \frac{\pi}{2}$, then the linear map $\cos\sqrt{X^*X} \in \mathrm{End}\ P$ is invertible and the resulting equality

$$\mathrm{im}\Big(v \ \longmapsto\ \cos\sqrt{X^*X}\,v + X\frac{\sin\sqrt{X^*X}}{\sqrt{X^*X}}\,v\Big) = \mathrm{im}\Big(v \ \longmapsto\ v + X\frac{\tan\sqrt{X^*X}}{\sqrt{X^*X}}\,v\Big)$$

implies the following explicit formulas for the difference element K and its inverse

$$K^{-1}X = X\,\frac{\tan\sqrt{X^*X}}{\sqrt{X^*X}} \qquad \Longleftrightarrow \qquad KX = X\,\frac{\arctan\sqrt{X^*X}}{\sqrt{X^*X}} \tag{49}$$

because in this way $\mathrm{knc}_P(K^{-1}X) = \exp_P X$. To calculate the pull back of the Fubini–Study metric g_{FS} on $\mathrm{Gr}_k V$ to T_pM via knc_P we embed $\mathrm{Gr}_k V$ isometrically into the real vector space of self adjoint endomorphisms of V with respect to the hermitean form h

$$\iota : \quad \mathrm{Gr}_k V \longrightarrow \mathrm{End}_{\mathrm{herm}} V, \qquad \hat{P} \longmapsto \mathrm{pr}_{\hat{P}} \tag{50}$$

where $\mathrm{pr}_{\hat{P}} = \mathrm{pr}^*_{\hat{P}}$ is the orthogonal projection onto $\hat{P}$, in turn the composition $\iota \circ \mathrm{knc}_P$ reads

$$\iota\Big(\mathrm{im}\,(v \longmapsto v + X\,v)\Big) = \begin{pmatrix} \mathrm{id} \\ X \end{pmatrix} \big(\mathrm{id} + X^*X\big)^{-1} \begin{pmatrix} \mathrm{id} & X^* \end{pmatrix} = \begin{pmatrix} Q & Q\,X^* \\ X\,Q & X\,Q\,X^* \end{pmatrix}$$

with $Q := (\mathrm{id} + X^*X)^{-1}$. In passing we observe that the very same calculation implies that the differential of the embedding ι in the chosen, but arbitrary point $P \in \mathrm{Gr}_k V$ is given by

$$\iota_{*,\,P} : \quad \mathrm{Hom}\,(P,\,P^\perp) \longrightarrow \mathrm{End}_{\mathrm{herm}} V, \qquad X \longmapsto \begin{pmatrix} 0 & X^* \\ X & 0 \end{pmatrix}$$

as $Q = \mathrm{id} + O(X^2)$, hence ι is an isometric embedding as claimed provided we choose the positive definite scalar product $G(F, \hat{F}) := \frac{1}{2}\,\mathrm{tr}_V F\hat{F}$ on $\mathrm{End}_{\mathrm{herm}} V$. Using the standard formula $\delta M = -M\,(\delta M^{-1})\,M$ for the variation of inverses we may calculate the differential

$$\begin{aligned} &(\iota \circ \mathrm{knc}_P)_{*,\,X} A \\ &= \begin{pmatrix} 0 \\ A \end{pmatrix} Q \begin{pmatrix} \mathrm{id} & X^* \end{pmatrix} - \begin{pmatrix} \mathrm{id} \\ X \end{pmatrix} Q(A^*X + X^*A)Q \begin{pmatrix} \mathrm{id} & X^* \end{pmatrix} + \begin{pmatrix} \mathrm{id} \\ X \end{pmatrix} Q \begin{pmatrix} 0 & A^* \end{pmatrix} \end{aligned}$$

of the composition $\iota \circ \mathrm{knc}_P : T_P\mathrm{Gr}_k V \longrightarrow \mathrm{End}_{\mathrm{herm}} V$ and find after a rather lengthy calculation better done separately for the 9 summands and simplifying $Q\,(\mathrm{id} + X^*X) = \mathrm{id}$:

$$\begin{aligned} (\mathrm{knc}^*_P g_{\mathrm{FS}})_X(A,\,B) &\stackrel{!}{=} ((\iota \circ \mathrm{knc}_P)^*G)_X(A,\,B) \\ &= \tfrac{1}{2}\,\mathrm{tr}_V((\iota \circ \mathrm{knc}_P)_{*,\,X}A\;(\iota \circ \mathrm{knc}_P)_{*,\,X}B) \\ &= \tfrac{1}{2}\,\mathrm{tr}_P(QA^*B + QB^*A - QA^*XQX^*B - QX^*AQB^*X). \end{aligned}$$

Note that the peculiar arrangement of the factors in the third and fourth summand make the result real as the trace of sums of hermitean matrices. Having calculated the pull back of the metric we are in the position to verify that the anchored holomorphic coordinates $\mathrm{knc}_P : T_P\mathrm{Gr}_k V \longrightarrow \mathrm{Gr}_k V$ proposed in equation (48) are actually the unique Kähler normal coordinates in the point $P \in \mathrm{Gr}_k V$. For this purpose we define $\theta_P \in C^\infty(T_P\mathrm{Gr}_k V)$ by

$$\theta_P(X) = \mathrm{tr}_P\,\log(\mathrm{id} + X^*X) = \log\,\det_P(\mathrm{id} + X^*X) \tag{51}$$

and observe that for given tangent vectors $A,\,B \in T_P\mathrm{Gr}_k V \cong \mathrm{Hom}\,(P, P^\perp)$

