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CRITICAL L-VALUES FOR SOME QUADRATIC TWISTS OF GROSS CURVES*

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Let $K = \mathbb{Q}(\sqrt{-q})$, where q is a prime congruent to 3 modulo 4. Let $A = A(q)$ denote the Gross curve [6]. Let $E = A^{(-\beta)}$ denote its quadratic twist, with $\beta = \sqrt{-q}$. The curve E has the nice explicit equation (see [5], equation (1.2))

$$y^2 = x^3 - 2^{-4}3^{-1}(j(\mathcal{O}_K)^{1/3}x + 2^{-5}3^{-3}(j(\mathcal{O}_K) - (12)^3)^{1/2}, \tag{1}$$

where it is understood that, in this equation, we take the real cube root of $j(\mathcal{O}_K)$, and the square root of $j(\mathcal{O}_K) - (12)^3$ lying in the upper half complex plane. Thus E is defined over the Hilbert class field H of K. Below we use Magma [1] to calculate the values $L(E/H, 1)$ for all such q's up to some reasonable ranges (different for $q \equiv 7 \bmod 8$ and $q \equiv 3 \bmod 8$). All these values are non-zero, and using the Birch and Swinnerton-Dyer conjecture, we can calculate hypothetical orders of $Ш(E/H)$ in these cases. Our calculations extend those given in [3] for the case $q = 7$.

1. The case $q \equiv 7 \bmod 8$. In this case we know, by a recent result of J. Coates and Y. Li ([5], Theorem 1.3), that $L(E/H, 1) \neq 0$. In the table below we calculate numerically these values for all such q up to 4663.

Now let us say a few words about the Magma implementation. The starting source for us was the article by M. Watkins [7], which gives some numerical examples (or rather hints) how to compute Grossencharacters and critical L-values (sections 5.4 and 6.1 deals with $\mathbb{Q}(\sqrt{-23})$, but of course we need to keep track of the effect of twisting). But it was not enough for us to write an algorithm calculating $L(E/H, 1)$. Watkins [8] corrected our algorithm (or better, he wrote a new one) and tested for $q = 23$ and 79. It was a starting point for us to make extensive numerical calculations. The algorithm uses the fact that L-series of an elliptic curve over H splits into factors corresponding to Grossencharacters twisted by Hilbert characters and its conjugates (so it uses the classical Hecke-Deuring theory linking elliptic curves with CM to Grossencharacters, with keeping track of the effect of twisting). Here are some more details. Assuming $2\mathcal{O}_K = \mathfrak{p}\mathfrak{p}^\star$, and choosing the sign of $\beta = \sqrt{-q}$ so that $\mathrm{ord}_{\mathfrak{p}}((1-\beta)/2) > 0$, we can check that E/H has good reduction outside the primes of H lying above $\mathfrak{p}$ (see [5]). Moreover, the Deuring Grossencharacter $\psi_{E/H}$ of E/H is then equal to $\rho \circ N_{H/K}$, where ρ is the Grossencharacter of K with conductor $\mathfrak{p}^2$ defined by

$$\rho(\mathfrak{a}) = \alpha, \quad \mathfrak{a}^h = \alpha\mathcal{O}_K, \quad \alpha \equiv 1 \bmod \mathfrak{p}^2.$$

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The algorithm computes the values $L(\rho\chi, 1)$, where χ runs over the characters of the ideal class group of K. Now thanks to the above formula and Deuring's theory it follows that $L(E/H, 1)$ will be given by the product of all the $L(\rho\chi, 1)$'s and their complex conjugates.

Now we know by Iwasawa theory that the Tate-Shafarevich group $Ш(E/H)$ is finite because $L(E/H, 1) \neq 0$. Below we will write down an explicit conjectural formula for the order of $Ш(E/H)$. Let h denote the class number of K, $m = \frac{q-1}{4} - \frac{h}{2}$, and let $\Omega(q)$ be a period defined in [6]:

$$\Omega(q) = \frac{1}{(2\pi)^m \cdot q^{\frac{h}{2}}} \prod_{\substack{0<c<q \\ \left(\frac{c}{q}\right)=1}} \Gamma\left(\frac{c}{q}\right).$$

The prime 2 splits in K, and we write $2\mathcal{O}_K = \mathfrak{p}\mathfrak{p}^\star$, where we have chosen the sign of $\beta = \sqrt{-q}$ so that $\mathrm{ord}_{\mathfrak{p}}((1-\beta)/2) > 0$. Then E is the quadratic twist of the Gross curve A/H by $H(\sqrt{-\beta})/H$. Let $\{v_1, ..., v_r\}$ be the set of primes of H lying above $\mathfrak{p}$, so that $r = h/j$, where j is the exact order of the class of $\mathfrak{p}$ in the ideal class group of K. It turned out that there are exactly 18 primes $q \leq 4663$ congruent to 7 modulo 8, for which $r > 1$.

