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RICCI-MEAN CURVATURE FLOWS IN GRADIENT SHRINKING RICCI SOLITONS*

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Abstract. It was proved by Huisken that a mean curvature flow converges to a self-shrinker in the Euclidean space after scaling when it develops a singularity of type I. In this paper, we study a coupled flow with a mean curvature flow and a Ricci flow, and generalize his result for this Ricci-mean curvature flow. Then, as a parabolic rescaling limit, we get a self-shrinker in a gradient shrinking Ricci soliton in the sense of Lott under some assumptions.

Key words. mean curvature flow, Ricci flow, self-similar solution, gradient soliton.

Mathematics Subject Classification. 53C42, 53C44.

1. Introduction. Let M^m and N^n be manifolds with $m \leq n$, $g = (\,g_t\,;\, t \in [0,T_1)\,)$ be a smooth 1-parameter family of Riemannian metrics on N and $F\colon M \times [0,T_2) \to N$ be a smooth 1-parameter family of immersions. Namely, $F_t\colon M \to N$ defined by $F_t(\,\cdot\,) := F(\cdot,t)$ is an immersion. We assume that $T_2 \leq T_1$. We say that the pair of g and F is a solution of the *Ricci-mean curvature flow* if it satisfies the following coupled equation of the Ricci flow and the mean curvature flow:

$$\frac{\partial g_t}{\partial t} = -2\mathrm{Ric}(g_t) \tag{1a}$$

$$\frac{\partial F_t}{\partial t} = H(F_t), \tag{1b}$$

where $H(F_t)$ denotes the mean curvature vector field of $F_t\colon M \to N$ computed by the ambient Riemannian metric g_t at the time t. Note that this coupling is partial in the sense that the Ricci flow equation (1a) does not depend on F.

1.1. Mean curvature flows in $\mathbb{R}^n$. Before stating our main results, we review the definition of self-similar solutions in $\mathbb{R}^n$ and the results due to Huisken [10]. On $\mathbb{R}^n$, we can naturally identify a point $x = (x^1,\dots,x^n) \in \mathbb{R}^n$ with a tangent vector $\overrightarrow{x} \in T_x\mathbb{R}^n$ by $\overrightarrow{x} := x^1(\partial/\partial x^1) + \cdots + x^n(\partial/\partial x^n)$. For an immersion $F\colon M \to \mathbb{R}^n$, we can define a section $\overrightarrow{F} \in \Gamma(M, F^*(T\mathbb{R}^n))$ by $\overrightarrow{F}(p) := \overrightarrow{F(p)}$ for all $p \in M$. Then, $F\colon M \to \mathbb{R}^n$ is called a *self-similar solution* if it satisfies

$$H(F) = \frac{\lambda}{2}\overrightarrow{F}^{\perp} \tag{2}$$

for some constant $\lambda \in \mathbb{R}$, where $\perp$ denotes the projection onto the normal bundle of M. A self-similar solution is called a *self-expander* or *self-shrinker* when $\lambda > 0$ or $\lambda < 0$, respectively.

Assume that M is compact and $F\colon M \times [0,T) \to \mathbb{R}^n$ be a mean curvature flow with the maximal time $T < \infty$. Here, a maximal time is that we can not extend the flow beyond the time. We always denote the second fundamental form of F_t by $A(F_t)$. We impose the following two conditions (A1) and (B1) on F.

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(A1) $\limsup_{t\to T}\big(\sqrt{T-t}\max_M|A(F_t)|\big)<\infty$. (Type I condition)
(B1) There exists a point p_0 in M such that $F_t(p_0)\to \mathrm{O}\in\mathbb{R}^n$ as $t\to T$.

For each $t\in(-\infty,T)$, let $\Phi_t\colon \mathbb{R}^n\to\mathbb{R}^n$ be a diffeomorphism of $\mathbb{R}^n$ defined by $\Phi_t(x):=x/\sqrt{T-t}$. Define the rescaled flow $\tilde{F}\colon M\times[-\log T,\infty)\to\mathbb{R}^n$ by $\tilde{F}_s:=\Phi_t\circ F_t$ with relation $s=-\log(T-t)$. Then, it satisfies the *normalized mean curvature flow* equation:

$$\frac{\partial\tilde{F}_s}{\partial s}=H(\tilde{F}_s)+\frac{1}{2}\overrightarrow{\tilde{F}_s}.$$

Huisken proved the following (cf. Proposition 3.4 and Theorem 3.5 in [10]).

THEOREM 1.1. *Under the assumptions (A1) and (B1), for each sequence $s_j\to\infty$, there exists a subsequence s_{j_k} such that the sequence of immersed submanifolds $\tilde{M}_{s_{j_k}}:=\tilde{F}_{s_{j_k}}(M)$ converges smoothly to an immersed nonempty limiting submanifold $\tilde{M}_\infty\subset\mathbb{R}^n$, and $\tilde{M}_\infty$ is a self-shrinker with $\lambda=-1$ in (2).*

1.2. Singularities of Ricci flows. On the other hand, there is also the notion of type I singularity for a Ricci flow $g=(\,g_t\,;\,t\in[0,T)\,)$ on N. Assume that $T<\infty$ is the maximal time. We say that g forms a singularity of type I if

$$\limsup_{t\to T}\big((T-t)\sup_N|\operatorname{Rm}(g_t)|\big)<\infty,$$

where $\operatorname{Rm}(g_t)$ denotes the Riemannian curvature tensor of g_t. In the Ricci flow case, a gradient shrinking Ricci soliton appears as a scaling limit of a singularity of type I (cf. [5, 19, 20]). Conversely, from a gradient shrinking Ricci soliton, we can construct a Ricci flow which develops a singularity of type I as follows. Let $(N,\tilde{g},\tilde{f})$ be an n-dimensional complete gradient shrinking Ricci soliton with

$$\operatorname{Ric}(\tilde{g})+\operatorname{Hess}\tilde{f}-\frac{1}{2}\tilde{g}=0. \tag{3}$$

As Hamilton's proof of Theorem 20.1 in [8], one can easily see that $R(\tilde{g})+|\nabla\tilde{f}|^2-\tilde{f}$ is a constant, where $R(\tilde{g})$ denotes the scalar curvature of $\tilde{g}$. Hence, by adding some constant to $\tilde{f}$ if necessary, we may assume that the potential function $\tilde{f}$ satisfies

$$R(\tilde{g})+|\nabla\tilde{f}|^2-\tilde{f}=0. \tag{4}$$

Fix a positive time $0<T<\infty$. Let $\{\Phi_t\colon N\to N\}_{t\in(-\infty,T)}$ be the 1-parameter family of diffeomorphisms with $\Phi_0=\mathrm{id}_N$ generated by the time-dependent vector field $V_t:=\frac{1}{T-t}\nabla\tilde{f}$. For $t\in(-\infty,T)$, define

$$g_t:=(T-t)\Phi_t^*\tilde{g}\quad\text{and}\quad f_t:=\Phi_t^*\tilde{f}. \tag{5}$$

Then, g_t satisfies the Ricci flow equation (1a).

1.3. Main Theorem. For an immersion $F\colon M\to N$, we get a section $(\nabla\tilde{f})\circ F\in\Gamma(M,F^*(TN))$, and we usually omit the symbol $\circ F$, for short.

DEFINITION 1.2. If an immersion $F\colon M\to N$ satisfies

$$H(F)=\lambda\nabla\tilde{f}^{\perp} \tag{6}$$

for some constant $\lambda \in \mathbb{R}$, we call it a *self-similar solution*. A self-similar solution is called a *self-expander* or *self-shrinker* when $\lambda > 0$ or $\lambda < 0$, respectively.

DEFINITION 1.3. If a 1-parameter family of immersions $\tilde{F}\colon M \times [0,S) \to N$ satisfies

$$\frac{\partial \tilde{F}_s}{\partial s} = H(\tilde{F}_s) + \nabla \tilde{f},$$

we call it a *normalized mean curvature flow*.

Assume that $F\colon M \times [0,T) \to N$ is a solution of Ricci-mean curvature flow (1b) along the Ricci flow $g = (\, g_t\, ;\, t \in [0,T)\,)$ constructed from $(N, \tilde{g}, \tilde{f})$ by (5). We will use the following two conditions (A2) and (B2).

(A2) $\limsup_{t\to T}\big(\sqrt{T-t}\max_M |A(F_t)|\big) < \infty$.

(B2) There exists a point $p_0 \in M$ such that $\ell_{F_t(p_0),t}$ converges to f pointwise on $N \times [0,T)$ as $t \to T$, where $f\colon N \times [0,T) \to \mathbb{R}$ is defined by (5) and $\ell_{*,\bullet}\colon N \times [0,\bullet) \to \mathbb{R}$ is the reduced distance based at $(*,\bullet)$.

