

AUTOMORPHISM GROUPS OF INOUE AND KODAIRA SURFACES*

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Abstract. We prove that automorphism groups of Inoue and primary Kodaira surfaces are Jordan.

Key words. automorphism group, Jordan property, Inoue surface, Kodaira surface.

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1. Introduction. It sometimes happens that groups of geometric origin are complicated and difficult to study. On the other hand, in many situations they are easier to access on the level of their finite subgroups. In particular, the following notion appeared to be very useful.

DEFINITION 1.1 (see [Pop11, Definition 2.1]). A group Γ is called *Jordan* (alternatively, we say that Γ *has Jordan property*) if there is a constant J such that for every finite subgroup $G \subset \Gamma$ there exists a normal abelian subgroup $A \subset G$ of index at most J .

An old result due to C. Jordan says that the group $\mathrm{GL}_n(\mathbb{C})$ is Jordan (see e.g. [CR62, Theorem 36.13]). S. Meng and D.-Q. Zhang proved in [MZ18] that an automorphism group of any projective variety is Jordan. T. Bandman and Yu. Zarhin proved a similar result for automorphism groups of quasi-projective surfaces in [BZ15], and also in some particular cases in arbitrary dimension in [BZ18]. J. H. Kim in [Kim18] generalized the result of [MZ18] to the case of automorphism groups of compact Kähler varieties. There are many further results on the Jordan property for diffeomorphism groups of smooth compact manifolds, see [Pop16], [CPS14], [Mun16], [Mun14], [Mun13], [Mun17], [Mun18], [Ye17], and references therein. There are also numerous results concerning Jordan property for groups of birational automorphisms of projective varieties, see [Ser09], [Pop11], [PS16], [PS14], [PS18a], [BZ17], [PS17], [PS18b], [Yas17].

Our goal is to study finite subgroups of automorphism groups of *compact complex surfaces*, that is, connected compact complex manifolds of dimension 2. Recall that such a surface is called *minimal* if it does not contain smooth rational curves with self-intersection equal to (-1) . There is a classification of minimal compact complex surfaces, known as Enriques–Kodaira classification, see e.g. [BHPVdV04, Chapter VI]. Two important classes of such surfaces are *Inoue surfaces* (see [Ino74]) and *primary Kodaira surfaces* (see [Kod64, §6], [BHPVdV04, § V.5]). The former are minimal compact complex surfaces of Kodaira dimension $-\infty$, vanishing algebraic dimension, vanishing second Betti number, and containing no curves. The latter are minimal compact complex surfaces with trivial canonical class and first Betti number equal to 3. Both of these classes do not contain projective or Kähler surfaces, so the methods of [MZ18] and [Kim18] do not provide an approach to their automorphism groups.

The main goal of this paper is to prove the following.

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THEOREM 1.2. *Let X be either an Inoue surface or a primary Kodaira surface. Then the automorphism group of X is Jordan.*

One feature of our proof that we find interesting to mention is that Inoue and Kodaira surfaces are treated by literally the same method which is based on the fact that they are diffeomorphic to solvmanifolds (cf. [Has05, Theorem 1]), and for which we never met a proper analog in the projective situation. It is possible that one can generalize this approach to higher dimensional solvmanifolds.

The plan of the paper is as follows. In §2 we collect some auxiliary facts. In §§3–6 we establish assertions about discrete groups (more precisely, Wang groups, see [Has05]) that will be used in our analysis of automorphism groups of surfaces. In §7 we prove Jordan property for automorphism groups of Inoue surfaces. In §8 we do the same for automorphism groups of primary Kodaira surfaces. Our main result, Theorem 1.2, is just a composition of Lemma 7.6 and 8.3.

NOTATION. For every group Γ we denote by $z(\Gamma)$ the center of Γ , and for a subgroup $\Gamma' \subset \Gamma$ we denote by $z(\Gamma', \Gamma)$ the centralizer of Γ' in Γ . Given a compact complex surface X , we denote by \mathcal{K}_X its canonical line bundle, by $b_i(X)$ its i -th Betti number, by $\chi_{\text{top}}(X)$ the topological Euler characteristic, by $\kappa(X)$ the Kodaira dimension, and by $a(X)$ the algebraic dimension of X . By $h^{p,q}(X)$ we denote the Hodge numbers $h^{p,q} = \dim H^q(X, \Omega_X^p)$, where Ω_X^p is the sheaf of holomorphic p -forms on X .

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2. Preliminaries. In this section we collect several auxiliary facts that will be used in the remaining parts of the paper.

We say that a group Γ has *bounded finite subgroups* if there exists a constant $B = B(\Gamma)$ such that for any finite subgroup $G \subset \Gamma$ one has $|G| \leq B$. If this is not the case, we say that Γ has *unbounded finite subgroups*. The following result is due to H. Minkowski (see e.g. [Ser07, Theorem 1]).

THEOREM 2.1. *For every n the group $\text{GL}_n(\mathbb{Q})$ has bounded finite subgroups.*

LEMMA 2.2. *Let*

$$1 \longrightarrow \Gamma' \longrightarrow \Gamma \longrightarrow \Gamma''$$

be an exact sequence of groups. Suppose that Γ' is Jordan and Γ'' has bounded finite subgroups. Then Γ is Jordan.

Proof. Obvious. \square

The following facts from number theory are well known to experts; we include their proofs for the reader’s convenience.

LEMMA 2.3. *The following assertions hold.*

- (i) *Let α be an algebraic integer such that for every Galois conjugate α' of α one has $|\alpha'| = 1$. Then α is a root of unity.*
- (ii) *Let n be a positive integer. Then there exists a constant $\varepsilon = \varepsilon(n)$ with the following property: if an algebraic integer α of degree n is such that for every Galois conjugate α' of α one has $1 - \varepsilon < |\alpha'| < 1 + \varepsilon$, then α is a root of unity.*

Proof. To prove assertion (i), fix an embedding $\mathbb{Q}(\alpha) \subset \mathbb{C}$. Then $\bar{\alpha}$ is a root of the minimal polynomial of α over \mathbb{Q} . Hence $\bar{\alpha}$ is an algebraic integer. On the other hand, one has $\bar{\alpha} = \alpha^{-1}$. Since both α and α^{-1} are algebraic integers, we conclude that all non-archimedean valuations of α equal 1. At the same time all archimedean valuations of α equal 1 by assumption. Therefore, assertion (i) follows from [Cas67, Lemma II.18.2].

