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# NEW IRREDUCIBLE TENSOR PRODUCT MODULES FOR THE VIRASORO ALGEBRA*

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**Abstract.** In this paper, we obtain a class of Virasoro modules by taking tensor products of the irreducible Virasoro modules $\Omega(\lambda, \alpha, h)$ defined in [CG1], with irreducible highest weight modules $V(\theta, h)$ or with irreducible Virasoro modules $\mathrm{Ind}_\theta(N)$ defined in [MZ2]. We obtain the necessary and sufficient conditions for such tensor product modules to be irreducible, and determine the necessary and sufficient conditions for two of them to be isomorphic. These modules are not isomorphic to any other known irreducible Virasoro modules.

**Key words.** Virasoro algebra, tensor products, non-weight modules, irreducible modules.

**Mathematics Subject Classification.** 17B10, 17B20, 17B65, 17B66, 17B68.

## 1. Introduction.

Let $\mathbb{C}, \mathbb{Z}, \mathbb{Z}_+$ and $\mathbb{N}$ be the sets of all complexes, all integers, all non-negative integers and all positive integers respectively. The **Virasoro algebra** Vir is an infinite dimensional Lie algebra over the complex numbers $\mathbb{C}$, with the basis $\{d_i, c \mid i \in \mathbb{Z}\}$ and defining relations

$$[d_i, d_j] = (j-i)d_{i+j} + \delta_{i,-j}\frac{i^3-i}{12}c, \quad i, j \in \mathbb{Z},$$

$$[c, d_i] = 0, \quad i \in \mathbb{Z}.$$

The algebra Vir is one of the most important Lie algebras both in mathematics and in mathematical physics, see for example [KR, IK] and references therein. The representation theory of the Virasoro algebra has been widely used in many physics areas and other mathematical branches, for example, quantum physics [GO], conformal field theory [FMS], vertex operator algebras [LL], and so on.

The theory of weight Virasoro modules with finite-dimensional weight spaces (called Harish-Chandra modules) is fairly well developed (see [KR, FF] and references therein). In particular, a classification of weight Virasoro modules with finite-dimensional weight spaces was given by Mathieu [M], and a classification of irreducible weight Virasoro modules with at least one finite dimensional nonzero weight space was given in [MZ1]. Later, many authors constructed several classes of simple non-Harish-Chandra modules, including simple weight modules with infinite-dimensional weight spaces (see [CGZ, CM, LLZ, LZ2]) and simple non-weight modules (see [BM, LGZ, LLZ, LZ1, MW, MZ1, TZ1, TZ2]).

In particular, taking tensor products of known irreducible modules is an efficient way to construct new irreducible modules and can help us understand the structures of the original modules. For example, the tensor products of irreducible highest

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weight modules and intermediate series modules were considered first by [Zh] and the irreducibility of these tensor modules are completely determined by [CGZ] and [R]. Recently, another class of tensor products between certain Omega modules defined and studied in [GLZ, LZ1] and some modules locally finite over a positive part defined in [MZ2] were studied in [TZ1, TZ2]. The irreducibilities and isomorphism classes of these modules are determined.

The purpose of the present paper is to construct new irreducible non-weight Virasoro modules by taking tensor products of irreducible Virasoro modules defined in [CG1] and [MZ2]. When considering modules for the W algebra $W(2,2)$ which are free of rank-1 when restricted to the 0 part of the algebra, the authors constructed a class of new irreducible $W(2,2)$-modules $\Omega(\lambda,\alpha,h)$ in [CG2]. Since the Virasoro algebra is a natural subalgebra of $W(2,2)$, one can regard $\Omega(\lambda,\alpha,h)$ as Virasoro modules. The explicit structures of these Virasoro modules are investigated in [CG1], and it is interesting that many of them remains irreducible as Virasoro modules.

The Vir-modules $\Omega(\lambda,\alpha,h)$ are quite different to and have more complicated structures than the previous modules $\Omega(\mu,b)$ define in [GLZ, LZ1], although we use similar notations for them. For example, the modules $\Omega(\mu,b)$ are only parameterized by two complexes $\mu$ and $b$, while the modules $\Omega(\lambda,\alpha,h)$ are parameterized by two complexes $\lambda,\alpha$ and an additional polynomial $h(t)\in\mathbb{C}[t]$; the modules $\Omega(\mu,b)$ are free of rank just 1 over the Cartan subalgebra, while the modules $\Omega(\lambda,\alpha,h)$ are free of infinite rank; the modules $\Omega(\mu,b)$ are irreducible if and only if $b\neq 0$, while the modules $\Omega(\lambda,\alpha,h)$ are irreducible if and only if $\deg(h)=1$ and $\alpha\neq 0$; the reducible module $\Omega(\mu,0)$ has a unique submodule which has codimensional 1, while the submodule structures of the module $\Omega(\lambda,\alpha,h)$ are much more complicated when they are reducible; the isomorphisms and automorphisms among the modules $\Omega(\mu,b)$ are almost trivial, while the isomorphisms and automorphisms among the modules $\Omega(\lambda,\alpha,h)$ are of various type (see Lemma 3.3 and Theorem 3.4).

In the present paper, we continue to study the Virasoro modules $\Omega(\lambda,\alpha,h)$. Our main tasks are to show that the irreducible ones of the modules $\Omega(\lambda,\alpha,h)$ are new Virasoro modules and to consider the tensor products of the modules $\Omega(\lambda,\alpha,h)$ and the modules with locally finite action of the positive part defined in [MZ2]. The organization of this paper is as follows. In section 2, we recall the definitions of the modules $\Omega(\lambda,\alpha,h)$, $V(\theta,h)$ and $\mathrm{Ind}_\theta(N)$ and some known results from [CG1] and [MZ2]. In section 3, we obtain the irreducibility of the tensor products $\Omega(\lambda,\alpha,h)\otimes V$, where $V=V(\theta,h)$ or $V=\mathrm{Ind}_\theta(N)$. Then we determine the necessary and sufficient conditions for two irreducible tensor modules to be isomorphic. In section 4, we compare the tensor products modules with all other known non-weight irreducible modules and prove that they are new irreducible Virasoro modules (in particular, the irreducible ones of $\Omega(\lambda,\alpha,h)$ are new). At last in Section 5, we reformulate these modules as modules induced from irreducible modules over some subalgebras of Vir.

In our subsequent paper [LGW], the main results in this paper are generalized to tensor products of several Omega modules (maybe different types) with the module $\mathrm{Ind}_\theta(N)$.

**2. Preliminaries.** Let us first recall the definition of the Virasoro modules $\Omega(\lambda,h,\alpha)$, $V(\theta,h)$ and $\mathrm{Ind}_\theta(N)$ and some basic properties of them. Denote by $\mathbb{C}[t,s]$ the polynomial ring in two variables $t$ and $s$.

Definition 2.1.

Fix any $\lambda\in\mathbb{C}^*=\mathbb{C}\setminus\{0\}$, $\alpha\in\mathbb{C}$ and $h(t)\in\mathbb{C}[t]$. Let $\Omega(\lambda,h,\alpha)=\mathbb{C}[t,s]$ as a vector space and we define the Vir-module action as follows:

$$d_m\big(f(t)s^i\big) = \lambda^m(s-m)^i\Big(\Big(s+mh(t)-m(m-1)\alpha\frac{h(t)-h(\alpha)}{t-\alpha}\Big)f(t)-m(t-m\alpha)f'(t)\Big),$$

$$c\big(f(t)s^i\big) = 0, \ \ \forall \ \ m \in \mathbb{Z}, i \in \mathbb{Z}_+,$$

where $f \in \mathbb{C}[t]$ and $f'(t)$ is the derivative of $f$ with respect to $t$.