REMARK 1.4. The condition (A2) corresponds to (A1) and (B2) to (B1). In (B2), $\ell_{F_t(p_0),t}$ is the reduced distance for the Ricci flow g based at $(F_t(p_0),t)$ introduced by Perelman. Here, we explain this briefly. Let (N, g_t) be a Ricci flow on $[0,T)$. For any curve $\gamma\colon [t_1,t_2] \to N$ with $0 \le t_1 < t_2 < T$, we define the $\mathcal{L}$-length of γ by

$$\mathcal{L}(\gamma) := \int_{t_1}^{t_2} \sqrt{t_2 - t}\big(R(g_t) + |\dot\gamma|^2\big)dt,$$

where $|\dot\gamma|$ is the norm of $\dot\gamma(t)$ measured by g_t. For a fixed point (p_2,t_2) in the space-time $N \times (0,T)$, we get the reduced distance $\ell_{p_2,t_2}\colon N \times [0,t_2) \to \mathbb{R}$ based at (p_2,t_2) defined by

$$\ell_{p_2,t_2}(p_1,t_1) := \frac{1}{2\sqrt{t_2-t_1}} \inf_{\gamma} \mathcal{L}(\gamma),$$

where the infimum is taken over all curves $\gamma\colon [t_1,t_2] \to N$ with $\gamma(t_1) = p_1$ and $\gamma(t_2) = p_2$. One can easily see that (B1) and (B2) are equivalent when $(N,\tilde{g},\tilde{f})$ is the Gaussian soliton $(\mathbb{R}^n, g_{\mathrm{st}}, \frac{1}{4}|x|^2)$.

If $(N,\tilde{g},\tilde{f})$ is compact (resp. non-compact), we assume that F satisfies (A2) (resp. (A2) and (B2)). As in the Euclidean case, we consider the rescaled flow $\tilde{F}\colon M \times [-\log T, \infty) \to N$ defined by

$$\tilde{F}_s := \Phi_t \circ F_t \quad \text{with} \quad s = -\log(T-t), \tag{7}$$

and we can see that $\tilde{F}$ becomes a normalized mean curvature flow in $(N,\tilde{g},\tilde{f})$ (cf. Proposition 4.4). Then, the main results in this paper are the following.

THEOREM 1.5. *Assume that $(N,\tilde{g},\tilde{f})$ is compact. Let $F\colon M \times [0,T) \to N$ be a Ricci-mean curvature flow along the Ricci flow (N, g_t) defined by (5). Assume that M is compact and F satisfies (A2). Let $\tilde{F}\colon M \times [-\log T,\infty) \to N$ be defined by (7). Then, for any sequence $s_1 < s_2 < \cdots < s_j < \cdots \to \infty$ and points $\{x_j\}_{j=1}^{\infty}$ in M, there exist sub-sequences s_{j_k} and x_{j_k} such that the family of immersion maps $\tilde{F}_{s_{j_k}}\colon M \to N$ from pointed manifolds (M, x_{j_k}) converges to an immersion map $\tilde{F}_\infty\colon M_\infty \to N$*

from some pointed manifold (M_∞, x_∞). Furthermore, M_∞ is a complete Riemannian manifold with metric $\tilde{F}_\infty^* \tilde{g}$ and $\tilde{F}_\infty$ is a self-shrinker in $(N, \tilde{g}, \tilde{f})$ with $\lambda = -1$, that is, $\tilde{F}_\infty$ satisfies

$$H(\tilde{F}_\infty) = -\nabla \tilde{f}^\perp.$$

Theorem 1.6. *Assume that $(N, \tilde{g}, \tilde{f})$ is non-compact and satisfies the assumption in Remark 1.7. Under the same setting in Theorem 1.5, assume that M is compact and F satisfies (A2) and (B2). Then, for any sequence of times s_j, the same statement as Theorem 1.5 holds, if we fix $x_j := p_0$ for all j.*

Remark 1.7. For a complete non-compact Riemannian manifold $(N, \tilde{g})$, we assume that there is an isometrically embedding $\Theta\colon N \to \mathbb{R}^L$ into some higher dimensional Euclidean space with

$$|\nabla^p A(\Theta)| \leq \tilde{D}_p < \infty$$

for some constants $\tilde{D}_p > 0$ for all $p \geq 0$. Under this assumption, one can see that $(N, \tilde{g})$ must have the bounded geometry.

The notion of the convergence of immersions from pointed manifolds is defined as follows. It is the immersion map version of the Cheeger–Gromov convergence.

Definition 1.8. Let (N, g) be a complete n-dimensional Riemannian manifold satisfying the assumption in Remark 1.7. Assume that for each $k \geq 1$ we have an m-dimensional pointed manifold (M_k, x_k) and an immersion $F_k\colon M_k \to N$. Then, we say that a sequence of immersions $\{F_k\colon M_k \to N\}_{k=1}^\infty$ converges to an immersion $F_\infty\colon M_\infty \to N$ from an m-dimensional pointed manifold (M_∞, x_∞) if there exist

(1) an exhaustion $\{U_k\}_{k=1}^\infty$ of M_∞ with $x_\infty \in U_k$ and
(2) a sequence of diffeomorphisms $\Psi_k\colon U_k \to V_k \subset M_k$ with $\Psi_k(x_\infty) = x_k$ such that the sequence of maps $F_k \circ \Psi_k\colon U_k \to N$ converges in C^∞ to $F_\infty\colon M_\infty \to N$ uniformly on compact sets in M_∞.

Remark 1.9. We see that Theorem 1.6 implies Theorem 1.1 in $\mathbb{R}^n$. Consider $\mathbb{R}^n$ as the Gaussian soliton with potential function $\tilde{f}(x) := \frac{1}{4}|x|^2$. Since $\overrightarrow{x} = 2\nabla \tilde{f}(x)$, Definition 1.2 coincides with (2) in $\mathbb{R}^n$. It is trivial that $(\mathbb{R}^n, g_{\mathrm{st}})$ satisfies the assumption in Remark 1.7. We take $T = 1$ for simplicity. Then we have $\Phi_t(x) = x/\sqrt{T-t}x$, $g_t \equiv g_{\mathrm{st}}$ and $f(x,t) = |x|^2/4(T-t)$. Since g_t is the trivial Ricci flow, the condition (A1) and (A2) coincides. Furthermore, one can easily see that, in this case, Perelman's reduced distance bases at $(*, \bullet)$ is given by $\ell_{*,\bullet}(x,t) := |x-*|^2/4(\bullet - t)$. Then, one can see that (B1) and (B2) are equivalent in $\mathbb{R}^n$, and Theorem 1.6 implies Theorem 1.1.

Example 1.10. Here, we consider *nontrivial* compact examples of self-similar solutions embedded in a compact gradient shrinking Ricci soliton, where nontrivial means that the dimension and codimension of the submanifold are greater than 0. A class of examples are given in a compact gradient shrinking *Kähler-Ricci soliton* $(N, \tilde{g}, \tilde{f})$. Let $M \subset N$ be a compact *complex* submanifold such that the gradient $\nabla \tilde{f}$ is tangent to M. Then, M is a compact self-similar solution for any λ, since $H = 0$ and $\nabla \tilde{f}^\perp = 0$ on M. Cao [2] and Koiso [12] (for notations and assumptions, see [13]) constructed examples of compact gradient shrinking Kähler Ricci solitons, denoted by N_k^n. By their construction, N_k^n is the total space of some complex $\mathbb{P}^1$-fibration and the gradient of the potential function is tangent to every $\mathbb{P}^1$-fiber. Hence, each $\mathbb{P}^1$-fiber is a compact self-similar solution with real dimension 2. Furthermore, it is known by [26] that a lens space $L(k;1)(r)$ with radius r is embedded in N_k^n as a nontrivial and non-minimal compact example of self-similar solutions.

Finally, we give some comments for Lagrangian self-similar solutions. For a Lagrangian immersion $F\colon L \to N$ in a Kähler manifold N with a Kähler form ω, a 1-form ω_H on L defined by $\omega_H(X) := \omega(H(F), F_*X)$ is called the mean curvature form. In Theorem 2.3.5 in [21], Smoczyk proved that there exists no compact Lagrangian self-similar solution with exact mean curvature form in $\mathbb{C}^n$. In his proof, it is proved that a compact Lagrangian self-similar solution with exact mean curvature form is a minimal submanifold in $\mathbb{C}^n$. However, there exists no compact minimal submanifold in $\mathbb{C}^n$. Hence, the assertion holds. As an analog of this theorem, we have the following theorem and its proof is given at the end of Section 4.