Now consider an algebraic integer α of degree n such that all its Galois conjugates have absolute values less than, say, 2. The absolute values of the coefficients of its minimal polynomial are bounded by some constant $C = C(n)$; for instance, one can take $C = 2^{2n}$. Consider the set of polynomials with integer coefficients

$$\mathcal{Q} = \{Q = x^n + a_{n-1}x^{n-1} + \dots + a_0 \mid |a_i| \leq C, \text{ and } Q \text{ is irreducible}\}.$$

The set \mathcal{Q} is finite. Put

$$\Pi = \{(x_1, \dots, x_n) \mid x_1, \dots, x_n \text{ are different roots of some polynomial } Q \in \mathcal{Q}\} \subset \mathbb{C}^n.$$

Then Π is a finite subset of \mathbb{C}^n ; furthermore, all algebraic integers of degree n such that all their Galois conjugates have absolute values at most 2 appear as coordinates of the points of Π . There is a number $\mu = \mu(n)$ such that for every point $P = (\alpha_1, \dots, \alpha_n) \in \Pi$ the inequalities $1 - \mu < |\alpha_i| < 1 + \mu$ for all i imply that $|\alpha_i| = 1$ for all i . In the latter case α_i are roots of unity by assertion (i). Thus it remains to put $\varepsilon = \min(\mu, 1)$ to prove assertion (ii). \square

LEMMA 2.4. *Let $M \in \text{GL}_n(\mathbb{Z})$ be a matrix. Suppose that for every C there is an integer $k > C$ such that there exists a matrix $R_k \in \text{GL}_n(\mathbb{Z})$ with $R_k^k = M$. Then all eigen-values of M are roots of unity.*

Proof. Let λ_k be an eigen-value of R_k . Then λ_k is an algebraic integer of degree at most n , because it is a root of the characteristic polynomial of the matrix R_k . Moreover, λ_k^k is an eigen-value of M . This means that

$$\sqrt[k]{l_{min}} \leq |\lambda_k| \leq \sqrt[k]{l_{max}}, \tag{2.5}$$

where l_{min} and l_{max} are the minimal and the maximal absolute values of the eigen-values of the matrix M , respectively. Both of the above bounds converge to 1 when k goes to infinity. All Galois conjugates of λ_k are also eigen-values of R_k , hence the inequality (2.5) holds for them as well. Therefore, for k large enough all eigen-values of R_k are roots of unity by Lemma 2.3(ii), and thus so are the eigen-values of M . \square

3. Lattices and semi-direct products. Consider the groups $\Gamma_0 \cong \mathbb{Z}^3$ and $\Gamma_1 \cong \mathbb{Z}$. Here and below we will use multiplicative notation for the operations in all arising groups, in particular, in Γ_0 and Γ_1 . Let γ be a generator of Γ_1 . Fix a basis $\{\delta_1, \delta_2, \delta_3\}$ in Γ_0 . Then $\text{End}(\Gamma_0)$ can be identified with $\text{Mat}_{3 \times 3}(\mathbb{Z})$, and so for any integral 3×3 -matrix $M = (m_{j,i})$ one can define its action on Γ_0 via

$$M(\delta_i) = \delta_1^{m_{1,i}} \delta_2^{m_{2,i}} \delta_3^{m_{3,i}}.$$

If $M \in \text{GL}_3(\mathbb{Z})$, this defines a semi-direct product $\Gamma = \Gamma_0 \rtimes \Gamma_1$.

The following facts are easy exercises in group theory.

LEMMA 3.1. *Suppose that the matrix M does not have eigen-values equal to 1. Then the following assertions hold:*

- (i) $[\Gamma, \Gamma] = \text{Im}(M - \text{Id}) \subset \Gamma_0$ is a free abelian subgroup of rank 3;
- (ii) one has $\Gamma_0 = z([\Gamma, \Gamma], \Gamma)$; in particular, Γ_0 is a characteristic subgroup of Γ ;
- (iii) the center $z(\Gamma)$ is trivial.

Proof. The subgroup $[\Gamma, \Gamma] \subset \Gamma_0$ is generated by the commutators

$$[\gamma, \delta_i] = \gamma \delta_i \gamma^{-1} \delta_i^{-1} = \delta_1^{m_{1,i}} \delta_2^{m_{2,i}} \delta_3^{m_{3,i}} \delta_i^{-1},$$

where $i = 1, 2, 3$. Therefore, it can be identified with the sublattice in Γ_0 that is the image of the operator $M - \text{Id}$. Since the latter operator is non-degenerate by assumption, we conclude that $[\Gamma, \Gamma]$ is a free abelian group of rank 3. This proves assertion (i).

Since Γ_0 has no torsion, by assertion (i) we have $z([\Gamma, \Gamma], \Gamma) = z(\Gamma_0, \Gamma) \supset \Gamma_0$. Thus, to prove assertion (ii) it is enough to show that no non-trivial power of γ is contained in the above centralizer. But if $\gamma^k \in z(\Gamma_0, \Gamma)$, then $(M - \text{Id})^k = 0$. This contradicts our assumptions, which proves assertion (ii).

By assertion (ii), we have $z(\Gamma) \subset \Gamma_0$. Considering the commutators with γ , we see that $z(\Gamma) \subset \text{Ker}(M - \text{Id})$. Thus, we have $z(\Gamma) = \{1\}$, which proves assertion (iii). \square

It also appears that one can easily describe all normal subgroups of finite index in Γ (and actually do it in a slightly more general setting).

LEMMA 3.2. *Let Δ_0 be an arbitrary group (and $\Gamma_1 \cong \mathbb{Z}$ as before be a cyclic group generated by an element γ). Consider a semi-direct product $\Delta = \Delta_0 \rtimes \Gamma_1$. Let $\Delta' \subset \Delta$ be a normal subgroup of finite index. Then*

- (i) $\Delta' = \Delta'_0 \rtimes \Gamma'_1$, where $\Delta'_0 = \Delta' \cap \Delta_0$, and $\Gamma'_1 \cong \mathbb{Z}$ is generated by $\gamma^k \delta'$ for some positive integer k and $\delta' \in \Delta_0$;
- (ii) Δ/Δ' has a normal subgroup of index k isomorphic to Δ_0/Δ'_0 .