For convenience, we define the following operators

$$F(f) = \frac{h(t)-h(\alpha)}{t-\alpha}f(t) - f'(t), \quad G(f) = h(\alpha)f + tF(f), \ \forall \ f \in \mathbb{C}[t]. \tag{2.1}$$

then the module action on $\Omega(\lambda, h, \alpha)$ can be rewritten as

$$d_m\big(f(t)s^i\big) = \lambda^m(s-m)^i\Big(sf + mG(f) - m^2\alpha F(f)\Big), \quad \forall \ m \in \mathbb{Z}, i \in \mathbb{Z}_+, f \in \mathbb{C}[t]. \tag{2.2}$$

THEOREM 2.2 ([CG1]). *The Virasoro module $\Omega(\lambda, \alpha, h)$ is simple if and only if $\deg(h) = 1$ and $\alpha \neq 0$.*

Let $U := U(\mathrm{Vir})$ be the universal enveloping algebra of the Virasoro algebra Vir. For any $\theta, h \in \mathbb{C}$, let $I(\theta, h)$ be the left ideal of $U$ generated by the set

$$\{d_i | i > 0\} \bigcup \{d_0 - h \cdot 1, c - \theta \cdot 1\}.$$

The Verma module with highest weight $(\theta, h)$ for Vir is defined as the quotient module $\bar{V}(\theta, h) := U/I(\theta, h)$. It is a highest weight module of Vir and has a basis consisting of all vectors of the form

$$d_{-1}^{k_{-1}} d_{-2}^{k_{-2}} \cdots d_{-n}^{k_{-n}} v_h, \ k_{-1}, k_{-2}, \cdots, k_{-n} \in \mathbb{Z}_+, n \in \mathbb{N},$$

where $v_h = 1 + I(\theta, h)$. Any nonzero scalar multiple of $v_h$ is called a highest weight vector of the Verma module. Then we have the irreducible highest weight module $V(\theta, h) = \bar{V}(\theta, h)/J$, where $J$ is the unique maximal proper submodule of $\bar{V}(\theta, h)$. For the structure of $V(\theta, h)$, please refer to [FF] or [A].

Denote by $\mathrm{Vir}_+$ the Lie subalgebra of Vir spanned by all $d_i$ with $i \geq 0$. For $n \in \mathbb{Z}_+$, denote by $\mathrm{Vir}_+^{(n)}$ the Lie subalgebra of Vir generated by all $d_i$ for $i > n$. For any $\mathrm{Vir}_+$ module $N$ and $\theta \in \mathbb{C}$, consider the induced module $\mathrm{Ind}(N) := U(\mathrm{Vir}) \otimes_{U(\mathrm{Vir}^+)} N$, and denote by $\mathrm{Ind}_\theta(N)$ the module $\mathrm{Ind}(N)/(c-\theta)\mathrm{Ind}(N)$. These modules are used to give a characterization of the irreducible Vir-modules such that the action of $d_k$ are locally finite for sufficiently large $k$.

THEOREM 2.3 ([MZ2]). *Assume that $N$ is an irreducible* $\mathrm{Vir}_+$*-module such that there exists $k \in \mathbb{N}$ satisfying the following two conditions:*

(a) *$d_k$ acts injectively on $N$;*

(b) *$d_i N = 0$ for all $i > k$.*

*Then for any $\theta \in \mathbb{C}$ the* Vir *module* $\mathrm{Ind}_\theta(N)$ *is simple.*

THEOREM 2.4 ([MZ2]). *Let $V$ be an irreducible* Vir *module. Then the following conditions are equivalent:*

(1) *There exists $k \in \mathbb{N}$ such that $V$ is a locally finite* $\mathrm{Vir}_+^{(k)}$*-module;*

(2) *There exists* $n \in \mathbb{N}$ *such that* $V$ *is a locally nilpotent* $\mathrm{Vir}_+^{(n)}$*-module;*
(3) *Either* $V$ *is a highest weight module or* $V \cong \mathrm{Ind}_\theta(N)$ *for some* $\theta \in \mathbb{C}$, $k \in \mathbb{N}$ *and an irreducible* $\mathrm{Vir}_+$*-module* $N$ *satisfying the conditions* (a) *and* (b) *in Theorem 2.3.*

In the rest of the paper, we will always fix some Vir-module $\Omega(\lambda, \alpha, h)$ with $\lambda \in \mathbb{C}^*, \alpha \in \mathbb{C}$ and $h \in \mathbb{C}[t]$, and an irreducible Vir-module $V$ such that each $d_k$ is locally finite (equivalently, locally nilpotent) on $V$ for any positive integer $k$ large enough. From Theorem 2.4, we know that either $V \cong V(\theta, h)$ for some $\theta, h \in \mathbb{C}$ or $V \cong \mathrm{Ind}_\theta(N)$ as described in Theorem 2.3.

**3. Irreducibility of the module** $\Omega(\lambda, \alpha, h) \otimes V$**.** In this section we will investigate the structure of the Virasoro module $\Omega(\lambda, \alpha, h) \otimes V$, where $V$ is the Virasoro module as stated at the end of Section 2. In particular, we will determine its irreducibility. The following technique lemma is similar to Proposition 3.2 of [CG1].

PROPOSITION 3.1. *Let* $W$ *be a subspace of* $\Omega(\lambda, \alpha, h) \otimes V$ *which is stable under the action of any* $d_m$ *for* $m$ *sufficiently large. Take any* $w = \sum_{i=0}^{r} a_i(t) s^i \otimes v_i \in W$ *for some* $a_i(t) \in \mathbb{C}[t]$ *and* $v_i \in V$*, then for any* $0 \le j \le r+2$ *we have*

$$\sum_{i=j-2}^{r} \left( \binom{i}{j} a_i s^{i-j+1} - \binom{i}{j-1} G(a_i) s^{i-j+1} - \binom{i}{j-2} \alpha F(a_i) s^{i-j+2} \right) \otimes v_i \in W, \quad (3.1)$$

*where the operators* $F, G$ *are defined in* (2.1) *and we make the convention that* $\binom{0}{0} = 1$ *and* $\binom{i}{j} = 0$ *whenever* $j > i$ *or* $j < 0$*. In particular,*
(1) *when* $j = 0$*, we have* $sw \in W$*;*
(2) *when* $j = r+1$*, we have* $G(a_r) \otimes v_r + \alpha F(a_{r-1}) \otimes v_{r-1} \in W$*;*
(3) *when* $j = r+2$*, we have* $\alpha F(a_r) \otimes v_r \in W$*.*

*Proof.* The element in (3.1) is just the coefficient of $m^j$ if one expands $d_m w$ as a polynomial in $m$. Then (3.1) follows by using the Vandermonde's determinant. □

THEOREM 3.2. *The module* $\Omega(\lambda, \alpha, h) \otimes V$ *is irreducible if and only if* $\Omega(\lambda, \alpha, h)$ *is irreducible, or more precisely, if and only if* $\deg(h) = 1$ *and* $\alpha \neq 0$*.*

*Proof.* We only need to prove the "if part". Suppose that $\Omega(\lambda, \alpha, h)$ is irreducible, then by Theorem 2.2, we have $\deg(h) = 1$ and $\alpha \neq 0$. Set $h(t) = \xi t + \eta$ for convenience.

