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LAPLACIAN COFLOW ON THE 7-DIMENSIONAL HEISENBERG GROUP*

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Abstract. We study the Laplacian coflow and the modified Laplacian coflow of G_2-structures on the 7-dimensional Heisenberg group. For the Laplacian coflow we show that the solution is always ancient, that is it is defined in some interval $(-\infty, T)$, with $0 < T < +\infty$. However, for the modified Laplacian coflow, we prove that in some cases the solution is defined only on a finite interval while in other cases the solution is ancient or eternal, that is it is defined on $(-\infty, \infty)$.

Key words. G_2-structure, Laplacian coflow.

Mathematics Subject Classification. Primary 53C15; Secondary 53C44, 53C30.

1. Introduction. A 7-dimensional manifold M carries a G_2-structure if M admits a globally defined 3-form φ, which is called G_2 form, that can be described locally as

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245},$$

with respect to some local basis $\{e^1, \dots, e^7\}$ of the 1-forms on M. Here, e^{127} stands for $e^1 \wedge e^2 \wedge e^7$, and so on. Such a 3-form φ determines a Riemannian metric g_φ and an orientation on M. If ∇ denotes the Levi-Civita connection of g_φ, one can view $\nabla\varphi$ as the torsion of the G_2-structure φ. Thus, if $\nabla\varphi = 0$, which is equivalent to $d\varphi = 0$ and $d \star_\varphi \varphi = 0$, where $\star_\varphi$ is the Hodge star operator with respect to g_φ, one says that the G_2-structure is torsion-free.

The different classses of G_2-structures can be described in terms of the exterior derivatives $d\varphi$ and $d \star_\varphi \varphi$ [2, 5]. If $d\varphi = 0$, then the G_2-structure is called *closed* (or *calibrated* in the sense of Harvey and Lawson [8]) and if φ is coclosed, that is if $\star_\varphi\varphi$ is closed, then the G_2-structure is called *coclosed* (or *cocalibrated* [8]).

Since Hamilton introduced the Ricci flow in 1982 [7], geometric flows have been an important tool in studying geometric structures on manifolds. The Laplacian flow for closed G_2-structures on a 7-manifold M has been introduced by Bryant in [2], and it is given by

$$\begin{cases} \frac{\partial}{\partial t}\varphi(t) = \Delta_t \varphi(t), \\ d\varphi(t) = 0, \\ \varphi(0) = \varphi, \end{cases}$$

where $\varphi(t)$ is a closed G_2 form on M, $\Delta_t = d\,d^* + d^* d$ is the Hodge Laplacian operator associated with the metric $g_{\varphi(t)}$ induced by the 3-form $\varphi(t)$, and φ is the initial closed G_2-structure. A short-time existence and uniqueness for this flow, in the case of compact manifolds, has been proved in [3]. Regarding the long-time behavior of the

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Laplacian flow on compact manifolds M, Lotay and Wei in [15] have proved recently that if the initial closed G_2 form φ is such that its torsion is sufficiently small (in a suitable sense), then the Laplacian flow of φ will exist for all time and converge to a torsion-free G_2-structure. Non-compact examples where the flow converges to a flat G_2-structure have been given in [4].

Shi-type derivative estimates for the Riemann curvature tensor and torsion tensor along the Laplacian flow have been determined in [14], and in [16] it is proved that for each fixed positive time $t \in (0, T]$, $(M, \varphi(t), g_{\varphi(t)})$ is real analytic. Consequently, any Laplacian soliton is real analytic. Moreover, solitons of the Laplacian flow of G_2-structures in the homogeneous case have been studied recently by Lauret in [13] using the bracket flow and the algebraic soliton approach.

Some work has also been done on other related flows of G_2-structures - such as the *Laplacian coflow*, or *flow*, for coclosed G_2-structures. This coflow has been originally proposed by Karigiannis, McKay and Tsui in [10] and, for an initial coclosed G_2 form φ with $\psi = \star_\varphi \varphi$, it is given by

$$\frac{\partial}{\partial t}\psi(t) = -\Delta_t \psi(t), \quad d\psi(t) = 0, \quad \psi(0) = \psi, \tag{1}$$

where $\psi(t)$ is the Hodge dual 4-form of a G_2-structure $\varphi(t)$, that is $\psi(t) = \star_t \varphi(t)$, Δ_t is the Hodge Laplacian operator with respect to the Remannian metric $g_{\varphi(t)}$. This flow preserves the condition of the G_2-structure being coclosed, that is $\psi(t)$ is closed for any t, and it was studied in [10] for two explicit examples of coclosed G_2-structures with symmetry, namely for warped products of an interval, or a circle, with a compact 6-manifold N which is taken to be either a nearly Kähler manifold or a Calabi-Yau manifold. Nevertheless, in [6] it was shown that the coflow (1) is not even a weakly parabolic flow, and that the symbol of the operator Δ_t, acting on $4-$forms, has a mixed signature. But no general result is known about the short time existence of the coflow (1).

A *modified Laplacian coflow* was introduced by Grigorian in [6]

$$\frac{\partial}{\partial t}\psi(t) = \Delta_t \psi(t) + 2d\Big(\big(A - \mathrm{Tr_t}(\tau(t))\big)\varphi(t)\Big), \quad d\psi(t) = 0, \quad \psi(0) = \psi, \tag{2}$$

where $\mathrm{Tr_t}(\tau(t))$ is the trace of the full torsion tensor $\tau(t)$ of the G_2-structure defined by $\varphi(t)$, and A is a fixed positive constant (see Section 3 for the details). Moreover, in [6] it is proved that the coflow (2) is weakly parabolic in the direction of closed forms $\psi(t)$ up to diffeomorphisms and, on compact manifolds, it has a unique solution $\psi(t)$ for the short time period $t \in [0, \epsilon)$, for some $\epsilon > 0$.

In [1], it is given a classification of 2-step nilpotent Lie groups admitting left invariant coclosed G_2-structures. In this paper, we study the coflows (1) and (2) in the case of the 7-dimensional Heisenberg group H.

As we mentioned before, there is not known any general result on the short time existence of solution for the coflow (1). Nevertheless, in Theorem 4, we show that the solution of the coflow (1) for any coclosed G_2-structure on the Heisenberg group is always *ancient*, that is it is defined on a time interval of the form $(-\infty, T)$, where $T > 0$ is a real number. To our knowledge, these are the first examples of non-compact manifolds having a coclosed G_2-structure for which the time interval of existence of the solution for (1) is not finite. However, we prove that the solution of the coflow (2) for some coclosed G_2 forms on H is defined only on a finite interval (Theorem 9) and, for other coclosed G_2 forms, the solution of (2) is *ancient* (Theorem 7, part i), and Theorem 8) or *eternal*, that is it is defined for all $t \in \mathbb{R}$ (Theorem 7, part ii)).

Moreover, considering the coflows (1) and (2) on the associated Lie algebra as a bracket flow on $\mathbb{R}^7$, in a similar way as Lauret did in [11] for the Ricci flow, we show that the underlying metrics $g(t)$ of the solution in Corollary 5 and Theorem 8 converge smoothly, up to pull-back by time-dependent diffeomorphisms, to a flat metric, as t goes to infinity. Indeed, by [11, Proposition 2.8] the convergence of the metrics in $\mathcal{C}^\infty$ uniformly on compact sets in $\mathbb{R}^7$ is equivalent to the convergence of the nilpotent Lie brackets $\mu(t)$ in the algebraic subset of nilpotent Lie brackets $\mathcal{N} \subset (\Lambda^2\mathbb{R}^7)^* \otimes \mathbb{R}^7$ with the usual vector space topology.

2. Coclosed G_2-structures on the Heisenberg group. A 7-dimensional manifold M is said to admit a G_2-*structure* if there is a reduction of the structure group of its frame bundle from $GL(7,\mathbb{R})$ to the exceptional Lie group G_2, which can actually be viewed naturally as a subgroup of $SO(7)$. Thus, a G_2-structure determines a Riemannian metric and an orientation on M. In fact, one can prove that the presence of a G_2-structure is equivalent to the existence of a differential 3-form φ (the G_2 form) on M, which induces the Riemannian metric g_φ given by

$$g_\varphi(X,Y)\,vol = \frac{1}{6}\iota_X\varphi \wedge \iota_Y\varphi \wedge \varphi, \tag{3}$$

for any vector fields X, Y on M, where vol is the volume form on M, and ι_X denotes the contraction by X. Let $\star_\varphi$ be the Hodge star operator determined by g_φ and the orientation induced by φ. We will always write ψ to denote the dual 4-form of a G_2-structure φ, that is

$$\psi = \star_\varphi\varphi.$$

A manifold M has a *coclosed* (or *cocalibrated*) G_2-*structure* if there is a G_2-structure on M such that the G_2 form φ is coclosed, that is $d\psi = 0$.

Now, let G be a 7-dimensional simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Then, a G_2-structure on G is *left invariant* if and only if the corresponding 3-form φ is left invariant. Thus, a left invariant G_2-structure on G corresponds to an element φ of $\Lambda^3(\mathfrak{g}^*)$ that can be written as

$$\varphi = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}, \tag{4}$$

with respect to some orthonormal coframe $\{e^1, \dots, e^7\}$ of the dual space $\mathfrak{g}^*$, where e^{127} stands for $e^1 \wedge e^2 \wedge e^7$, and so on. So the dual form $\psi = \star_\varphi\varphi$ has the following expression

$$\psi = e^{1234} + e^{1256} + e^{1367} + e^{1457} + e^{2357} - e^{2467} + e^{3456}. \tag{5}$$

Note that in order to recover the left invariant G_2 form φ from the 4-form $\star_\varphi\varphi$ we need to fix an orientation of $\mathfrak{g}$. In fact, the stabilizer of $\star_\varphi\varphi$ in $GL(7,\mathbb{R})$ is $G_2 \times \mathbb{Z}_2$ since the matrix $-Id$ preserves the form $\star_\varphi\varphi$, and so the latter fails to determine the overall orientation.

Recall that the seven dimensional Heisenberg group H is the simply connected nilpotent Lie group whose Lie algebra $\mathfrak{h}$ is defined by

$$\mathfrak{h} = \left(0,0,0,0,0,0,\frac{\sqrt{6}}{6}(e^{12} + e^{34} + e^{56})\right). \tag{6}$$

This notation means that the dual space $\mathfrak{h}^*$ is spanned by $\{e^1, \dots, e^7\}$ satisfying

$$de^i = 0, \quad 1 \le i \le 6, \qquad de^7 = \frac{\sqrt{6}}{6}(e^1 \wedge e^2 + e^3 \wedge e^4 + e^5 \wedge e^6).$$

3. On the coflows of cococlosed G_2-structures. Here we show the expression of each one of the coflows (1) and (2) in terms of the intrinsic torsion forms of a coclosed G_2-structure [2, 6].