Conjecture 1. $\#(Ш(E/H)) = L(E/H, 1)2^{h+6-2r}/(\Omega(q)^2\sqrt{q})$.

One easily checks that in case $q = 7$, the formula from Conjecture 1 is equivalent to (2.11) from [3]. The above conjecture agrees with the Birch and Swinnerton-Dyer conjecture for E/H. In particular, the set of bad primes of E/H is precisely $\{v_1, ..., v_r\}$, and the factor 2^{-2r} in the above formula takes account of the fact that the Tamagawa factor at each of these primes is 4.

John Coates informed one of us (A. D.) that one should be able to use the Iwasawa theory being developed in [4] to prove the above conjecture.

Below we use Conjecture 1 to calculate $\#(Ш(E/H))$ (i.e. the analytic order of $Ш(E/H)$) for all primes q congruent to 7 modulo 8 up to 4663.

q	h	$L(E/H,1)$	$\sqrt{\#(Ш(E/H))}$
7	1	0.30903153751765917103	1
23	3	0.79196294535428296044	1
31	3	0.35288571505654851763	1
47	5	3.25049251883301426121	3
71	7	0.10920125590289049507	1
79	5	0.02577591231345318312	1
103	5	0.84244014254446144514	13
127	5	0.33138747507581642444	17
151	7	0.00899919291175804982	5
167	11	338.84342541058916626822	2049
191	13	0.07538843930533773444	81
199	9	0.00178784908116291475	9
223	7	0.66858391145992740299	289
239	15	0.02401256252449269664	311
263	13	0.24799355777337639904	1767
271	11	0.00495300516895988511	127
311	19	0.08289319536914465106	12559
359	19	0.00008262935013341212	2057
367	9	0.02393861560648477609	1679
383	17	1058.78512825720370837609	9090067
431	21	5.38876192180481196261	2039928
439	15	0.00002907103487183395	1279
463	7	0.08500423491817571054	11663
479	25	1483.07868786791841546796	1746287691
487	7	0.14694723669623207042	22807
503	21	260759.24737728583571680044	2880463783
599	25	0.00000001076991986883	162285
607	13	0.10795424186869623536	884605
631	13	0.00004385443164140780	44425
647	23	0.00000607641351529086	4925391
719	31	0.00530561250904147645	31646320057
727	13	0.00299892234779012135	1113693
743	21	1498.05565627050935641062	332146468299
751	15	0.00000000896688802629	10512