Let $W$ be a nonzero submodule of $\Omega(\lambda, \alpha, h) \otimes V$. It is enough to show $W = \Omega(\lambda, \alpha, h) \otimes V$. Take any nonzero element $w = \sum_{i=0}^{r} a_i s^i \otimes v_i \in W$ with $a_i \in \mathbb{C}[t], v_i \in V$ such that $r \in \mathbb{Z}_+$ is minimal. By Proposition 3.1, we have $\alpha F(a_r) \otimes v_r \in W$. Since $F(a_r) = \xi a_r - a_r' \neq 0$, by the minimality of $r$, we have $r = 0$ and hence $a_0 \otimes v_0 \in W$. Fix this $v_0$ and we denote

$$X = \{a \in \mathbb{C}[t, s] \mid a \otimes v_0 \in W\}.$$

By Proposition 3.1 (3.1) and (3.1), we see that $f \in X \cap \mathbb{C}[t]$ implies that $F(f) = \xi f - f', G(f) = h(\alpha) f + \xi t f - t f' \in X \cap \mathbb{C}[t]$, or, equivalently, $f', tf \in X$. Using this, we can easily deduce that $\mathbb{C}[t] \subseteq X$ from $0 \neq a_0 \in X \cap \mathbb{C}[t]$. Now Proposition 3.1 (3.1) indicates that $X$ is stable under the multiplication by $s$. Hence $X = \mathbb{C}[t, s] = \Omega(\lambda, \alpha, h)$. Now let

$$Y = \{v \in V \mid \Omega(\lambda, \alpha, h) \otimes v \in W\}.$$

Again $Y$ is nonzero and the module action

$$d_i(a \otimes v) = d_i a \otimes v + a \otimes d_i v, \quad \forall\, a \in \mathbb{C}[t, s], v \in Y$$

implies that $Y$ is a submodule of $V$. Hence $Y = V$ and $W = \Omega(\lambda, \alpha, h) \otimes V$, as desired. ∎

Then we can determine the necessary and sufficient conditions for two such irreducible modules to be isomorphic. Before doing this, we first construct some isomorphisms.

Given any $\lambda, \alpha_1, \alpha_2 \in \mathbb{C}^*$, $h_1 = \xi_1 t + \eta_1, h_2 = \xi_2 t + \eta_2 \in \mathbb{C}[t]$ with $\alpha_1 \xi_1 = \alpha_2 \xi_2 \neq 0$, we have the irreducible modules $\Omega(\lambda, \alpha_1, h_1)$ and $\Omega(\lambda, \alpha_2, h_2)$. We define the following sequences $\{b_i, i \in \mathbb{Z}_+\}$ of complex numbers inductively by

$$b_0 = 1, \quad b_1 = 0, \quad \text{and} \quad b_{i+1} = i b_i + i(\eta_2 - \eta_1) b_{i-1}, \ \forall\, i \in \mathbb{N}.$$

Then we have the following sequences of polynomials in the variable $x$:

$$g_n(x) = \sum_{i=0}^{n} \binom{n}{i} b_{n-i} x^i, \ \forall\, n \in \mathbb{Z}_+. \tag{3.2}$$

The following identities can be easily calculated:

$$g_n'(x) = n g_{n-1}(x), \tag{3.3}$$

$$\big(g_{n+1}(x) - x g_n(x)\big) - n\big(g_n(x) - x g_{n-1}(x)\big) = (\eta_2 - \eta_1) n g_{n-1}(x). \tag{3.4}$$

Now we can define a linear map via

$$\phi:\ \Omega(\lambda, \alpha_1, h_1)\ \to\ \Omega(\lambda, \alpha_2, h_2),\ \phi(s^i h_1^n) = s^i g_n(h_2),\ \forall\, n, i \in \mathbb{Z}_+. \tag{3.5}$$

Lemma 3.3. *Let notations as above, then $\phi$ is an isomorphism of* Vir*-modules.*

*Proof.* Denote the operators in (2.1) as $F_i$ and $G_i, i = 1, 2$ for corresponding modules, then it is easy to see that, for all $n \in \mathbb{Z}_+, i = 1, 2$,

$$F_i(h_i^n) = \xi_i h_i^n - n\xi_i h_i^{n-1}, \text{ and } G_i(h_i^n) = h_i^{n+1} + (\alpha_i \xi_i - n) h_i^n + n\eta_i h_i^{n-1}. \tag{3.6}$$

Claim 1. $\phi(\alpha_1 F_1(h_1^n)) = \alpha_2 F_2(g_n(h_2))$ and $\phi(G_1(h_1^n)) = G_2(g_n(h_2))$ for all $n \in \mathbb{Z}_+$.

This can be verified straightforward. For example, the second formula follows from (3.3), (3.4) and the following calculations:

$$\begin{aligned}\phi(G_1(h_1^n)) =& \phi\big(h_1^{n+1} + (\alpha_1\xi_1 - n)h_1^n + n\eta_1 h_1^{n-1}\big)\\ =& g_{n+1}(h_2) + (\alpha_2\xi_2 - n) g_n(h_2) + n\eta_1 g_{n-1}(h_2)\end{aligned}$$

and

$$\begin{aligned}G_2(g_n(h_2)) =& h_2 g_n(h_2) + \alpha_2\xi_2 g_n(h_2) - h_2 g_n'(h_2) + \eta_2 g_n'(h_2)\\ =& h_2 g_n(h_2) + \alpha_2\xi_2 g_n(h_2) - n h_2 g_{n-1}(h_2) + n\eta_2 g_{n-1}(h_2).\end{aligned}$$

Now we see that

$$\begin{aligned}\phi(d_m(s^ih_1^n)) =& \phi(\lambda^m(s-m)^i(sh_1^n + mG_1(h_1^n) - m^2\alpha_1F_1(h_1^n)))\\ =& \lambda^m(s-m)^is\big(\phi(h_1^n) + m\phi(G_1(h_1^n)) - m^2\phi(\alpha_1F_1(h_1^n))\big)\\ =& \lambda^m(s-m)^i\big(sg_n(h_2) + mG_2(g_n(h_2)) - m^2\alpha_2F_2(g_n(h_2))\big)\\ =& d_m(s^ig_n(h_2)) = d_m(\phi(s^ih_1^n)).\end{aligned}$$

That is, $\phi$ is a nonzero homomorphism between the irreducible modules $\Omega(\lambda, \alpha_1, h_1)$ and $\Omega(\lambda, \alpha_2, h_2)$ and hence an isomorphism. □

Theorem 3.4. *Let $\lambda_i \in \mathbb{C}^*$, $\deg(h_i) = 1$ and $\alpha_i \neq 0$, where $i = 1, 2$. Let $V_1, V_2$ be two irreducible modules over* Vir *such that the action of $d_k$ is locally finite on both of them for sufficiently large $k \in \mathbb{Z}_+$. Then $\Omega(\lambda_1, \alpha_1, h_1) \otimes V_1$ and $\Omega(\lambda_2, \alpha_2, h_2) \otimes V_2$ are isomorphic as* Vir *modules if and only if $\lambda_1 = \lambda_2, \alpha_1\xi_1 = \alpha_2\xi_2$ and $V_1 \cong V_2$ as* Vir *modules. Moreover, any such isomorphism is of the form:*

$$\begin{aligned}&\phi \otimes \tau : \quad \Omega(\lambda_1, \alpha_1, h_1) \otimes V_1 \to \Omega(\lambda_1, \alpha_2, h_2) \otimes V_2,\\ &f \otimes v \mapsto \phi(f) \otimes \tau(v),\ \forall\ f \in \mathbb{C}[t, s], v \in V_1,\end{aligned}$$

*where $\phi$ is defined as in* (3.5) *and $\tau$ is an isomorphism between $V_1$ and $V_2$.*

*Proof.* The sufficiency of the theorem follows from Lemma 3.3. We need only to prove the necessity. Let $\varphi$ be a Vir-module isomorphism from $\Omega(\lambda_1, \alpha_1, h_1) \otimes V_1$ to $\Omega(\lambda_2, \alpha_2, h_2) \otimes V_2$. Take a nonzero element $v \in V_1$. Suppose

$$\varphi(1 \otimes v) = \sum_{i=0}^{n} a_is^i \otimes w_i,$$

where $a_i \in \mathbb{C}[t], w_i \in V_2$ with $a_n \neq 0, w_n \neq 0$.