Let M be a 7-dimensional manifold with a G_2-structure defined by a 3-form φ. Denote by ψ the 4-form $\psi = \star_\varphi \varphi$, where $\star_\varphi$ is the Hodge star operator of the metric g_φ induced by φ. Let $(\Omega^*(M), d)$ be the de Rham complex of differential forms on M. Then, Bryant in [2] proved that the forms $d\varphi$ and $d\psi$ are such that

$$\begin{cases} d\varphi = \tau_0\, \psi + 3\, \tau_1 \wedge \varphi + \star_\varphi \tau_3, \\ d\psi = 4\tau_1 \wedge \psi - \star_\varphi \tau_2, \end{cases} \tag{7}$$

where $\tau_0 \in \Omega^0(M), \tau_1 \in \Omega^1(M), \tau_2 \in \Omega^2_{14}(M)$ and $\tau_3 \in \Omega^3_{27}(M)$. Here $\Omega^2_{14}(M)$ and $\Omega^3_{27}(M)$ are the spaces

$$\Omega^2_{14}(M) = \{\alpha \in \Omega^2(M) \mid \alpha \wedge \varphi = - \star_\varphi \alpha\},$$

$$\Omega^3_{27}(M) = \{\beta \in \Omega^3(M) \mid \beta \wedge \varphi = 0 = \beta \wedge \star_\varphi \varphi\}.$$

The differential forms τ_i $(i = 0, 1, 2, 3)$ that appear in (7), are called the *intrinsic torsion forms* of φ. According to Grigorian [6] the *full torsion tensor* τ of φ is the tensor field on M given by

$$\tau = \frac{1}{4}\tau_0\, g_\varphi - \imath_{\tau_1}\varphi - \frac{1}{3} j_\varphi(\tau_3) + \frac{1}{2}\tau_2,$$

where $\imath_{\tau_1}$ denotes the contraction by τ_1 using the metric g_φ induced by φ (that is, if U is the vector field on M such that $\tau_1 = \imath_U g$, then $\imath_{\tau_1}\varphi = \imath_U \varphi$) and $j_\varphi : \Omega^3(M) \longrightarrow S^2(M)$ is the map defined by

$$j_\varphi(\gamma)(X, Y) = \star_\varphi\Big((\imath_X \varphi) \wedge (\imath_Y \varphi) \wedge \gamma\Big),$$

where $\gamma \in \Omega^3(M)$, and X, Y are vector fields on M [2]. In particular, by [2] j_φ is an isomorphism between the space $\Omega^3_{27}(M)$ and the space $S^2_0(M)$ of trace-free symmetric 2-tensors on M.

Recall that φ defines a coclosed G_2-structure on M if ψ is closed, that is $d\psi = 0$. In this case, (7) implies that the forms τ_1 and τ_2 vanish, and so the full torsion tensor τ has the following expression

$$\tau = \frac{1}{4}\tau_0\, g_\varphi - \frac{1}{3} j_\varphi(\tau_3).$$

Since $\tau_3 \in \Omega^3_{27}(M)$, the trace of $j_\varphi(\tau_3)$ vanishes. Therefore, $\mathrm{Tr}(\tau)$ of τ is given by

$$\mathrm{Tr}(\tau) = \frac{1}{4}\tau_0\, \mathrm{Tr}(g_\varphi) = \frac{7}{4}\tau_0. \tag{8}$$

Lemma 1. *Let M be a 7-dimensional manifold with a coclosed G_2 form φ. Denote by τ_0 and τ_3 the torsion forms of φ. Then, the torsion forms $\widetilde{\tau_0}$ and $\widetilde{\tau_3}$ of $-\varphi$ satisfy*

$$\widetilde{\tau_0} = -\tau_0, \quad \widetilde{\tau_3} = \tau_3. \tag{9}$$

Proof. Using (7), we see that $\widetilde{\tau_0} = -\tau_0$ and $\widetilde{\tau_3} = \tau_3$ since $\star_{-\varphi} = -\star_\varphi$. □

PROPOSITION 2. *Let M be a 7-dimensional manifold with a coclosed G_2 form φ. Then, the coflow* (1) *for φ has the following expression*

$$\text{(C)}\quad \begin{aligned}&\frac{\partial}{\partial t}\psi(t) = -d\big(\tau_0(t)\big)\wedge\varphi(t) - \big(\tau_0(t)\big)^2\psi(t) - \tau_0(t)\star_t \tau_3(t) - d\tau_3(t),\\ &d\psi(t) = 0, \quad \varphi(0) = \varphi,\end{aligned}$$

and the modified coflow (2) *is expressed as*

$$\text{(G)}\quad \begin{aligned}\frac{\partial}{\partial t}\psi(t) &= \tau_0(t)\left(2A - \frac{5}{2}\tau_0(t)\right)\psi(t) + \left(2A - \frac{5}{2}\tau_0(t)\right) *_t \tau_3(t) + d\tau_3(t)\\ &\quad + \frac{5}{2}\varphi(t)\wedge d\tau_0(t),\\ d\psi(t) &= 0, \quad \varphi(0) = \varphi,\end{aligned}$$

where $\tau_0(t)$ and $\tau_3(t)$ are the torsion forms of $\varphi(t)$ (according with (7)*), $\star_t$ is the Hodge star operator with respect to the Riemannian metric $g_{\varphi(t)}$ induced by $\varphi(t)$ and A is a fixed positive constant.*

Proof. Since the solution $\psi(t)$ to the coflow (1), if it exists, remains closed and (2) preserves the closedness of $\psi(t) = \star_t\varphi(t)$, by (7) and the vanishing of the torsion forms $\tau_1(t)$ and $\tau_2(t)$ of $\varphi(t)$,

$$d\varphi(t) = \tau_0(t)\psi(t) + \star_t\tau_3(t).$$

Hence,

$$\begin{aligned}\Delta_t\psi(t) &= d\,d^*\psi(t) = d\star_t d\varphi(t) = d\star_t\Big(\tau_0(t)\psi(t) + \star_t\tau_3(t)\Big)\\ &= d(\tau_0(t))\wedge\varphi(t) + \tau_0(t)^2\psi(t) + \tau_0(t)\star_t\tau_3(t) + d\tau_3(t),\end{aligned}$$

and

$$\begin{aligned}2d\Big(\big(A - \mathrm{Tr}(\tau(t))\big)\varphi(t)\Big) &= 2d\left((A - \frac{7}{4}\tau_0(t))\varphi(t)\right)\\ &= -\frac{7}{2}d(\tau_0(t))\wedge\varphi(t) + \left(2A - \frac{7}{2}\tau_0(t)\right)(\tau_0(t)\psi(t) + \star_t\tau_3(t)).\end{aligned}$$

Thus,

$$\begin{aligned}\Delta_t\psi + 2d\Big(\big(A - \mathrm{Tr}(\tau(t))\big)\varphi(t)\Big) &= -\frac{5}{2}d(\tau_0(t))\wedge\varphi(t) + \tau_0(t)\left(2A - \frac{5}{2}\tau_0(t)\right)\psi(t)\\ &\quad + \left(2A - \frac{5}{2}\tau_0(t)\right)\star_t\tau_3(t) + d\tau_3(t),\end{aligned}$$

and the Proposition follows. □

REMARK 1. Note that (9) and Proposition 2 imply that the solution of the coflow (G) for φ (if such a solution exists) changes when the initial coclosed G_2 form is $-\varphi$ instead of φ (see Theorem 7 and Theorem 8). However, the study of the coflow (C) is independent of whether the initial condition is φ or $-\varphi$.

REMARK 2. By [6], since $\mathrm{Tr}(\tau(t)) = \frac{7}{4}\tau_0(t)$, as long as the condition $0 \leq \frac{7}{4}\tau_0(t) \leq \frac{4}{3}A$ holds for the time of existence, we have the following inequality for the volume

$$A \int_M \frac{7}{4}\,\tau_0(t)\,\mathrm{vol} \geq \int_M \frac{3}{4}\Big(\frac{7}{4}\,\tau_0(t)\Big)^2\,\mathrm{vol}.$$

4. Explicit solutions for the Laplacian coflow. In this section we study the Laplacian coflow on the seven dimensional Heisenberg Lie group H with structure equations (6).

Let φ_0 be a left invariant coclosed G_2-structure on H. Denote by g_0 the underling metric and by $\psi_0 = \star_0\varphi_0$ its Hodge dual.

Let $\eta = \|e^7\|_0^{-1}e^7$. Clearly $\|\eta\|_0 = 1$ and $d\eta \in \Lambda^2\mathrm{Ker}(\eta)^*$ is a non-degenerate two-form on $\mathrm{Ker}(\eta)$. Moreover, $\mathrm{Ker}(\eta)^* = Span\langle e^1, \dots, e^6\rangle$ and the 1-forms e^j, $j = 1, \dots, 6$, are all closed. If we identify $\mathfrak{h}$ with $\mathrm{Ker}(\eta) \oplus \mathfrak{z}$, being $\mathfrak{z} = [\mathfrak{h}, \mathfrak{h}] = Span\langle e_7\rangle$ the commutator of $\mathfrak{h}$, then every four-form $\psi \in \Lambda^4\mathfrak{h}^*$ has a unique decomposition as

$$\psi = \psi^{(4)} + \psi^{(3)} \wedge \eta, \tag{10}$$

where $\psi^{(i)} \in \Lambda^i\mathrm{Ker}(\eta)^*$, $i = 3, 4$, are closed forms.

Denote by $\star_0$ and $*_0$ the Hodge operators on $\mathfrak{h}$ and $\mathrm{Ker}(\eta)$, respectively. Note that the G_2-structure φ_0 defines an $\mathrm{SU}(3)$-structure (ω_0, ρ_0) on $\mathrm{Ker}(\eta)$. Using this fact, the four-form $\psi_0 = \star_0\varphi_0$ on $\mathfrak{h}$ can be written as

$$\psi_0 = \frac{1}{2}\omega_0^2 + \widehat{\rho}_0 \wedge \eta,$$

where $\widehat{\rho}_0 = J_0\rho_0$, and J_0 is the almost complex structure induced by (ω_0, ρ_0). Indeed, if $x_0 \in \mathfrak{h}$ is the vector defined by

$$g_0(x_0, y) = \eta(y),$$

for every $y \in \mathfrak{h}$, then

$$\mathrm{Ker}(\eta) = \{y \in \mathfrak{h} \mid g_0(x_0, y) = 0\} = \mathrm{Span}\langle x_0\rangle^{\perp_0},$$

and we can apply Proposition 4.5 in [17] to define the $\mathrm{SU}(3)$-structure (ω_0, ρ_0).

For a general $\mathrm{SU}(3)$-structure on a real vector space we have the following result.

LEMMA 3. *Let (ω, ρ) be a linear $\mathrm{SU}(3)-$structure on $\mathbb{R}^6$, and let $\alpha \in \Lambda^2(\mathbb{R}^6)^*$. Then the following inequalities hold*

1. $\|\alpha\|^2 + \|\frac{1}{2}\omega^2 \wedge \alpha\|^2 = \|\alpha \wedge \omega\|^2 \leq 4\|\alpha\|^2$;
2. $\|\alpha^3\|^2 \leq 6\|\alpha\|^6$,

where $\|\cdot\|$ is the norm induced by the scalar product defined by the $\mathrm{SU}(3)$-structure (ω, ρ).

Proof. Let us fix an orthonormal basis $\{e^1, \dots, e^6\}$ of $(\mathbb{R}^6)^*$ so that $\omega = e^{12} + e^{34} + e^{56}$, and write $\alpha = \sum_{1 \leq h<k\leq 6} a_{hk}e^{hk}$. Then,

$$\|\alpha\|^2 = \sum_{1\leq h<h\leq 6}^{6} a_{hk}^2. \tag{11}$$

On the other hand,

$$\begin{aligned}\omega\wedge\alpha &= e^{12}\wedge\left(a_{34}e^{34}+a_{35}e^{35}+a_{36}e^{36}+a_{45}e^{45}+a_{46}e^{46}+a_{56}e^{56}\right)\\ &\quad +e^{34}\wedge\left(a_{12}e^{12}+a_{15}e^{15}+a_{16}e^{16}+a_{25}e^{25}+a_{26}e^{26}+a_{56}e^{56}\right)\\ &\quad +e^{56}\wedge\left(a_{12}e^{12}+a_{13}e^{13}+a_{14}e^{14}+a_{23}e^{23}+a_{24}e^{24}+a_{34}e^{34}\right).\end{aligned}$$

Thus,

$$\|\omega\wedge\alpha\|^2=\|\alpha\|^2+(a_{12}+a_{34}+a_{56})^2=\|\alpha\|^2+\left\|\frac{1}{2}\,\omega^2\wedge\alpha\right\|^2. \tag{12}$$

Moreover,

$$\left\|\frac{1}{2}\omega^2\wedge\alpha\right\|^2=\|*(\omega)\wedge\alpha\|^2=(\omega|\alpha)^2\le\|\omega\|^2\|\alpha\|^2=3\|\alpha\|^2.$$

This equality together with (11) and (12) imply the first part of the Lemma.