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q	h	$L(E/H,1)$	$\sqrt{\#(Ш(E/H))}$
823	9	0.00102716234866514602	469855
839	33	0.12347121525795507868	9315485111867
863	21	4.28771850132666368851	240267975371
887	29	19477038.03896518654448808873	30785347392739103
911	31	0.00000312260876667640	98895319091
919	19	0.00000000001480555834	36741
967	11	0.09804744238584120050	67762715
983	27	80292384.19994893868777292158	238138622744502833
991	17	0.00004920969572684842	91561037
1031	35	0.06252031110591168794	1595084268489133
1039	23	0.00000369632068956271	1196802971
1063	19	1.67498940035011217030	54030361471
1087	9	0.00390824946108466869	22866381
1103	23	0.00000572944338432493	80104082513
1151	41	0.16613607401358903141	1320694330164429335
1223	35	0.00194039562173745137	9307326019643999
1231	27	0.00000000093497039710	3758406353
1279	23	0.00000000287686176036	1814619877
1303	11	0.00021448455521015721	157522307
1319	45	28941477.30398577348268950039	494146202273056454638285
1327	15	0.00028916732527519815	1982319913
1367	25	28601.01936469536020713962	1839953001047559952
1399	27	0.00000001208663958591	210925423356
1423	9	0.01738234231379547068	1715895524
1439	39	0.00000006661042814296	30644209290657623
1447	23	0.00013884708802240947	811815992737
1471	23	0.00000000029882112555	5374292551
1487	37	6576514753.03121783833652353519	10937565775616401748256581
1511	49	0.00055060129228412947	25127641761490803096975
1543	19	0.00176400518665963757	1034981247929
1559	51	0.87726903612746376846	2738379667079823194690949
1567	15	0.20108355789279651145	1575131870837
1583	33	0.13269489544044152769	15757834130321482863
1607	27	0.55194730530618115640	349749435110817399
1663	17	0.00119021469463015769	513229201123
1759	27	0.00000000101913474126	1344751145077
1783	17	0.00024486558886747311	1885276250643
1823	45	9746102.85279061269213579273	1808382408390942813565214784
1831	19	0.00000000000009270267	880904745
1847	43	1207965.54129407491879223532	4921203932360352431818054853
1871	45	0.00000467957783395582	211817576664143143791049
1879	27	0.00000055405311625279	359706350854836
1951	33	0.00000000000191765783	71833637298811
1999	27	0.00000000047804213103	21632373948999
2039	45	0.00000000367413120950	7999274603520740597625
2063	45	2766645.80955923764536501056	1341376803998421383216338486800
2087	35	164.98809160497598617071	7549276579545660794505221
2111	49	0.00109168602722010353	4327178311989716140475456427
2143	13	0.01476049903300890724	12322652032019
2207	39	8356.57170711443065175664	5015914871462321771139549955
2239	35	0.00000000000004393777	11915930073079139
2287	29	0.00766729713419227393	6824315445622624657
2311	29	0.00000000187662881760	20015174351579955
2351	63	552470425.54422669425872718713	2816977737852577527309898650097367443659
2383	29	0.00002630161314175636	1162866812268136121
2399	59	0.04141467915645522162	13235014758527507066087807686368973
2423	33	250.16970151756207630008	281045797696507991427800573
2447	37	23.82978667301734246096	2092852385170614860214887843
2503	21	0.02648336374331768867	100742172570243491
2543	35	382.95959917323265302185	10157978602722623458338530496
2551	41	0.00000000000008248960	60637243039930628065
2591	57	0.00000000000238017737	336275233499026658311829296679
2647	15	0.03372248437933683851	6798505360748663
2663	43	822.32375569716777860250	76481720773755781866561562664447
2671	23	0.00000000000003162935	66697260897735
2687	51	194688424.12336279094797547720	17321740551070983627343715625416872652
2711	53	0.37421143332764690649	38629854923911845038305204026873649
2719	41	0.00000000000083400226	1406443138529321307393
2767	21	0.31592061458094422698	3800789493866165232
2791	39	0.00000000000054815909	897017319101036106209
2879	57	0.00000013893708474299	10273597469646245008601935022410689
2887	25	0.00000727449204224937	4682281660959493201
2903	59	469211113283732.46988840553969223592	400977393247118763374214959073343661518898573
2927	31	64.29980720973840221765	27260274502712067357277082151
2999	73	360.95151202061450399689	3943185992249268695560714545947130529724467545
3023	47	511953545751.66177107863339231208	6789457020754215256411685440174304374421
3079	41	0.00000000023207164115	2141735462113012348434191
3119	69	0.00602643768003535676	1049324059291363104640660260593606354947228
3167	53	260425057231369.09972689242486572823	1799402000099564565183145128420460525695159379
3191	69	14.95753417523273502334	628315606611285122323089495662377335179881369
3271	27	0.00000001241846857681	243743805808037264960
3319	41	0.00000000000000931719	33642982599580090033407
3343	19	1.58620548101554534230	202212319807498314139
3359	69	1.67365772978370467795	4049500549734207136741889632539175584626802773
3391	37	0.00000000047269143915	3101068731835541701552139
3407	57	28819363.52788452185843622630	14560069825940266325266106631738472810251876
3463	19	13.77281385527310381528	8436425008239920177883
3511	41	0.00000000001434615685	11283089506997160477570507
3527	65	7976156006.93765520950738207851	177909675370561108532941247559470051694366241471
3559	45	0.00000000000454262476	2360984303797804985269627200
3583	29	0.03920861329093863272	5633558142934297942544477
3607	19	36.90684229826591202567	6563789114878102952373
3623	45	26.02198199101404503659	1367770373787759276795854320370634923
3631	43	0.00000000000255623850	427172018608825201653180173
3671	81	68467807.61967835918310312766	117843219225119087861773569889261690958967498129448976645
3719	67	0.00000000387780125945	2100976923197097618349171255793507783224063
			Continued on next page

q	h	$L(E/H,1)$	$\sqrt{\#(Ш(E/H))}$
3727	31	0.00004718946253999069	24449127078644229586732 93
3767	39	0.14327587591455739687	19175435293484578149783919131747899
3823	29	0.00002507286357856734	1124176121982007178 61969
3847	23	0.00001533345082846095	862130992415445336857
3863	61	15711737.48318757316819605069	5383943677731735916930843963677275939761416 47921
3911	83	0.00000000001135189188	144993271979803991679156248599573671855885463636 29
3919	39	0.00000000044261630153	36691961918662203454631354 68
3943	27	0.00000000002095427812	2473366856748486 8147
3967	33	159215.22350765659455009473	1699553443719169549240379327572
4007	57	0.91990088235951608333	244794838022929311845106053989885634485786 89
4079	85	0.00000127446973332763	321963672249515003702405195900159942517697859377925616
4111	39	0.00000000000002095214	33609809857361575194185877
4127	49	264235.35416434781417718265	8507300729794243618442185373646968515974841
4159	31	0.00000000108677317318	813947892469212648852 3677
4231	51	0.00000000799222631822	429957063340261892270549743969 4423
4271	65	0.00000290128128404136	282831869678333181715682199779445845014898674 47
4327	19	0.00017450275228592056	1066852592208273415311
4391	79	0.00001251482541320985	111218267845699128837861765031864433022917071402312905
4423	33	1.13324073727367674386	5723955223602671057559964476 52
4447	17	0.00000077492121857457	398726853667472262 31
4463	55	115564.52263908769265426894	727808874976081978023034234242783649996543852 53
4519	29	0.00000000000000004313	3597009679993991314033
4567	33	0.00000017729987000499	1945455740103468168767260495
4583	61	551441731499.78356036276104299918	14389734137882571707194551338101810084628886102126808 83
4591	49	0.00000000000000079237	16108217968515978652127771698061
4639	51	0.00000000000000004301	10187734894995520568067890682547 2
4663	33	0.00001145384710376496	116950444281625739633416250 0