$$\begin{aligned} \frac{\partial^2}{\partial A\,\partial B}\,\theta_P(X) &= \frac{\partial}{\partial A}\,\mathrm{tr}_P(Q\,(B^*X + X^*B)) \\ &= \mathrm{tr}_P(Q\,(B^*A + A^*B) - Q\,(A^*X + X^*A)\,Q\,(B^*X + X^*B)) \end{aligned}$$

due to the standard logarithmic derivative $\delta\log(\det M) = \mathrm{tr}(M^{-1}\delta M)$ of the determinant and the definition $Q := (\mathrm{id} + X^*X)^{-1}$ of Q. Comparing this result with the formula for the pull back $\mathrm{knc}^*_P g_{\mathrm{FS}}$ of the Fubini–Study metric from $\mathrm{Gr}_k V$ to $T_P\mathrm{Gr}_k V$ we conclude

$$(\mathrm{knc}^*_P g_{\mathrm{FS}})_X(A,\, B) = \frac{1}{4}\Big(\frac{\partial^2}{\partial A\,\partial B}\,\theta_P(X) + \frac{\partial^2}{\partial IA\,\partial IB}\,\theta_P(X)\Big)$$

because the troublesome terms with either none or both of A or B starred drop out in averaging over A, B and IA, IB. In consequence the function θ_P is some potential function for the Riemannian metric $\mathrm{knc}^*_P g_{\mathrm{FS}}$ on $T_P\mathrm{Gr}_k V$ and since it evidently satisfies the normalization constraint imposed on the unique normal potential it is actually equal to this potential, in turn $\mathrm{knc}_P : T_P\mathrm{Gr}_k V \longrightarrow \mathrm{Gr}_k V$ are Kähler normal coordinates as claimed.

Although the formula for the difference element of the complex Grassmannians obtained in this way is quite explicit, it is more useful to recast it in terms of the curvature of $\mathrm{Gr}_k V$. According to our discussion of the Lie algebra of Killing vector fields on symmetric spaces the curvature of the complex Grassmannians can be calculated from the Lie algebra of Killing vector fields on $\mathrm{Gr}_k V$ by means of equation (36), more precisely we find

$$R_{U,\,V}W = -\,[[U - U^*,\, V - V^*],\, W - W^*] = U\,V^*W - V\,U^*W - W\,U^*V + W\,V^*U$$

for all U, V, $W \in \mathrm{Hom}\,(P, P^\perp)$ with adjoints U^*, V^*, $W^* \in \mathrm{Hom}\,(P^\perp, P)$. The complex structure on the Grassmannians $\mathrm{Gr}_k V$ is now defined by declaring the isomorphism $T_P\mathrm{Gr}_k V \cong \mathrm{Hom}\,(P, P^\perp)$ of real vector spaces to be complex linear $IX := iX$ and so the formula for the curvature becomes for the arguments $U = IX = W$ and $V = X\,(X^*X)^k$

$$(\mathrm{ad}^2 IX)\Big(X\,(X^*X)^k\Big) := R_{iX,\,X(X^*X)^k}\,iX = -\,4\,X\,(X^*X)^{k+1}$$

for all $k \geq 0$. In consequence equation (49) can be written in the form:

$$KX = \frac{\arctan(-\,\frac{1}{4}\,\mathrm{ad}^2 IX)^{\frac{1}{2}}}{(-\,\frac{1}{4}\,\mathrm{ad}^2 IX)^{\frac{1}{2}}}\,X = \frac{\mathrm{artanh}(\frac{1}{2}\,\mathrm{ad}\,IX)}{\frac{1}{2}\,\mathrm{ad}\,IX}\,X. \tag{52}$$

This calculation was the motivation for the authors to look for a proof of Lemma 5.3 describing the difference element for arbitrary hermitean symmetric spaces by exactly this formula. In the same vein the normal potential for the complex Grassmannians can be rewritten in terms of the Lie algebra structure in order to reflect Corollary 5.4:

$$(\mathrm{knc}^*_P\theta_P)(X) = \mathrm{tr}_P\log(\mathrm{id} + X^*X) \;\stackrel{!}{=}\; g_P\Big(X,\; \frac{\log(\mathrm{id} - \frac{1}{4}\,\mathrm{ad}^2 IX)}{-\,\frac{1}{4}\,\mathrm{ad}^2 IX}\,X\Big). \tag{53}$$

The second family of examples of Kähler manifolds discussed in this appendix are the real Grassmannians of oriented planes in a real vector space V, the universal covering spaces of the standard real Grassmannians $\mathrm{Gr}_2 V$ of planes in V, elements of $\mathrm{Gr}^{\mathrm{or}}_2 V$ are thus 2–dimensional subspaces $P \subset V$ endowed with an orientation determining the sense of counterclockwise rotation. Choosing a positive definite scalar product $g : V \times V \longrightarrow \mathbb{R}$ we may identify the vector space tangent to $\mathrm{Gr}^{\mathrm{or}}_2 V$ in an oriented plane $P \in \mathrm{Gr}^{\mathrm{or}}_2 V$ with the vector space

$$\mathrm{Hom}\,(P,\; P^\perp) \;\stackrel{\cong}{\longrightarrow}\; T_P\mathrm{Gr}^{\mathrm{or}}_2 V, \qquad X \;\longmapsto\; \frac{d}{dt}\Big|_0\;\mathrm{im}\,(\mathrm{id} + t\,X : \quad P \longrightarrow V)$$

of linear maps from P to its orthogonal complement $P^\perp$ and define the Fubini–Study metric