To prove 2. note that the spectral theorem guarantees the existence of an orthonormal basis of 1-forms $\{f^1,\dots,f^6\}$ such that $\alpha=\lambda_1 f^{12}+\lambda_2 f^{34}+\lambda_3 f^{56}$, for some real numbers λ_i with $i=1,2,3$. Indeed, any real skew-symmetric matrix can be diagonalized by a unitary matrix. Since the eigenvalues of a real skew-symmetric matrix are imaginary, it is possible to transform it to a block diagonal form by an orthogonal transformation. Therefore,

$$\|\alpha\|^2=\lambda_1^2+\lambda_2^2+\lambda_3^2,$$

and

$$\alpha^3=6\lambda_1\lambda_2\lambda_3 f^{123456}.$$

Thus,

$$\|\alpha^3\|^2=36\lambda_1^2\lambda_2^2\lambda_3^2,$$

and 2. follows. □

THEOREM 4. *Let H be the seven dimensional Heisenberg group whose Lie algebra is defined by* (6). *Then, for any left invariant coclosed* G_2 *form* φ_0, *the solution* ϕ_t *of the Laplacian coflow* (1) *with initial condition* $\psi_0=\star_0\varphi_0$ *is given by*

$$\psi(t)=\frac{1}{2}\omega(t)^2+\widehat{\rho}(t)\wedge\frac{1}{\varepsilon_t}\,\eta,$$

where $6\,\varepsilon_t^2=*_0(\omega(t)^3)$, *and* $\omega(t)$ *and* $\widehat{\rho}(t)$ *are forms on* $\mathrm{Ker}(\eta)$, *given respectively by*

$$\omega(t)=\lambda_1(t)f^{12}+\lambda_2(t)f^{34}+\lambda_3(t)f^{56},$$

$$\widehat{\rho}(t)=\sqrt{\lambda_1(t)\lambda_2(t)\lambda_3(t)}\left(-f^{246}+f^{136}+f^{145}+f^{235}\right),$$

with respect to some g_0-*orthonormal frame* $\{f_1,\dots,f_6\}$ *of* $\mathrm{Ker}(\eta)$, *and the functions* $\lambda_i(t)$, $i=1,2,3$, *satisfy*

$$\begin{cases}\lambda_1'(t)=-\frac{\lambda_2(t)\lambda_3(t)+n_2n_3\lambda_1^2(t)}{\lambda_1(t)^2\lambda_2(t)^2\lambda_3(t)^2},\\ \lambda_2'(t)=-\frac{\lambda_1(t)\lambda_3(t)+n_1n_3\lambda_2^2(t)}{\lambda_1(t)^2\lambda_2(t)^2\lambda_3(t)^2},\\ \lambda_3'(t)=-\frac{\lambda_1(t)\lambda_2(t)+n_1n_2\lambda_3^2(t)}{\lambda_1(t)^2\lambda_2(t)^2\lambda_3(t)^2},\\ \lambda_1(0)=\omega(0)(f_1,f_2),\ \lambda_2(0)=\omega(0)(f_3,f_4),\ \lambda_3(0)=\omega(0)(f_5,f_6),\end{cases} \tag{13}$$

with $n_j \in \{1, -1\}$*. In particular, the solution is ancient with singular time* $0 < T < \frac{4}{\sqrt[3]{6\,||d\eta||_0^2}}$.

Proof. We are going to show that the system (1) turns out to be equivalent to the system of ODEs given by (13). But first let us observe that the initial $\psi_0 = \star_0\varphi_0$ is H-invariant and the system (1) is invariant by diffeomorphisms, whence H-invariant too, and therefore the system (1) reduces to a system of ODEs on $\Lambda^4\mathfrak{h}^*$. This ensures the existence of a unique H-invariant solution ψ_t of (1) for short times. Now let ε_t be the norm $\|\eta\|_t$ of η with respect to the metric induced by $\psi(t)$. We can write

$$\psi(t) = \frac{1}{2}\omega(t)^2 + \widehat{\rho}(t) \wedge \frac{1}{\varepsilon_t}\eta,$$

where the pair $(\omega(t), \rho(t))$ defines an SU(3)−structure on Ker(η) and $\widehat{\rho}(t) = J_t\rho(t)$. In fact, if $x_t \in \mathfrak{h}$ is the vector defined by $g_t(x_t, y) = \eta(y)$, for any $y \in \mathfrak{h}$, then $\mathrm{Ker}(\eta) = \{y \in \mathfrak{h} \,|\, g_t(x_t, y) = 0\}$ is the orthogonal complement of the span of x_t with respect to g_t. Thus we can apply Proposition 4.5 in [17].

With respect to the decomposition (10) we have

$$\psi(t) = \psi^{(4)}(t) + \psi^{(3)}(t) \wedge \eta,$$

so,

$$\psi^{(4)}(t) = \frac{1}{2}\omega(t)^2, \quad \psi^{(3)}(t) = \frac{1}{\varepsilon_t}\widehat{\rho}(t).$$

Moreover, the forms $\omega(t) \in \Lambda^2\mathrm{Ker}(\eta)^*$ and $\widehat{\rho}(t) \in \Lambda^3\mathrm{Ker}(\eta)^*$ are closed. Since $\frac{d}{dt}\psi(t)$ is exact, the cohomology class of $\psi(t)$ is fixed by the flow, and hence

$$\frac{d}{dt}\psi(t) = \frac{d}{dt}\psi^{(4)}(t) + \frac{d}{dt}\psi^{(3)}(t) \wedge \eta \in d\Lambda^3\mathfrak{h}^* \subseteq \Lambda^4\mathrm{Ker}(\eta)^*.$$

Therefore, $\frac{d}{dt}\psi^{(3)}(t) = 0$ and

$$\widehat{\rho}(t) = \varepsilon_t\psi^{(3)}(0) = \varepsilon_t\widehat{\rho}_0.$$

Consequently, the almost complex structure J_t defined by $\rho(t)$ does not change along the flow, i.e. $J_t \equiv J_0$, where J_0 is the almost complex structure defined by ρ_0. Thus $\rho(t) = -J_0\widehat{\rho}(t) = \varepsilon_t\rho_0$ and

$$\frac{1}{6}\omega(t)^3 = \frac{1}{4}\rho(t) \wedge \widehat{\rho}(t) = \varepsilon_t^2 *_0 (1), \tag{14}$$

where in the first equality we used the fact that $(\omega(t), \rho(t))$ defines an SU(3)-structure on Ker(η).

Now let us compute the Laplacian of $\psi(t)$ with respect to the metric g_t:

$$\begin{aligned} d \star_t d \star_t \psi(t) &= d \star_t d \star_t (\tfrac{1}{2}\omega(t)^2 + \widehat{\rho}(t) \wedge \varepsilon_t^{-1}\eta) \\ &= d \star_t d(\omega(t) \wedge \varepsilon_t^{-1}\eta + \rho(t)) \\ &= d \star_t (\varepsilon_t^{-1}\omega(t) \wedge d\eta) \\ &= d\left(\varepsilon_t^{-2} *_t (\omega(t) \wedge d\eta) \wedge \eta\right) \\ &= \varepsilon_t^{-2} *_t (\omega(t) \wedge d\eta) \wedge d\eta. \end{aligned}$$

On the other hand we have

$$\frac{d}{dt}\psi_t = \frac{d}{dt}\left(\frac{1}{2}\omega(t)^2\right).$$

Thus, by $\frac{d}{dt}\psi(t) = -\Delta_t\psi(t)$ we obtain

$$\frac{1}{2}\left(\frac{d}{dt}\omega(t)^2\right) = -\varepsilon_t^{-2} *_t (\omega(t)\wedge d\eta)\wedge d\eta. \tag{15}$$

We observe that, being $d\psi(t) = 0$,

$$0 = \varepsilon_t d\psi(t) = d\left(\varepsilon_t\frac{1}{2}\omega(t)^2 + \hat\rho(t)\wedge\eta\right) = \widehat{\rho}(t)\wedge d\eta.$$

Since $\hat\rho$ is the imaginary part of a $(3,0)$-form, η must be of type $(1,1)$ and hence it is J_0-invariant, i.e. $J_0(d\eta) = d\eta$.

Fixing a frame $(x_1,\dots,x_6)$ of $\mathrm{Ker}(\eta)$ and using $*_t\omega(t) = \frac{1}{2}\omega(t)^2$ we get

$$\begin{aligned}
*_t(d\eta\wedge\omega(t)) &= \textstyle\sum_{1\le i<j\le 6} *_t((d\eta)_{ij}x^{ij}\wedge\omega(t))\\
&= -\textstyle\sum_{1\le i<j\le 6}(d\eta)^{ij}x_i\lrcorner x_j\lrcorner *_t\omega(t)\\
&= -\textstyle\sum_{1\le i<j\le 6}(d\eta)^{ij}x_i\lrcorner x_j\lrcorner \frac{1}{2}\omega(t)^2\\
&= -\textstyle\sum_{1\le i<j\le 6}(d\eta)^{ij}x_i\lrcorner((x_j\lrcorner\omega(t))\wedge\omega(t)).
\end{aligned}$$

Moreover

$$\begin{aligned}
x_i\lrcorner((x_j\lrcorner\omega(t))\wedge\omega(t)) &= \omega(t)(x_j,x_i)\omega(t) - (x_j\lrcorner\omega(t))\wedge(x_i\lrcorner\omega(t))\\
&= -\omega(t)_{ij}\omega(t) + (x_i\lrcorner\omega(t))\wedge(x_j\lrcorner\omega(t)).
\end{aligned}$$

Now, taking into account the fundamental relation $\omega(t)(x,y) = g_t(x,J_0y) = -[(J_0)^*(x\lrcorner g)](y)$, we have

$$x_i\lrcorner\omega(t) = -(J_0)^*\left(\sum_m g_{im}(t)x^m\right),\quad x_j\lrcorner\omega(t) = -(J_0)^*\left(\sum_n g_{jn}(t)x^n\right).$$

Therefore,

$$\begin{aligned}
*_t(d\eta\wedge\omega(t)) &= \textstyle\sum_{1\le i<j\le 6}(d\eta)^{ij}\left[\omega(t)_{ij}\,\omega(t) - x_i\lrcorner\omega(t)\wedge x_j\lrcorner\omega(t)\right]\\
&= \textstyle\sum_{1\le i<j\le 6}(d\eta)^{ij}\left[\omega_{ij}(t)\,\omega(t) - \sum_{m,n=1}^6 g_{im}(J_0^*x^m)\wedge g_{jn}(J_0^*x^n)\right]\\
&= \textstyle\sum_{1\le i<j\le 6}(d\eta)^{ij}\omega_{ij}(t)\,\omega(t) - \sum_{m,n=1}^6(d\eta)_{mn}J_0^*x^m\wedge J_0^*x^n\\
&= (d\eta,\omega(t))_t\,\omega(t) - J_0^*(d\eta)\\
&= (d\eta,\omega(t))_t\,\omega(t) - d\eta,
\end{aligned}$$

where $(\cdot,\cdot)_t$ denotes the scalar product induced by g_t and $\lrcorner$ is the contraction.