Claim 1. $n = 0, \lambda_1 = \lambda_2$ and $\alpha_1\xi_1 = \alpha_2\xi_2$.

There is a positive integer $K$ such that $d_m(v) = d_m(w_i) = 0$ for all $m \geq K$ and $0 \leq i \leq n$. Taking any $m \geq K$, we have

$$(\lambda_1^{-m-1}d_{m+1} - \lambda_1^{-m}d_m)(1 \otimes v) = \big(h_1(t) - 2m\xi_1\alpha_1\big)(1 \otimes v).$$

Replacing $m$ with another $l \geq K$ and making the difference of them, we get

$$\Big(\big(\lambda_1^{-l-1}d_{l+1} - \lambda_1^{-l}d_l\big) - \big(\lambda_1^{-m-1}d_{m+1} - \lambda_1^{-m}d_m\big)\Big)(1 \otimes v) = 2(m-l)\xi_1\alpha_1(1 \otimes v).$$

Then applying $\varphi$, we obtain,

$$
\begin{aligned}
&2(m-l)\xi_1\alpha_1 \sum_{i=0}^{n} a_i s^i \otimes w_i \\
=&\Big(\big(\lambda_1^{-l-1}d_{l+1} - \lambda_1^{-l}d_l\big) - \big(\lambda_1^{-m-1}d_{m+1} - \lambda_1^{-m}d_m\big)\Big) \sum_{i=0}^{n} a_i s^i \otimes w_i \\
=&\sum_{i=0}^{n}(\lambda_2/\lambda_1)^{l+1}(s-l-1)^i\big(sa_i + (l+1)G_2(a_i) - (l+1)^2\alpha_2F_2(a_i)\big) \otimes w_i \\
&\quad - \sum_{i=0}^{n}(\lambda_2/\lambda_1)^{l}(s-l)^i\big(sa_i + lG_2(a_i) - l^2\alpha_2F_2(a_i)\big) \otimes w_i \\
&- \sum_{i=0}^{n}(\lambda_2/\lambda_1)^{m+1}(s-m-1)^i\big(sa_i + (m+1)G_2(a_i) - (m+1)^2\alpha_2F_2(a_i)\big) \otimes w_i \\
&\quad + \sum_{i=0}^{n}(\lambda_2/\lambda_1)^{m}(s-m)^i\big(sa_i + mG_2(a_i) - m^2\alpha_2F_2(a_i)\big) \otimes w_i.
\end{aligned}
$$

Comparing the coefficients of $s^{n+1} \otimes w_n$ in the above equation, we can deduce that

$$\big((\lambda_2/\lambda_1)^l - (\lambda_2/\lambda_1)^m\big)(\lambda_2/\lambda_1 - 1)a_n = 0, \ \forall\, m, l \geq K,$$

forcing $\lambda_1 = \lambda_2$. Then the previous equation can be simplified as

$$
\begin{aligned}
&2(m-l)\xi_1\alpha_1 \sum_{i=0}^{n} a_i s^i \otimes w_i \\
=&\sum_{i=0}^{n}(s-l-1)^i\big(sa_i + (l+1)G_2(a_i) - (l+1)^2\alpha_2F_2(a_i)\big) \otimes w_i \\
&\quad - \sum_{i=0}^{n}(s-l)^i\big(sa_i + lG_2(a_i) - l^2\alpha_2F_2(a_i)\big) \otimes w_i \\
&- \sum_{i=0}^{n}(s-m-1)^i\big(sa_i + (m+1)G_2(a_i) - (m+1)^2\alpha_2F_2(a_i)\big) \otimes w_i \\
&\quad + \sum_{i=0}^{n}(s-m)^i\big(sa_i + mG_2(a_i) - m^2\alpha_2F_2(a_i)\big) \otimes w_i.
\end{aligned}
$$

Regard it as a polynomial in $m, l \geq K$ with coefficients in $\Omega_2(\lambda_2, \alpha_2, h_2)$. If $n \geq 1$, considering the coefficients of $l^{n+1}$ gives $\alpha_2 F(a_n) = \alpha_2(\xi_2 a_n - a_n') = 0$, contradicting the fact $a_n \neq 0$. So $n = 0$. Hence we have

$$2(m-l)\alpha_1\xi_1 a_0 \otimes w_0 = 2(m-l)\alpha_2 F_2(a_0) \otimes w_0, \ \forall\, m, l \geq K,$$

that is, $\alpha_1\xi_1 a_0 = \alpha_2\xi_2 a_0 - \alpha_2 a_0'$. Therefore we obtain that $\alpha_1\xi_1 = \alpha_2\xi_2$ and $a_0 \in \mathbb{C}$. Without loss of generality, we assume that $a_0 = 1$. Denote $w_0 = w$ and $\lambda_1 = \lambda_2 = \lambda$ in what follows.

Claim 2. There exist polynomials $g_j(h_2)$ in $h_2$ such that $\varphi(h_1^j \otimes v) = g_j(h_2) \otimes w, \ \forall\, j \in \mathbb{Z}_+$.

The claim is clear true for $j = 0$ with $g_0(h_2) = 1$. Now suppose the claim holds for non-negative integers no larger than some $j \in \mathbb{Z}_+$, then for $m \geq K$, we have

$$\begin{aligned}\varphi(d_m(h_1^j \otimes v)) =&\varphi\Big(\lambda^m\big(sh_1^j + mG_1(h_1^j) - m^2\alpha_1 F_1(h_1^j)\big) \otimes v\Big)\\ =&d_m\varphi(h_1^j \otimes v) = d_m(g_j(h_2) \otimes w)\\ =&\lambda^m\big(sg_j(h_2) + mG_2(g_j(h_2)) - m^2\alpha_2 F_2(g_j(h_2))\big) \otimes w.\end{aligned}$$

Regarding the expressions in the above equation as polynomials in $m$ and comparing the coefficients of $m^2$ and $m$, we deduce by (3.6) that

$$\begin{aligned}\varphi(\alpha_1 F_1(h_1^j) \otimes v) = \alpha_1\varphi\Big(\big(\xi_1 h_1^j - j\xi_1 h_1^{j-1}\big) \otimes v\Big)\\ =\alpha_2 F_2(g_j(h_2)) \otimes w = \alpha_2\big(\xi_2 g_j(h_2) - \xi_2 g_j'(h_2)\big) \otimes w\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}\varphi(G_1(h_1^j) \otimes v) = \varphi\Big(\big(h_1^{j+1} + (\alpha_1\xi_1 - j)h_1^j + j\eta_1 h_1^{j-1}\big) \otimes v\Big)\\ =G_2(g_j(h_2)) \otimes w = \Big(h_2 g_j(h_2) + \alpha_2\xi_2 g_j(h_2) - h_2 g_j'(h_2) + \eta_2 g_j'(h_2)\Big) \otimes w,\end{aligned} \tag{3.8}$$

where $g_j'$ is the derivative of the polynomial $g_j$. Then we see that $\varphi(h_1^{j+1} \otimes v) = g_{j+1}(h_2) \otimes w$ for a suitable polynomial $g_{j+1}$. The claim follows by induction.

Claim 3. $\varphi(s^i h_1^j \otimes v) = s^i g_j(h_2) \otimes w$, $\forall\ i, j \in \mathbb{Z}_+$, where $g_j$ are defined as in (3.2).

From the equations (3.7) and (3.8) and noticing $\alpha_1\xi_1 = \alpha_2\xi_2$, we deduce

$$g_j'(h_2) = j g_{j-1}(h_2), \tag{3.9}$$

and

$$g_{j+1}(h_2) - j g_j(h_2) + \eta_1 j g_{j-1}(h_2) = h_2 g_j(h_2) - j h_2 g_{j-1}(h_2) + \eta_2 j g_{j-1}(h_2). \tag{3.10}$$

There exists a unique sequence of polynomials $g_j(x)$ in $x$, $j \in \mathbb{Z}_+$ satisfying (3.9), (3.10) and $g_0(x) = 1$, which are just those defined in (3.2), thanks to (3.3) and (3.4).