Proof. The subgroup Δ'/Δ'_0 has finite index in $\Gamma_1 \cong \mathbb{Z}$. Thus it is generated by the image of an element γ^k for some positive integer k . Choose a preimage $\theta = \gamma^k \delta$ in Δ' , where $\delta \in \Delta'_0$. Let Γ'_1 be the subgroup of Δ' generated by θ . Then Δ' is generated by its subgroups Δ'_0 and $\Gamma'_1 \cong \mathbb{Z}$. This proves assertion (i).

Consider the image $\bar{\Delta}_0$ of Δ_0 in the quotient group Δ/Δ' . It is isomorphic to Δ_0/Δ'_0 . Furthermore, the quotient $(\Delta/\Delta')/\bar{\Delta}_0$ maps isomorphically to the quotient of Γ_1 by the subgroup generated by γ^k . Thus, $\bar{\Delta}_0$ is a normal subgroup of index k in Δ/Δ' . This proves assertion (ii). \square

LEMMA 3.3. *Let $\Gamma = \Gamma_0 \rtimes \Gamma_1$ be a semi-direct product defined by a matrix M as above. Suppose that the matrix M does not have eigen-values equal to 1, and at least one of its eigen-values is not a root of unity. Then there exists a constant $\nu = \nu(\Gamma)$ with the following property.*

Let $\hat{\Gamma} = \hat{\Gamma}_0 \rtimes \hat{\Gamma}_1$, where $\hat{\Gamma}_0 \cong \mathbb{Z}^3$ and $\hat{\Gamma}_1 \cong \mathbb{Z}$. Suppose that $\hat{\Gamma}$ contains Γ as a normal subgroup. Then the group $G = \hat{\Gamma}/\Gamma$ is finite and has a normal abelian subgroup of index at most ν .

Proof. The group G is finite for obvious reasons. By Lemma 3.1(ii) we have $\Gamma_0 = \Gamma \cap \hat{\Gamma}_0$. Thus by Lemma 3.2 there is a positive integer k with the following properties: first, the subgroup $\Gamma_1 \subset \Gamma$ is generated by an element $\hat{\gamma}^k \hat{\delta}$, where $\hat{\gamma}$ is a generator of $\hat{\Gamma}_1$, and $\hat{\delta}$ is an element of $\hat{\Gamma}_0$; second, the group G contains a normal abelian subgroup of index k . Note that the subgroup Γ_0 is normal in $\hat{\Gamma}$, because Γ is normal in $\hat{\Gamma}$, while Γ_0 is a characteristic subgroup of Γ by Lemma 3.1(ii). Let $R \in \text{GL}_3(\mathbb{Z})$ be the matrix that defines the semi-direct product $\hat{\Gamma} = \hat{\Gamma}_0 \rtimes \hat{\Gamma}_1$. Considering the action of the element $\hat{\gamma}$ on the lattice $\hat{\Gamma}_0 \cong \mathbb{Z}^3$ and its sublattice $\Gamma_0 \cong \mathbb{Z}^3$, we

see that R^k is conjugate to M . Thus k is bounded by some constant ν that depends only on M (that is, only on Γ) by Lemma 2.4. \square

4. Heisenberg groups. Let r be a positive integer. Consider a group

$$\mathcal{H}(r) = \langle \delta_1, \delta_2, \delta_3 \mid [\delta_i, \delta_3] = 1, [\delta_1, \delta_2] = \delta_3^r \rangle. \tag{4.1}$$

One can think about $\mathcal{H}(r)$ as the group of all matrices

$$\begin{pmatrix} 1 & a & \frac{c}{r} \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \in \text{GL}_3(\mathbb{Q}),$$

where a, b , and c are integers. One can choose the generators so that the element δ_1 corresponds to $a = 1, b = c = 0$, the element δ_2 corresponds to $a = 0, b = 1, c = 0$, and the element δ_3 corresponds to $a = b = 0, c = 1$. The group $\mathcal{H}(1)$ is known as the discrete Heisenberg group. The center $z(\mathcal{H}(r)) \cong \mathbb{Z}$ is generated by δ_3 , while the commutator $[\mathcal{H}(r), \mathcal{H}(r)]$ is generated by δ_3^r . For the quotient group $\bar{\mathcal{H}}(r) = \mathcal{H}(r)/z(\mathcal{H}(r))$ one has $\bar{\mathcal{H}}(r) \cong \mathbb{Z}^2$.

LEMMA 4.2. *Every subgroup of finite index in $\mathcal{H}(r)$ is isomorphic to $\mathcal{H}(r')$ for some positive integer r' . Every subgroup of infinite index in $\mathcal{H}(r)$ is abelian.*

Proof. Let Δ be a subgroup of $\mathcal{H}(r)$. Denote by Δ_z the intersection $\Delta \cap z(\mathcal{H}(r))$, and let $\bar{\Delta} \cong \Delta/\Delta_z$ be the image of Δ in $\bar{\mathcal{H}}(r)$.

Suppose that Δ has finite index in $\mathcal{H}(r)$. Then Δ_z has finite index in $z(\mathcal{H}(r))$, so that it is generated by δ_3^k for some positive integer k . Furthermore, $\bar{\Delta}$ has finite index in $\bar{\mathcal{H}}(r) \cong \mathbb{Z}^2$, so that $\bar{\Delta} \cong \mathbb{Z}^2$. Choose generators $\theta_i, i = 1, 2$, of the group $\bar{\Delta}$, and let $\theta_i, i = 1, 2$, be their preimages in Δ . Then Δ is generated by θ_1, θ_2 , and δ_3^k . One has $[\theta_1, \theta_2] = \delta_3^d$ for some integer d . In particular, we see that d is divisible by k . This implies that $\Delta \cong \mathcal{H}(r')$ for $r' = \frac{|d|}{k}$.