Therefore we can reformulate (15) as

$$\frac{d}{dt}\left(\frac{1}{2}\omega(t)^2\right) = -\varepsilon_t^{-2}\left[(d\eta,\omega(t))_t\,\omega(t)\wedge d\eta - d\eta\wedge d\eta\right], \tag{16}$$

with $\omega(t)\in\Lambda^2\mathrm{Ker}(\eta)^*$. Define on $\mathrm{Ker}(\eta)$ the following bilinear form

$$h(x,y) = d\eta(x,J_0y),\quad x,y\in\mathrm{Ker}(\eta).$$

Since $J_0(d\eta) = d\eta$, we have that h is symmetric. If we consider $\mathrm{Ker}(\eta)$ as complex vector space through J_0, then both g_0 and h define on $\mathrm{Ker}(\eta)$ sesquilinear forms g^c and h^c, respectively. Clearly g^c is positive definite while h^c is non-degenerate with mixed signature. By the spectral theorem there exists a complex g^c-orthonormal basis $\{k_1, k_2, k_3\}$ of the complex vector space $\mathrm{Ker}(\eta)$ such that

$$h^c(k_i, k_j) = \delta_{ij}\frac{n_i}{l_i}, \quad n_i \in \{1, -1\}\,,\ l_i > 0.$$

Therefore, if $\left\{k^1, k^2, k^3\right\}$ is the dual basis of $\{k_1, k_2, k_3\}$, putting $f^i = \sqrt{l_i}\, k^i$, for $i = 1, 2, 3$, we get

$$d\eta = \sum_{i=1}^{3} n_i f^i \wedge J_0 f^i, \quad \omega_0 = \sum_{i=1}^{3} l_i f^i \wedge J_0 f^i.$$

In order to find $\omega(t)$, let us suppose that it is given by

$$\omega(t) = \sum_{i=1}^{3} \lambda_i(t) f^i \wedge J_0 f^i, \tag{17}$$

where $\lambda_1(t), \lambda_2(t)$ and $\lambda_3(t)$ are positive functions such that $\lambda_j(0) = l_j$. Then $\omega(t)$ is a non-degenerate 2-form of type $(1, 1)$ with respect to J_0, and

$$\frac{d}{dt}\left(\frac{1}{2}\omega(t)^2\right) = \sum_{i<j}\left\{\lambda_i'(t)\lambda_j(t) + \lambda_i(t)\lambda_j'(t)\right\} f^i \wedge J_0 f^i \wedge f^j \wedge J_0 f^j.$$

Using (14) and (17) we have

$$\varepsilon_t^2 = \frac{1}{6} *_0 \omega(t)^3 = \lambda_1(t)\lambda_2(t)\lambda_3(t).$$

Now, from the equation (16) and $h^c(f_i, f_j) = \delta_{ij} n_i$ we get

$$\sum_{i<j}\left\{\lambda_i'(t)\lambda_j(t) + \lambda_i(t)\lambda_j'(t)\right\} f^i \wedge J_0 f^i \wedge f^j \wedge J_0 f^j =$$
$$-\frac{1}{\varepsilon_t^2}\sum_{i<j}\left[\left(\frac{n_1}{\lambda_1(t)} + \frac{n_2}{\lambda_2(t)} + \frac{n_3}{\lambda_3(t)}\right)(n_i\lambda_j(t) + n_j\lambda_i(t))) - 2n_i n_j\right] f^i \wedge J_0 f^i \wedge f^j \wedge J_0 f^j.$$

This is the system of ordinary differential equations given by (13). This system does indeed have a unique solution, which, in turn, defines a non-degenerate 2-form compatible with J_0 by (17). Such a form has to satisfy (16) by construction. Therefore, we find out that the solution of (1) is given by

$$\psi(t) = \frac{1}{2}\omega(t)^2 + \widehat{\rho}(t) \wedge \frac{1}{\varepsilon_t}\,\eta,$$

where $\widehat{\rho}(t) = \varepsilon_t\widehat{\rho}(0)$ and $\omega(t)$ as in (17), with the functions $\lambda_i(t)$, $i = 1, 2, 3$, solving (13).

Let (τ, T) be the maximal interval of existence of $\psi(t)$, where $-\tau, T \in [-\infty, +\infty]$. We want to prove that $T < +\infty$ and $\tau = -\infty$. Computing the derivative of $6\,\varepsilon_t^2 = *_0(\omega(t)^3)$ we get

$$12\,\varepsilon_t\varepsilon_t' = 3 *_0\left(\frac{d}{dt}\omega(t) \wedge \omega(t)^2\right).$$

Then,

$$
\begin{aligned}
\varepsilon_t' &= -\tfrac{\varepsilon_0^2}{4\varepsilon_t^3} *_0 \left[*_t\left(\omega(t)\wedge d\eta\right)\wedge\omega(t)\wedge d\eta\right]\\
&= -\tfrac{\varepsilon_0^2}{4\varepsilon_t^3} *_0 \left(\|\omega(t)\wedge d\eta\|_t^2 *_t(1)\right)\\
&= -\tfrac{1}{4\varepsilon_t^3} *_0 \left(\|\omega(t)\wedge d\eta\|_t^2\,\varepsilon_t^2\, *_0(1)\right)\\
&= -\tfrac{1}{4\varepsilon_t}\|\omega(t)\wedge d\eta\|_t^2.
\end{aligned}
$$

This implies that $\varepsilon_t' < 0$ and also the existence of $\lim_{t\to T}\varepsilon_t = \varepsilon_T \geq 0$. Note that $\varepsilon_t^{-1}\eta$ is the unit vector orthogonal to $\mathrm{Ker}(\eta)$ such that $\star_t(1) = *_t(1)\wedge\left(\varepsilon_t^{-1}\eta\right)$, thus

$$
*_t(1) = \star_t\left(\frac{1}{\varepsilon_t}\eta\right), \tag{18}
$$

where $\star_t$ is the Hodge star operator with respect to the metric induced by $\psi(t)$. Moreover, the volume form $*_0(1)$ is proportional to the 6-form $d\eta^3$, since both are non-zero 6-forms on $\mathrm{Ker}(\eta)$. Therefore, we can write

$$
*_0(1) = \frac{1}{6\delta_0}(d\eta)^3, \tag{19}
$$

where

$$
\delta_0 = \frac{1}{6}\|(d\eta)^3\|_0.
$$

Thus, using (18), (19) and $*_t(1) = \frac{1}{6}\omega(t)^3 = \varepsilon_t^2 *_0(1)$, we obtain

$$
\star_t\left(\frac{1}{\varepsilon_t}\eta\right) = *_t(1) = \varepsilon_t^2 *_0(1) = \frac{1}{6}\frac{\varepsilon_t^2}{\delta_0}(d\eta)^3,
$$

that is

$$
\star_t\left(\varepsilon_t^{-1}\eta\right) = \frac{1}{6}\frac{\varepsilon_t^2}{\delta_0}d\eta^3.
$$

Taking the square norm of the previous expression gives

$$
1 = \|\star_t\left(\varepsilon_t^{-1}\eta\right)\|_t^2 = \left\|\frac{1}{6}\frac{\varepsilon_t^2}{\delta_0}d\eta^3\right\|_t^2,
$$

whence

$$
\|d\eta^3\|_t^2 = 36\frac{\delta_0^2}{\varepsilon_t^4}.
$$

Now by Lemma 3 we have

$$
36\frac{\delta_0^2}{\varepsilon_t^4} = \|d\eta^3\|_t^2 \leq 6\|d\eta\|_t^6,
$$

and

$$
\|\omega(t)\wedge d\eta\|_t^2 \geq \|d\eta\|_t^2.
$$

Therefore, we can estimate ε_t' as follows

$$\varepsilon_t' = -\frac{1}{4\varepsilon_t}\|\omega(t)\wedge d\eta\|_t^2 \le -\frac{1}{4\varepsilon_t}\|d\eta\|_t^2 \le -\frac{1}{4\varepsilon_t}\left(\frac{1}{6}\|(d\eta)^3\|_t^2\right)^{\frac{1}{3}} = -\frac{\sqrt[3]{6\,\delta_0^2}}{4\varepsilon_t\sqrt[3]{\varepsilon_t^4}} = -\frac{C}{\varepsilon_t^{1+\frac{4}{3}}},$$

with $C = \frac{\sqrt[3]{6\,\delta_0^2}}{4}$. As a consequence, if $t \in (0, T)$, we get

$$\varepsilon_t - 1 = \int_0^t \varepsilon_s' ds \le -C\int_0^t \frac{1}{\varepsilon_s^{1+\frac{4}{3}}} ds \ \Rightarrow\ t = \int_0^t ds \le \int_0^t \frac{1}{\varepsilon_s^{1+\frac{4}{3}}} ds \le \frac{1-\varepsilon_t}{C},$$

where we have used that $\varepsilon_s < 1$ if $s \in (0, t)$. So

$$T \le \frac{1-\varepsilon_T}{C} = 4\frac{1-\varepsilon_T}{\sqrt[3]{6\,\delta_0^2}} \le \frac{4}{\sqrt[3]{6\,\delta_0^2}}.$$

It remains to show that $\tau = -\infty$. Firstly, we prove that it is true if h is positive or negative definite. Note that in this case $n_i n_j = 1$, for every $i, j = 1, 2, 3$. So $\lambda_i'(t) < 0$, for any $i = 1, 2, 3$. Define

$$f(t) = \lambda_1(t) + \lambda_2(t) + \lambda_3(t).$$

Then it is clear that the solution exists as long as $f(t) < +\infty$ and, consequently, $f(\tau) = +\infty$. Now observe that by (13)

$$\begin{aligned} f''(t) &= -\tfrac{d}{dt}\left(\textstyle\sum_{a,b,c} \frac{\lambda_a^2+\lambda_b\lambda_c(t)}{\lambda_1^2\lambda_2^2\lambda_3^2}\right) \\ &= -\textstyle\sum_{a,b,c} \frac{(2\lambda_a\lambda_a'+\lambda_b'\lambda_c+\lambda_b\lambda_c')\lambda_1^2\lambda_2^2\lambda_3^2 - 2\sum_{i,j,k}(\lambda_i\lambda_i'(t)\lambda_j^2\lambda_k^2)(\lambda_a^2+\lambda_b\lambda_c)}{\lambda_1^4\lambda_2^4\lambda_3^4}, \end{aligned}$$

where (a, b, c) and $(i, j, k) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$. But, from (13), for any (a, b, c), it follows

$$2\lambda_a\lambda_a'\lambda_1^2\lambda_2^2\lambda_3^2 - 2(\lambda_a\lambda_a'\lambda_b^2\lambda_c^2)\lambda_a^2 = 0,$$

and

$$\begin{aligned} \lambda_b'\lambda_c\lambda_1^2\lambda_2^2\lambda_3^2 - 2(\lambda_b\lambda_b'\lambda_a^2\lambda_c^2)\lambda_b\lambda_c &= \lambda_b'\lambda_1^2\lambda_2^2\lambda_3^2(\lambda_c - 2\lambda_c) > 0, \\ \lambda_c'\lambda_b\lambda_1^2\lambda_2^2\lambda_3^2 - 2(\lambda_c\lambda_c'\lambda_a^2\lambda_b^2)\lambda_b\lambda_c &= \lambda_c'\lambda_1^2\lambda_2^2\lambda_3^2(\lambda_b - 2\lambda_b) > 0. \end{aligned}$$

Therefore $f''(t) < 0$, for $t \in (\tau, T)$. But, for $t \in (\tau, 0)$,

$$f(0) - f(t) = \int_t^0 f'(s)ds \ge \int_t^0 f'(0)ds = -tf'(0).$$

Thus,

$$f(t) \le f(0) + tf'(0),$$

which means $\tau = -\infty$.