Theorem 1.11. *Let (N, g, f) be a gradient shrinking Kähler Ricci soliton and $F\colon L \to N$ be a compact Lagrangian self-similar solution with exact mean curvature form. Then, $F\colon L \to N$ is a minimal Lagrangian immersion.*

Much more properties of Lagrangian self-similar solutions in gradient shrinking Kähler Ricci solitons are found in [27].

1.4. Relation to previous literature. Recently, there has been some studies in Ricci-mean curvature flows. One of main streams of the study is to generalize results established for mean curvature flows in a fixed Riemannian manifolds to Ricci-mean curvature flows. For example, some results for Lagrangian mean curvature flows can be generalized (cf. [9, 14]). The result that the Gauss map of a mean curvature flow in $\mathbb{R}^n$ is a harmonic map heat flow is also generalized for Ricci-mean curvature flow by [11].

Another main stream of the study is to generalize Huisken's monotonicity formula in $\mathbb{R}^n$ to Ricci-mean curvature flows. In this direction, Lott considered a mean curvature flow in a gradient Ricci soliton in Section 5 in [15], and a certain kind of monotonicity formula is obtained in gradient steady soliton case. He also gave a definition of a self-similar solution for hypersurfaces in a gradient Ricci soliton. Our definition of a self-similar solution (Definition 1.2) coincides with Lott's one for hypersurfaces. In Remark 5 in [15], he pointed out the existence of an analog of a monotonicity formula in gradient shrinking soliton case. Actually, a monotonicity formula for a mean curvature flow moving in a gradient shrinking Ricci soliton was also given by Magni, Mantegazza and Tsatis [16, Proposition 3.1] more directly. In this paper, we reintroduce their monotonicity formula in Section 4.

In this paper, a self-similar solution appears as a local model of singularities of a Ricci-mean curvature flow and it can be considered as a weighted minimal submanifold by Proposition 4.2. The study of weighted minimal submanifolds in Riemannian manifolds with potential is widely spread. For example, self-similar solutions are defined in Riemannian cone manifolds in [6]. This work is generalized and developed in, for example, [1, 7, 23, 24].

1.5. Organization of this paper. The rest of this paper is organized as follows. In Section 2, we prove Theorem 1.5 and 1.6, after reviewing the proof of Theorem 1.1. In this proof, we use lemmas and propositions proved in the following sections. In Section 3, we introduce some general formulas for the first variation of a certain kind of weighted volume functional. In Section 4, we study some properties of Ricci-mean curvature flows along Ricci flows constructed from gradient shrinking Ricci solitons, and introduce the monotonicity formula. Furthermore, we give an analog of Stone's estimate.

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2. Proofs of main theorems. In this section, we give proofs of Theorem 1.5 and 1.6. First, we review the proof of Theorem 1.1. The key results to prove Theorem 1.1 are the following (i), (ii) and (iii).

(i) The monotonicity formula for the weighted volume functional (cf. Theorem 3.1 and Corollary 3.2 in [10]).

The weighted volume functional is defined by $\int_{\tilde{M}} e^{-|x|^2/4} d\mu_{\tilde{M}}$ for a submanifold $\tilde{M}$ in $\mathbb{R}^n$. This result corresponds to Proposition 4.5 and 4.6. For a submanifold $\tilde{M}$ (or immersion $\tilde{F}\colon M \to N$) in a gradient shrinking Ricci soliton $(N, \tilde{g}, \tilde{f})$, we consider the weighted volume functional $\int_M e^{-\tilde{f}} d\mu(\tilde{F}^*\tilde{g})$. The monotonicity formula decides the profile of the limiting submanifold $\tilde{M}_\infty$ if it exists.

(ii) Uniform estimates for all derivatives of the second fundamental form of $\tilde{M}_{s_j}$ (cf. Proposition 2.3 in [10]).

This result corresponds to Proposition 4.9. It is proved by the parabolic maximum principle for the evolution equation of $|\tilde{\nabla}^k \tilde{A}_s|^2$. This result implies the subconvergence of $\tilde{M}_{s_j}$ to some limiting submanifold $\tilde{M}_\infty$.

(iii) A uniform estimate for the second derivative of the weighted volume functional. The core of its proof is Stone's estimate (cf. Lemma 2.9 in [22]).

In this paper, we prepare an analog of Stone's estimate in Lemma 4.7, and by combining Proposition 4.9 we prove Proposition 4.10, an analog of the result (iii).

Remark 2.1. The uniform estimate noted in (iii) is necessary. In general, if we know $\frac{d}{ds}\mathcal{F}(s) \le 0$ for some smooth non-negative function $\mathcal{F}\colon [0,\infty) \to [0,\infty)$, we can say that $\mathcal{F}$ is monotone decreasing and converges to some value as $s \to \infty$. However, we can not say that $\frac{d}{ds}\mathcal{F}(s_j) \to 0$ for any sequence $s_1 < s_2 < \cdots \to \infty$. If we further know that $|\frac{d^2}{d^2 s}\mathcal{F}(s)| \le C$ uniformly, then we can say that. In our situation, $\mathcal{F}(s)$ is the weighted volume of $\tilde{M}_s$. This argument is pointed out right before Lemma 3.2.7 in [17].

Proof of Theorem 1.5. First, we prove the existence of a smooth manifold M_∞ and a smooth map $\tilde{F}_\infty\colon M_\infty \to N$. Next, we show that this $\tilde{F}_\infty$ is a self-shrinker by using the monotonicity formula (16) in Proposition 4.6.

By Proposition 4.9, for all $k = 0, 1, 2, \ldots$, there exist constants $C_k > 0$ such that

$$|\tilde{\nabla}^k A(\tilde{F}_s)| \le C_k \quad \text{on} \quad M \times [-\log T, \infty).$$

Since N is compact, by a standard argument as the Arzelà–Ascoli theorem (see [25] for detail), we get a sub-sequence j_k, a pointed manifold (M_∞, x_∞) and an immersion map $\tilde{F}_\infty\colon M_\infty \to N$ with a complete Riemannian metric $F_\infty^*\tilde{g}$ on M_∞ such that $\tilde{F}_{s_{j_k}}\colon (M, x_{j_k}) \to N$ converges to $\tilde{F}_\infty\colon (M_\infty, x_\infty) \to N$ in the sense of Definition 1.8 as $k \to \infty$. We denote $\tilde{F}_{s_{j_k}}$ by $\tilde{F}_k$ for short. Then, there exist an exhaustion $\{U_k\}_{k=1}^\infty$ of M_∞ with $x_\infty \in U_k$ and a sequence of diffeomorphisms $\Psi_k\colon U_k \to V_k := \Psi_k(U_k) \subset M$ with $\Psi_k(x_\infty) = x_{j_k}$ such that $\Psi_k^*(\tilde{F}_k^*\tilde{g})$ converges in C^∞ to $\tilde{F}_\infty^*\tilde{g}$ uniformly on compact sets in M_∞, and furthermore the sequence of maps $\tilde{F}_k \circ \Psi_k\colon U_k \to N$ converges in C^∞ to $F_\infty\colon M_\infty \to N$ uniformly on compact sets in M_∞.

Let $K \subset M_\infty$ be any compact set. Then, there exists k_0 such that $K \subset U_k$ for all $k \geq k_0$. Since $\tilde{F}_k \circ \Psi_k \colon U_k \to N$ converges to $F_\infty \colon M_\infty \to N$ in C^∞ uniformly on K for $k \geq k_0$, we have

$$\begin{aligned} &\int_K \left| H(\tilde{F}_k \circ \Psi_k) + \nabla \tilde{f}^{\perp_{\tilde{F}_k \circ \Psi_k}} \right|_{\tilde{g}}^2 e^{-\tilde{f} \circ (\tilde{F}_k \circ \Phi_k)} d\mu((\tilde{F}_k \circ \Phi_k)^* \tilde{g}) \\ \to &\int_K \left| H(\tilde{F}_\infty) + \nabla \tilde{f}^{\perp_{\tilde{F}_\infty}} \right|_{\tilde{g}}^2 e^{-\tilde{f} \circ \tilde{F}_\infty} d\mu(\tilde{F}_\infty^* \tilde{g}) \end{aligned} \tag{8}$$

as $k \to \infty$, where $\perp_{\tilde{F}_\infty}$ denotes the normal projection with respect to $\tilde{F}_\infty$. Since $\Psi_k \colon U_k \to V_k \subset M$ is a diffeomorphism, it is clear that

$$\begin{aligned} &\int_K \left| H(\tilde{F}_k \circ \Psi_k) + \nabla \tilde{f}^{\perp_{\tilde{F}_k \circ \Psi_k}} \right|_{\tilde{g}}^2 e^{-\tilde{f} \circ (\tilde{F}_k \circ \Psi_k)} d\mu((\tilde{F}_k \circ \Psi_k)^* \tilde{g}) \\ = &\int_{\Psi_k(K)} \left| H(\tilde{F}_k) + \nabla \tilde{f}^{\perp_{\tilde{F}_k}} \right|_{\tilde{g}}^2 e^{-\tilde{f} \circ \tilde{F}_k} d\mu(\tilde{F}_k^* \tilde{g}) \\ \leq &\int_M \left| H(\tilde{F}_k) + \nabla \tilde{f}^{\perp_{\tilde{F}_k}} \right|_{\tilde{g}}^2 e^{-\tilde{f} \circ \tilde{F}_k} d\mu(\tilde{F}_k^* \tilde{g}) \\ = &- \frac{d}{ds}\bigg|_{s = s_{j_k}} \int_M e^{-\tilde{f} \circ \tilde{F}} \, d\mu(\tilde{F}^* \tilde{g}), \end{aligned} \tag{9}$$

where we used the monotonicity formula (16) in the last equality. Then, letting $k \to \infty$, the most right hand side of (9) converges to 0 by Proposition 4.10 with Remark 2.1. Then, combining (8) completes the proof. ∎