$$g_{\mathrm{FS}}: \quad \mathrm{Hom}\,(P,\, P^\perp) \,\times\, \mathrm{Hom}\,(P,\, P^\perp) \;\longrightarrow\; \mathbb{R}, \qquad (X,\, Y) \;\longmapsto\; \mathrm{tr}_P(X^*Y)$$

in complete analogy to the complex Grassmannians, where X^* is now the adjoint of X with respect to the scalar product g. In difference to the complex Grassmannians however $\mathrm{Hom}\,(P,\, P^\perp)$ is not a priori a complex vector space so that it is impossible to define an almost complex structure I on $\mathrm{Gr}_2^{\mathrm{or}}V$ simply by multiplication with i. Nevertheless every oriented plane $P \subset V$ carries a unique isometry $J \in \mathbf{O}(P, g)$ satisfying $J^2 = -\mathrm{id}_P$, namely the rotation by $+90°$. In turn the tangent space $T_P\mathrm{Gr}_2^{\mathrm{or}}V = \mathrm{Hom}\,(P,\, P^\perp)$ to the Grassmannian of oriented planes in V becomes a complex vector space by precomposing

$$I: \quad \mathrm{Hom}\,(P,\; P^\perp) \;\longrightarrow\; \mathrm{Hom}\,(P,\; P^\perp), \qquad X \;\longmapsto\; XJ$$

with $J \in \mathbf{O}(P, g)$, moreover this complex structure on $\mathrm{Hom}\,(P, P^\perp)$ is orthogonal with respect to the Fubini–Study metric $g_{\mathrm{FS}}(IX, IY) = \mathrm{tr}_P(J^*X^*YJ) = g_{\mathrm{FS}}(X, Y)$ due to $J^* = J^{-1} = -J$. Evidently no similar construction exists for the real Grassmannians of oriented or unoriented subspaces of V of dimensions other than 2.

Leaving the question of integrability of the almost complex structure I on $\mathrm{Gr}_2^{\mathrm{or}}V$ aside for the moment we recall that the complex bilinear extension of the scalar product g to the complexification $V_{\mathbb{C}} := V \otimes_{\mathbb{R}} \mathbb{C}$ of the real vector space V defines the complex quadric

$$Q_g(V) := \{\ \ [p] \in \mathbb{P}\,V_{\mathbb{C}} \ \ |\ \ v \text{ isotropic vector with } g(p,\, p) = 0 \ \ \}\ \subset\ \mathbb{P}\,V_{\mathbb{C}}$$

of isotropic lines in $V_{\mathbb{C}}$, which is a smooth complex submanifold of $\mathbb{P}\,V_{\mathbb{C}}$ and thus a Kähler manifold itself endowed with the restriction of the Fubini–Study metric associated to the positive definite hermitean form $h : \overline{V_{\mathbb{C}}} \times V_{\mathbb{C}} \longrightarrow \mathbb{C}$, $(v, w) \longmapsto g(\overline{v}, w)$, arising from g and the real structure on $V_{\mathbb{C}}$.

Evidently the real and imaginary part of every vector $p = (\mathrm{Re}\,p) + i(\mathrm{Im}\,p) \in V_{\mathbb{C}}$ representing an isotropic line $[p] \in Q_g(V)$ are orthogonal vectors of the same length in V and vice versa. In this way the Grassmannian $\mathrm{Gr}_2^{\mathrm{or}}V$ of oriented planes in V embeds canonically into a complex projective space and becomes the quadric $Q_g(V) \subset \mathbb{P}\,V_{\mathbb{C}}$ of isotropic lines. More precisely the canonical embedding sends an oriented plane $P \subset V$ to the isotropic line spanned by the vector $p := e_1 - ie_2$ encoding an oriented orthonormal basis $e_1,\, e_2$ for P:

$$\iota: \quad \mathrm{Gr}_2^{\mathrm{or}}V \;\longrightarrow\; Q_g(V) \;\subset\; \mathbb{P}\,V_{\mathbb{C}}, \qquad P \;\longmapsto\; [p].$$

With P being a plane in V all its oriented orthonormal bases are related by rotations, thus the representative vector $p \in V_{\mathbb{C}}$ is only defined up to multiplication by an element of S^1. Independent of this S^1–ambiguity in the choice of $p \in V_{\mathbb{C}}$ the following identities hold true

$$\begin{aligned} g(\overline{p},\, Fp) \quad &= \quad \mathrm{tr}_P F \\ |\; g(p,\, Fp)\; |^2 \quad &= \quad \mathrm{tr}_P^2 F - 4\,\mathrm{det}_P F = 2\,\mathrm{tr}_P(F^2) - \mathrm{tr}_P^2 F \end{aligned} \tag{54}$$

for every symmetric endomorphism $F \in \mathrm{End}\ P$ as the reader may easily verify using the matrix coefficients of F in the orthonormal basis $e_1,\, e_2$. In terms of the identification of the vector space tangent to $\mathbb{P}\,V_{\mathbb{C}}$ in the point $\iota(P) = [p]$ with the vector space of complex linear maps from the line $\mathbb{C}p$ to its orthogonal complement $\{\,p\,\}^\perp$ with respect to the hermitean form h the differential of ι in an oriented plane $P \in \mathrm{Gr}_2^{\mathrm{or}}V$ can be written

$$\iota_{*,\,P}: \quad \mathrm{Hom}\,(P,\ P^\perp) \longrightarrow \mathrm{Hom}\,(\mathbb{C}\,p,\ \{\,p\,\}^\perp), \qquad X \longmapsto X|_{\mathbb{C}p}$$