In order to prove that $\tau = -\infty$ if h is indefinite, we proceed by contradiction as follows. Suppose by contradiction that $\tau > -\infty$ and that $\lambda_1(t), \lambda_2(t)$ and $\lambda_3(t)$ are all bounded near τ. Then, we can find a sequence $t_n \to \tau$ for which all $\lambda_i(t_n)$ converge. If the limits of $\lambda_i(t_n)$ are non-zero we can restart the flow past τ, contradicting the maximality of the solution. Therefore, if $\tau > -\infty$, at least one of the $\lambda_i(t_n)$ has to go to zero for $t_n \to \tau$. Since $\lambda_1(t)\lambda_2(t)\lambda_3(t) = 1/6 *_0 (\omega_t^3) = \varepsilon_t^2$ decreases, we get also a contradiction. Indeed,

$$0 = \lambda_1(\tau)\lambda_2(\tau)\lambda_3(\tau) = \lim_{t\to\tau} \varepsilon_t^2 \geq \epsilon_0^2 > 0.$$

Therefore, if $\tau > -\infty$, there is at least one $\lambda_i(t)$ $(i = 1, 2, 3)$ which is unbounded. Suppose now that $\lambda_2(t)$ is unbounded. Then, choosing a sequence of negative times $\{t_n\}$ converging to τ and such that $\lambda_2(t_n) - \lambda_2(t_{n-1})$ diverges, it follows that

$$\begin{aligned} \lambda_2(t_n) - \lambda_2(t_{n-1}) &= -\int_{t_{n-1}}^{t_n} \frac{\lambda_1(s)\lambda_3(s)-\lambda_2(s)^2}{\lambda_1(s)^2\lambda_2(s)^2\lambda_3(s)^2} ds \\ &= -\int_{t_{n-1}}^{t_n} \left(\frac{1}{\lambda_1(s)\lambda_3(s)\lambda_2(s)^2} - \frac{1}{\lambda_1(s)^2\lambda_3(s)^2}\right) ds \\ &= \left(-\frac{1}{\varepsilon^2(\bar t_n)\lambda_2(\bar t_n)} + \frac{1}{\lambda_1^2(\bar t_n)\lambda_3^2(\bar t_n)}\right)(t_n - t_{n-1}) \to +\infty, \end{aligned}$$

where $\bar t_n \in (t_n, t_{n-1})$. Hence $\lambda_1(\bar t_n)\lambda_3(\bar t_n) \to 0$. Indeed $\varepsilon(\bar t_n)^2\lambda_2(\bar t_n)$ stays away from zero. But for n large we get a contradiction

$$0 > \frac{\lambda_2(t_n) - \lambda_2(t_{n-1})}{t_n - t_{n-1}} = -\frac{\lambda_1(\bar t_n)\lambda_3(\bar t_n) - \lambda_2^2(\bar t_n)}{\lambda_1^2(\bar t_n)\lambda_2^2(\bar t_n)\lambda_3^2(\bar t_n)} > 0.$$

Thus $\lambda_2(t)$ must be bounded. The same argument shows that $\lambda_3(t)$ must be bounded as well. So, the only possibility is that $\lambda_1(t)$ is unbounded whereas both $\lambda_2(t)$ and $\lambda_3(t)$ are bounded. As done previously choose $\{t_n\}$ so that $t_n \to \tau$ and

$$\lambda_1(t_n) - \lambda_1(t_{n-1}) = -\int_{t_{n-1}}^{t_n} \frac{\lambda_2(s)\lambda_3(s) + \lambda_1(s)^2}{\lambda_1(s)^2\lambda_2(s)^2\lambda_3(s)^2} ds \to +\infty.$$

Then there exists $\bar t_n \in (t_n, t_{n-1})$ such that $\lambda_2(\bar t_n)\lambda_3(\bar t_n) \to 0$. We can certainly assume that $\lambda_2(\bar t_n)\lambda_3(\bar t_n)$ decreases in n by choosing a suitable subsequence which we will still denote by t_n. Then,

$$0 > \lambda_2(t_n)\lambda_3(t_n) - \lambda_2(t_{n-1}\lambda_3)(t_{n-1}) = \left(\frac{d}{dt}(\lambda_2\lambda_3)\right)(s_n)(t_n - t_{n-1}),$$

for some $s_n \in (t_n, t_{n-1})$. On the other hand, by (13), it turns out that

$$\frac{d}{dt}\left(\lambda_2(t)\lambda_3(t)\right) = -\frac{\lambda_1(t)(\lambda_2(t)^2 + \lambda_3(t)^2) - \lambda_2(t)\lambda_3(t)(\lambda_2(t) + \lambda_3(t))}{\lambda_1(t)^2\lambda_2(t)^2\lambda_3(t)^2}.$$

Since

$$\lambda_1(t_n) \to +\infty, \quad \frac{\lambda_2(t_n)\lambda_3(t_n)(\lambda_2(t_n) + \lambda_3(t_n))}{\lambda_2^2(t_n) + \lambda_3^2(t_n)} \to 0,$$

we obtain that $\frac{d}{dt}\left(\lambda_2\lambda_3\right)(s_n) < 0$, for n large. Then we get the following contradiction:

$$0 > \lambda_2\lambda_3(t_n) - \lambda_2\lambda_3(t_{n-1}) = \left(\frac{d}{dt}\lambda_2\lambda_3\right)(s_n)(t_n - t_{n-1}) > 0.$$

Thus also $\lambda_1(t)$ must be bounded. But we have already proved that, assuming $\tau > -\infty$, at least one λ must be unbounded. To avoid any contradiction it must be $\tau = -\infty$. This completes the proof. □

We now solve the coflow (1) on the 7-dimensional Heisenberg group when the initial coclosed G_2 form is equal to φ_i $(i = 1, 2)$, where φ_1 and φ_2 are defined by

$$\varphi_1 = e^{127} + e^{347} + e^{567} + e^{135} - e^{146} - e^{236} - e^{245}. \tag{20}$$

and

$$\varphi_2 = e^{127} - e^{347} - e^{567} + e^{135} - e^{146} + e^{236} + e^{245}, \tag{21}$$

respectively. Note that φ_1 and φ_2 induce the same metric and orientation, namely they are SO(7)-equivalent via the special orthogonal transformation

$$R = \mathrm{diag}(1, 1, 1, -1, -1, -1, 1).$$

Moreover, their dual 4-forms are given respectively by the closed forms

$$\star\varphi_1 = e^{1234} + e^{1256} + e^{3456} - e^{2467} + e^{1367} + e^{1457} + e^{2357}$$

and

$$\star\varphi_2 = -e^{1234} - e^{1256} + e^{3456} - e^{2467} - e^{1367} - e^{1457} + e^{2357}.$$

We will show in the next section that the behavour of the solution for the modified coflow is different.

Corollary 5. *The solution of the Laplacian coflow* (1) *on* H *with the initial coclosed* G_2 *form* φ_1, *defined by* (20), *is given by*

$$\varphi(t) = \frac{1}{y(t)}\,(e^{127} + e^{347} + e^{567}) + y(t)^3\,(e^{135} - e^{146} - e^{236} - e^{245}), \quad t \in \left(-\infty, \frac{3}{5}\right), \tag{22}$$

where $y = y(t)$ *is the positive function*

$$y(t) = \sqrt[10]{1 - \frac{5}{3}t}. \tag{23}$$

The underlying metrics g_t *of this solution converge smoothly, up to pull-back by time-dependent diffeomorphisms, to a flat metric, uniformly on compact sets in* H *as* t *goes to* $-\infty$.

Proof. For each $t \in (-\infty, \frac{3}{5})$, we consider the basis $\{f^1(t), \dots, f^7(t)\}$ of left invariant 1-forms on H defined by

$$\begin{aligned} f^i &= f^i(t) = y(t)\,e^i, \qquad 1 \le i \le 6,\\ f^7 &= f^7(t) = y(t)^{-3}\,e^7, \end{aligned} \tag{24}$$

where the function $y = y(t)$ is given by (23). Then, $f^i(0) = e^i$, for $i \in \{1, \cdots, 7\}$, and the structure equations of H, with respect to the basis $\{f^1(t), \dots, f^7(t)\}$, are

$$df^i = 0, \quad 1 \le i \le 6, \qquad df^7 = \frac{\sqrt{6}}{6}\,y(t)^{-5}(f^{12} + f^{34} + f^{56}). \tag{25}$$

Now, for any t, the 3-form $\varphi(t)$ defined by (22) has the following expression

$$\varphi(t) = f^{127} + f^{347} + f^{567} + f^{135} - f^{146} - f^{236} - f^{245}. \tag{26}$$

Note that $\varphi(0) = \varphi_1$ and, for any t, the 3-form $\varphi(t)$ on H induces the metric g_t such that the coframe $\{f^1(t), \dots, f^7(t)\}$ of $\mathfrak{h}^*$ is orthonormal. Denote by $\star_t$ the Hodge star operator determined by g_t. Using (4), (5) and (25), we have $d \star_t \varphi(t) = 0$, where the 4-form

$$\star_t \varphi(t) = f^{1234} + f^{1256} + f^{1367} + f^{1457} + f^{2357} - f^{2467} + f^{3456}.$$

So, in terms of the coframe $\{e^1, \dots, e^7\}$ of $\mathfrak{h}^*$, $\star_t \varphi(t)$ has the following expression

$$\star_t \varphi(t) = y(t)^4 (e^{1234} + e^{1256} + e^{3456}) + e^{1367} + e^{1457} + e^{2357} - e^{2467}.$$

Thus,

$$\frac{d}{dt} (\star_t \varphi(t)) = 4y(t)^3\, y'(t)\, (e^{1234} + e^{1256} + e^{3456}). \tag{27}$$

Moreover, using (25) and (26), we have

$$\begin{aligned} -\Delta_t \star_t \varphi(t) &= -d \star_t d\varphi(t) = -\tfrac{\sqrt{6}}{3} y(t)^{-5}\, d \star_t \left(f^{1234} + f^{1256} + f^{3456}\right) \\ &= -\tfrac{2}{3} y(t)^{-10} \left(f^{1234} + f^{1256} + f^{3456}\right), \end{aligned}$$

or, equivalently,

$$-\Delta_t \star_t \varphi(t) = -\tfrac{2}{3} y(t)^{-6} \left(e^{1234} + e^{1256} + e^{3456}\right).$$

The last equality and (27) prove that (22) is the solution of the coflow (1) when the function $y = y(t)$ is given by (23).