Now suppose that Δ has infinite index in $\mathcal{H}(r)$. If Δ_z is trivial, then $\Delta \cong \bar{\Delta}$ is abelian. So we may assume that Δ_z is not trivial. This means that Δ_z has finite index in $z(\mathcal{H}(r)) \cong \mathbb{Z}$, and $\bar{\Delta}$ has infinite index in $\bar{\mathcal{H}}(r)$. The latter implies that $\bar{\Delta} \cong \mathbb{Z}$. Thus Δ is a central extension of a cyclic group, so it is abelian. \square

Note that a subgroup in $\mathcal{H}(r)$ generated by δ_1^a, δ_2 , and δ_3^c is isomorphic to $\mathcal{H}(\frac{ar}{c})$. Hence any group $\mathcal{H}(r')$ is realized as a subgroup of a given group $\mathcal{H}(r)$. We will be interested in properties of normal subgroups of $\mathcal{H}(r)$.

LEMMA 4.3. *Let $\Gamma_0 \subset \mathcal{H}(r)$ be a normal subgroup of finite index, and put $G_0 = \mathcal{H}(r)/\Gamma_0$. Then there are integers $a_1, a_2, a_3, b_1, b_2, b_3$ with $a_1b_2 - a_2b_1 \neq 0$, and $c > 0$ such that δ_3^c generates the intersection of Γ_0 with $z(\mathcal{H}(r))$, and*

$$\Gamma_0 = \langle \delta_1^{a_1} \delta_2^{a_2} \delta_3^{a_3}, \delta_1^{b_1} \delta_2^{b_2} \delta_3^{b_3}, \delta_3^c \rangle.$$

Moreover, the number c divides $r \text{gcd}(a_1, a_2, b_1, b_2)$, one has

$$\Gamma_0 \cong \mathcal{H} \left(\frac{r|a_1b_2 - a_2b_1|}{c} \right),$$

and the group G_0 contains a normal abelian subgroup of index at most $\text{gcd}(a_1, b_1)$.

Proof. Since G_0 is finite, the image $\bar{\Gamma}_0$ of Γ_0 in $\bar{\mathcal{H}}(r)$ is isomorphic to \mathbb{Z}^2 . Choose the vectors (a_1, a_2) and (b_1, b_2) in $\bar{\mathcal{H}}(r) \cong \mathbb{Z}^2$ generating $\bar{\Gamma}_0$. The group Γ_0 contains

the elements $\zeta = \delta_1^{a_1} \delta_2^{a_2} \delta_3^{a_3}$ and $\xi = \delta_1^{b_1} \delta_2^{b_2} \delta_3^{b_3}$ for some integers a_3 and b_3 . The subgroup of Γ_0 generated by ζ and ξ maps surjectively to $\bar{\Gamma}_0$. Hence Γ_0 is generated by ζ , ξ , and the intersection $\Gamma_0 \cap z(\mathcal{H}(r))$. The latter is a subgroup of $z(\mathcal{H}(r)) \cong \mathbb{Z}$, and thus is generated by some element of the form δ_3^c .

Since the subgroup Γ_0 is normal, we have $\delta_3^{rb_1} = [\delta_1, \zeta] \in \Gamma_0$, so that c divides ra_2 . Similarly, we see that c divides ra_1 , rb_1 , and rb_2 , and thus also divides $r \gcd(a_1, a_2, b_1, b_2)$.

It is easy to compute that $[\zeta, \xi] = \delta_3^{r(a_1b_2 - a_2b_1)}$. Therefore, one has

$$\Gamma_0 \cong \mathcal{H} \left(\frac{r|a_1b_2 - a_2b_1|}{c} \right).$$

Let F be the subgroup of $\mathcal{H}(r)$ generated by the elements δ_2 and δ_3 , and \check{F} be its image in G_0 . The subgroup F is a normal abelian subgroup of $\mathcal{H}(r)$, hence \check{F} is a normal abelian subgroup of G_0 . Let $f: \mathcal{H}(r) \rightarrow G_0/\check{F}$ be the natural projection. Then the group G_0/\check{F} is generated by $f(\delta_1)$, and one has

$$f(\delta_1^{a_1}) = f(\zeta) = 1 = f(\xi) = f(\delta_1^{b_1}).$$

Hence $[G_0 : \check{F}] = |G_0/\check{F}|$ is bounded from above by (and actually equals) the number $\gcd(a_1, b_1)$. \square

An immediate consequence of Lemma 4.3 is the following boundedness result.

COROLLARY 4.4. *Let $\Gamma_0 \subset \hat{\Gamma}_0$ be a normal subgroup, where $\Gamma_0 \cong \mathcal{H}(r_1)$ and $\hat{\Gamma}_0 \cong \mathcal{H}(r_2)$. Then the quotient group $G_0 = \hat{\Gamma}_0/\Gamma_0$ is finite, and G_0 contains a normal abelian subgroup of index at most r_1 .*

Proof. The group G_0 is finite for obvious reasons. By Lemma 4.3 there are integers a_1, a_2, b_1, b_2 with $a_1b_2 - a_2b_1 \neq 0$, and $c > 0$ such that c divides $r \gcd(a_1, a_2, b_1, b_2)$, one has

$$r_1 = \frac{r_2|a_1b_2 - a_2b_1|}{c},$$

and G_0 contains a normal abelian subgroup of index at most $\gcd(a_1, b_1)$. On the other hand, one has

$$\begin{aligned} \frac{r_2|a_1b_2 - a_2b_1|}{c} &\geq \frac{r_2 \gcd(a_1, b_1) \gcd(a_2, b_2)}{c} \\ &\geq \frac{r_2 \gcd(a_1, a_2, b_1, b_2)}{c} \gcd(a_1, b_1) \\ &\geq \gcd(a_1, b_1). \end{aligned}$$

\square

5. Heisenberg groups and semi-direct products. Consider the groups $\Gamma_0 \cong \mathcal{H}(r)$ and $\Gamma_1 \cong \mathbb{Z}$. Let γ be a generator of Γ_1 . Consider a semi-direct product $\Gamma = \Gamma_0 \rtimes \Gamma_1$. The action of γ on Γ_0 gives rise to its action on

$$\bar{\Gamma}_0 = \Gamma_0/z(\Gamma_0) \cong \mathbb{Z}^2,$$

which is given by a matrix $M \in \text{GL}_2(\mathbb{Z})$ if we fix a basis in $\bar{\Gamma}_0$ (cf. [Osi15] for a detailed description of the automorphism group of the discrete Heisenberg group).