in fact $\iota_{*,\,P}: \frac{d}{dt}\big|_0 \mathrm{im}(\mathrm{id} + tX) \longmapsto \frac{d}{dt}\big|_0 [p + t\,Xp]$. In particular the embedding ι is actually an holomorphic embedding with $\iota_{*,\,P}(IX) = XJ|_{\mathbb{C}p} = iX|_{\mathbb{C}p}$ due to $Jp = ip$ so that the almost complex structure I on $\mathrm{Gr}_2^{\mathrm{or}}V$ is necessarily integrable. However ι is not an isometric embedding for the Fubini–Study metric $g^{\mathbb{P}\,V_\mathbb{C}}$ on the target, to be precise

$$g_{[p]}^{\mathbb{P}\,V_\mathbb{C}}(\,X|_{\mathbb{C}p}\,,\ Y|_{\mathbb{C}p}) := \frac{h(Xp,\,Yp)}{h(p,\,p)} = \frac{1}{2}\,g(X\overline{p},\,Yp) = \frac{1}{2}\,\mathrm{tr}_P(X^*Y)$$

equals half the Fubini–Study metric $\frac{1}{2}\,g_{\mathrm{FS}}$ on $\mathrm{Gr}_2^{\mathrm{or}}V$. Interestingly this observation is sufficient to construct the Kähler normal coordinates for the real Grassmannian $\mathrm{Gr}_2^{\mathrm{or}}V$ by setting:

$$\iota \circ \mathrm{knc}_p: \quad \mathrm{Hom}\,(P,\ P^\perp) \longrightarrow Q_g(V), \qquad X \longmapsto [p + X\,p - \tfrac{1}{4}\,g(Xp, Xp)\,\overline{p}].$$

The S^1–ambiguity of the isotropic vector $p \in V_\mathbb{C}$ representing the oriented plane $P \in \mathrm{Gr}_2^{\mathrm{or}}V$ has no bearance on the complex line spanned by the vector $p + Xp - \frac{1}{4}g(Xp, Xp)\overline{p}$, moreover the image vector is isotropic due to $Xp \in P^\perp \otimes_\mathbb{R} \mathbb{C}$ and $g(\overline{p}, p) = 2$. Last but not least we observe that the map $\iota \circ \mathrm{knc}_P : \mathrm{Hom}\,(P, P^\perp) \longrightarrow Q_g(V)$ is covered by the complex quadratic polynomial $\mathrm{Hom}\,(P, P^\perp) \longrightarrow V_\mathbb{C} \setminus \{\,0\,\}$, $X \longmapsto p + Xp - \frac{1}{4}g(Xp, Xp)\overline{p}$, and in turn is holomorphic, recall that after all we have $(IX)p := XJp = i(Xp)$.

The implicitly defined map $\mathrm{knc}_P : \mathrm{Hom}\,(P, P^\perp) \longrightarrow \mathrm{Gr}_2^{\mathrm{or}}V$ arising from $\iota \circ \mathrm{knc}_P$ is thus well–defined and holomorphic, to verify that knc_P are the Kähler normal coordinates it thus suffices to find a local Kähler potential for g_{FS} satisfying the normalization condition of Definition 2.6. For this purpose we pull back the Kähler normal potential of $\mathbb{P}\,V_\mathbb{C}$ in the point $\iota(P) = [p]$ back to $\mathrm{Hom}\,(P, P^\perp)$ and multiply by 2 to account for the homothety $(\iota \circ \mathrm{knc}_P)^* g^{\mathbb{P}\,V_\mathbb{C}} = \frac{1}{2}\,g_{\mathrm{FS}}$. According to our discussion (53) of the complex Grassmannians

$$\theta_{[p]}^{\mathbb{P}\,V_\mathbb{C}}([p + q]) = \log\Big(1 + \frac{h(q,\,q)}{h(p,\,p)}\Big)$$

is the Kähler normal potential for $g^{\mathbb{P}\,V_\mathbb{C}}$ whenever $q \in \{\,p\,\}^\perp$ holds true, in turn we conclude

$$\begin{aligned} 2\,(\iota \circ \mathrm{knc}_P)^*\theta^{\mathbb{P}\,V_\mathbb{C}}(X) &= 2\,\theta_{[p]}^{\mathbb{P}\,V_\mathbb{C}}([p + X\,p - \tfrac{1}{4}\,g(Xp, Xp)\,\overline{p}]) \\ &= 2\,\log\Big(1 + \tfrac{1}{2}\,g(X\overline{p}, Xp) + \tfrac{1}{16}\,|\,g(Xp, Xp)\,|^2\Big) \\ &= 2\,\log\Big(1 + \tfrac{1}{2}\,\mathrm{tr}_P(X^*X) + \tfrac{1}{8}\,\mathrm{tr}_P(X^*X)^2 - \tfrac{1}{16}\,\mathrm{tr}_P^2(X^*X)\Big) \end{aligned}$$

by using $h(p, p) = 2$ and the identities (54). The result is actually a power series invariant under Der_I due to $(IX)^*(IX) = -J^*(X^*X)J$ and thus satisfies the normalization constraint required by Definition 2.6, in consequence $\mathrm{knc}_P : \mathrm{Hom}\,(P, P^\perp) \longrightarrow \mathrm{Gr}_2^{\mathrm{or}}V$ are the unique Kähler normal coordinates of $\mathrm{Gr}_2^{\mathrm{or}}V$ in the point $P \in \mathrm{Gr}_2^{\mathrm{or}}V$.