2. The case $q \equiv 3 \,\mathbf{mod}\, 8$. In this case the curve E defined by the equation (1) is also defined over H. Here the prime 2 is inert in K, and the curve E will always have good reduction outside the set of the primes of H lying above 2 (assuming $q > 3$). The Deuring Grossencharacter $\psi_{E/H}$ of E/H is then equal to $\rho \circ N_{H/K}$, where ρ is the Grossencharacter of K with conductor $4\mathcal{O}_K$ defined by

$$\rho(\mathfrak{a}) = \alpha, \quad \mathfrak{a}^h = \alpha\mathcal{O}_K, \quad \alpha \equiv 1 \bmod 4\mathcal{O}_K.$$

The algorithm computes the values $L(\rho\chi, 1)$, where χ runs over the characters of the ideal class group of K. Again, thanks to the above formula and Deuring's theory it follows that $L(E/H, 1)$ will be given by the product of all the $L(\rho\chi, 1)$'s and their complex conjugates.

Our numerical calculations (given in the table below) lead to the following conjecture (see [5], Conjecture 1.8).

CONJECTURE 2. *For all primes q with $q \equiv 3 \bmod 8$, we have $L(E/H, 1) \neq 0$.*

As it is remarked in ([5], p. 1534), in contrast to the proof of Theorem 1.5 there, the authors see no way at present for attacking such a conjecture using Iwasawa theory.

In this case, we propose the following conjectural formula for the order of $Ш(E/H)$. Now, the Tamagawa factor at each prime of bad reduction is 1, and the following conjecture agrees with the Birch and Swinnerton-Dyer conjecture for E/H.

CONJECTURE 3. $\#(Ш(E/H)) = L(E/H, 1)2^{2h}/(\Omega(q)^2\sqrt{q})$.

Below we use Conjecture 3 to calculate $\#(Ш(E/H))$ (i.e. the analytic order of $Ш(E/H)$) for all primes q congruent to 3 modulo 8 up to 11131.

q	h	$L(E/H,1)$	$\sqrt{\#(Ш(E/H))}$
11	1	1.73845792121760807790	1
19	1	1.00717576250064706853	1
43	1	1.27416027648354776885	2
59	3	3.27291981598555587930	5
67	1	1.22243364144817892444	3
83	3	0.03764760689032642372	1
107	3	1.03936693115122887083	9
131	5	0.00697158270425921776	2
139	3	0.06083483360363034315	3
163	1	1.23339224060989144293	10
179	5	0.06195226194655516123	17