Now suppose the claim holds for some $i, j \in \mathbb{Z}_+$. For $m \geq K$, we consider

$$\begin{aligned}\varphi(d_m(s^i h_1^j \otimes v)) =&\varphi\Big(\lambda^m(s-m)^i\big(sh_1^j + mG_1(h_1^j) - m^2\alpha_1 F_1(h_1^j)\big) \otimes v\Big)\\ =&d_m(\varphi(s^i h_1^j \otimes v)) = d_m(s^i g_j(h_2) \otimes w)\\ =&\lambda^m(s-m)^i\big(sg_j(h_2) + mG_2(g_j(h_2)) - m^2\alpha_2 F_2(g_j(h_2))\big) \otimes w.\end{aligned}$$

Regard the expressions in the above equation as polynomials in $m$ and comparing the constant terms, we deduce $\varphi(s^{i+1}h_1^j \otimes v) = s^{i+1}g_j(h_2) \otimes w$. The assertion follows by induction.

Claim 4. There exists a Vir-module isomorphism $\tau : V_1 \to V_2$ such that $\varphi(f \otimes v) = \phi(f) \otimes \tau(v)$ for all $f \in \mathbb{C}[t, s], v \in V_1$, where $\phi : \Omega(\lambda, \alpha_1, h_1) \to \Omega(\lambda, \alpha_2, h_2)$ is the Vir-module isomorphism defined by (3.5).

Set $\tau(v) = w$ as in the previous arguments. It is obvious that $\tau$ is a bijective linear map and we have $\varphi(f \otimes v) = \phi(f) \otimes \tau(v)$ for all $f \in \mathbb{C}[t,s], v \in V_1$. Applying $d_m$, we have

$$\begin{aligned}\varphi\big(d_m(f\otimes v)\big) =&\varphi\big((d_m f)\otimes v + f\otimes (d_m v)\big)\\ =&\phi(d_m f)\otimes \tau(v) + \phi(f)\otimes\tau(d_m v)\\ = d_m\varphi(f\otimes v) =&d_m\big(\phi(f)\otimes\tau(v)\big)\\ =&\big(d_m\phi(f)\big)\otimes\tau(v)+\phi(f)\otimes\big(d_m\tau(v)\big).\end{aligned}$$

Since $\varphi$ and $\phi$ are both Vir-module homomorphisms, we have $\tau(d_m v) = d_m\tau(v)$, that is, $\tau$ is a Vir-module isomorphism. □

Note that the proof of Theorem 3.4 is also valid if $V$ is a 1-dimensional trivial module. So we obtain a similar result for the module $\Omega(\lambda,\alpha,h)$ as a corollary.

COROLLARY 3.5. *Let $\lambda_i \in \mathbb{C}^*$, $\deg(h_i) = 1$ and $\alpha_i \neq 0$. Then $\Omega(\lambda_1,\alpha_1,h_1)$ and $\Omega(\lambda_2,\alpha_2,h_2)$ are isomorphic if and only if $\lambda_1 = \lambda_2, \alpha_1\xi_1 = \alpha_2\xi_2$ and the isomorphism are just nonzero multiples of $\phi$ defined in* (3.5).

**4. The irreducible tensor modules are new.** In this section we will compare the irreducible tensor modules with all other known non-weight irreducible Virasoro modules in [LZ1, LLZ, MZ2, MW] and [TZ1, TZ2]. Note that modules in [MW] and [TZ1] are special cases of modules in [TZ2] respectively.

For any $r \in \mathbb{Z}_+, l, m \in \mathbb{Z}$, we recall from [LLZ] the operators

$$\omega_{l,m}^{(r)} = \sum_{i=0}^{r}\binom{r}{i}(-1)^{r-i}d_{l-m-i}d_{m+i} \in U(\mathrm{Vir}).$$

LEMMA 4.1. *Let $\Omega(\lambda,\alpha,h)$ and $V$ be the irreducible* Vir*-modules as before. Then*

(1) *For any integer $n$, the action of $d_n$ on $\Omega(\lambda,\alpha,h)$ or $\Omega(\lambda,\alpha,h)\otimes V$ is not locally finite.*
(2) *Suppose $h(t) = \xi t + \eta, \xi \neq 0$. For any $f(t,s) \in \Omega(\lambda,\alpha,h)$, we have*

$$\omega_{l,m}^{(r)}(f(t,s)) = 0,\ \forall\ l,m,r \in \mathbb{Z}, r > 4,$$

$$\omega_{l,m}^{(4)}(f(t,s)) = 24\lambda^l\alpha^2\big(\xi^2 - 2\xi\frac{\partial}{\partial t} + \frac{\partial^2}{\partial t^2}\big)f(t,s-l) \neq 0,\ \forall\ l,m \in \mathbb{Z}.$$

(3) *For any integer $r > 4$, there exists $v \in V$ and $m, l \in \mathbb{Z}$ such that*

$$\omega_{l,-m}^{(r)}(f\otimes v) \neq 0,\ \forall\ f \in \mathbb{C}[s,t]\setminus\{0\}.$$

*Proof.* (1). It is clear that $d_n^k(f), k \in \mathbb{Z}_+$ are linearly independent in $\mathbb{C}[t,s]$ for any $n \in \mathbb{Z}$ and $f \in \mathbb{C}[t,s]$. So the assertion follows easily.

(2). For any $f(t) \in \mathbb{C}[t]$ and $j \in \mathbb{Z}_+$, by (2.2), we have

$$\begin{aligned}
\omega_{l,m}^{(r)}(s^j f) =& \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} d_{l-m-i} d_{m+i}(s^j f) \\
=& \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} d_{l-m-i} \lambda^{m+i} (s-m-i)^j \cdot \\
& \big(sf + (m+i)G(f) - (m+i)^2 \alpha F(f)\big) \\
=& \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} \lambda^{l} (s-l)^j \cdot \\
& \Big((s-l+m+i)\big(sf + (l-m-i)G(f) - (l-m-i)^2 \alpha F(f)\big) \\
& + (m+i)\big(sG(f) + (l-m-i)G^2(f) - (l-m-i)^2 \alpha FG(f)\big) \\
& - (m+i)^2 \alpha \big(sF(f) + (l-m-i)GF(f) - (l-m-i)^2 \alpha F^2(f)\big)\Big).
\end{aligned}$$

Using the following identity

$$\sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} i^j = 0, \ \forall\ j, r \in \mathbb{Z}_+ \text{ with } j < r, \tag{4.1}$$

we can easily deduce that $\omega_{l,m}^{(r)}\big(s^j f(t)\big) = 0$ provided $r > 4$ and

$$\begin{aligned}
\omega_{l,m}^{(4)}(s^j f) =& \sum_{i=0}^{4} \binom{4}{i} (-1)^{4-i} i^4 \lambda^l (s-l)^j \alpha^2 F^2(f) \\
=& 24 \lambda^l (s-l)^j \alpha^2 \big(\xi^2 f - 2\xi f' + f''\big).
\end{aligned}$$

The result of (2) follows from linearity.
(3). Fix any $r > 4$. Take $v$ to be a highest weight vector in $V$ if $V$ is a highest weight module, otherwise $v$ can be any nonzero vector in $V$. From [FF] and [MZ2] we know that the vectors $v, d_{-2}v, d_{-3}v, \cdots, d_{-r-2}v$ are linearly independent in $V$ and there exists $K \in \mathbb{N}$ such that these vectors are annihilated by $d_m$ for all $m > K$. For any $l > K$ and $m = r+2$, we have $\omega_{l,-m}^{(r)}(f) = 0$ for any $0 \neq f \in \mathbb{C}[s,t]$ by (2) and

$$\begin{aligned}
\omega_{l,-m}^{(r)}(f \otimes v) =& \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} d_{l+m-i} d_{-m+i}(f \otimes v) \\
=& \sum_{i=0}^{r} \binom{r}{i} (-1)^{r-i} d_{l+m-i}(f) \otimes d_{-m+i}(v)
\end{aligned}$$

which is nonzero since $d_{-m+i}v, i = 0, 1, \cdots, r$ are linearly independent. □

Theorem 4.2. *The* Vir*-modules $\Omega(\lambda, \alpha, h)$ or $\Omega(\lambda, \alpha, h) \otimes V$ is not isomorphic to any irreducible module defined in [MZ2, LZ1, LLZ] or in [TZ2].*

*Proof.* For any irreducible modules in [MZ2], there exists a positive integer $n$ such that $d_n$ acts locally finitely. So neither our module $\Omega(\lambda, \alpha, h)$ nor $\Omega(\lambda, \alpha, h) \otimes V$ is isomorphic to any irreducible modules constructed in [MZ2] by Lemma 4.1 (1).