LEMMA 5.1. *The following assertions hold.*

- (i) One has $\gamma\delta_3\gamma^{-1} = \delta_3^{\det M}$.
- (ii) The center $z(\Gamma)$ is trivial if and only if $\det M = -1$.
- (iii) One has $[\Gamma, \Gamma] \subset \Gamma_0$.

Proof. For $i = 1, 2$ one has

$$\gamma\delta_i\gamma^{-1} = \delta_1^{m_{1,i}}\delta_2^{m_{2,i}}\delta_3^{p_i},$$

where $M = (m_{i,j})$, and p_i are some integers. Obviously, we have $\gamma\delta_3\gamma^{-1} = \delta_3^t$ for some integer t . Therefore

$$\begin{aligned} \delta_3^{rt} &= \gamma\delta_3^r\gamma^{-1} = \gamma\delta_1\delta_2\delta_1^{-1}\delta_2^{-1}\gamma^{-1} = \delta_1^{m_{1,1}}\delta_2^{m_{2,1}}\delta_1^{m_{1,2}}\delta_2^{m_{2,2}}\delta_2^{-m_{2,1}}\delta_1^{-m_{1,1}}\delta_2^{-m_{2,2}}\delta_1^{-m_{1,2}} \\ &= \delta_3^{r(m_{1,1}m_{2,2}-m_{1,2}m_{2,1})} = \delta_3^{r \det M}, \end{aligned}$$

which implies assertion (i). Assertion (ii) easily follows from assertion (i). Assertion (iii) is obvious. \square

We will need the following notation. Let Υ be a group, and Δ be its subset. Denote

$$\text{rad}(\Delta, \Upsilon) = \{g \in \Upsilon \mid g^k \in \Delta \text{ for some positive integer } k \}.$$

If Δ is invariant with respect to some automorphism of Υ , then $\text{rad}(\Delta, \Upsilon)$ is invariant with respect to this automorphism as well. If a group Υ has no torsion and $\Delta \subset \Delta'$ is a pair of subgroups in Υ such that the index $[\Delta' : \Delta]$ is finite, then $\Delta' \subset \text{rad}(\Delta, \Upsilon)$.

LEMMA 5.2. *Suppose that the matrix M does not have eigen-values equal to 1. Then the following assertions hold:*

- (i) one has

$$[\Gamma, \Gamma]/z([\Gamma, \Gamma]) = \text{Im}(M - \text{Id}) \subset \bar{\Gamma}_0$$

is a free abelian group of rank 2;

- (ii) $[\Gamma, \Gamma] \cong \mathcal{H}(r')$ for some r' , and $[\Gamma, \Gamma]$ is a subgroup of finite index in Γ_0 ;
- (iii) $\Gamma_0 = \text{rad}([\Gamma, \Gamma], \Gamma)$; in particular, Γ_0 is a characteristic subgroup of Γ .

Proof. The subgroup $[\Gamma, \Gamma]/z([\Gamma, \Gamma]) \subset \bar{\Gamma}_0$ is generated by the images of the commutators $[\gamma, \delta_i]$, where $i = 1, 2$. Therefore, it can be identified with the sublattice in Γ_0 that is the image of the operator $M - \text{Id}$. Since the latter operator is non-degenerate by assumption, we conclude that $[\Gamma, \Gamma]/z([\Gamma, \Gamma])$ is a free abelian group of rank 2. This proves assertion (i).

Since $[\Gamma, \Gamma]/z([\Gamma, \Gamma])$ has finite index in $\bar{\Gamma}_0$ by assertion (i), and $[\Gamma, \Gamma]$ contains the element $\delta_3^{\det M}$ by Lemma 5.1, we conclude that $[\Gamma, \Gamma]$ is a subgroup of finite index in Γ_0 . Therefore, it is isomorphic to a group $\mathcal{H}(r')$ for some positive integer r' by Lemma 4.2. This gives assertion (ii).

Since $[\Gamma, \Gamma]$ has finite index in Γ_0 , we see that $\Gamma_0 \subset \text{rad}([\Gamma, \Gamma], \Gamma)$. Since $\Gamma/\Gamma_0 \cong \mathbb{Z}$ has no torsion, we see that the opposite inclusion holds as well. This gives assertion (iii). \square

REMARK 5.3. If the matrix M does not have eigen-values equal to 1, then the group Γ does not contain subgroups isomorphic to \mathbb{Z}^3 . Indeed, suppose that $\Delta \subset \Gamma$ is such a subgroup. Then both the intersection $\Delta_z = \Delta \cap z(\Gamma)$ and the image $\bar{\Delta}_0$ of $\Delta \cap \Gamma_0$ in $\bar{\Gamma}_0$ are non-trivial. Now the contradiction is obtained by taking a commutator of

a non-trivial element of Δ_z with a preimage in Δ of a non-trivial element of $\bar{\Delta}_0$ and using the fact that the operator $M - \text{Id}$ is non-degenerate.

LEMMA 5.4. *Suppose that the eigen-values of the matrix M are not roots of unity. Then there exists a constant $\nu = \nu(\Gamma)$ with the following property. Let $\hat{\Gamma} = \hat{\Gamma}_0 \rtimes \hat{\Gamma}_1$, where $\hat{\Gamma}_0 \cong \mathcal{H}(\hat{r})$ and $\hat{\Gamma}_1 \cong \mathbb{Z}$, and suppose that $\Gamma \subset \hat{\Gamma}$ is a normal subgroup. Then the group $G = \hat{\Gamma}/\Gamma$ is finite and has a normal abelian subgroup of index at most ν .*