Next, we give the proof of the non-compact version of the above theorem.

Proof of Theorem 1.6. We will prove that $\tilde{F}_{s_j}(p_0)$ is a bounded sequence in $(N, \tilde{g})$. For any t_1, t_2 with $0 \leq t_1 < t_2 < T$, we can consider $\{F_t(p_0)\}_{t \in [t_1, t_2]}$ as a curve in N joining $F_{t_1}(p_0)$ and $F_{t_2}(p_0)$. Hence, we have

$$\begin{aligned} \ell_{F_{t_2}(p_0), t_2}(F_{t_1}(p_0), t_1) \leq &\frac{1}{2\sqrt{t_2 - t_1}} \int_{t_1}^{t_2} \sqrt{t_2 - t} \left(R(g_t) + \left| \frac{\partial F_t}{\partial t} \right|^2 \right) dt \\ = &\frac{1}{2\sqrt{t_2 - t_1}} \int_{t_1}^{t_2} \sqrt{t_2 - t} \big(R(g_t) + |H(F_t)|^2 \big) dt. \end{aligned}$$

By the assumption (A2), $(T-t)|H(F_t)|^2$ is bounded, and it is clear that $(T-t)R(g_t) = R(g_0)$ and it is also bounded by the assumption in Remark 1.7. Hence, we have $R(g_t) + |H(F_t)|^2 \leq \frac{C}{T-t}$ for some $C > 0$ and

$$\ell_{F_{t_2}(p_0), t_2}(F_{t_1}(p_0), t_1) \leq \frac{C}{2\sqrt{t_2 - t_1}} \int_{t_1}^{t_2} \frac{\sqrt{t_2 - t}}{T - t} dt \leq C \frac{\sqrt{T - t_1}}{\sqrt{t_2 - t_1}}.$$

By the assumption (B2) with $t_2 \to T$, we have $f(F_{t_1}(p_0), t_1) \leq C$. Since

$$f(F_t(p_0), t) = f_t(F_t(p_0)) = \tilde{f}(\tilde{F}_s(p_0)),$$

this bound means that $\tilde{f}(\tilde{F}_s(p_0)) \leq C$ for all $s \in [-\log T, \infty)$. In [3, Theorem 1.1], Cao and Zhou proved that there exist positive constants C_1 and C_2 such that

$$\frac{1}{4}(r - C_1)^2 \leq \tilde{f} \leq \frac{1}{4}(r + C_2)^2$$

on N, where $r(q) = d_{\tilde{g}}(q_0, q)$ is the distance function from some fixed point q_0 in N. Hence, we have

$$d_{\tilde{g}}(q_0, \tilde{F}_s(p_0)) \leq 2\sqrt{C} + C_1,$$

that is, $\tilde{F}_s(p_0)$ moves in a bounded region in N. Hence, we can use again a similar argument as the Arzelà–Ascoli theorem (see [25] for detail). Then, the remainder of the proof is completely same as the proof of the case that N is compact. □

3. Monotonicity formulas. In this section, we introduce some general formulas which are useful in the following sections. Let M^m and N^n be manifolds, and assume that $m \leq n$ and M is compact. We denote the space of all immersion maps from M to N by $\mathfrak{Imm}(M, N)$ and the space of all Riemannian metrics on N by $\mathfrak{Met}(N)$. Consider the following functional:

$$\mathcal{F}\colon C^\infty(M) \times \mathfrak{Imm}(M, N) \times C^\infty(N)_{>0} \times \mathfrak{Met}(N) \to \mathbb{R}$$
$$\mathcal{F}(u, F, \rho, g) := \int_M u\, F^*\!\rho\, d\mu(F^*g),$$

where u is a smooth function on M and ρ is a positive smooth function on N. We denote diffeomorphism groups of M and N by $\mathrm{Diff}(M)$ and $\mathrm{Diff}(N)$, respectively. Then, $\mathcal{F}$ has some elementary symmetric properties.

Remark 3.1. For $\varphi \in \mathrm{Diff}(M)$ and $\psi \in \mathrm{Diff}(N)$, we have

$$\mathcal{F}(\varphi^* u, \psi^{-1} \circ F \circ \varphi, \psi^* \rho, \psi^* g) = \mathcal{F}(u, F, \rho, g),$$

and for a positive constant $\lambda > 0$ we have

$$\mathcal{F}(\lambda^{n-m} u, F, \lambda^{-n} \rho, \lambda^2 g) = \mathcal{F}(u, F, \rho, g).$$

Let $p := (u, F, \rho, g)$ be a point in $C^\infty(M) \times \mathfrak{Imm}(M, N) \times C^\infty(N)_{>0} \times \mathfrak{Met}(N)$ and $v := (w, V, k, h)$ be a tangent vector of $C^\infty(M) \times \mathfrak{Imm}(M, N) \times C^\infty(N)_{>0} \times \mathfrak{Met}(N)$ at p. Namely, $w \in C^\infty(M)$, $V \in \Gamma(M, F^*(TN))$, $k \in C^\infty(N)$ and $h \in \mathrm{Sym}^2(N)$. Then, we calculate $\delta_v \mathcal{F}(p)$, the first variation of $\mathcal{F}$ at p in the direction v.

Proposition 3.2. *We have*

$$\begin{aligned}
\delta_v \mathcal{F}(p) = &- \int_M u\, g(V + \nabla f^{\perp_F}, H(F) + \nabla f^{\perp_F})\, F^*\!\rho\, d\mu(F^*g) \\
&+ \int_M u\, F^* \Big(\Delta_g \rho + k + \frac{1}{2} \rho \operatorname{tr} h \Big) d\mu(F^*g) \\
&+ \int_M \Big(w - \Delta_{F^*g} u - g(V, F_* \nabla u) \\
&\qquad + u \operatorname{tr}^{\perp_F} \Big(\mathrm{Hess}\, f - \frac{1}{2} h \Big) \Big) F^*\!\rho\, d\mu(F^*g),
\end{aligned} \tag{10}$$

where we define f by $\rho = (4\pi\tau)^{-\frac{n}{2}} e^{-f}$ for a positive function $\tau = \tau(t)$ (which depends only on t).

Remark 3.3. Even though there is an ambiguity of the choice of a function τ, the gradient and Hessian of f do not depend on the choice of τ.

NOTATION 3.4. By $\perp_F$, we denote the normal projection with respect to the orthogonal decomposition $F^*(TN) = F_*(TM) \oplus T^{\perp_F} M$ defined by the immersion F, and by $\mathrm{tr}^{\perp_F}$ we denote the normal trace, that is, for a 2-tensor η on N and a point $p \in M$, $(\mathrm{tr}^{\perp_F} \eta)(p)$ is defined by

$$(\mathrm{tr}^{\perp_F} \eta)(p) := \sum_{j=1}^{n-m} \eta(F(p))(\nu_j, \nu_j),$$

where $\{\nu_j\}_{j=1}^{n-m}$ is an orthonormal basis of $T_p^{\perp_F} M$.