Then we consider the irreducible non-weight Vir-module $A_b$ defined in [LZ1]. From the proof of Theorem 9 in [LLZ] or the argument in the proof of Corollary 4 in [TZ1], we have

$$\omega_{l,m}^{(r)}(A_b) = 0, \ \forall\, l, m \in \mathbb{Z}, r \geq 3. \tag{4.2}$$

Combining this with Lemma 4.1 (2) and (3), we see easily that $\Omega(\lambda, \alpha, h) \ncong A_b$ and $\Omega(\lambda, \alpha, h) \otimes V \ncong A_b$.

Now we recall the irreducible non-weight Virasoro modules defined in [LLZ]. Let $M$ be an irreducible module over the Lie algebra $\mathfrak{a}_k := \mathrm{Vir}_+/\mathrm{Vir}_+^{(k)}, k \in \mathbb{N}$ such that the action of $\bar{d_k} := d_k + \mathrm{Vir}_+^{(k)}$ on $M$ is injective, where $\mathrm{Vir}_+^{(k)} = \{d_i \mid i > k\}$ and $\mathrm{Vir}_+ = \mathrm{span}\{d_i \mid i \in \mathbb{Z}_+\}$. For any $\beta \in \mathbb{C}[t^{\pm 1}] \setminus \mathbb{C}$, the Vir-module structure on $\mathcal{N}(M, \beta) = M \otimes \mathbb{C}[t^{\pm 1}]$ is defined by

$$d_m \cdot (v \otimes t^n) = (n + \textstyle\sum_{i=0}^{k} \frac{m^{i+1}}{(i+1)!} \bar{d}_i) v \otimes t^{n+m} + v \otimes (\beta t^{m+n}),$$

$$c \cdot (v \otimes t^n) = 0, \ \forall\, m, n \in \mathbb{Z}.$$

From the computation in (6.7) of [LLZ] we see that

$$\begin{aligned} &\omega_{l,m}^{(r)}(\mathcal{N}(M, \beta)) = 0, \ \forall\, l, m \in \mathbb{Z}, r > 2k+2, \\ &\omega_{l,m}^{(2k+2)}(w \otimes t^i) = (2k+2)!(-1)^{k+1}(\bar{d_k}^2 w) \otimes t^{i+l} \neq 0, \ \forall\, l, m \in \mathbb{Z}. \end{aligned} \tag{4.3}$$

Combining the first equation of (4.3) with Lemma 4.1 (3), we see that $\mathcal{N}(M, \beta)$ is not isomorphic to $\Omega(\lambda, \alpha, h) \otimes V$. Similarly, combining the second equation of (4.3) with Lemma 4.1 (2), we see that $\Omega(\lambda, \alpha, h)$ is not isomorphic to $\mathcal{N}(M, \beta)$ provided $k \geq 2$. If $k = 1$, we see that $u$ and $\omega_{l,m}^{(4)}(u)$ are linearly independent for any $u \in \mathcal{N}(M, \beta)$ provided $l \neq 0$, while $\omega_{l,m}^{(4)}(1) = 24\alpha^2\xi^2$ in $\Omega(\lambda, \alpha, h)$ by Lemma 4.1 (2). We see $\Omega(\lambda, \alpha, h) \ncong \mathcal{N}(M, \beta)$ in this case.

Finally, we take any irreducible module in [TZ2], say, $T = \big(\bigotimes_{i=1}^{k} \Omega(\mu_i, b_i)\big) \otimes W$, where $\mu_1, \cdots, \mu_k \in \mathbb{C}^*$ are distinct, $b_1, \cdots, b_k \in \mathbb{C}^*$ and $W$ is a Vir-module such that $d_n$ acts locally finitely for sufficiently large $n \in \mathbb{N}$. As vector spaces $\Omega(\mu_i, b_i) = \mathbb{C}[s_i]$ and the Vir-action on $\Omega(\mu_i, b_i)$ is given by

$$d_m f(s_i) = \mu_i^m (s_i + m b_i) f(s_i - m), \ \forall\, m \in \mathbb{Z}.$$

Then $T$ is just the tensor product of $\Omega(\mu_1, b_1), \cdots, \Omega(\mu_k, b_k)$ and $W$. If $W$ is 1-dimensional and $k = 1$, then this tensor product module is just $\Omega(\mu_1, b_1)$, which is just a special case of some module $A_b$ we treated previously. So we assume that $k \geq 2$ or $\dim(W) \geq 2$ in the following.

It was shown in Proposition 7 of [TZ2] that there exist $l, m \in \mathbb{Z}$ and $w \in W$ such that

$$\omega_{l,-m}^{(r)}(1 \otimes \cdots \otimes 1 \otimes w) \neq 0, \ \forall\, r > 4.$$

We remark that this formula holds for $r > 2$ actually. However, it does not matter for us in the present argument. In deed, this formula, in whichever version, together with Lemma (4.1) (2) implies that $\Omega(\lambda, \alpha, h) \ncong T = \big(\bigotimes_{i=1}^{k} \Omega(\mu_i, b_i)\big) \otimes W$.

For any nonzero elements $\sum_{i=0}^{n} a_i(t)s^i \otimes v_i \in \Omega(\lambda, \alpha, h) \otimes V$ with $v_i \in V$, there exists $K \in \mathbb{N}$ such that $d_j v_i = 0$ for all $j > K$ and $i = 0, \cdots, n$. Then by Lemma 4.1 (2), we have

$$\omega_{l,m}^{(4)} \sum_{i=0}^{n} a_i(t)s^i \otimes v_i = 24\lambda^l \alpha^2 \sum_{i=0}^{n} (\xi^2 a_i(t) - 2\xi a_i'(t) + a_i''(t))(s-l)^i \otimes v_i \neq 0, \quad (4.4)$$

for all $l - m - 4 > K$ and $m > K$, and

$$\omega_{l,m}^{(5)} \sum_{i=0}^{5} a_i(t)s^i \otimes v_i = 0, \ \forall\, l - m - 5 > K, m > K. \quad (4.5)$$

If $k = 1$, write $s = s_1$ for short. Then for any element $\sum_{i=0}^{n'} s^i \otimes w_i \in \Omega(\mu_1, b_1) \otimes W$ with $w_i \in W$, there exists $K' \in \mathbb{N}$ such that $d_j w_i = 0$ for all $j > K'$ and $i = 0, \cdots, n'$. Since $\Omega(\mu_1, b_1)$ is a special case of $A_b$, using (4.2) we get

$$\omega_{l,m}^{(4)} \sum_{i=0}^{n'} s^i \otimes w_i = \sum_{i=0}^{n'} \omega_{l,m}^{(4)} s^i \otimes w_i = 0, \ \forall\, l - m - 4 > K', m > K'. \quad (4.6)$$

The formulas (4.4) and (4.6) indicate that $\Omega(\lambda, \alpha, h) \otimes V$ is not isomorphic to $\Omega(\mu_1, b_1) \otimes W$.