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q	h	$L(E/H,1)$	$\sqrt{\#(Ш(E/H))}$
211	3	0.00822152771334987023	3
227	5	101.51957725718817748183	1524
251	7	0.32436912039697336340	343
283	3	0.66685505848109321080	53
307	3	0.78609012909015461469	73
331	3	0.14907707956876846359	49
347	5	0.08080910761305406985	250
379	3	0.33390038219003027566	117
419	9	1.67741166405082133222	40225
443	5	0.00285060561331757659	156
467	7	0.35140804797178265726	8190
491	9	0.01884246766690787210	11387
499	3	0.46414182882437918271	395
523	5	0.02170338307342622816	242
547	3	0.41073019842431356261	316
563	9	2217.38322210227025102910	7489893
571	5	0.29829944260019244950	1956
587	7	0.25898213029907196441	31143
619	5	0.00509394359043077170	479
643	3	0.60354061296945788027	893
659	11	1.79783553470780040531	4006249
683	5	0.00159958302432158035	1447
691	5	0.10350577224179846609	4018
739	5	0.10706905313737192318	4561
787	5	0.01395924800465447269	1425
811	7	0.17939743855163310334	60156
827	7	154.81828584920445528361	7220288
859	7	0.00003489503731419296	1202
883	3	1.46744585216089836663	4031
907	3	0.59650901915093688786	2888
947	5	0.03074953029629406725	62879
971	15	0.04674641781017625752	846868715
1019	13	0.23524712311785247652	481254336
1051	5	0.15568884564868710538	48000
1091	17	0.00217725207751161362	4271088999
1123	5	0.07364280939031303020	23322
1163	7	0.37473402025732946137	6435488
1171	7	0.00082674907382398778	33427
1187	9	8.57806631683350786688	196575884
1259	15	0.04793034848358401730	23807653018
1283	11	8.90099754034007899869	2309889447
1291	9	0.00241815361075660441	1270411
1307	11	16150.89301909056086676059	186585360146
1427	15	1.56491207854150961696	208776957205
1451	13	5.78796172237250084560	198287772553
1459	11	0.02184103208570492546	81647642
1483	7	3.19082650762232731055	6018956
1499	13	0.00107052328295353035	3762443612
1523	7	13.14472327791761386093	432126977
1531	11	0.00934308869252333838	91517708
1571	17	77374.81778094942048006157	4227257131159937
1579	9	0.04967065171792227175	18842731
1619	15	0.01588367050528775720	287671778919
1627	7	1.14234502669777570952	8445329
1667	13	3015.72287790923136695989	5415661616355
1699	11	0.00000951105549164743	7161466
1723	5	0.59636225379566621680	627691
1747	5	0.63250816243545283913	545202
1787	7	0.85334738271374115107	618910949
1811	23	0.06992899432395535990	16061273092160342
1867	5	2.38591921224867532372	6297927
1907	13	6.99689763033106016118	3165747533075
1931	21	90.38028537368310596632	258729934559477369
1979	23	4142.79431454502235459380	2198667664214704926 3
1987	7	0.00236988066983797578	2280430
2003	9	0.01742633984133732059	2517289989
2011	7	0.00053306208627154529	2638203
2027	11	0.14670352596641644330	58540694152
2083	7	0.05598543472754948453	4585113
2099	19	0.03659270303763838186	2787484250243039
2131	13	0.08176766229504099310	89178286597
2179	7	0.00000665857111220152	527657
2203	5	1.91715188964021586919	15965226
2243	15	455169.94065435589748159377	81753149480991824
2251	7	0.02300309205779565400	64040332
2267	11	7.28105312585640396026	4870756173147
2339	19	0.97878697785561566356	59954610410365278
2347	5	1.60476528633349239697	7286700
2371	13	0.00000800495831506833	3115285126
2411	23	504.85388397709402086354	362000857310467738783
2459	19	0.93923549011641697155	117489182977955129
2467	7	0.09990402861216668317	100745543
2531	17	0.00014747927341115960	306040338160927
2539	11	0.00011234010687514967	3085577520
2579	21	15539.25782562482093805923	758227028237459575815
2659	13	0.14349748145910982056	1922642144666
2683	5	0.35221319600572141285	14543262
2699	15	0.00003740841184509724	50729716969650
2707	7	0.17461587946412744900	348539119
2731	11	0.00000353700779516742	321349458
2803	9	0.78284831007160905719	6224866339
2819	21	0.27671024228777928761	204772771725896988 69
2843	15	10.14431840386199781185	4022140478178599
2851	11	0.00056503632880051287	11870467976
2939	29	0.00624713894355604616	10526813787500331295354 7
2963	13	1.32077115136644560933	319550765817053
			Continued on next page