We study the behavior of the underlying metric g_t of the solution $\varphi(t)$ in the limit for $t \to -\infty$. The limit can be computed fixing the G_2-structure and changing the Lie bracket as in [12]. If we evolve the Lie brackets $\mu(t)$ instead of the 3-form defining the G_2-structure, the corresponding bracket flow has a solution for every t. Indeed, if we fix on $\mathbb{R}^7$ the 3-form $f^{127} + f^{347} + f^{567} + f^{135} - f^{146} - f^{236} - f^{245}$, then the basis $(f_1(t), \dots, f_7(t))$ defines, for every $t < 3/5$, a nilpotent Lie algebra with bracket $\mu(t)$ such that $\mu(0)$ is the Lie bracket of $\mathfrak{h}$. Moreover, the solution converges to the null bracket corresponding to the abelian Lie algebra. For this, let $\{f_1(t), \dots, f_7(t)\}$ be the basis dual to $\{f^1(t), \dots, f^7(t)\}$ (defined by (24)). Then, the equations (25) imply that all the Lie brackets $[f_i(t), f_j(t)]$ $(1 \le i \le j \le 7)$ vanish excepting

$$[f_1(t), f_2(t)] = [f_3(t), f_4(t)] = [f_5(t), f_6(t)] = -\frac{\sqrt{6}}{6}\, y(t)^{-5} f_7(t).$$

Thus, all the Lie brackets $[f_i(t), f_j(t)]$ tend to zero as t goes to $-\infty$. □

In a similar way we can prove the following

Corollary 6. *The solution of the Laplacian coflow* (1) *on H with initial coclosed G_2 form φ_2, defined by* (21)*, is ancient and it is given by*

$$\varphi(t) = \frac{y(t)}{z(t)^2}\, e^{127} - \frac{1}{y(t)}\, e^{347} - \frac{1}{y(t)}\, e^{567} + y(t) z(t)^2 \left(e^{135} - e^{146} + e^{236} + e^{245}\right), \tag{28}$$

where the functions $y = y(t)$ and $z = z(t)$ satisfy

$$\begin{cases} \frac{d}{dt} y(t) = -\frac{1}{12} \frac{y(t)^4 + z(t)^4}{y(t)^5 z(t)^8}, & \frac{d}{dt} z(t) = \frac{1}{12} \frac{z(t)^2 - y(t)^2}{y(t)^4 z(t)^7}, \\ y(0) = 1, & z(0) = 1. \end{cases} \tag{29}$$

5. Explicit solutions for the modified Laplacian coflow. We study the modified Laplacian coflow (2) for each of the coclosed G_2 forms φ_i, $i = 1, 2$, defined respectively by (20) and (21), on the 7-dimensional Heisenberg group. In particular, we prove that the solution of (2) for φ_1 is ancient only if the positive constant A, that appears in (2), take values in a certain open interval, while the solution of (2) for $-\varphi_1$ is ancient for any A. However, we prove that the solution of (2) for φ_2 is never ancient.

THEOREM 7. *The solution of the modified Laplacian coflow* (2) *for the coclosed* G_2 *form* φ_1*, defined by* (20)*, is given by*

$$\varphi(t) = \frac{1}{y(t)}\left(e^{127} + e^{347} + e^{567}\right) + y(t)^3\left(e^{135} - e^{146} - e^{236} - e^{245}\right), \tag{30}$$

where the function $y = y(t)$ *satisfies*

$$\begin{cases} \frac{d}{dt}y(t) = \frac{2A\sqrt{6}\,y(t)^5 - 1}{12\,y(t)^9}, \\ y(0) = 1. \end{cases} \tag{31}$$

Moreover,

i) *if* $0 < A < \frac{1}{2\sqrt{6}}$*, then* $t \in (-\infty, T)$*, with* $T = -\frac{1}{10\,A^2}\Big(2\sqrt{6}\,A + \log\big(1 - 2\sqrt{6}\,A\big)\Big) > 0$*. Therefore, in this case, the solution* (30) *is ancient;*

ii) *if* $A \geq \frac{1}{2\sqrt{6}}$*, then* $t \in (-\infty, +\infty)$*, that is, the solution* (30) *is eternal.*

Proof. By the Picard-Lindelöf Theorem, there exists a maximal open interval I, containing 0, and a smooth function $y : I \to (0, +\infty)$, which is the unique solution of (31).

To prove that (30) is the solution to the coflow (2) for φ_1, we proceed as follows. As in the proof of Theorem 5 , for each $t \in I$, we consider the basis $\{f^1(t), \dots, f^7(t)\}$ of left invariant 1-forms on H defined by

$$\begin{aligned} f^i &= f^i(t) = y(t)\,e^i, \quad i = 1, \dots, 6, \\ f^7 &= f^7(t) = y(t)^{-3}\,e^7, \end{aligned}$$

where the function $y = y(t)$ now satisfies (31). Then, $f^i(0) = e^i$, for $i \in \{1, \cdots, 7\}$, and the structure equations of H, with respect to the basis $\{f^1(t), \dots, f^7(t)\}$, are

$$df^i = 0, \quad 1 \leq i \leq 6, \qquad df^7 = \frac{\sqrt{6}}{6}\,y^{-5}(t)(f^{12} + f^{34} + f^{56}). \tag{32}$$

Moreover, for any $t \in I$, the 3-form $\varphi(t)$ defined by (30) has the following expression

$$\varphi(t) = f^{127} + f^{347} + f^{567} + f^{135} - f^{146} - f^{236} - f^{245}. \tag{33}$$

So, $\varphi(0) = \varphi_1$ and, for any $t \in I$, the 3-form $\varphi(t)$ on H induces the metric g_t such that the coframe $\{f^1(t), \dots, f^7(t)\}$ of $\mathfrak{h}^*$ is orthonormal. Denote by $\star_t$ the Hodge star operator determined by g_t. Using (4), (5) and (32), we have $d \star_t \varphi(t) = 0$, where $\star_t\varphi(t)$ is given by

$$\star_t\varphi(t) = f^{1234} + f^{1256} + f^{1367} + f^{1457} + f^{2357} - f^{2467} + f^{3456}.$$

Thus, in terms of the coframe $\{e^1, \dots, e^7\}$ of $\mathfrak{h}^*$, the 4-form $\star_t\varphi(t)$ has the following expression

$$\star_t\varphi(t) = y(t)^4(e^{1234} + e^{1256} + e^{3456}) + e^{1367} + e^{1457} + e^{2357} - e^{2467}.$$

This implies

$$\frac{d}{dt}\star_t \varphi(t) = 4\, y(t)^3\, y'(t)(e^{1234} + e^{1256} + e^{3456}),$$

that is

$$\frac{d}{dt}\star_t \varphi(t) = \frac{2A\sqrt{6}\, y(t)^5 - 1}{3\, y(t)^6}(e^{1234} + e^{1256} + e^{3456}), \tag{34}$$

since the function $y = y(t)$ satisfies (31).

On the other hand, by (7) we know that the torsion forms $\tau_i(t)$ $(i = 0, 1, 2, 3)$ of $\varphi(t)$ are such that $\tau_1(t) = 0 = \tau_2(t)$ since $d(\star_t\varphi(t)) = 0$. Then, from (32), (33) and (7), we have

$$d\varphi(t) = \frac{\sqrt{6}}{3\, y(t)^5}\,(f^{1234} + f^{1256} + f^{3456}) = \tau_0(t)\star_t \varphi(t) + \star_t\tau_3(t), \tag{35}$$

where

$$\begin{aligned}
\tau_3(t) &= \frac{\sqrt{6}}{7\, y(t)^5}(-f^{135} + f^{146} + f^{236} + f^{245}) + \frac{4\sqrt{6}}{21y(t)^5}(f^{127} + f^{347} + f^{567}),\\
\star_t\tau_3(t) &= \frac{\sqrt{6}}{7y(t)^5}(-f^{1367} - f^{1457} - f^{2357} + f^{2467}) + \frac{4\sqrt{6}}{21y(t)^5}(f^{1234} + f^{1256} + f^{3456}),
\end{aligned}$$

and

$$\tau_0(t) = \frac{\sqrt{6}}{7y(t)^5}.$$

So, according with the first equality of (35),

$$\begin{aligned}
\Delta_t \star_t \varphi(t) + 2d\Big((A - \tfrac{7}{4}\tau_0)\varphi(t)\Big) &= d\star_t d(\varphi(t)) + 2(A - \tfrac{7}{4}\tau_0)d\varphi(t)\\
&= \tfrac{2A\sqrt{6}\, y(t)^5 - 1}{3\, y(t)^{10}}\left(f^{1234} + f^{1256} + f^{3456}\right),
\end{aligned}$$

that is

$$\Delta_t \star_t \varphi(t) + 2d\Big((A - \tfrac{7}{4}\tau_0)\varphi(t)\Big) = \tfrac{2A\sqrt{6}\, y(t)^5 - 1}{3\, y(t)^{6}}\left(e^{1234} + e^{1256} + e^{3456}\right).$$

The last equality, together with (8) and (34), show that (30) solves the modified Laplacian coflow (2) for φ_1.

In order to show that the solution $\varphi(t)$, given by (30), is ancient, we analyse the behaviour of the function $y = y(t)$ according with the values of the positive constant A. If $A = \frac{1}{2\sqrt{6}}$, then $y(t) \equiv 1$ solves (31) for all $t \in (-\infty, +\infty)$. Assume $A \neq \frac{1}{2\sqrt{6}}$ and observe that the constant function $\widehat{y}(t) \equiv \left(2\sqrt{6}A\right)^{-1/5}$ satisfies the differential equation that appears in (31), which is autonomous. Consequently any solution $y(t)$ having $y'(t_0) = 0$ at some time t_0 satisfies $y(t_0) = \widehat{y}(t_0)$, giving $y \equiv \widehat{y}$. Hence, the solution $y = y(t)$ of the system (31) is monotone and it must satisfy either $y(t) > \widehat{y}(t)$ or $y(t) < \widehat{y}(t)$ for any $t \in I$, according to the value of A. In other words, if $2\sqrt{6}\,A < 1$ then $y(0) < \widehat{y}(0)$, so $y(t) < \widehat{y}(t)$, and similarly $y(t) > \widehat{y}(t)$ if $2\sqrt{6}\,A > 1$.

Now, we rewrite the differential equation that appears in (31) as

$$\left(\frac{\sqrt{6}}{A} y(t)^4 + \frac{\sqrt{6}}{A} \frac{y(t)^4}{2\sqrt{6}Ay(t)^5 - 1} \right) y'(t) = 1.$$

Integrating this equation from 0 to t, we have

$$t = \frac{\sqrt{6}}{5A}(y(t)^5 - 1) + \frac{1}{10A^2} \log \left| \frac{1 - 2\sqrt{6}Ay(t)^5}{1 - 2\sqrt{6}A} \right|. \tag{36}$$

This equation allows us to understand the behaviour of the solution at its singular times. Indeed the limits of $y(t)$ must be singular values of (36); otherwise, through a trivial compactness argument, we could restart the flow, violating the maximality of solutions. So, if $2\sqrt{6}\,A < 1$ then $y = y(t)$ decreases from $\left(2\sqrt{6}\,A\right)^{-1/5}$ to 0 as t goes from $-\infty$ to $-\frac{2A\,\sqrt{6}+\log\left(1-2\sqrt{6}\,A\right)}{10A^2}$. Otherwise, if $2\sqrt{6}\,A > 1$, then $y = y(t)$, which now is an increasing function, goes from $\left(2\sqrt{6}\,A\right)^{-1/5}$ to $+\infty$ as t goes from $-\infty$ to $+\infty$. In particular, we have that the definition interval I of the function $y = y(t)$ is

$$I = (-\infty, -\tfrac{2\sqrt{6}\,A + \log(1-2\sqrt{6}\,A)}{10A^2}), \quad \text{if A} < \tfrac{1}{2\sqrt{6}},$$

and

$$I = (-\infty, +\infty), \quad \text{if A} \geq \tfrac{1}{2\sqrt{6}}.$$

□

Remark 3. In a similar way as in the proof of Theorem 5, one can check that the Riemannian curvature $R(g_t)$ of the metric g_t induced by (30) is such that

$$\|R(g_t)\|^2_{g_t} = \frac{23}{48} y(t)^{-20},$$

and so, in the case iii) (corresponding to $A > \frac{1}{2\sqrt{6}}$) $\lim_{t\to+\infty} R(g_t) = 0$.