In order to calculate the difference elements K and K^{-1} for the real Grassmannians $\mathrm{Gr}_2^{\mathrm{or}} V$ we still have to compare the formula for knc_P established above with the analoguous formula for the Riemannian exponential map. The calculations can be done in complete analogy to the case of the complex Grassmannians, because the Riemannian exponential is still covered by a suitable version of the matrix exponential, and the final results reads:

$$\iota(\exp_P X) = \Big[(\cos\sqrt{X^*X})\,p + X\,\frac{\sin\sqrt{X^*X}}{\sqrt{X^*X}}\,p\Big]. \tag{55}$$

Solving the equation $\exp_P Y = \mathrm{knc}_P X$ with respect to the tangent vector $Y = KX$ for a given argument vector $X \in \mathrm{Hom}\,(P, P^\perp)$ is thus equivalent to solving the equation

$$\Big[(\cos\sqrt{Y^*Y})\,p + Y\,\frac{\sin\sqrt{Y^*Y}}{\sqrt{Y^*Y}}\,p\Big] = \Big[p + X\,p - \frac{1}{4}\,g(Xp, Xp)\,\overline{p}\Big]$$

for points in $\mathbb{P}\,V_{\mathbb{C}}$, which we may decouple into two independent equations

$$\tau\Big((\cos\sqrt{Y^*Y})\,p\Big) = p - \frac{1}{4}\,g(Xp, Xp)\,\overline{p} \qquad\qquad \tau\Big(Y\,\frac{\sin\sqrt{Y^*Y}}{\sqrt{Y^*Y}}\,p\Big) = X\,p \tag{56}$$

by taking the orthogonal decomposition $V = P \oplus P^\perp$ into account and introducing a non–zero slack variable $\tau \in \mathbb{C}^*$. Applying the complex bilinear scalar product g with $\overline{p}$ to the first equation and using the identities (54) we find that $\tau \in \mathbb{R}^+$ is actually positive

$$\tau\,(\mathrm{tr}_P \cos\sqrt{Y^*Y}) = g\Big(\overline{p},\; p - \frac{1}{4}\,g(Xp, Xp)\,\overline{p}\Big) = 2$$

and thus:

$$X = K^{-1}Y = \frac{2}{\mathrm{tr}_P(\cos\sqrt{Y^*Y})}\;Y\,\frac{\sin\sqrt{Y^*Y}}{\sqrt{Y^*Y}}\;. \tag{57}$$

Unluckily we wanted to solve the equation $\exp_P Y = \mathrm{knc}_P X$ for Y and not for X, hence we regress to the decoupled equations (56) and read the identities (54) backwards to obtain

$$\begin{aligned}
\tau^2\,(\mathrm{tr}_P \sin^2\sqrt{Y^*Y}) &= g\Big(\tau\,(Y\,\frac{\sin\sqrt{Y^*Y}}{\sqrt{Y^*Y}})\,\overline{p},\; \tau\,(Y\,\frac{\sin\sqrt{Y^*Y}}{\sqrt{Y^*Y}})\,p\Big)\\
&= g(X\,\overline{p},\; X\,p)
\end{aligned}$$

$$\begin{aligned}
\tau^2\,(\mathrm{tr}_P \cos^2\sqrt{Y^*Y}) &= g(\overline{p} - \tfrac{1}{4}\,g(X\overline{p}, X\overline{p})\,p,\; p - \tfrac{1}{4}\,g(Xp, Xp)\,\overline{p})\\
&= 2 + \tfrac{1}{8}\,|\,g(Xp,\, Xp)\,|^2 = 2 + \tfrac{1}{4}\,\mathrm{tr}_P(X^*X)^2 - \tfrac{1}{8}\,\mathrm{tr}_P^2(X^*X).
\end{aligned}$$

With $\mathrm{tr}_P(\sin^2\sqrt{Y^*Y} + \cos^2\sqrt{Y^*Y}) = 2$ and $g(X\overline{p}, Xp) = \mathrm{tr}_P(X^*X)$ we can solve for τ:

$$\tau = \tau(X) = \Big(1 + \tfrac{1}{2}\,\mathrm{tr}_P(X^*X) + \tfrac{1}{8}\,\mathrm{tr}_P(X^*X)^2 - \tfrac{1}{16}\,\mathrm{tr}_P^2(X^*X)\Big)^{\frac{1}{2}}.$$

Incidentally we observe that this expression is exactly the argument of the logarithm in the formula $(\mathrm{knc}_P^*\theta_P^{\mathrm{FS}})(X) = 4\,\log\,\tau(X)$ for the Kähler normal potential of the real

Grassmannian $\mathrm{Gr}_2^{\mathrm{or}} V$. In any case we conclude by solving the second of the equations (56) that:

$$Y = KX = \frac{X}{\tau(X)} \frac{\arcsin \sqrt{\frac{X^* X}{\tau^2(X)}}}{\sqrt{\frac{X^* X}{\tau^2(X)}}} . \tag{58}$$

Although this formula has little to no resemblance to the formula of Theorem 5.3, it can be shown by explicit power series expansion that both formulas actually amount to the same.