q	h	$L(E/H,1)$	$\sqrt{\#(Ш(E/H))}$
2971	11	0.03190494829015117475	324491996326
3011	21	0.26180404016133656775	3395023855916546489 6
3019	7	0.04312030585447679517	1802300809
3067	7	0.50159638459199363857	383973387
3083	13	66.44365839794315801657	15879087340845386
3163	9	0.00000102202290136419	76089107
3187	7	0.21740590516706865825	569775265
3203	11	7.58674726638825494993	378391105040096
3251	31	1.49283692874036018378	2081449420722283950255 06250
3259	9	0.01121435077834574673	21243653932
3299	27	0.00000054505993369606	403911133854039617472
3307	9	5.24935939959546086799	190357002279
3323	17	7353.12213893642943109960	56954453746238785334
3331	15	0.00013628808312750543	12824146340774
3347	11	0.00693687813140206147	15997504388590
3371	21	0.00000905587936280291	1798520024157799359
3467	19	0.14118916689423367305	2179702892746704617
3491	23	1521.81323914654475213527	221691728204379185381267 7
3499	11	0.00000048245603947392	12562962745
3539	23	0.00000000024192865064	502868601762633637
3547	9	0.51723990155998270507	64134211184
3571	15	0.00001877157784780692	5298557564473
3643	9	0.19035162141447944342	196656106527
3659	29	0.52058334069619626047	2243459184984706616165720 59
3691	13	0.00010142020688338228	2190791538023
3739	11	0.00021638831563001181	281259932931
3779	31	8.11514180382761610358	2931962725595777637940231 0856
3803	15	617.93079127720341400907	6125701430298575603
3851	25	0.60046598691752433358	1242978436326110671361250
3907	7	0.03397552970283538243	1708500805
3923	23	0.06601853781292555224	1616051104915262246246 6
3931	11	0.00034265756041784101	281879748512
3947	17	1.75074743805447665390	28850358440768828838
4003	13	0.00530457688125282635	13618654340552
4019	19	14.89793441535627886639	1020172843078608753309 6
4027	9	0.06278002336060817839	27507874104
4051	11	0.00000004836198044776	32711170970
4091	33	0.00015048845055910840	1311910319936381687227227 5835
4099	15	0.00009653216942583305	135256029258383
4139	19	0.00018105359839238306	80074451508066303468
4211	23	0.00001204796805032161	4490981695211565226634
4219	15	0.03461154251357343869	12962166563665575
4243	9	0.00066320718752067075	4758420837
4259	35	0.32159852619627858305	1675950477293011639148264 1893012
4283	21	8773.29312623157893512197	15356437897411686412796 99
4339	17	0.00136435844805308578	41415425722868069
4363	9	0.00169672983900952063	12072493829
4451	29	0.52221992024973798611	5724346890410456077621917 1927
4483	9	345.39066341572129604016	90594797052049
4507	13	50.23492940730687592262	8000361069297882
4523	21	0.47925191687294653824	1269668961826947922246 3
4547	17	0.00237327326500158548	3365011327793496632
4603	7	0.67370851258063680284	37507414772
4643	13	0.05750243316638869235	143101155917879067
4651	17	0.00001544101918254929	15235214493740502
4691	21	0.00000333535197828509	2621016573927853606584
4723	9	0.95386187065080598582	5224661311515
4787	25	196299950.09431612703528249287	1407786352476914766515861 602276
4931	35	0.00000002277116284325	4480079863031129057155676 52356
4987	9	0.08774530421699530811	1780109232599
5003	15	10706.38070422879767696786	3439511694539863522013 9
5011	21	0.00050859723642713633	911599106036522307 20
5051	29	8.49904519468908134660	1141170772780262770088006 751376
5059	19	0.00156551345932179882	98676291037454285 09
5099	39	0.20398131340164447660	1184616873402209140039591 463547587231
5107	7	0.41759384988925603185	40655661145
5147	19	9.89685243132721839321	1104803380907477889807 62
5171	35	0.36860175091251048590	7458685927940450075817149 649785164
5179	11	0.00241753460843546585	432857713333757
5227	15	0.02590550552009961454	28726219588915216
5323	15	0.63466138387854294475	101661479910418479
5347	13	48.38774889967386398841	60377540719495011
5387	23	46969768.16794671957479225493	5986189185208903010040976 6615
5419	13	0.00155877501356795214	3346444025253455
5443	9	0.00772315292672455124	259851370654
5483	17	0.00000850543167234778	12838230594323889723
5507	23	1196283.61113816912742549060	3089770105624374530373859 49809
5531	23	0.00787997073850461781	1149287163481763661129444 26
5563	15	0.17610681503569755604	289650044515887808
5651	31	0.00001430611962530720	1053842731936249031364945 295419
5659	19	0.00000095193065835809	2612065297147014949
5683	11	44.34550258638220009604	12037895727172693
5779	13	0.00020942032710240135	1281689314164362
5827	15	0.00301443397675326391	76300991821550264
5843	25	5443491913.33884952923429281132	9651942905302234307183506 05440606
5851	21	0.00045098763594953089	4466493231539670036837
5867	21	2.55857313302494265980	6273825474355118152363875
5923	7	0.58161114692979764321	287690477472
5939	35	603122.99403995879567858207	1661965438061076810127871 901025304745853
5987	15	110.46116247014125681999	1386564075494413587343 1
6011	27	4.86492240549697162717	8721683005751328093023655 3759140
6043	9	0.00721258251692584701	635377400757
6067	15	0.00083214765715847858	66156492166979308
6091	15	0.00004657801554426414	37563167157316048
6131	31	0.47402035862479912750	1058880862406647363504965 2220685900
6163	11	0.00109962773126547504	212505143679808