In the following theorem we study the modified Laplacian coflow (2) when the initial coclosed G_2 form on the 7-dimensional Heisenberg group is equal to $-\varphi_1$, where φ_1 is defined by (20).

Theorem 8. *The solution of the modified Laplacian coflow* (2) *with initial coclosed* G_2 *form* $-\varphi_1$ *is ancient and it is given by*

$$\varphi(t) = -\frac{1}{y(t)}\left(e^{127} + e^{347} + e^{567}\right) - y(t)^3\left(e^{135} - e^{146} - e^{236} - e^{245}\right), \tag{37}$$

where $t \in (-\infty, T)$, *with* $T = \frac{\sqrt{6}}{5A}\Big(1 - (2A\sqrt{6})^{-1}\log\big(2A\sqrt{6}+1\big)\Big)$, *and the function* $y = y(t)$ *satisfies*

$$\begin{cases} \frac{d}{dt}y(t) = -\frac{2A\sqrt{6}y(t)^5+1}{12y(t)^9}, \\ y(0) = 1. \end{cases} \tag{38}$$

The underlying metrics g_t of this solution converge smoothly, up to pull-back by time-dependent diffeomorphisms, to a flat metric, uniformly on compact sets in H as t goes to $-\infty$.

Proof. By the Picard-Lindelöf Theorem, there exists a maximal open interval I, containing 0, and a smooth function $y : I \to (0, +\infty)$, which is the unique solution of (38).

To prove that (37) is the solution of the coflow (2) for $-\varphi_1$, we proceed as follows. As in the proof of Theorem 7, for each $t \in I$, we consider the basis $\{f^1(t), \dots, f^7(t)\}$ of left invariant 1-forms on H defined by

$$f^i = f^i(t) = y(t)\, e^i, \quad i = 1, \dots, 6$$
$$f^7 = f^7(t) = y(t)^{-3}\, e^7,$$

where the function $y = y(t)$ now satisfies (38). Then, $f^i(0) = e^i$, for $i \in \{1, \cdots, 7\}$, and the structure equations of H, with respect to the basis $\{f^1(t), \dots, f^7(t)\}$, are

$$df^i = 0, \quad 1 \le i \le 6, \qquad df^7 = \frac{\sqrt{6}}{6}\, y(t)^{-5}(f^{12} + f^{34} + f^{56}). \tag{39}$$

Now, for any $t \in I$, the 3-form $\varphi(t)$ defined by (37) has the following expression

$$\varphi(t) = -(f^{127} + f^{347} + f^{567} + f^{135} - f^{146} - f^{236} - f^{245}). \tag{40}$$

So, $\varphi(0) = -\varphi_1$ and, for any $t \in I$, the metric g_t induced by $\varphi(t)$ is such that the coframe $\{f^1(t), \dots, f^7(t)\}$ of $\mathfrak{h}^*$ is orthonormal. Denote by $\star_t$ the Hodge star operator determined by g_t. Using (39), we have $d \star_t \varphi(t) = 0$, where $\star_t \varphi(t)$ is given by

$$\star_t \varphi(t) = f^{1234} + f^{1256} + f^{1367} + f^{1457} + f^{2357} - f^{2467} + f^{3456}.$$

Then, in terms of the coframe $\{e^1, \dots, e^7\}$ of $\mathfrak{h}^*$, the 4-form $\star_t \varphi(t)$ has the following expression

$$\star_t \varphi(t) = y(t)^4(e^{1234} + e^{1256} + e^{3456}) + e^{1367} + e^{1457} + e^{2357} - e^{2467}.$$

Therefore,

$$\tfrac{d}{dt} \star_t \varphi(t) = 4y(t)^3\, y'(t)(e^{1234} + e^{1256} + e^{3456}),$$

that is,

$$\tfrac{d}{dt} \star_t \varphi(t) = -\tfrac{2A\sqrt{6}\, y(t)^5 + 1}{3\, y(t)^6}(e^{1234} + e^{1256} + e^{3456}), \tag{41}$$

since the function $y = y(t)$ satisfies (38).

On the other hand, by (7) we know that the torsion forms $\tau_i(t)$ $(i = 0, 1, 2, 3)$ of $\varphi(t)$ are such that $\tau_1(t) = 0 = \tau_2(t)$ since $d(\star_t \varphi(t)) = 0$. Then, from (39), (40) and using again (7), we have

$$d\varphi(t) = -\frac{\sqrt{6}}{3\, y(t)^5}\,(f^{1234} + f^{1256} + f^{3456}) = \tau_0(t) \star_t \varphi(t) + \star_t \tau_3(t), \tag{42}$$

where

$$\tau_3(t) = \frac{\sqrt{6}}{7\, y(t)^5}(-f^{135} + f^{146} + f^{236} + f^{245}) + \frac{4\sqrt{6}}{21 y(t)^5}(f^{127} + f^{347} + f^{567}),$$

$$\star_t \tau_3(t) = \frac{\sqrt{6}}{7 y(t)^5}(-f^{1367} - f^{1457} - f^{2357} + f^{2467}) + \frac{4\sqrt{6}}{21 y(t)^5}(f^{1234} + f^{1256} + f^{3456}),$$

and

$$\tau_0(t) = -\frac{\sqrt{6}}{7y(t)^5}.$$

Then, according with the first equality of (42),

$$\begin{aligned} \Delta_t \star_t \varphi(t) + 2d\Big((A - \tfrac{7}{4}\tau_0)\varphi(t)\Big) &= d \star_t d(\varphi(t)) + 2(A - \tfrac{7}{4}\tau_0)d\varphi(t) \\ &= -\tfrac{2A\sqrt{6}\,y(t)^5+1}{3y(t)^{10}}\left(f^{1234} + f^{1256} + f^{3456}\right), \end{aligned}$$

or, equivalently,

$$\Delta_t \star_t \varphi(t) + 2d\Big((A - \tfrac{7}{4}\tau_0)\varphi(t)\Big) = -\tfrac{2A\sqrt{6}\,y(t)^5+1}{3y(t)^6}\Big(e^{e^{1234}+1256} + e^{3456}\Big).$$

The last equality, together with (8) and (41), show that (37) solves the modified Laplacian flow (2) for $-\varphi_1$.

To show that the solution $\varphi(t)$, given by (37), is ancient, we study the behaviour of the function $y = y(t)$. To this end, we rewrite the differential equation that appears in (38) as

$$-\frac{12\,y(t)^9}{2A\sqrt{6}\,y(t)^5 + 1}\,y' = 1.$$

Integrating this equation from 0 to t we obtain

$$\frac{\sqrt{6}}{5A}\left(1 - y^5(t)\right) + \frac{1}{10A^2}\log\left(\frac{2A\sqrt{6}\,y^5(t) + 1}{2A\sqrt{6} + 1}\right) = t. \tag{43}$$

Clearly $y'(t) < 0$ since the function $y = y(t)$ satisfies the differential equation that appears in (38). Then, (43) implies that the function $y = y(t)$ decreases from $+\infty$ to 0 as t goes from $-\infty$ to $\frac{\sqrt{6}}{5A}\Big(1 - \frac{1}{2A\sqrt{6}}\log\big(2A\sqrt{6} + 1\big)\Big)$.

To study the behaviour of the underlying metric g_t of the solution (37) for $t \to -\infty$, we proceed in a similar way as in the proof of Theorem 5. □

Concerning the modified Laplacian coflow (2) for the coclosed G_2 form φ_2 on the 7-dimensional Heisenberg group H we have the following.

Theorem 9. *The solution of the modified Laplacian coflow* (2) *with initial coclosed* G_2*-structure* φ_2 *is defined on a bounded interval, and it is given by*

$$\varphi(t) = \frac{y(t)}{z(t)^2}e^{127} - y(t)^{-1}\left(e^{347} + e^{567}\right) + y(t)z(t)^2\left(e^{135} - e^{146} + e^{236} + e^{245}\right), \tag{44}$$

where the functions $y = y(t)$ *and* $z = z(t)$ *satisfy*

$$\begin{cases} \frac{d}{dt}y(t) = \frac{2A\sqrt{6}\,y(t)z(t)^6 + 2z(t)^2 + y(t)^2}{12\,y(t)^3z(t)^8}, & \frac{d}{dt}z(t) = -\frac{2A\sqrt{6}\,y(t)z(t)^4 + 1}{12\,y(t)^2z(t)^7}, \\ y(0) = 1, & z(0) = 1. \end{cases} \tag{45}$$

Proof. By the Picard–Lindelöf Theorem, there exists a maximal open interval I, containing 0, and two smooth functions $y, z : I \to (0, +\infty)$, which are the unique solution of (45).

We first prove that (44) is the solution of the coflow (2) for φ_2. As in the proof of Theorem 6, for each $t \in I$, we consider the basis $\{f^1(t), \dots, f^7(t)\}$ of left invariant 1-forms on H defined by

$$
\begin{aligned}
f^i &= f^i(t) = y(t)\, e^i, && i = 1, 2,\\
f^i &= f^i(t) = z(t)\, e^i, && i = 3, \dots, 6,\\
f^7 &= f^7(t) = y(t)^{-1} z(t)^{-2}\, e^7,
\end{aligned}
$$

where the functions $y = y(t)$ and $z = z(t)$ satisfy now (45). Then, $f^i(0) = e^i$, for $i \in \{1, \cdots, 7\}$, and the structure equations of H, with respect to the basis $\{f^1(t), \dots, f^7(t)\}$, are

$$
\begin{aligned}
df^i &= 0, \quad 1 \le i \le 6,\\
df^7 &= \frac{\sqrt{6}}{6}\, y(t)^{-1} z(t)^{-2} \Big(y(t)^{-2} f^{12} + z(t)^{-2} f^{34} + z(t)^{-2} f^{56} \Big).
\end{aligned}
\tag{46}
$$

Moreover, for any $t \in I$, the 3-form $\varphi(t)$ defined by (44) has the following expression

$$
\varphi(t) = f^{127} - f^{347} - f^{567} + f^{135} - f^{146} + f^{236} + f^{245}. \tag{47}
$$

So $\varphi(0) = \varphi_2$ and, for any $t \in I$, the 3-form $\varphi(t)$ on H induces the metric g_t such that $\{f^1(t), \dots, f^7(t)\}$ of $\mathfrak{h}^*$ is an orthonormal basis of $\mathfrak{h}^*$. Denote by $\star_t$ the Hodge operator determined by g_t. Using (4), (5) and (46), we have $d \star_t \varphi(t) = 0$, where $\star_t \varphi(t)$ is given by

$$
\star_t \varphi(t) = -f^{1234} - f^{1256} - f^{1367} - f^{1457} + f^{2357} - f^{2467} + f^{3456}.
$$

Thus, in terms of the coframe $\{e^1, \dots, e^7\}$ of $\mathfrak{h}^*$, the 4-form $\star_t \varphi(t)$ has the following expression

$$
\star_t \varphi(t) = y(t)^2 z(t)^2 (-e^{1234} - e^{1256}) - e^{1367} - e^{1457} + e^{2357} - e^{2467} + z(t)^4 e^{3456}.
$$

Therefore,

$$
\frac{d}{dt}\,(\star_t \varphi(t)) = 2\Big(y(t) z(t)^2 y'(t) + y(t)^2 z(t) z'(t) \Big)(-e^{1234} - e^{1256}) + 4 z(t)^3 z'(t)\, e^{3456},
$$

that is

$$
\begin{aligned}
\frac{d}{dt} \star_t \varphi(t) = {} & \frac{A\sqrt{6}\big(y(t)^3 z(t)^2 - y(t) z(t)^4\big) - 1}{3\, y(t)^2 z(t)^4} (e^{1234} + e^{1256})\\
& - \frac{2A\sqrt{6}\, y(t) z(t)^4 + 1}{3\, y(t)^2 z(t)^4} e^{3456},
\end{aligned}
\tag{48}
$$

since the functions $y = y(t)$ and $z = z(t)$ satisfy (45).