Proof. Let $\{F_s : M \to N\}_{s\in(-\epsilon,\epsilon)}$ be a smooth 1-parameter family of immersions with $F_0 = F$ and $(\partial F_s/\partial s)|_{s=0} = V$. Let $u_s := u + sw$, $\rho_s := \rho + sk$ and $g_s := g + sh$. Then, $p(s) := (u_s, F_s, \rho_s, g_s)$ is a curve in $C^\infty(M) \times \mathfrak{Imm}(M, N) \times C^\infty(N)_{>0} \times \mathfrak{Met}(N)$ with $p(0) = p$ and $\dot p(0) = v$. Then, the first variation of $\mathcal{F}$ at p in the direction v is calculated as

$$\delta_v \mathcal{F}(p) = \frac{d}{ds}\bigg|_{s=0} \mathcal{F}(p(s)) = \frac{d}{ds}\bigg|_{s=0} \int_M u_s\, F_s^*\rho_s\, d\mu(F_s^* g_s),$$

and we have

$$\begin{aligned} &\frac{d}{ds}\bigg|_{s=0} \int_M u_s\, F_s^*\rho_s\, d\mu(F_s^* g_s) \\ &= \int_M w\, F^*\rho\, d\mu(F^* g) + \int_M u\, g(V, \nabla\rho)\, d\mu(F^* g) + \int_M u\, F^*k\, d\mu(F^* g) \qquad (11) \\ &\quad + \int_M u\, F^*\rho \left(\frac{d}{ds}\bigg|_{s=0} d\mu(F_s^* g) \right) + \int_M u\, F^*\rho \left(\frac{d}{ds}\bigg|_{s=0} d\mu(F^* g_s) \right). \end{aligned}$$

It is well-known that the first variation of the induced measure $d\mu(F_s^* g)$ is given by

$$\frac{d}{ds}\bigg|_{s=0} d\mu(F_s^* g) = \{\mathrm{div}_{F^*g}\, F_*^{-1}(V^{\top_F}) - g(H(F), V)\} d\mu(F^* g).$$

On the right hand side, we decompose V as $V = V^{\top_F} + V^{\perp_F} \in F_*(TM) \oplus T^{\perp_F} M$, and we take the divergence of $F_*^{-1}(V^{\top_F})$ on a Riemannian manifold $(M, F^* g)$.

On the other hand, $F^* g_s$ is a time-dependent metric on M. Since $g_s = g + sh$, we have $F^* g_s = F^* g + sF^* h$. Thus, the derivation of $F^* g_s$ is $F^* h$ at $s = 0$. In such a situation, it is also well-known that the first variation of the induced measure $d\mu(F^* g_s)$ of a time-dependent metric on M is given by

$$\frac{d}{ds}\bigg|_{s=0} d\mu(F^* g_s) = \frac{1}{2}\, \mathrm{tr}(F^* h) d\mu(F^* g),$$

where the trace is taken with respect to a metric $F^* g$ on M. By the divergence formula on $(M, F^* g)$, we have

$$\begin{aligned} &\int_M u\, F^*\rho\, \mathrm{div}_{F^*g}\, F_*^{-1}(V^{\top_F}) d\mu(F^* g) \\ &= -\int_M g(V, F_*\nabla u)\, F^*\rho\, d\mu(F^* g) - \int_M u\, g(V, \nabla \rho^{\top_F})\, d\mu(F^* g). \end{aligned}$$

Since $\nabla\rho = -\rho\nabla f$, we have

$$\begin{aligned}&\int_M u\, g(V, \nabla\rho)\, d\mu(F^*g) + \int_M u\, F^*\rho\left(\frac{d}{ds}\bigg|_{s=0} d\mu(F_s^* g)\right)\\ &= -\int_M g(V, F_*\nabla u)\, F^*\rho\, d\mu(F^*g) - \int_M u\, g(V, H(F) + \nabla f^{\perp_F})\, F^*\rho\, d\mu(F^*g).\end{aligned}$$

By the trivial identity $\mathrm{tr}(F^*h) = F^*(\mathrm{tr}\, h) - \mathrm{tr}^{\perp_F} h$, we have

$$\begin{aligned}&\int_M u\, F^*k\, d\mu(F^*g) + \int_M u\, F^*\rho\left(\frac{d}{ds}\bigg|_{s=0} d\mu(F^* g_s)\right)\\ &= \int_M uF^*\left(k + \frac{1}{2}\rho\, \mathrm{tr}\, h\right) d\mu(F^*g) - \int_M \frac{1}{2} u\, F^*\rho(\mathrm{tr}^{\perp_F} h) d\mu(F^*g).\end{aligned}$$

Furthermore, one can easily see that

$$\begin{aligned}F^*(\Delta_g\rho) &= \Delta_{F^*g}(F^*\rho) - g(H(F), \nabla\rho) + \mathrm{tr}^{\perp_F}(\mathrm{Hess}\,\rho)\\ &= \Delta_{F^*g}(F^*\rho) + F^*\rho\, g(\nabla f^{\perp_F}, H(F) + \nabla f^{\perp_F}) - F^*\rho\, \mathrm{tr}^{\perp_F}(\mathrm{Hess}\, f).\end{aligned}$$

Hence, we have

$$\begin{aligned}&\int_M u\, F^*k\, d\mu(F^*g) + \int_M u\, F^*\rho\left(\frac{d}{ds}\bigg|_{s=0} d\mu(F^* g_s)\right)\\ &= \int_M uF^*\left(\Delta_g\rho + k + \frac{1}{2}\rho\, \mathrm{tr}\, h\right) d\mu(F^*g) - \int_M \frac{1}{2} u\, F^*\rho(\mathrm{tr}^{\perp_F} h) d\mu(F^*g)\\ &\quad - \int_M uF^*(\Delta_g\rho) d\mu(F^*g)\\ &= \int_M u\, F^*\left(\Delta_g\rho + k + \frac{1}{2}\rho\, \mathrm{tr}\, h\right) d\mu(F^*g)\\ &\quad - \int_M u\, g(\nabla f^{\perp_F}, H(F) + \nabla f^{\perp_F})\, F^*\rho\, d\mu(F^*g)\\ &\quad + \int_M \left(-\Delta_{F^*g} u + u\, \mathrm{tr}^{\perp_F}(\mathrm{Hess}\, f - \frac{1}{2}h)\right) F^*\rho\, d\mu(F^*g),\end{aligned} \tag{12}$$

where we used

$$\int_M u\Delta_{F^*g}(F^*\rho) d\mu(F^*g) = \int_M (\Delta_{F^*g} u)\, F^*\rho\, d\mu(F^*g).$$

Finally, by combining equations (11)-(12), we get the formula (10). □

By (10), we get a monotonicity formula for Ricci-mean curvature flows.

Proposition 3.5. *Assume that the pair $g = (g_t\,;\, t \in [0, T_1))$ and $F\colon M \times [0, T_2) \to N$ is a solution of Ricci-mean curvature flow with $T_2 \leq T_1$. Further assume that $\rho\colon N \times [0, T_1) \to \mathbb{R}^+$ and $u\colon M \times [0, T_2) \to \mathbb{R}_{\geq 0}$ on M satisfy the following coupled equations:*

$$\frac{\partial \rho_t}{\partial t} = -\Delta_{g_t}\rho_t + R(g_t)\rho_t \tag{13a}$$

$$\frac{\partial u_t}{\partial t} = \Delta_{F_t^* g_t} u_t - u_t\, \mathrm{tr}^{\perp_{F_t}}\big(\mathrm{Ric}(g_t) + \mathrm{Hess}\, f_t\big), \tag{13b}$$

where we define f by $\rho = (4\pi\tau)^{-\frac{n}{2}}e^{-f}$ for a positive function $\tau = \tau(t)$. Then, for all $t \in (0, T_2)$,

$$\frac{d}{dt}\mathcal{F}(u_t, F_t, \rho_t, g_t) = -\int_M u_t \Big| H(F_t) + \nabla f_t^{\perp_{F_t}} \Big|^2_{g_t} F_t^* \rho_t d\mu(F_t^* g_t) \le 0. \tag{14}$$

Proof. Since g_t is a solution of the Ricci flow (1a), $h = -2\mathrm{Ric}(g_t)$ in (10). Furthermore, in this case, $g(V, F_{t*}\nabla u_t) = 0$ since $V = H(F_t)$ is normal and $F_{t*}\nabla u_t$ is tangential. Then, (14) is clear by Proposition 3.2. □

Remark 3.6. The equation (13a) is called the conjugate heat equation for the Ricci flow, and the equation (13b) is a linear heat equation with time-dependent potential $\mathrm{tr}^{\perp_{F_t}}\big(\mathrm{Ric}(g_t) + \mathrm{Hess}\, f_t\big)$ on M.

By a straightforward computation, we can prove the following.

Proposition 3.7. *Assume that $\tau(t) = T - t$. Let $u\colon M \times [0, T) \to \mathbb{R}$ be a solution for (13b). Define $v\colon M \times [0, T) \to \mathbb{R}$ by $u = (4\pi\tau)^{\frac{n-m}{2}} v$. Then, v satisfies*

$$\frac{\partial v}{\partial t} = \Delta_{F^*g} v - v\,\mathrm{tr}^{\perp_F}(\mathrm{Ric} + \mathrm{Hess}\, f - \frac{g}{2\tau}), \tag{13b$'$}$$

and the converse is also true.