Now suppose that $k \geq 2$. Given any $0 \neq w \in W$, there exists $K'' \in \mathbb{N}$ such that $d_j w = 0$ for all $j \geq K''$. Then for any $l - m \geq K'' + 6$ and $m \geq K''$, we have

$$\omega_{l,m}^{(5)}(1 \otimes 1 \cdots \otimes 1 \otimes w) = \sum_{i=0}^{5} \binom{5}{i} (-1)^{5-i} d_{l-m-i} d_{m+i} (1 \otimes 1 \cdots \otimes 1) \otimes w.$$

The coefficient of $s_1 \otimes s_2 \otimes 1 \cdots \otimes 1 \otimes w$ is

$$\begin{aligned} &\sum_{i=0}^{5} \binom{5}{i} (-1)^{5-i} \left(\mu_1^{l-m-i} \mu_2^{m+i} + \mu_1^{m+i} \mu_2^{l-m-i}\right) \\ =&(\mu_2 - \mu_1)^5 \left(\mu_1^{l-m-5} \mu_2^{m} - \mu_1^{m} \mu_2^{l-m-5}\right), \end{aligned} \quad (4.7)$$

which is nonzero for infinitely many $l$ and $m$ with $l > 2m + 5$ and $m \geq K''$. Therefore, $\omega_{l,m}^{(5)}(1 \otimes 1 \cdots \otimes 1 \otimes w) \neq 0$ for infinitely many $l$ and $m$ with $l > 2m + 5, m \geq K''$. This together with (4.5) shows that $\Omega(\lambda, \alpha, h) \otimes V \not\cong T$. □

COROLLARY 4.3. *Let $\lambda, \alpha \in \mathbb{C}^*$, $h \in \mathbb{C}[t]$ with $\deg(h) = 1$, and $V$ is an irreducible* Vir*-module such that any $d_k$ is locally finite on $V$ for sufficiently large $k$. Then both $\Omega(\lambda, \alpha, h)$ and $\Omega(\lambda, \alpha, h) \otimes V$ are new non-weight irreducible* Vir *modules.*

**5. Reformulations of $\Omega(\lambda, \alpha, h) \otimes V$.** In this section, we will reformulate the tensor modules $\Omega(\lambda, \alpha, h) \otimes V$ as certain induced modules when $V$ is an irreducible highest weight module or an irreducible Whittaker module. The idea of this section is from Section 5 of [TZ1]. We first recall the definition of the Whittaker modules for Vir.

Fix any $n \in \mathbb{Z}_+$ and $\mathbf{a} = (a_n, \cdots, a_{2n}) \in \mathbb{C}$. Recall that $\mathrm{Vir}_+^{(n-1)} = \mathrm{span}\{d_i \mid i \geq n\}$. We define a $\mathrm{Vir}_+^{(n-1)}$-module structure on $\mathbb{C}$ as follows:

$$d_i \cdot 1 = a_i, \ \forall\, n \leq i \leq 2n, \quad \text{and} \quad d_i \cdot 1 = 0, \ \forall\, i > 2n.$$

Now we have the Vir-module:

$$V_{\mathbf{a},\theta}=U(\mathrm{Vir})\otimes_{U(\mathrm{Vir}_+^{(n-1)})}\mathbb{C}/(c-\theta)U(\mathrm{Vir})\otimes_{U(\mathrm{Vir}_+^{(n-1)})}\mathbb{C}.$$

In the following, we will still write $x\cdot 1$ instead of $x\otimes 1$ for any $x\in U(\mathrm{Vir})$. When $n\geq 1$, $V_{\mathbf{a},\theta}$ is just the Whittaker modules studied in [LGZ] and when $n=0$, the module $V_{a_0,\theta}$ is the Verma module with highest weight $a_0$ and central charge $\theta$ (See [KR]).

Fix any $\lambda\in\mathbb{C}^*$. As in [MW, TZ1], let $\mathfrak{b}_{\lambda,n+1}=\mathrm{span}\{d_k-\lambda^{k-n}d_n\mid k\geq n+1\}$ be the subalgebra of Vir. For any $\alpha\in\mathbb{C}, h\in\mathbb{C}[t]$, we can define a $\mathfrak{b}_{\lambda,n+1}$-module structure on $\mathbb{C}[t]$ as follows

$$(d_k-\lambda^{k-n}d_n)\circ f=(k-n)\lambda^k\big(G(f)-(k+n)\alpha F(f)\big)+(a_k-\lambda^{k-n}a_n)f, \qquad (5.1)$$

where we make the convention that $a_k=0$ provided $k>2n$. Denote this module as $\mathbb{C}[t]_{\alpha,h,\mathbf{a}}$.

Proposition 5.1. *The* $\mathfrak{b}_{\lambda,n+1}$*-module* $\mathbb{C}[t]_{\alpha,h,\mathbf{a}}$ *is irreducible if and only if* $\alpha\neq 0$ *and* $\deg(h)=1$.

*Proof.* When $\alpha=0$, then (5.1) becomes

$$(d_k-\lambda^{k-n}d_n)\circ f=(k-n)\lambda^k G(f)+(a_k-\lambda^{k-n}a_n)f,$$

and it is easy to see that $t^i\mathbb{C}[t]$ is a $\mathfrak{b}_{\lambda,n+1}$-submodule of $\mathbb{C}[t]_{\alpha,h,\mathbf{a}}$ for any $i\in\mathbb{Z}_+$. When $h\in\mathbb{C}$, then (5.1) becomes

$$(d_k-\lambda^{k-n}d_n)\circ f=(k-n)\lambda^k\big(h(\alpha)f-tf'+(k+n)\alpha f'\big)+(a_k-\lambda^{k-n}a_n)f,$$

and the subspace of $\mathbb{C}[t]$ consisting of all polynomials with degree no larger than $i$ is a $\mathfrak{b}_{\lambda,n+1}$-submodule for any $i\in\mathbb{Z}_+$. When $\alpha\neq 0$ and $\deg(h)\geq 2$, we have that $F(\mathbb{C}[t])=\{F(f)\mid f\in\mathbb{C}[t]\}$ is a proper $\mathfrak{b}_{\lambda,n+1}$-submodule of $\mathbb{C}[t]_{\alpha,h,\mathbf{a}}$ (cf. Theorem 3.8 of [CG1]).

Now suppose that $\alpha\neq 0$ and $h=\xi t+\eta$ for some $\xi,\eta\in\mathbb{C}$ with $\xi\neq 0$. Let $W$ be a nonzero submodule of $\mathbb{C}[t]_{\alpha,h,\mathbf{a}}$. Take any nonzero $f\in W$, by (5.1), we have $G(f)-(k+n)\alpha F(f)\in W$ for all $k\geq n+1$. By the Vandermonde's determinant, we obtain that $F(f)=\xi f-f'\in W$ and $G(f)=h(\alpha)f+t(\xi f-f')\in W$, which implies $f'\in W$ and $t(\xi f-f')\in W$. Then we can deduce that $1\in W$ by downward induction on the degree of $f$ and $\mathbb{C}[t]\subseteq W$ by upward induction. Hence $\mathbb{C}[t]_{\alpha,h,\mathbf{a}}$ is an irreducible $\mathfrak{b}_{\lambda,n+1}$-module. □

Now fix any $\alpha\neq 0$ and $h=\xi t+\eta$ for some $\xi,\eta\in\mathbb{C}$ with $\xi\neq 0$. We can form the induced Vir-module as

$$\mathrm{Ind}_{\theta,\lambda}(\mathbb{C}[t]_{\alpha,h,\mathbf{a}})=U(\mathrm{Vir})\otimes_{U(\mathfrak{b}_{\lambda,n+1})}\mathbb{C}[t]_{\alpha,h,\mathbf{a}}/(c-\theta)U(\mathrm{Vir})\otimes_{U(\mathfrak{b}_{\lambda,n+1})}\mathbb{C}[t]_{\alpha,h,\mathbf{a}}.$$

Then we have the following:

Theorem 5.2. *Let notation as above, then* $\Omega(\lambda,\alpha,h)\otimes V_{\boldsymbol{a},\theta}\cong\mathrm{Ind}_{\theta,\lambda}(\mathbb{C}[t]_{\alpha,h,\boldsymbol{a}})$.