In the last part of this appendix we want to discuss Kähler normal coordinates for a particularly interesting family of hermitean symmetric spaces: The twistor spaces of orthogonal complex structures on real vector spaces of even dimension and the closely related twistor spaces of quaternionic linear orthogonal complex structures on quaternionic vector spaces. Starting with the former we consider a real vector space V of even dimension $2n$ endowed with a positive definite scalar product $g : V \times V \longrightarrow \mathbb{R}$ and define its twistor space as:

$$\mathfrak{T}(V,\, g) := \{ J \in \mathrm{End}\, V \ \mid\ J \text{ orthogonal endomorphism with } J^2 = -\mathrm{id}_V \}.$$

Since an endomorphism squaring to $-\mathrm{id}_V$ is orthogonal, if and only if it is skew symmetric with respect to g, the twistor space is actually a submanifold $\mathfrak{T}(V, g) \subset \mathfrak{so}(V, g)$ of the Lie algebra of skew symmetric endomorphisms of V. In particular we may identify the tangent space $T_J\mathfrak{T}(V, g) \subset \mathfrak{so}(V, g)$ of the twistor space in a point $J \in \mathfrak{T}(V, g)$ with the vector space

$$\Sigma^1_{\mathrm{skew}}(J) := \{ X \in \mathfrak{so}(V,\, g) \ \mid\ X \text{ skew symmetric and } XJ + JX = 0 \}$$

of skew symmetric endomorphisms X of V anticommuting with J by differentiation:

$$T_J\mathfrak{T}(V,\, g) \ \xrightarrow{\ \cong\ }\ \Sigma^1_{\mathrm{skew}}(J), \qquad \left.\frac{d}{dt}\right|_0 J_t \ \longmapsto\ \dot{J}_0.$$

In fact $\dot{J}_0 \in \Sigma^1_{\mathrm{skew}}(J)$ anticommutes with $J = J_0$, because $J_t^2 = -\mathrm{id}_V$ for all t. The Riemannian metric of choice on $\mathfrak{T}(V, g) \subset \mathfrak{so}(V, g)$ is simply the restriction of the standard scalar produce $g(X, Y) = -\frac{1}{2}\,\mathrm{tr}_V(XY)$ on $\mathfrak{so}(V, g)$, moreover the almost complex structure on $\mathfrak{T}(V, g)$ in a point $J \in \mathfrak{T}(V, g)$ is the right multiplication with J on the tangent space

$$I_J : \quad \Sigma^1_{\mathrm{skew}}(J) \ \longrightarrow\ \Sigma^1_{\mathrm{skew}}(J), \qquad X \ \longmapsto\ XJ$$

because XJ still is skew and anticommutes with J. Since $\mathfrak{T}(V, g) \subset \mathfrak{so}(V, g)$ is actually a union of two adjoint orbits the Riemannian exponential is easily seen to be covered by the matrix exponential for the Lie group $\mathbf{SO}(V,\, g)$, the only subtlety here is that the Lie algebra element corresponding to the tangent vector $X \in \Sigma^1_{\mathrm{skew}}(J)$ is not X itself, but $-\frac{1}{2}\,XJ$, due to the identity $[-\frac{1}{2}\,XJ,\, J] = X$. In this way we find the explicit formula

$$\exp_J : \quad \Sigma^1_{\mathrm{skew}}(J) \ \longrightarrow\ \mathfrak{T}(V,\, g), \qquad X \ \longmapsto\ e^{-\frac{1}{2}XJ}\, J\, e^{+\frac{1}{2}XJ}$$

for the Riemannian exponential of the twistor space $\mathfrak{T}(V, g)$, and since X anticommutes with J we may simplify this to read $\exp_J X = e^{-XJ}J = (\cosh\, X)J + (\sinh\, X)$

using the observation $(-XJ)^2 = X^2$. In order to establish the integrability of the almost complex structure I and calculate the Kähler normal coordinates it turns out to be convenient to identify the twistor space $\mathfrak{T}(V, g)$ with the Grassmannian of Lagrangian subspaces for the complex bilinear extension of the scalar product g on V to $V_\mathbb{C} = V \otimes_\mathbb{R} \mathbb{C}$, which is a complex submanifold of the Grassmannian $\mathrm{Gr}_n V_\mathbb{C}$ of n–dimensional subspaces of $V_\mathbb{C}$ defined by:

$$\mathrm{LGr}_n V_\mathbb{C} = \{ L \in \mathrm{Gr}_n V_\mathbb{C} \mid L \text{ isotropic with respect to } g \}.$$

Explicitly the diffeomorphism between $\mathfrak{T}(V, g)$ and $\mathrm{LGr}_n V_\mathbb{C}$ reads

$$\iota : \quad \mathfrak{T}(V,\, g) \stackrel{\cong}{\longrightarrow} \mathrm{LGr}_n V_\mathbb{C}, \qquad J \longmapsto V_J^{1,\,0}$$

where $V_J^{1,0} \subset V_\mathbb{C}$ is the eigenspace for the eigenvalue $+i$ of the complex linear extension of the orthogonal complex structure $J \in \mathfrak{T}(V, g)$ to $V_\mathbb{C}$. In passing we remark that $\mathrm{LGr}_n V_\mathbb{C}$ can also be realized as the quadratic projective variety of pure spinors.

The argument demonstrating the surjectivity of ι uses the canonical real structure on the complexification $V_\mathbb{C} = V \otimes_\mathbb{R} \mathbb{C}$ in the form of an involution $L \longmapsto \overline{L}$ of $\mathrm{LGr}_n V_\mathbb{C}$ satisfying $L \cap \overline{L} = \{0\}$, after all g is positive definite on the real subspace $V \subset V_\mathbb{C}$. We may thus associate to every Lagrangian subspace $L \in \mathrm{LGr}_n V_\mathbb{C}$ the endomorphism $J := i\,\mathrm{pr}_L - i\,\mathrm{pr}_{\overline{L}}$ of $V_\mathbb{C} = L \oplus \overline{L}$, which commutes by construction with the real structure and thus comes from a complex structure $J \in \mathrm{End}\, V$ of the underlying real vector space V. With its eigenspaces L and $\overline{L}$ being isotropic subspaces J is automatically skew symmetric and thus orthogonal with respect to g, moreover we find $V_J^{1,0} = L$ so that the orthogonal complex structure $J \in \mathfrak{T}(V, g)$ is a preimage of the subspace L we started with.