Example 3.8. If the ambient space is a Euclidean space, that is, $(N, g) = (\mathbb{R}^n, g_{\mathrm{st}})$, we can reduce Huisken's monotonicity formula from (14).

4. Mean curvature flows in gradient shrinking Ricci solitons. In this section, we recall some definitions and properties of gradient shrinking Ricci solitons and self-similar solutions (cf. Definition 1.2), and prove the monotonicity formula for a Ricci-mean curvature flow along a Ricci flow constructed from a gradient shrinking Ricci soliton and also prove an analog of Stone's estimate.

Recall that if an n-dimensional Riemannian manifold $(N, \tilde{g})$ and a function $\tilde{f}$ on N satisfies the equation (3), it is called a gradient shrinking Ricci soliton. In this paper, we assume that $(N, \tilde{g})$ is a complete Riemannian manifold. Then, by the result due to Zhang [28], it follows that $\nabla\tilde{f}$ is a complete vector field on N. As Theorem 20.1 in Hamilton's paper [8], one can easily see that $R(\tilde{g}) + |\nabla\tilde{f}|^2 - \tilde{f}$ is a constant. Hence, by adding some constant to $\tilde{f}$ if necessary, we can assume that the potential function $\tilde{f}$ satisfying (3) also satisfy the equation (4). As a special case of a more general result for complete ancient solutions by Chen [4] (cf. Corollary 2.5), we can see that $(N, \tilde{g}, \tilde{f})$ must have the nonnegative scalar curvature $R(\tilde{g}) \ge 0$. Hence, we have

$$0 \le |\nabla\tilde{f}|^2 \le \tilde{f} \quad \text{and} \quad 0 \le R(\tilde{g}) \le \tilde{f}.$$

Fix a positive time $T > 0$ arbitrary. Let $\{\Phi_t\colon N \to N\}_{t\in(-\infty,T)}$ be the 1-parameter family of diffeomorphisms with $\Phi_0 = \mathrm{id}_N$ generated by the time dependent vector field $V(t) := \frac{1}{T-t}\nabla\tilde{f}$. For $t \in (-\infty, T)$, define

$$g_t := (T-t)\Phi_t^*\tilde{g}, \quad f_t := \Phi_t^*\tilde{f}, \quad \rho_t := (4\pi(T-t))^{-\frac{n}{2}} e^{-f_t}.$$

Then, by the standard calculation, one can prove the following (cf. [18]).

Proposition 4.1. *The metrics g is the solution of the Ricci flow on the time interval $(-\infty, T)$ with $g_0 = T\tilde{g}$, and ρ and f satisfy the following equations:*

$$\frac{\partial \rho}{\partial t} = -\Delta_g \rho + R(g)\rho$$
$$\mathrm{Ric}(g) + \mathrm{Hess}\, f - \frac{g}{2(T-t)} = 0.$$
$$R(g) + |\nabla f|^2 - \frac{f}{T-t} = 0.$$

Recall that an immersion map $F\colon M \to N$ is called a self-similar solution if it satisfies the equation (6), and it is called shrinking when $\lambda < 0$, steady when $\lambda = 0$ and expanding when $\lambda > 0$. A self-similar solution corresponds to a minimal submanifold in a conformal rescaled ambient space. It easily follows from the formula telling that the mean curvature vector field in the conformal rescaling $(N, e^{2\varphi}\tilde{g})$ is given by $e^{-2\varphi}(H(F) - m\nabla\varphi^{\perp})$. Thus, we have the following.

Proposition 4.2. *Let $F\colon M^m \to N^n$ be an immersion map in a gradient shrinking Ricci soliton $(N, \tilde{g}, \tilde{f})$. Then, the following two conditions are equivalent.*

(1) *F is a self-similar solution with coefficient λ.*

(2) *F is a minimal immersion with respect to a metric $e^{2\lambda\tilde{f}/m}\tilde{g}$ on N.*

From a self-shrinker, we can construct a solution of Ricci-mean curvature flow.

Proposition 4.3. *Let $\tilde{F}\colon M \to N$ be a self-shrinker with $\lambda = -1$. For a fixed time $T > 0$, let Φ_t and g_t be defined as above, and define $\Psi_t := \Phi_t^{-1}$. Then, $F\colon M \times [0, T) \to N$ defined by $F(p, t) := \Psi_t(\tilde{F}(p))$ satisfies*

$$\left(\frac{\partial F}{\partial t}\right)^{\perp} = H(F_t),$$

in the Ricci flow (N, g_t) defined on $t \in [0, T)$, that is, F becomes a solution of the Ricci-mean curvature flow in (N, g_t) up to a time-dependent re-parametrization of M.

Proof. By differentiating the identity $\Phi_t \circ \Psi_t = \mathrm{id}_N$, we have

$$\frac{\partial \Psi_t}{\partial t} = -\Psi_{t*}\left(\frac{1}{T-t}\nabla\tilde{f}\right).$$

Since $H(\tilde{F}) = -\nabla\tilde{f}^{\perp}$, more precisely $H(\tilde{F}) = -\nabla\tilde{f}^{\perp_{\tilde{F},\tilde{g}}}$ (note that the notion of the normal projection depends on an immersion map and an ambient metric), we have

$$\left(\frac{\partial F}{\partial t}\right)^{\perp_{F_t, g_t}} = \left(-\Psi_{t*}\left(\frac{1}{T-t}\nabla\tilde{f}\right)\right)^{\perp_{F_t, g_t}} = \frac{1}{T-t}\Psi_{t*}(H(\tilde{F})) = H(F_t),$$

where $H(\tilde{F})$ is the mean curvature vector field with respect to the metric $\tilde{g}$ and $H(F_t)$ is the one with respect to the metric g_t. □

There exists a one to one correspondence between Ricci-mean curvature flows in (N, g_t) and normalized mean curvature flows (cf. Definition 1.3) in $(N, \tilde{g})$. The following is clear.

Proposition 4.4. *For a fixed time $T > 0$, let Φ_t and g_t be defined as above. If $F\colon M \times [0, T) \to N$ is a Ricci-mean curvature flow along the Ricci flow (N, g_t),* *then the rescaled flow* $\tilde{F}\colon M\times[-\log T,\infty)\to N$ *defined by the equation* (7) *becomes a normalized mean curvature flow in* $(N,\tilde{g})$*, that is, it satisfies*

$$\frac{\partial\tilde{F}}{\partial s}=H(\tilde{F})+\nabla\tilde{f}.$$

Conversely, if $\tilde{F}\colon M\times[-\log T,\infty)\to N$ *is a normalized mean curvature flow in* $(N,\tilde{g})$*, then the flow* $F\colon M\times[0,T)\to N$ *defined by* (7) *becomes a Ricci-mean curvature flow along the Ricci flow* (N,g_t)*.*

Here, the monotonicity formula for a Ricci-mean curvature flow moving along the Ricci flow (N,g_t) is almost clear by Proposition 3.5.

PROPOSITION 4.5. *For a fixed time* $T>0$*, let* g_t*,* f_t*, and* ρ_t *be defined as above, and define* $u_t:=(4\pi(T-t))^{\frac{n-m}{2}}$*. If* $F\colon M\times[0,T)\to N$ *is a Ricci-mean curvature flow along the Ricci flow* (N,g_t) *and* M *is compact, then we have the monotonicity formula:*

$$\frac{d}{dt}\int_M u\,F^*\rho\,d\mu(F^*g)=-\int_M u\Big|H(F)+\nabla f^{\perp_F}\Big|_g^2F^*\rho\,d\mu(F^*g)\le 0. \tag{15}$$

Proof. By Proposition 4.1, we see that ρ satisfies the conjugate heat equation (13a). To see that u satisfies the equation (13b), we use the equivalent equation (13b′). In this case, by Proposition 4.1, the equation (13b′) becomes

$$\frac{\partial v}{\partial t}=\Delta_{F^*g}v,$$

the standard heat equation on M, where u and v are related by $u=(4\pi(T-t))^{\frac{n-m}{2}}v$. Then, $v\equiv 1$ is a trivial solution of (13b′). Hence, $u_t=(4\pi(T-t))^{\frac{n-m}{2}}$ becomes a solution of (13b). Thus, by Proposition 3.5, we have the above monotonicity formula (15). □

By Proposition 4.5, we can deduce the following monotonicity formula of the weighted volume functional for a normalized mean curvature flow, immediately.