*Proof.* From the PBW Theorem, we see that the module $\mathrm{Ind}_{\theta,\lambda}(\mathbb{C}[t^{\pm 1}]_{\alpha,h,\mathbf{a}})$ has a basis

$$A=\{d_l^{k_l}d_{l+1}^{k_{l+1}}\cdots d_n^{k_n}\otimes t^i\;\big|\;k_l,\cdots,k_n\in\mathbb{Z}_+,l\in\mathbb{Z},l\leq n,i\in\mathbb{Z}_+\},$$

and the module $\Omega(\lambda,\alpha,h)\otimes V(\mathbf{a},\theta)$ has a basis

$$B=\{t^i s^{k_n}\otimes(d_l^{k_l}d_{l+1}^{k_{l+1}}\cdots d_{n-1}^{k_{n-1}}\cdot 1)\bigm| k_l,\cdots,k_n\in\mathbb{Z}_+, l\in\mathbb{Z}, l\le n-1, i\in\mathbb{Z}_+\}.$$

Now we can define the following linear map:

$$\begin{aligned}\phi:\quad \mathrm{Ind}_{\theta,\lambda}(\mathbb{C}[t]_{\alpha,h,\mathbf{a}}) &\to \Omega(\lambda,\alpha,h)\otimes V(\mathbf{a},\theta)\\ d_l^{k_l}\cdots d_n^{k_n}\otimes t^i &\mapsto d_l^{k_l}\cdots d_n^{k_n}(t^i\otimes 1).\end{aligned}$$

Claim 1. $\phi$ is a Vir-module homomorphism.

We first have the following observation by (5.1):

$$(d_j-\lambda^{j-n}d_n)(t^i\otimes 1)=\big((d_j-\lambda^{j-n}d_n)\circ t^i\big)\otimes 1,\quad \forall\, j>n. \tag{5.2}$$

Then for any $x=d_l^{k_l}d_{l-1}^{k_{l-1}}\cdots d_n^{k_n}, l\le n, i\in\mathbb{Z}_+$ and $j>n$, we have

$$\begin{aligned}\phi(xd_j\otimes t^i)&=\phi(\lambda^{j-n}xd_n\otimes t^i+x\big(d_j-\lambda^{j-n}d_n\big)\otimes t^i)\\ &=\phi\big(\lambda^{j-n}xd_n\otimes t^i+x\otimes(d_j-\lambda^{j-n}d_n)\circ t^i\big)\\ &=\lambda^{j-n}xd_n(t^i\otimes 1)+x\big((d_j-\lambda^{j-n}d_n)\circ t^i\otimes 1\big)\\ &=xd_j(t^i\otimes 1).\end{aligned}$$

Then for any $j\in\mathbb{Z}$, by the PBW Theorem, we can write $d_jx$ as

$$d_jx=\sum_{r=1}^{p}c_rx_r+\sum_{r=1}^{q}b_ry_rd_{j_r},\quad c_r,b_r\in\mathbb{C}$$

such that $x_r\otimes t^i, y_r\otimes t^i\in A$ and $j_r>n$. Then we deduce

$$\begin{aligned}\phi(d_jx\otimes t^i)&=\phi\Big(\sum_{r=1}^{p}c_rx_r\otimes t^i+\sum_{r=1}^{q}b_ry_rd_{j_r}\otimes t^i\Big)\\ &=\sum_{r=1}^{p}c_rx_r(t^i\otimes 1)+\sum_{r=1}^{q}b_ry_rd_{j_r}(t^i\otimes 1)\\ &=d_jx(t^i\otimes 1)=d_j\phi(x\otimes t^i).\end{aligned}$$

Claim 2. $\phi$ is a Vir-module isomorphism.

It is clear that $t^i\otimes 1\in\mathrm{Im}(\phi)$ for all $i\in\mathbb{Z}_+$. Then from $d_n^j(t^i\otimes 1)\in\mathrm{Im}(\phi)$ we can deduce that $t^is^j\otimes 1\in\mathrm{Im}(\phi)$ for all $j\in\mathbb{Z}_+$ inductively. Then applying elements of the form $d_l^{k_l}d_{l+1}^{k_{l+1}}\cdots d_n^{k_n}$ on $\Omega(\lambda,\alpha,h)\otimes 1$, we can deduce that $\Omega(\lambda,\alpha,h)\otimes V_{\mathbf{a},\theta}\in\mathrm{Im}(\phi)$ and $\phi$ is surjective.

To obtain the injectivity of $\phi$, we first define a total order on $B$ as follows

$$t^{k_{n+1}}s^{k_n}\otimes(d_l^{k_l}\cdots d_{n-1}^{k_{n-1}}\cdot 1)<t^{k'_{n+1}}s^{k'_n}\otimes(d_m^{k'_m}\cdots d_{n-1}^{k'_{n-1}}\cdot 1)$$

if and only if there exists $r\le n+1$ such that $k_i=k'_i$ for all $i<r$ and $k_r<k'_r$, where we have made the convenience that $k_i=0$ for $i<l$ and $k'_i=0$ for $i<m$. By simple computations we can obtain that

$$\begin{aligned}&d_l^{k_l}\cdots d_{n-1}^{k_{n-1}}d_n^{k_n}(t^i\otimes 1)\\ &=t^is^{k_n}\otimes(d_l^{k_l}\cdots d_{n-1}^{k_{n-1}}\cdot 1)+\text{ lower terms },\ \forall\, k_l,\cdots,k_n,i\in\mathbb{Z}_+.\end{aligned}$$

Since $B$ is a basis of the module $\Omega(\lambda, \alpha, h) \otimes V$, so is the set

$$\phi(A) = \{d_l^{k_l} \cdots d_{n-1}^{k_{n-1}} d_n^{k_n}(t^i \otimes 1) \mid k_l, \cdots, k_n, i \in \mathbb{Z}_+, l \leq n\}.$$

Therefore, $\phi$ is injective, as desired. □

Corollary 5.3. *Suppose that* $n \in \mathbb{N}$. *The module* $\mathrm{Ind}_{\theta,\lambda}(\mathbb{C}[t]_{\alpha,h,\boldsymbol{a}})$ *is irreducible if and only if* $\deg(h) = 1, \alpha \neq 0$ *and* $a_{2n-1}^2 + a_{2n}^2 \neq 0$.

*Proof.* By the Theorem 5.2, we know that the module $\mathrm{Ind}_{\theta,\lambda}(\mathbb{C}[t]_{\alpha,h,\mathbf{a}})$ is irreducible if and only if $\Omega(\lambda, \alpha, h) \otimes V_{\mathbf{a},\theta}$ is irreducible, if and only if the module $\Omega(\lambda, \alpha, h)$ and $V_{\mathbf{a},\theta}$ are both irreducible. From Theorem 2.2 we know that $\Omega(\lambda, \alpha, h)$ is simple if and only if $\deg(h) = 1$ and $\alpha \neq 0$. And $V_{\mathbf{a},\theta}$ is an irreducible if and only if $a_{2n-1} \neq 0$ or $a_{2n} \neq 0$ by the Theorem 7 in [LGZ]. So we complete the proof. □

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