On the other hand, let us consider the torsion forms $\tau_i(t)$ $(i = 0, 1, 2, 3)$ of $\varphi(t)$. By (7), $\tau_1(t) = 0 = \tau_2(t)$ since $d(\star_t \varphi(t)) = 0$. Then, from (46), (47) and using again (7), we have

$$
\begin{aligned}
d\varphi(t) &= \frac{\sqrt{6}}{6} y(t)^{-1} z(t)^{-2} \Big(\big(z(t)^{-2} - y(t)^{-2}\big)(f^{1234} + f^{1256}) - 2 z(t)^{-2} f^{3456} \Big)\\
&= \tau_0(t) \star_t \varphi(t) + \star_t \tau_3(t),
\end{aligned}
\tag{49}
$$

where

$$\begin{aligned}\tau_3(t) &= -\tfrac{\sqrt{6}\left(5y(t)^2+z(t)^2\right)}{21\,y(t)^3z(t)^4}f^{127}+\tfrac{\sqrt{6}\left(3y(t)^2-5z(t)^2\right)}{42\,y(t)^3z(t)^4}(f^{347}+f^{567})\\ &\quad+\tfrac{\sqrt{6}\left(2y(t)^2-z(t)^2\right)}{21\,y(t)^3z(t)^4}(f^{135}-f^{146}+f^{236}+f^{245}),\end{aligned}$$

$$\begin{aligned}\star_t\tau_3(t) &= -\tfrac{\sqrt{6}\left(5y(t)^2+z(t)^2\right)}{21\,y(t)^3z(t)^4}f^{3456}+\tfrac{\sqrt{6}\left(3y(t)^2-5z(t)^2\right)}{42\,y(t)^3z(t)^4}(f^{1234}+f^{1256})\\ &\quad+\tfrac{\sqrt{6}\left(2y(t)^2-z(t)^2\right)}{21\,y(t)^3z(t)^4}(-f^{1367}-f^{1457}+f^{2357}-f^{2467}),\end{aligned}$$

and

$$\tau_0(t)=-\frac{\sqrt{6}}{21y(t)^3z(t)^4}\big(2y(t)^2-z(t)^2\big).$$

Then, according with the first equality of (49),

$$\begin{aligned}\Delta_t\star_t\varphi(t)+2d\Big((A-\tfrac{7}{4}\tau_0)\varphi(t)\Big) &= d\star_t d(\varphi(t))+2(A-\tfrac{7}{4}\tau_0)d\varphi(t)\\ &= \tfrac{A\sqrt{6}\left(y(t)^3z(t)^2-y(t)z(t)^4\right)-1}{3\,y(t)^4z(t)^6}\big(f^{1234}+f^{1256}\big)\\ &\quad-\tfrac{2A\sqrt{6}y(t)z(t)^4+1}{3\,y(t)^2z(t)^8}f^{3456},\end{aligned}$$

or, equivalently,

$$\begin{aligned}\Delta_t\star_t\varphi(t)+2d\Big((A-\tfrac{7}{4}\tau_0)\varphi(t)\Big) &= \tfrac{A\sqrt{6}\left(y(t)^3z(t)^2-y(t)z(t)^4\right)-1}{3\,y(t)^2z(t)^4}\big(e^{1234}+e^{1256}\big)\\ &\quad-\tfrac{2A\sqrt{6}y(t)z(t)^4+1}{3\,y(t)^2z(t)^4}e^{3456}.\end{aligned}$$

The last equality, together with (8) and (48), show that (44) solves the modified Laplacian flow (2) for φ_2.

To prove that (44) is defined on a bounded interval, we will show that $t_+=\sup(I)<+\infty$ and $t_-=\inf(I)>-\infty$. On the one hand, we know that the functions $y=y(t)$ and $z=z(t)$ are positive. Then, the system (45) implies that $z'(t)<0<y'(t)$, for any $t\in I$. Therefore, the function $z=z(t)$ is decreasing, and $y=y(t)$ is increasing. Thus, there exist

$$\lim_{t\to t_-}y(t)=y_-\in[0,1)\quad\text{and}\quad\lim_{t\to t_+}z(t)=z_+\in[0,1).$$

Now, using (45), it is straightforward to verify that the function $z=z(t)$ satisfies

$$z''=-\tfrac{1}{144\,y^6z^{15}}\Big(24A^2(3y^4z^8-y^2z^{10})+2A\sqrt{6}\,(9y^3z^4-4yz^6)+5y^2-4z^2\Big),$$

for any $t\in I$. Note that in the last equality, the functions $(3y^4z^8-y^2z^{10})=y^2z^8(3y^2-z^2)$, $(9y^3z^4-4yz^6)=yz^4(9y^2-4z^2)$ and $(5y^2-4z^2)$ are positive functions in $(0,t_+)$. Indeed, their values at $t=0$ are positive, and $z=z(t)$ decreases while $y=y(t)$ increases in $(0,t_+)$. Therefore, $z''(t)<0$, for $t\in(0,t_+)$. Thus, $z'(t)<z'(0)<0$, for any $t\in(0,t_+)$. Now, we choose a sequence $\{t_n\}\subset I$ of positive times converging to t_+. Then,

$$z(t_n)-1=\int_0^{t_n}z'(t)\,dt<\int_0^{t_n}z'(0)\,dt<z'(0)\,t_n.$$

So, $t_n < \frac{z(t_n)-1}{z'(0)}$ and, consequently, $t_+ \leq \frac{z_+-1}{z'(0)} < +\infty$.

Using again (45), we have

$$\begin{aligned} -144y^7z^{16}y'' = {} & 48A^2(z^{12}y^2 - z^{10}y^4) + 2A\sqrt{6}\,(10z^8y - 11z^6y^3 - 8z^4y^5) \\ & + 12z^4 - 4z^2y^2 - 7y^4. \end{aligned} \tag{50}$$

Then, it is possible to show that $y''(t) < 0$ in some neighbourhood of t_-. Indeed, the functions $z^{12}y^2 - z^{10}y^4$ and

$$(12z^4 - 4z^2y^2 - 7y^4) = 4z^2(z^2 - y^2) + (8z^4 - 7y^4)$$

are both positive on $(t_-, 0)$, since the functions $z^2 - y^2$ and $8z^4 - 7y^4$ are both decreasing. Moreover, the solution is maximal for t going to t_-. Therefore, the limits $\lim_{t\to t_-} z(t) = z_-$ and $\lim_{t\to t_-} y(t) = y_-$ cannot be both finite and different from zero, otherwise we can restart the flow. As a consequence, since $y'(t) > 0$ and $z'(t) < 0$, for any $t \in I$, we get that either $z_- < +\infty$ (and consequently $y_- = 0$) or $z_- = +\infty$.

In the first case, the leading term (as polynomial in z) of the right side of (50) is $12z^4$, so it must be positive in a neighbourhood of t_-. On the other hand $-144y^7z^{16} < 0$, so $y''(t) < 0$ in some neighbourhood of t_-. In the other case (i.e. when $z_- = +\infty$),

$$\lim_{t\to t_-} (10z^8 - 11z^6y^2 - 8z^4y^4) = +\infty$$

since $z_- = +\infty$ and y is bounded. Therefore $y(10z^8 - 11z^6y^2 - 8z^4y^4)$ is positive in some neighbourhood of t_-. Hence, in both cases, it follows that $y'' < 0$ for $t \in (t_-, \bar{t})$, for some $\bar{t} \in (t_-, 0)$, i.e. that $y'(t) > y'(\bar{t})$, for $t \in (t_-, \bar{t})$. Now, we choose a sequence of negative times $\{t_n\} \subset (t_-, \bar{t})$ converging to t_-. Then,

$$y(\bar{t}) - y(t_n) = \int_{t_n}^{\bar{t}} y'(t)\,dt > \int_{t_n}^{\bar{t}} y'(\bar{t})\,dt = (\bar{t} - t_n)\,y'(\bar{t}).$$

It follows that $t_n > \frac{y(t_n)-y(\bar{t})}{y'(\bar{t})} + \bar{t}$. So, $t_- \geq \frac{y_- - y(\bar{t})}{y'(\bar{t})} + \bar{t} > -\infty$. □

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REFERENCES

[1] L. Bagaglini, M. Fernández and A. Fino, *Coclosed* G_2*-structures inducing nilsolitons*, Forum Math., 30 (2018), pp. 109–128.

[2] R. L. Bryant, *Some remarks on* G_2*-structures*, Proceedings of Gökova Geometry-Topology Conference 2005, pp. 75–109, Gökova Geometry/Topology Conference (GGT), Gökova, 2006.

[3] R. L. Bryant and F. Xu, *Laplacian Flow for Closed* G_2*-Structures: Short Time Behavior*, arXiv: 1101.2004 [math.DG].

[4] M. Fernández, A. Fino and V. Manero, *Laplacian flow of closed* G_2*-structures inducing nilsolitons*, J. Geom. Anal., 26 (2016), pp. 1808–1837.

[5] M. Fernández, A. Gray, *Riemannian manifolds with structure group* G_2, Ann. Mat. Pura Appl., 132 (1982), pp. 19–45.

[6] S. Grigorian, *Short-time behavior of a modified Laplacian coflow of* G_2*-structures*, Adv. Math., 248 (2013), pp. 378–415.
[7] R. S. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Diff. Geom., 17 (1982), pp. 255–306.
[8] R. Harvey and H. B. Lawson, *Calibrated geometries*, Acta Math., 148 (1982), pp. 47–157.
[9] N. Hitchin, *The geometry of three-forms in six and seven dimensions*, J. Diff. Geom., 55 (2000), pp. 547–576.
[10] S. Karigiannis, B. McKay and M. P. Tsui, *Soliton solutions for the Laplacian coflow of some* G_2*-structures with symmetry*, Diff. Geom. Appl., 30 (2012), pp. 318–333.
[11] J. Lauret, *Einstein solvmanifolds and nilsolitons*, in New Developments in Lie Theory and Geometry, Contemp. Math., 491, Amer. Math. Soc, Providence, RI, 2009, pp. 1–35.
[12] J. Lauret, *The Ricci flow for simply connected nilmanifolds*, Comm. Anal. Geom., 19:5 (2011), pp. 831–854.
[13] J. Lauret, *Laplacian flow of homogeneous* G_2*-structures and its solitons*, Proc. London Math., 114 (2017), pp. 527–560.
[14] J. Lotay and Y. Wei, *Laplacian flow for closed* G_2*-structures: Shi-type estimates, uniqueness and compactness*, Geom. Funct. Anal., 27:1 (2017), pp. 165–233.
[15] J. Lotay and Y. Wei, *Stability of torsion-free* G_2*-structures along the Laplacian flow*, J. Differential Geom., 111:3 (2019), pp. 495–526.
[16] J. Lotay and Y. Wei, *Laplacian flow for closed* G_2*-structures: real analyticity*, Comm. Anal. Geom., 27 (2019), pp. 73–109.
[17] F. Schulte-Hengesbach, *Half-flat structures on Lie groups*, PhD Thesis (2010), Hamburg, available at http://www.math.uni-hamburg.de/home/schulte-hengesbach/diss.pdf.