Proof. The group G is finite for obvious reasons (cf. Lemma 4.2). By Lemma 5.2(iii) we have $\Gamma_0 = \Gamma \cap \hat{\Gamma}_0$. Thus by Lemma 3.2 there is a positive integer k with the following properties: the subgroup $\Gamma_1 \subset \Gamma$ is generated by an element $\hat{\gamma}^k \hat{\delta}$, where $\hat{\gamma}$ is a generator of $\hat{\Gamma}_1$, and $\hat{\delta}$ is an element of $\hat{\Gamma}_0$; and the group G contains a normal subgroup $G_0 \cong \hat{\Gamma}_0/\Gamma_0$ of index k . Note that the subgroup Γ_0 is normal in $\hat{\Gamma}$, because Γ is normal in $\hat{\Gamma}$, while Γ_0 is a characteristic subgroup of Γ by Lemma 5.2(iii). Let $R \in \text{GL}_2(\mathbb{Z})$ be the matrix that defines the semi-direct product $\hat{\Gamma} = \hat{\Gamma}_0 \rtimes \hat{\Gamma}_1$. Considering the action of the element $\hat{\gamma}$ on the lattice $\hat{\Gamma}_0/z(\hat{\Gamma}_0) \cong \mathbb{Z}^2$ and its sublattice $\Gamma_0/z(\Gamma_0) \cong \mathbb{Z}^2$, we see that R^k is conjugate to M . Thus k is bounded by some constant that depends only on M (that is, only on Γ) by Lemma 2.4. On the other hand, the group G_0 contains a normal abelian subgroup of index at most r by Corollary 4.4, and the assertion easily follows. \square

6. Heisenberg groups and direct products. Consider the groups $\Gamma_0 \cong \mathcal{H}(r)$ and $\Gamma_1 \cong \mathbb{Z}$, and put $\Gamma = \Gamma_0 \times \Gamma_1$. One has

$$z(\Gamma) = \langle \delta_3, \gamma \rangle \cong \mathbb{Z}^2,$$

and $\bar{\Gamma} = \Gamma/z(\Gamma) \cong \mathbb{Z}^2$.

Unlike the situation in §3 and §5, the subgroup Γ_0 is not characteristic in Γ . Indeed, let δ_1, δ_2 , and δ_3 be the generators of Γ_0 as in (4.1), and γ be a generator of Γ_1 . Define an automorphism ψ of Γ by

$$\psi(\delta_1) = \delta_1\gamma, \quad \psi(\delta_2) = \delta_2, \quad \psi(\delta_3) = \delta_3, \quad \psi(\gamma) = \gamma,$$

cf. [Osi15]. Then ψ does not preserve Γ_0 . However, the following weaker uniqueness result holds.

LEMMA 6.1. *Let $\Gamma'_0 \subset \Gamma$ be a normal subgroup isomorphic to $\mathcal{H}(r')$ for some positive integer r' . Suppose that $\varsigma: \Gamma/\Gamma'_0 \cong \mathbb{Z}$. Then the natural projection $\Gamma'_0 \rightarrow \Gamma_0$ is an isomorphism. In particular, one has $r = r'$.*

Proof. Put $\Upsilon_1 = \Gamma_1/\Gamma'_0 \cap \Gamma_1$ and $\Upsilon_0 = \Gamma_0/\varsigma(\Gamma'_0)$. Then $\Gamma/\Gamma'_0 \cong \Upsilon_0 \times \Upsilon_1$. Therefore, either Υ_0 is trivial and $\Upsilon_1 \cong \mathbb{Z}$, or $\Upsilon_0 \cong \mathbb{Z}$ and Υ_1 is trivial. In the former case ς provides an isomorphism from Γ'_0 to Γ_0 . In the latter case $\Gamma_1 \subset \Gamma'_0$ and the group $\varsigma(\Gamma'_0) \cong \Gamma'_0/\Gamma_1$ is abelian by Lemma 4.2. Thus the group Γ'_0 is abelian as well, which is a contradiction. \square

LEMMA 6.2. *Suppose that Γ is a normal subgroup in a group $\hat{\Gamma} = \hat{\Gamma}_0 \times \hat{\Gamma}_1$, where $\hat{\Gamma}_0 \cong \mathcal{H}(\hat{r})$ and $\hat{\Gamma}_1 \cong \mathbb{Z}$. Then the group $G = \hat{\Gamma}/\Gamma$ is finite and has a normal abelian subgroup of index at most r .*

Proof. The group G is finite for obvious reasons. Put $\Gamma'_0 = \Gamma \cap \hat{\Gamma}_0$ and $G_0 = \hat{\Gamma}_0/\Gamma'_0$. By Lemma 3.2 one has $\Gamma = \Gamma'_0 \rtimes \Gamma'_1$, where $\Gamma'_1 \cong \mathbb{Z}$ is generated by $\hat{\gamma}^k \hat{\delta}$ for some positive integer k , a generator $\hat{\gamma}$ of $\hat{\Gamma}_1$, and an element $\hat{\delta} \in \hat{\Gamma}_0$. Thus $\Gamma/\Gamma'_0 \cong \mathbb{Z}$. Since Γ'_0 is

a subgroup of finite index in $\hat{\Gamma}_0$, by Lemma 4.2 one has $\Gamma'_0 \cong \mathcal{H}(r')$ for some r' . By Lemma 6.1 we know that $r' = r$. Thus G_0 contains a normal abelian subgroup N of index at most r by Corollary 4.4. On the other hand, G is generated by G_0 and the image $\bar{\gamma}$ of $\hat{\gamma}$. Since $\bar{\gamma}$ is a central element in G , the group generated by N and $\bar{\gamma}$ is a normal abelian subgroup of index at most r in G . \square

7. Inoue surfaces. In this section we study automorphism groups of Inoue surfaces.

Inoue surfaces are quotients of $\mathbb{C} \times \mathbb{H}$, where \mathbb{H} is the upper half-plane, by certain infinite discrete groups. They were introduced by M. Inoue [Ino74]. These surfaces contain no curves and their invariants are as follows:

$$a(X) = 0, \quad b_1(X) = 1, \quad b_2(X) = 0, \quad h^{1,0}(X) = 0, \quad h^{0,1}(X) = 1.$$

LEMMA 7.1. *Let X be an Inoue surface, and $G \subset \text{Aut}(X)$ be a finite subgroup. Then the action of G on X is free, and the quotient $\hat{X} = X/G$ is again an Inoue surface.*

Proof. Assume that the action of G on X is not free. Let g be an element of G that acts on X with fixed points. To get a contradiction we may assume that the order of g is prime. Since X contains no curves, the fixed point locus of G consists of a finite number of points. Denote by n the number of such points. By the topological Lefschetz fixed point formula, one has

$$n = \sum_{i=0}^4 (-1)^i \text{tr}_{H^i(X, \mathbb{R})} g^* = 2 - 2 \text{tr}_{H^1(X, \mathbb{R})} g^*.$$

Hence the action of g^* on $H^1(X, \mathbb{R}) \cong \mathbb{R}$ is not trivial. This is possible only if g is of order 2 and $n = 4$.