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q	h	$L(E/H,1)$	$\sqrt{\#(Ш(E/H))}$
6203	17	16956.91608309776740115902	17758388633281304578194836
6211	15	0.00006166346655870246	143325584038005751
6299	43	2.05775857985552601858	1388856212566707742241274725107204020014609
6323	21	0.15649673018585925176	5097012104044281804792939
6379	17	0.00000844841271024925	2826990910532861162
6427	9	0.04949970413006985692	3642618058167
6451	17	0.00000000015961944529	4513598851205200
6491	31	0.00000661570599992748	26061044582689107840212347433528
6547	11	0.00049212375667197424	28027440443663
6563	23	331287.05252863171559172025	875165582628365469003240470630
6571	15	0.00113860246370370679	6769437817419744845
6619	13	0.01464489531673771186	228222213173218843
6659	23	0.00004608564886020156	3960701905118801624128927055
6691	21	0.00076792513778405932	44437713772698490604378
6763	9	0.02842453885205402194	12452915791145
6779	39	4.86957703804081099214	69700200603218027480784056371040211086176
6803	19	3887.42979178406639019887	18785753293448529197999195649
6827	17	14.01183337740201353984	3988442850286746357347994
6883	9	0.00267729911739092469	8323175699073
6899	35	0.00001267331994634709	216639728383074935163216743914068125
6907	17	0.00034176219626136351	7191713364116285807
6947	29	9807687.61625312683632853546	1396721892374430002050625825823542808
6971	45	144.95259877715406861536	16300433495176531353395464824906912329257404117
7019	43	0.00006851095988046843	419823130126477043502027992927631092428380
7027	11	0.00126334189165518431	372846808979421
7043	23	79.57658531732701207353	42967244325677025460819829585
7187	25	1941.80715024988578872060	111790555942583682538467190725164
7211	35	0.00000013542177935543	92297726998496877729409205982930880
7219	15	0.00005574430738945572	1293006824668014040
7243	13	12.33651259315172930941	1756240279022402156
7283	25	2.49002172917008268374	10685692353679063484178785461948
7307	25	62171.26156087145465015128	1419081250825229997211797415503825
7331	33	0.00000000032502361559	219455139795728684736426909205613
7411	25	0.00015095236624874787	283024230782339795658408815
7451	35	93950.78490910382748157143	1553319522388195483106386505386846710579688
7459	15	0.00003036649237987241	2337968558206904465
7499	33	0.00000135124978504196	60792334311439101075808649223338741
7507	11	0.00010611225020431393	506903678439547
7523	35	33522.02988333545407545411	15470691108260141377130066889443447728251
7547	15	0.60032452144273867428	258920923898858718493581
7603	11	0.00653892833763582491	1434870589808453
7643	29	179270.99125915725948310242	2288294534095276709215678772272550333
7691	43	1936270.78489178696965784529	3624801748457692072742156116148403862768295124739
7699	27	0.00000699702201774193	2304258410337272194320291613
7723	9	0.01895970678708005242	46425972727345
7867	11	1.61910857940903382775	969216189901440388
7883	17	0.23552172495493403317	28080473969786372085402707
7907	21	0.00175081994226267973	568887022906905104307160000
7963	13	0.00002874140087673655	1773163400695878
8011	25	0.00000000000000681929	5972318120577118158596
8059	21	0.00000000008523807585	384252254712642261955
8123	21	2383.30719868549773975742	18940429636369330011648211950850
8147	37	12176247.61615672604210432982	108543217201536511344104411121816314211436357
8171	21	0.00001153050501312074	43709209245428902074487555857
8179	25	0.00000000000408693701	245379074777785127542855
8219	35	0.00003219371470307854	320575044136527674847638735156405728284
8243	21	0.00003184119831052584	169936812859198942971657899
8291	47	13.82276752176918097863	144216236301930044357917532496691060210143449492929
8363	35	232.06334304127124142727	15312049368913584760165020656408258883985
8387	21	1545.33342862710955449647	646162618485014801463564772032
8419	19	0.00000012646740535682	7680287320591089475912
8443	11	1.17893537822797473748	35478826850611232
8467	15	0.00021223474007323571	1265829019458818985
8539	17	0.00000478356805956157	2275177608844149620235
8563	9	2.03683466086338426663	5420212257148597
8627	21	3853.12162724028605983782	198171648305412280420765428378075
8699	35	0.00002481823232900755	11240439275853225767486298164708391703500
8707	15	0.05248611135357774651	544957212659766295149
8731	17	0.00003500145787893966	1576791412901313685993
8747	21	484018.07264403016140460258	288450573402627146790548293522904
8779	15	0.00000030582300865004	3039062315648643705
8803	9	0.01552030193111727803	126149700575776
8819	49	4034780.63025405890013810127	23921217585664253728980536953688630289914216264814750750
8867	27	2.17002992458273671577	128281233388318287503394675078625248
8923	19	159.33124314583988814840	14174087757188097620330473
8963	29	1985418.83735640580162530103	1118129255137682166151656920634902507171
8971	19	0.00000000983100532208	2258409802656205931201
9011	33	0.00335393090412302635	3045909543390800207752459587622105801155
9043	15	0.10664966467865754486	247161131467830177656
9059	39	13.77046293947696022352	1591196881664881653149442733268746342271235079
9067	9	0.00731970822333483793	111048528074239
9091	21	0.00000000236926748020	236934738121647151313216
9187	21	32.52092107710830419611	366682928077533301738876375
9203	31	1023606763.23245034828055208642	10450577199144964797897931682954499342069972
9227	25	0.03059726342701622948	61065081813711319346093946656236
9283	11	0.00312668644337035262	114634424721388682
9323	29	158279113.16389204676979182791	198428451440095157640040934380936648153653
9371	49	0.00000000381610952189	9868042349887290226589618012608800982902276255554
9403	11	0.00062444568609082732	9557020329789716
9419	35	0.00004080832458224277	7755628001204338180900900374435880983024
9467	41	85262.97994248194021607026	712735551990570229824239164893859490851236314718
9491	45	1.93118062921922617356	699904157096559596488802074953836695863315457589115
9539	55	0.00000011262029208767	1854017230259878781685683081695866642830423627047547360
9547	13	4.55503594583652426666	636512933633007687775
9587	23	100123.63380876703245300853	36563316420884623249966367262977278
9619	19	0.00000012220071093949	5176494409557611828238