PROPOSITION 4.6. *If* $\tilde{F}\colon M\times[-\log T,\infty)\to N$ *is a normalized mean curvature flow in* $(N,\tilde{g},\tilde{f})$ *and* M *is compact, then we have the monotonicity formula:*

$$\frac{d}{ds}\int_M e^{-\tilde{f}\circ\tilde{F}}\,d\mu(\tilde{F}^*\tilde{g})=-\int_M\Big|H(\tilde{F})+\nabla\tilde{f}^{\perp_{\tilde{F}}}\Big|_{\tilde{g}}^2e^{-\tilde{f}\circ\tilde{F}}d\mu(\tilde{F}^*\tilde{g})\le 0. \tag{16}$$

Proof. In this proof, we follow the notations in Proposition 4.4. It is clear that $f_t\circ F_t=\tilde{f}\circ\tilde{F}_s$ and $F_t^*g_t=(T-t)\tilde{F}_s^*\tilde{g}$. Hence, we have

$$\begin{aligned}u_t\,F_t^*\rho_t\,d\mu(F_t^*g_t)=&(4\pi(T-t))^{-\frac{m}{2}}e^{-f_t\circ F_t}\,d\mu(F_t^*g_t)\\=&(4\pi)^{-\frac{m}{2}}e^{-\tilde{f}\circ\tilde{F}_s}\,d\mu(\tilde{F}_s^*\tilde{g})\,.\end{aligned}$$

Since $H(\tilde{F}_s)=(T-t)\Phi_{t*}H(F_t)$ and $\nabla\tilde{f}=(T-t)\Phi_{t*}\nabla f_t$, we have

$$(T-t)\Big|H(F_t)+\nabla f_t^{\perp_{F_t}}\Big|_{g_t}^2=\Big|H(\tilde{F}_s)+\nabla\tilde{f}^{\perp_{\tilde{F}_s}}\Big|_{\tilde{g}}^2.$$

Thus, by the equality (15), one can easily see that the equality (16) holds. □

To prove the main theorems, we need the following key lemma. Its proof is an analog of the proof of Stone's estimate (cf. Lemma 2.9 in [22]). Stone considered the weight $e^{-\sqrt{\tilde{f}}}$ in the Euclidean case, where $\tilde{f} := |x|^2/4$. However we consider the weight $e^{-\frac{\tilde{f}}{2}}$, since $-\frac{\tilde{f}}{2}$ is a smooth function and we can apply Proposition 3.2.

LEMMA 4.7. *Assume that $(N, \tilde{g})$ has bounded geometry. If $\tilde{F}\colon M\times[-\log T, \infty) \to N$ is a normalized mean curvature flow in $(N, \tilde{g}, \tilde{f})$ and M is compact, then there exists a constant $C > 0$ such that*

$$\int_M e^{-\frac{\tilde{f}}{2}\circ\tilde{F}}\, d\mu(\tilde{F}^*\tilde{g}) \le C.$$

uniformly on $[-\log T, \infty)$.

Proof. In this proof, we follow the notations in Proposition 4.4. As the proof of Proposition 4.6, we have

$$\int_M e^{-\frac{\tilde{f}}{2}\circ\tilde{F}_s}\, d\mu(\tilde{F}_s^*\tilde{g}) = (4\pi)^{\frac{m}{2}} \int_M u_t\, F_t^*\bar{\rho}_t\, d\mu(F_t^* g_t),$$

where

$$\bar{\rho}_t := \frac{1}{(4\pi(T-t))^{\frac{n}{2}}} e^{-\frac{f_t}{2}} \quad \text{and} \quad u_t := (4\pi(T-t))^{\frac{n-m}{2}}.$$

By Proposition 3.2, we have

$$\begin{aligned}
&\frac{d}{dt}\int_M u_t\, F_t^*\bar{\rho}_t\, d\mu(F_t^* g_t)\\
&= -\int_M u_t \left| H(F_t) + \frac{1}{2}\nabla f_t^{\perp_{F_t}}\right|_{g_t}^2 F_t^*\bar{\rho}_t\, d\mu(F_t^* g_t)\\
&\quad + \int_M u_t F_t^*\left(\frac{\partial\bar{\rho}_t}{\partial t} + \Delta_{g_t}\bar{\rho}_t - R(g_t)\bar{\rho}_t\right) d\mu(F_t^* g_t)\\
&\quad + \int_M \left(\frac{\partial u_t}{\partial t} - \Delta_{F_t^* g_t} u_t + u_t \operatorname{tr}^{\perp_{F_t}}\left(\frac{1}{2}\operatorname{Hess} f_t + \operatorname{Ric}(g_t)\right)\right) F_t^*\bar{\rho}_t\, d\mu(F_t^* g_t)\,.
\end{aligned}$$

By $\frac{\partial f}{\partial t} = |\nabla f|^2$, $|\nabla f|^2 = \frac{f}{T-t} - R(g)$ and $\Delta_g f = -R(g) + \frac{n}{2(T-t)}$, we can compute $\frac{\partial\bar{\rho}}{\partial t}$ and $\Delta_g\bar{\rho}$. Then, we can see that

$$\frac{\partial\bar{\rho}}{\partial t} + \Delta_g\bar{\rho} - R(g)\bar{\rho} = \bar{\rho}\left(\frac{n}{4(T-t)} - \frac{f}{4(T-t)} - \frac{1}{4}R(g)\right) \le \frac{\bar{\rho}}{4(T-t)}(n-f).$$

Since u satisfies $\frac{\partial u_t}{\partial t} - \Delta_{F_t^* g_t} u_t + u_t \operatorname{tr}^{\perp_{F_t}} (\operatorname{Hess} f_t + \operatorname{Ric}(g_t)) = 0$, we have

$$\frac{\partial u_t}{\partial t} - \Delta_{F_t^* g_t} u_t + u_t \operatorname{tr}^{\perp_{F_t}}\left(\frac{1}{2}\operatorname{Hess} f_t + \operatorname{Ric}(g_t)\right) = -\frac{1}{2} u_t \operatorname{tr}^{\perp_{F_t}} \operatorname{Hess} f_t.$$

By $\operatorname{Hess} f_t = \frac{1}{2(T-t)} g_t - \operatorname{Ric}(g_t)$, we have

$$-\frac{1}{2} u_t \operatorname{tr}^{\perp_{F_t}} \operatorname{Hess} f_t = u_t \left(-\frac{n-m}{4(T-t)} + \frac{1}{2}\operatorname{tr}^{\perp_{F_t}} \operatorname{Ric}(g_t)\right).$$

It is clear that $\mathrm{tr}^{\perp_{F_t}} \mathrm{Ric}(g_t) \le (n-m)|\mathrm{Ric}(g_t)|_{g_t} = (n-m)\frac{|\mathrm{Ric}(\tilde{g})|_{\tilde{g}}}{T-t} \le \frac{C''}{T-t}$, where $C'' := (n-m)\max_N |\mathrm{Ric}(\tilde{g})|_{\tilde{g}}$ is a bounded constant since $(N, \tilde{g})$ has bounded geometry. Thus, we have

$$\frac{d}{dt}\int_M u_t\, F_t^*\bar{\rho}_t\, d\mu(F_t^* g_t) < \frac{1}{4(T-t)}\int_M \Big(C_0 - f_t \circ F_t\Big) u_t\, F_t^*\bar{\rho}_t\, d\mu(F_t^* g_t),$$

where $C_0 := m + 4C'' + 1$. Since $s = -\log(T-t)$, we have

$$\frac{d}{ds}\int_M e^{-\frac{\tilde{f}}{2}\circ\tilde{F}_s}\, d\mu(\tilde{F}_s^*\tilde{g}) = (4\pi)^{\frac{m}{2}}(T-t)\frac{d}{dt}\int_M u_t\, F_t^*\bar{\rho}_t\, d\mu(F_t^* g_t)\,.$$

Hence, we have

$$\frac{d}{ds}\int_M e^{-\frac{\tilde{f}}{2}\circ\tilde{F}_s}\, d\mu(\tilde{F}_s^*\tilde{g}) < \frac{1}{4}\int_M \Big(C_0 - \tilde{f}\circ\tilde{F}_s\Big) e^{-\frac{\tilde{f}}{2}\circ\tilde{F}_s}\, d\mu(\tilde{F}_s^*\tilde{g})\,.$$