Consider the quotient \check{X} of X by the cyclic group generated by g . We see that \check{X} has exactly 4 singular points which are Du Val of type A_1 . Let $Y \rightarrow \check{X}$ be the minimal resolution of singularities. Then

$$c_1(Y)^2 = c_1(\mathcal{K}_{\check{X}})^2 = \frac{1}{2}c_1(X)^2 = 0,$$

and $\chi_{\text{top}}(Y) = 4 + \chi_{\text{top}}(\check{X}) = 6$. This contradicts the Noether's formula, see e.g. [BHPVdV04, §I.5].

Therefore, the action of G on X is free, and the quotient morphism $X \rightarrow \hat{X} = X/G$ is an unramified finite cover. This implies that $\chi_{\text{top}}(\hat{X}) = 0$. Furthermore, one has

$$b_2(\hat{X}) = \text{rk } H^2(X, \mathbb{Z})^G = 0,$$

and so $b_1(\hat{X}) = 1$. Therefore, by Enriques–Kodaira classification \hat{X} is a minimal surface of class VII. Clearly, the surface \hat{X} contains no curves, so that in particular $b_2(\hat{X}) = 0$. Thus \hat{X} is either a Hopf surface or an Inoue surface (see [Bog77] and [Tel94], cf. [BBK17]). Since every Hopf surface contains a curve (see [Kod66, Theorem 32]), we conclude that \hat{X} is an Inoue surface. \square

There are three types of Inoue surfaces: S_M , $S^{(+)}$, and $S^{(-)}$. They are distinguished by the type of their fundamental group $\Gamma = \pi_1(X)$, see [Ino74]:

type	generators	relations
S_M	$\delta_1, \delta_2, \delta_3, \gamma$	$[\delta_i, \delta_j] = 1, \gamma\delta_i\gamma^{-1} = \delta_1^{m_{1,i}}\delta_2^{m_{2,i}}\delta_3^{m_{3,i}}, (m_{j,i}) \in \text{SL}_3(\mathbb{Z})$
$S^{(\pm)}$	$\delta_1, \delta_2, \delta_3, \gamma$	$[\delta_i, \delta_3] = 1, [\delta_1, \delta_2] = \delta_3^r, \gamma\delta_i\gamma^{-1} = \delta_1^{m_{1,i}}\delta_2^{m_{2,i}}\delta_3^{p_i}$ for $i = 1, 2,$ $\gamma\delta_3\gamma^{-1} = \delta_3^{\pm 1}, (m_{j,i}) \in \text{GL}_2(\mathbb{Z}), \det(m_{j,i}) = \pm 1$

In the notation of §3 and §5, one has $\Gamma \cong \Gamma_0 \rtimes \Gamma_1$, where $\Gamma_1 \cong \mathbb{Z}$, while $\Gamma_0 \cong \mathbb{Z}^3$ for Inoue surfaces of type S_M , and $\Gamma_0 \cong \mathcal{H}(r)$ for Inoue surfaces of types $S^{(\pm)}$. In the former case the matrix $M \in \text{SL}_3(\mathbb{Z})$ that defines the semi-direct product has eigenvalues α, β , and $\bar{\beta}$, where $\alpha \in \mathbb{R}, \alpha > 1$, and $\beta \notin \mathbb{R}$. In the latter case the matrix $M \in \text{GL}_2(\mathbb{Z})$ that defines the action of \mathbb{Z} on $\mathcal{H}(r)/z(\mathcal{H}(r)) \cong \mathbb{Z}^2$ has real eigenvalues α and β , where $\alpha > 1$ and $\alpha\beta = \pm 1$ depending on whether Γ is of type $S^{(+)}$ or $S^{(-)}$, see [Ino74, §§2–4].

LEMMA 7.2. *Let Γ be a group of one of the types $S_M, S^{(+)}$, or $S^{(-)}$. Then*

- (i) Γ is of type S_M if and only if Γ contains a characteristic subgroup isomorphic to \mathbb{Z}^3 ;
- (ii) Γ is of type $S^{(+)}$ if and only if Γ contains no subgroups isomorphic to \mathbb{Z}^3 and $z(\Gamma) \neq \{1\}$;
- (iii) Γ is of type $S^{(-)}$ if and only if Γ contains no subgroups isomorphic to \mathbb{Z}^3 and $z(\Gamma) = \{1\}$.

Proof. This follows from Lemmas 3.1(ii) and 5.1(ii) and Remark 5.3. \square

COROLLARY 7.3. *Let X be an Inoue surface, and $G \subset \text{Aut}(X)$ be a finite subgroup. Then the action of G on X is free, and the following assertions hold.*

- (i) If X is of type S_M , then so is X/G ;
- (ii) If X is of type $S^{(-)}$, then so is X/G ;
- (iii) If X is of type $S^{(+)}$, then X/G is of type $S^{(+)}$ or $S^{(-)}$.

Proof. Put $\hat{X} = X/G$. Then the action of G on X is free, and X is an Inoue surface by Lemma 7.1. Put $\hat{\Gamma} = \pi_1(\hat{X})$. Then $\hat{\Gamma}$ is a group of one of the types $S_M, S^{(+)}$, or $S^{(-)}$, and $\Gamma \subset \hat{\Gamma}$ is a normal subgroup of finite index. Now everything follows from Lemma 7.2. \square

LEMMA 7.4. *Let X be an Inoue surface of type S_M . Then the group $\text{Aut}(X)$ is Jordan.*

Proof. Let $G \subset \text{Aut}(X)$ be a finite subgroup, and put $\hat{X} = X/G$. By Corollary 7.3 the action of G on X is free, and \hat{X} is also an Inoue surface of type S_M . Put $\Gamma = \pi_1(X)$ and $\hat{\Gamma} = \pi_1(\hat{X})$. Then Γ is a normal subgroup of $\hat{\Gamma}$, and $\hat{\Gamma}/\Gamma \cong G$; moreover, both Γ and $\hat{\Gamma}$ are semi-direct products as in §3. Now it follows from Lemma 3.3 that there is a constant ν that depends only on Γ (that is, only on X), such that G has a normal abelian subgroup of index at most ν . \square

Now we deal with Inoue surfaces of types $S^{(+)}$ and $S^{(-)}$.