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q	h	$L(E/H,1)$	$\sqrt{\#(Ш(E/H))}$
9643	11	0.00824565865962835292	516445608289196035
9739	13	0.00000131331766917028	3753135247337182744
9787	11	0.26005393619869399226	362054016521593344
9803	37	5930542264.64370531353643671288	3056997114849536281991895391097536748139790872620
9811	21	0.00000000057298287826	127050303874152535296768
9851	45	0.00000000000000654205	18104457473462858976938437755791606152672401
9859	21	0.00000242965527955730	28849640043711849588277760
9883	17	3.13476671971555114510	242746734273754376709691
9907	15	0.00104023793825974741	106327569146508318840
9923	25	0.00000242632276300466	2387019558186614148757292853595
9931	23	0.00000000000063190232	807612994249556860279774
10067	21	518.08680600795693371589	342288622319158792879184584048129
10091	39	0.00000478306284603539	33197255379943602421429579685905522266248888
10099	25	0.00000000001780723538	52822827121201782525001225
10139	55	73803.78400959271904263740	33150521228939817885342694776696740143379123299513891375082064
10163	39	67991.83407099603855149887	440513245214192270403084087547571959843972111425
10211	43	36.21163226407725524787	2018436122269325072524514224058369879912299468812420
10243	15	0.08816299542047442521	1839984519589335519209
10259	43	652.29068417959804070259	718692371932632451954249149920382743582780046485373
10267	15	0.00014937068406596087	382939450567034881777
10331	55	0.00062202944015487748	6664989351016927856864678973892159602391574133103405618788
10427	31	43.44997691536163284782	2845156404392349071355292880344541653644
10459	15	0.00028151807209305816	4605303213234756184743
10499	41	507989.87585259730539676742	36955062681702947421893940479768814784946674217168545
10531	27	0.00000000000146797238	3351769586174993494323439275
10627	9	0.00017318804748393473	398694874516979
10651	15	0.00011268583573677928	12553194111400362264301
10667	39	25506036465736.67223882308179488976	44811079267440631723932527212656541784290245322216891
10691	45	3520.56248376976938840747	232694011311673350014283799021469164888464834922982077
10723	15	0.00000092340639534283	21063173152441592413
10739	37	0.00000000004343365251	63654774159741723864112646151198426204132
10771	21	0.00006625182262625102	6376645704546135373491729472
10859	45	0.00001929088385542839	144220156391029884341565008365356176802926959516032
10867	23	0.00238767061834684639	10525038031112725310622398635
10883	23	515416.17141491515842049134	87346650878073993681932895672040241977
10891	19	0.00000016242472588156	11884817420032518255654205
10939	27	0.00000000901518555777	384273411889144154980660519291
10979	35	0.00368603736420883092	3439622056704005260743458738409400370601464092
10987	11	0.00349823665853601300	20844399361215150
11003	27	2285808.76686708228105121178	215352519101955536146438795127413473169151
11027	27	1.79342141479218417410	1218295757801391198254967804816697855
11059	25	0.00000017029649146888	339791808371879703974813638561
11083	15	0.00029395864104112152	69130942815185383643
11131	25	0.00000000011466927704	779081246955355170396973552

After all the calculations were finished, John Coates informed us that for q congruent to 3 modulo 8, one should actually take the square root of the $j(\mathcal{O}_K) - 12^3$ with negative imaginary part to get the appropriate quadratic twist of the Gross curve A (see the formula (2.2) of the paper by Buhler and Gross [2]). Hence, we should work with the conjugate of the equation (1) under the Galois group of H over $\mathbb{Q}$. We have checked for a few small q congruent to 3 modulo 8, that the L-values, torsion parts and Tamagawa numbers of the two conjugate curves are the same. As a consequence, the analytic orders of Tate-Shafarevich groups of these curves are the same. John Coates expects that Tate-Shafarevich groups of these curves are actually isomorphic, but all is not totally clear theoretically.

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