Let us now turn to the calculation of the differential of ι in order to show that it is an holomorphic embedding $\mathfrak{T}(V, g) \longrightarrow \mathrm{Gr}_n V_\mathbb{C}$. For a curve $t \longmapsto J_t$ of orthogonal complex structures on V representing a tangent vector $X = \frac{d}{dt}\big|_0 J_t \in \Sigma^1_{\mathrm{skew}}(J)$ in $J = J_0$ we find

$$\iota_{*,\,J}\Big(\frac{d}{dt}\Big|_0 J_t \Big) = \frac{d}{dt}\Big|_0 \mathrm{im}\Big(V_J^{0,1} \longrightarrow V_{J_t}^{0,1},\ v \longmapsto \frac{1}{2}(v - iJ_t v) \Big) \ \widehat{=}\ -\frac{i}{2}\,\frac{d}{dt}\Big|_0 J_t$$

and hence conclude that the differential of $\iota : \mathfrak{T}(V, g) \longrightarrow \mathrm{Gr}_n V_\mathbb{C}$ reads:

$$\iota_{*,\,J} : \quad \Sigma^1_{\mathrm{skew}}(J) \longrightarrow \mathrm{Hom}\,(V_J^{1,0},\ V_J^{0,1}), \qquad X \longmapsto -\frac{i}{2}\, X|_{V_J^{1,0}}\,.$$

In particular $\iota_{*,\,J}(IX) = -\frac{i}{2} XJ|_{V_J^{1,0}} = i\,\iota_{*,\,J}(X)$ due to $J = +i$ on $V_J^{1,0}$ making ι a holomorphic embedding as claimed, necessarily then the almost complex structure I on the twistor space $\mathfrak{T}(V, g)$ is integrable. Moreover the pull back of the Fubini–Study metric on $\mathrm{Gr}_n V_\mathbb{C}$ via ι equals $\frac{1}{4}$ times the chosen Riemannian metric on $\mathfrak{T}(V, g) \subset \mathfrak{so}(V, g)$, because

$$(\iota^* g_{\mathrm{FS}})_J(X,\ Y) = \mathrm{Re}\, \mathrm{tr}_{V_J^{1,0}}((-\tfrac{i}{2}X)^*(-\tfrac{i}{2}Y)) = -\tfrac{1}{8}\,\mathrm{tr}_V(XY) \tag{59}$$

recall here that $X^* = -X$ is skew symmetric and that $\overline{\mathrm{tr}_{V_J^{1,0}} F} = \mathrm{tr}_{V_J^{0,1}} F$ for every endomorphism $F \in \mathrm{End}\, V$ of the underlying real vector space V.

For the next step we need to discuss another characterization of symmetric spaces as manifolds endowed with a binary operation $*$ generalizing in a sense the multiplication of a Lie group, [Ber] is a very good reference for this point of view. The binary operation $*: \mathrm{Gr}_n V_{\mathbb{C}} \times \mathrm{Gr}_n V_{\mathbb{C}} \longrightarrow \mathrm{Gr}_n V_{\mathbb{C}}$ associated to the complex Grassmannian $\mathrm{Gr}_n V_{\mathbb{C}}$ is most easily defined in terms of a suitable modification of the embedding (50) used previously

$$\mathrm{inv}: \quad \mathrm{Gr}_n V_{\mathbb{C}} \longrightarrow \mathrm{End}_{\,\mathrm{herm}} V_{\mathbb{C}}, \qquad \hat{P} \longmapsto 2\,\mathrm{pr}_{\hat{P}} - \mathrm{id}$$

which identifies $\mathrm{Gr}_n V_{\mathbb{C}}$ with the set of self adjoint involutions of $V_{\mathbb{C}}$ with zero trace, to wit

$$\mathrm{inv}(P * \hat{P}) := \mathrm{inv}(P) \circ \mathrm{inv}(\hat{P}) \circ \mathrm{inv}(P)$$

where the right hand side is still a self adjoint involution with zero trace. For obvious reasons the composition $\mathrm{inv} \circ \iota : \mathfrak{T}(V, g) \longrightarrow \mathrm{End}_{\,\mathrm{herm}} V_{\mathbb{C}}$ maps J to $\mathrm{inv}(V_J^{1,0}) = -iJ$ so that ι is actually a homomorphism of symmetric spaces in the sense:

$$\iota(J) * \iota(\hat{J}) = \mathrm{inv}^{-1}((-iJ)\,(-i\hat{J})\,(-iJ)) = \iota(J * \hat{J}) := \iota(J\,\hat{J}^{-1}\,J).$$

Like every other homomorphism of symmetric spaces ι is thus a totally geodesic embedding, in turn Corollary 4.9 tells us that the Kähler normal coordinates for the twistor space $\mathfrak{T}(V, g)$ are simply the restriction of the Kähler normal coordinates of the complex Grassmannian:

$$\iota \circ \mathrm{knc}_J : \quad \Sigma^1_{\mathrm{skew}}(J) \longrightarrow \mathrm{LGr}_n V_{\mathbb{C}}, \qquad X \longmapsto \mathrm{knc}_{V_J^{1,0}}(\iota_{*,\,J} X).$$