LEMMA 7.5. *Let X be an Inoue surface of type $S^{(+)}$ or $S^{(-)}$. Then the group $\text{Aut}(X)$ is Jordan.*

Proof. Let $G \subset \text{Aut}(X)$ be a finite subgroup, and put $\hat{X} = X/G$. By Corollary 7.3 the action of G on X is free, and \hat{X} is also an Inoue surface of type $S^{(+)}$ or $S^{(-)}$. Put $\Gamma = \pi_1(X)$ and $\hat{\Gamma} = \pi_1(\hat{X})$. Then Γ is a normal subgroup of $\hat{\Gamma}$, and $\hat{\Gamma}/\Gamma \cong G$; moreover, both Γ and $\hat{\Gamma}$ are semi-direct products as in §5. Now it follows from

Lemma 5.4 that there is a constant ν that depends only on Γ (that is, only on X), such that G has a normal abelian subgroup of index at most ν . \square

We summarize the results of Lemmas 7.4 and 7.5 as follows.

COROLLARY 7.6. *Let X be an Inoue surface. Then the group $\text{Aut}(X)$ is Jordan.*

REMARK 7.7. There are certain types of minimal compact complex surfaces of class VII whose automorphism groups are studied in details, for instance, hyperbolic and parabolic Inoue surfaces, see [Pin84] and [Fuj09]. Note that surfaces of both of these types have positive second Betti numbers (and thus they are not to be confused with Inoue surfaces we deal with in this section).

8. Kodaira surfaces. In this section we study automorphism groups of Kodaira surfaces. Our approach here is similar to what happens in §7.

Recall (see e.g. [BHPVdV04, § V.5]) that a Kodaira surface is a compact complex surface of Kodaira dimension 0 with odd first Betti number. There are two types of Kodaira surfaces: primary and secondary ones. A primary Kodaira surface is a compact complex surface with the following invariants [Kod64, §6]:

$$\chi_X \sim 0, \quad a(X) = 1, \quad b_1(X) = 3, \quad b_2(X) = 4, \quad \chi_{\text{top}}(X) = 0, \quad h^{0,1}(X) = 2, \quad h^{0,2}(X) = 1.$$

A secondary Kodaira surface is a quotient of a primary Kodaira surface by a free action of a finite cyclic group.

Let X be a primary Kodaira surface. The universal cover of X is isomorphic to \mathbb{C}^2 , and the fundamental group $\Gamma = \pi_1(X)$ has the following presentation:

$$\Gamma = \langle \delta_1, \delta_2, \delta_3, \gamma \mid [\delta_1, \delta_2] = \delta_3^r, \quad [\delta_i, \delta_3] = [\delta_i, \gamma] = 1 \rangle, \tag{8.1}$$

where r is a positive integer [Kod64, §6]. In the notation of §6 one has $\Gamma \cong \mathcal{H}(r) \times \mathbb{Z}$.

Denote by $\overline{\text{Aut}}(X) \subset \text{Aut}(X)$ the subgroup that consists of all elements acting trivially on $H^*(X, \mathbb{Q})$ and $H^*(B, \mathbb{Q})$.

LEMMA 8.2 (cf. Lemma 7.1). *Let X be a primary Kodaira surface, and $G \subset \overline{\text{Aut}}(X)$ be a finite subgroup. Then the action of G on X is free, and the quotient $\hat{X} = X/G$ is again a primary Kodaira surface.*

Proof. Let $\phi: X \rightarrow B$ be the algebraic reduction of X . Then B is an elliptic curve, and ϕ is a principal elliptic fibration [Kod64, §6], [BHPVdV04, § V.5]. Furthermore, ϕ is equivariant with respect to $\text{Aut}(X)$.

Let g be an element of G . Since the curve B is elliptic, we see that g acts on B without fixed points. This means that there are no fibers of ϕ that consist of points fixed by g . On the other hand, every curve on X is a fiber of $\phi: X \rightarrow B$. Indeed, otherwise one can construct a curve on X with a positive self-intersection, which would imply that X is projective, see [BHPVdV04, § IV.6]. Hence there are no curves that consist of points fixed by g on X at all. Now the topological Lefschetz fixed point formula shows that the number of points on X fixed by g equals

$$\sum_{i=0}^4 (-1)^i \text{tr}_{H^i(X, \mathbb{R})} g^* = \sum_{i=0}^4 b_i(X) = 0.$$

Therefore, the action of G on X is free, so that \hat{X} is a smooth surface and the quotient morphism $X \rightarrow \hat{X}$ is an unramified finite cover. Hence $\kappa(\hat{X}) = \kappa(X) = 0$. Moreover, we have

$$c_1(\hat{X})^2 = c_1(X)^2 = 0.$$

This means that the surface \hat{X} is minimal. Since $G \subset \overline{\text{Aut}}(X)$, we have

$$b_1(\hat{X}) = b_1(X) = 3.$$

Therefore, \hat{X} is a primary Kodaira surface by Kodaira–Enriques classification. \square

LEMMA 8.3. *Let X be a primary Kodaira surface. Then the group $\text{Aut}(X)$ is Jordan.*

Proof. Let $G \subset \text{Aut}(X)$ be a finite subgroup. By Theorem 2.1 and Lemma 2.2 we can assume that $G \subset \overline{\text{Aut}}(X)$. Put $\hat{X} = X/G$. It follows from Lemma 8.2 that G acts freely on X , and \hat{X} is a primary Kodaira surface. Put $\Gamma = \pi_1(X)$ and $\hat{\Gamma} = \pi_1(\hat{X})$. Then Γ is a normal subgroup of $\hat{\Gamma}$, and $\hat{\Gamma}/\Gamma \cong G$; moreover, both Γ and $\hat{\Gamma}$ are as in §6. Now it follows from Lemma 6.2 that there is a constant r that depends only on Γ (that is, only on X), such that G has a normal abelian subgroup of index at most r . \square

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