In fact it is easily verified that the image subspace

$$\mathrm{knc}_{V_J^{1,0}}(-\tfrac{i}{2}X) = \mathrm{im}\Big(\ \mathrm{id} - \tfrac{i}{2}X \ :\ V_J^{1,0} \longrightarrow V_{\mathbb{C}}\ \Big) \ \overset{!}{\in}\ \mathrm{LGr}_n V_{\mathbb{C}}$$

is actually a Lagrangian subspace of $V_{\mathbb{C}}$. It remains to find the orthogonal complex structure corresponding to this image subspace. For this purpose we recall that the square X^2 of the skew symmetric endomorphism $X \in \Sigma^1_{\mathrm{skew}}(J)$ is diagonalizable with non–positive eigenvalues so that the endomorphisms $\mathrm{id} \pm \frac{1}{2}XJ$ are always invertible due to:

$$4\,(\mathrm{id} + \tfrac{1}{2}\,XJ)\,(\mathrm{id} - \tfrac{1}{2}\,XJ) = 4 - (XJ)^2 = 4 - X^2.$$

Hence we may write the image of $\mathrm{id} - \frac{i}{2}X : V_J^{1,0} \longrightarrow V_{\mathbb{C}}$ as the image of the endomorphism

$$\begin{aligned}(\mathrm{id} - \tfrac{i}{2}\,X) \circ \mathrm{pr}_{V_J^{1,0}} &= \tfrac{1}{2}\,(\mathrm{id} - \tfrac{i}{2}\,X)\,(\mathrm{id} - i\,J)\\ &= \tfrac{1}{2}\,(\mathrm{id} - i\,[(J + \tfrac{1}{2}X)\,(\mathrm{id} - \tfrac{1}{2}XJ)^{-1}]) \circ (\mathrm{id} - \tfrac{1}{2}\,X\,J)\\ &= \mathrm{pr}_{V^{1,0}_{\mathrm{knc}_J X}} \circ (\mathrm{id} - \tfrac{1}{2}\,X\,J)\end{aligned}$$

of $V_{\mathbb{C}}$ where the orthogonal complex structure $\mathrm{knc}_J X \in \mathfrak{T}(V, g)$ is given explicitly by:

$$\mathrm{knc}_J X = 4\,\frac{(J + \frac{1}{2}X)\,(\mathrm{id} + \frac{1}{2}XJ)}{4 - X^2} = \frac{4 + X^2}{4 - X^2}\,J + \frac{4\,X}{4 - X^2}\,. \tag{60}$$

The reader may find it amusing to verify the slightly surprising statement $\mathrm{knc}_J X \in \mathfrak{T}(V, g)$ directly. Comparing this result with the formula for the exponential map of the twistor space

$$\exp_J(KX) = (\cosh\,KX)\,J + (\sinh\,KX) \ \overset{!}{=}\ \frac{4 + X^2}{4 - X^2}\,J + \frac{4\,X}{4 - X^2} = \mathrm{knc}_J X$$

we find $\exp(KX) = \cosh KX + \sinh KX = \frac{(2+X)\,(2+X)}{(2+X)\,(2-X)}$ and thus conclude:

$$KX = \log\frac{2+X}{2-X} = 2\,\mathrm{artanh}(\tfrac{1}{2}X) \qquad \Longleftrightarrow \qquad K^{-1}X = 2\,\tanh(\tfrac{1}{2}X). \tag{61}$$

Last but not least the normal potential of the twistor space equals the restriction of the Kähler potential of the complex Grassmannian $\mathrm{Gr}_n V_{\mathbb{C}}$ to the Kähler submanifold $\mathrm{LGr}_n V_{\mathbb{C}} \cong \mathfrak{T}(V, g)$

$$\theta_J(X) = 4\,\mathrm{tr}_{V_J^{1,0}} \log\Big(\mathrm{id} + (-\tfrac{i}{2}X)^*(-\tfrac{i}{2}X)\Big) = 2\,\mathrm{tr}_V \log\Big(\mathrm{id} - \tfrac{1}{4}\,X^2\Big)$$

where the factor 4 accounts for the homothety $\iota^* g_{\mathrm{FS}} = \frac{1}{4}\,g$. Needless to say this formula for the potential is compatible with the description $g_J(X, X) = -\frac{1}{2}\,\mathrm{tr}_V(X^2)$ of the Riemannian metric induced on the union $\mathfrak{T}(V, g) \subset \mathfrak{so}(V, g)$ of two adjoint orbits.

The discussion of the last family of hermitean symmetric spaces considered in this appendix, the quaternionic twistor spaces, can be kept very short, because we may interprete a quaternionic vector space endowed with a positive definite quaternionic hermitean form h as a real vector space V of dimension $4n$ endowed with a scalar multiplication $\mathbb{H} \times V \longrightarrow V$ by quaternions and a compatible positive definite scalar product $g = \mathrm{Re}\,h$ in the sense $g(qv, w) = g(v, \bar{q}w)$ for all $q \in \mathbb{H}$ and all $v, w \in V$. Under this reinterpretation the quaternionic twistor space of all orthogonal complex structures on V commuting with $\mathbb{H}$

$$\mathfrak{T}^{\mathbb{H}}(V,\, g) := \{J \in \mathrm{End}_{\mathbb{H}} V \;\mid\; J \text{ orthogonal endomorphism with } J^2 = -\mathrm{id}_V\}$$

becomes a symmetric subspace $\mathfrak{T}^{\mathbb{H}}(V, g) \subset \mathfrak{T}(V, g)$ of the real twistor space associated to V, because $J * \hat{J}$ commutes with the scalar multiplication by $\mathbb{H}$, whenever so do $J, \hat{J} \in \mathfrak{T}^{\mathbb{H}}(V, g)$. In consequence the quaternionic twistor space is a totally geodesic Kähler submanifold and all the formulas pertaining to $\mathfrak{T}(V, g)$ apply verbatim to $\mathfrak{T}^{\mathbb{H}}(V, g)$.

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