Here, we divide M into time-dependent three pieces by $M_{1,s} := \tilde{F}_s^{-1}(\{\tilde{f} \le C_0\})$, $M_{2,s} := \tilde{F}_s^{-1}(\{C_0 < \tilde{f} \le 2C_0\})$ and $M_{3,s} := \tilde{F}_s^{-1}(\{2C_0 < \tilde{f}\})$. On each component, we have

$$\begin{aligned}
&\int_{M_{1,s}} \Big(C_0 - \tilde{f}\circ\tilde{F}_s\Big) e^{-\frac{\tilde{f}}{2}\circ\tilde{F}_s}\, d\mu(\tilde{F}_s^*\tilde{g}) \le C_0 \int_{M_{1,s}} e^{-\frac{\tilde{f}}{2}\circ\tilde{F}_s}\, d\mu(\tilde{F}_s^*\tilde{g}),\\
&\int_{M_{2,s}} \Big(C_0 - \tilde{f}\circ\tilde{F}_s\Big) e^{-\frac{\tilde{f}}{2}\circ\tilde{F}_s}\, d\mu(\tilde{F}_s^*\tilde{g}) \le 0,\\
&\int_{M_{3,s}} \Big(C_0 - \tilde{f}\circ\tilde{F}_s\Big) e^{-\frac{\tilde{f}}{2}\circ\tilde{F}_s}\, d\mu(\tilde{F}_s^*\tilde{g}) \le -C_0 \int_{M_{3,s}} e^{-\frac{\tilde{f}}{2}\circ\tilde{F}_s}\, d\mu(\tilde{F}_s^*\tilde{g})\,.
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\frac{d}{ds}\int_M e^{-\frac{\tilde{f}}{2}\circ\tilde{F}_s}\, d\mu(\tilde{F}_s^*\tilde{g})\\
<&\frac{C_0}{4}\Big(\int_{M_{1,s}} e^{-\frac{\tilde{f}}{2}\circ\tilde{F}_s}\, d\mu(\tilde{F}_s^*\tilde{g}) - \int_{M_{3,s}} e^{-\frac{\tilde{f}}{2}\circ\tilde{F}_s}\, d\mu(\tilde{F}_s^*\tilde{g})\Big).
\end{aligned} \tag{17}$$

On the other hand, by the monotonicity formula (cf. Proposition 4.6), we have $\int_M e^{-\tilde{f}\circ\tilde{F}_s}\, d\mu(\tilde{F}_s^*\tilde{g}) \le C'$, where C' is the value of the left hand side at the initial time $s = -\log T$. We further define a region in M by $M_{4,s} := \tilde{F}_s^{-1}(\{\tilde{f} \le 2C_0\}) = M_{1,s} \cup M_{2,s}$. Since $e^{-\frac{\tilde{f}}{2}} = e^{\frac{\tilde{f}}{2}} e^{-\tilde{f}} \le e^{\frac{C_0}{2}} e^{-\tilde{f}}$ on $M_{1,s}$, we have

$$\int_{M_{1,s}} e^{-\frac{\tilde{f}}{2}\circ\tilde{F}_s}\, d\mu(\tilde{F}_s^*\tilde{g}) \le e^{\frac{C_0}{2}} \int_M e^{-\tilde{f}\circ\tilde{F}_s}\, d\mu(\tilde{F}_s^*\tilde{g}) \le e^{\frac{C_0}{2}} C' =: C_1.$$

As on $M_{1,s}$, we have

$$\int_{M_{4,s}} e^{-\frac{\tilde{f}}{2}\circ\tilde{F}_s}\, d\mu(\tilde{F}_s^*\tilde{g}) \le e^{C_0} C' =: C_2. \tag{18}$$

Hence, by the inequality (17), we see that for each $s \in [-\log T, \infty)$ we must have either

$$\frac{d}{ds}\int_M e^{-\frac{\tilde{f}}{2}\circ\tilde{F}_s}\, d\mu(\tilde{F}_s^*\tilde{g}) < 0 \quad \text{or} \quad \int_{M_{3,s}} e^{-\frac{\tilde{f}}{2}\circ\tilde{F}_s}\, d\mu(\tilde{F}_s^*\tilde{g}) \le C_1.$$

Since $M = M_{3,s} \cup M_{4,s}$ and we have the bound (18), we see that for each $s \in [-\log T, \infty)$ we must have either

$$\frac{d}{ds}\int_M e^{-\frac{\tilde{f}}{2}\circ\tilde{F}_s}\,d\mu(\tilde{F}_s^*\tilde{g}) < 0 \quad \text{or} \quad \int_M e^{-\frac{\tilde{f}}{2}\circ\tilde{F}_s}\,d\mu(\tilde{F}_s^*\tilde{g}) \le C_1 + C_2.$$

This condition implies that

$$\int_M e^{-\frac{\tilde{f}}{2}\circ\tilde{F}_s}\,d\mu(\tilde{F}_s^*\tilde{g}) \le \max\{C_1 + C_2, C_3\} =: C,$$

where C_3 is the value of the left hand side at the initial time $s = -\log T$. ∎

Remark 4.8. In Section 1, we consider the condition (A2) for a Ricci-mean curvature flow $F\colon M \times [0,T) \to N$ along the Ricci flow g_t. Note that if M is compact this condition is equivalent to that there exists a constant $C_0 > 0$ such that

$$\max_M |A(F_t)|_{g_t} \le \frac{C_0}{\sqrt{T-t}} \quad \text{on} \quad [0,T).$$

Then, we can prove the following uniform estimates for higher derivatives of the second fundamental form.

Proposition 4.9. *Let $(N, \tilde{g}, \tilde{f})$ be a gradient shrinking Ricci soliton with bounded geometry. For a fixed time $T > 0$, let Φ_t and g_t be defined as above, and let $F\colon M \times [0,T) \to N$ be a Ricci-mean curvature flow along the Ricci flow (N, g_t). Assume that M is compact and F satisfies the condition (A2). Let $\tilde{F}$ be the normalized mean curvature flow defined by (7). Then, for all $k = 0, 1, 2, \dots,$ there exist constants $C_k > 0$ such that*

$$|\tilde{\nabla}^k A(\tilde{F}_s)|_{\tilde{g}} \le C_k \quad \text{on} \quad M \times [-\log T, \infty),$$

where $\tilde{\nabla}$ is the connection defined by the Levi–Civita connection on $(N, \tilde{g})$ and the one on $(M, \tilde{F}_s^*\tilde{g})$.

The proof is standard by the iteration and parabolic maximum principle. The computation of $(\frac{\partial}{\partial s} - \Delta)|A(F_t)|_{g_t}$ is straightforward but tedious. Readers who want to know its detail can find a rigorous proof in [25]. Combining Lemma 4.7 and Proposition 4.9, we can deduce the following uniform bound of the second derivative of the weighted volume. Its proof is also straightforward long computation and is included in [25].

Lemma 4.10. *Let $(N, \tilde{g}, \tilde{f})$ be a gradient shrinking Ricci soliton with bounded geometry. For a fixed time $T > 0$, let Φ_t and g_t be defined as above, and let $F\colon M \times [0,T) \to N$ be a Ricci-mean curvature flow along the Ricci flow (N, g_t). Assume that M is compact and F satisfies the condition (A2). Let $\tilde{F}$ be the normalized mean curvature flow defined by (7). Then, there exists a constant $C' > 0$ such that*

$$\left|\frac{d^2}{ds^2}\int_M e^{-\tilde{f}\circ\tilde{F}_s}\,d\mu(\tilde{F}_s^*\tilde{g})\right| = \left|\frac{d}{ds}\int_M \left|H(\tilde{F}_s) + \nabla\tilde{f}^{\perp_{\tilde{F}_s}}\right|_{\tilde{g}}^2 e^{-\tilde{f}\circ\tilde{F}_s} d\mu(\tilde{F}_s^*\tilde{g})\right| \le C'$$

uniformly on $[-\log T, \infty)$.

Finally, we give the proof of Theorem 1.11.

Proof of Theorem 1.11. We denote the Kähler form and the complex structure on (N, g, f) by ω and J respectively. Since $F\colon L \to N$ is a self-similar solution, F satisfies

$$H(F) = \lambda \nabla f^{\perp}$$

for some constant $\lambda \in \mathbb{R}$. Then, by the definition of the mean curvature form ω_H, for a tangent vector X on L, we have

$$\omega_H(X) = \omega(H(F), F_*X) = \lambda\omega(\nabla f^{\perp}, F_*X) = \lambda\omega(\nabla f, F_*X),$$

where we used the Lagrangian condition in the last equality. Since the mean curvature form is exact, there exists a smooth function θ on L such that $\omega_H = d\theta$. Let $\{e_i\}_{i=1}^n$ be an orthonormal local frame on L with respect to the metric F^*g. Since ω and J are parallel, we have

$$\begin{aligned}\Delta\theta =& \nabla_{e_i}\omega_H(e_i) - \omega_H(\nabla_{e_i}e_i)\\ =& \lambda\nabla_{e_i}\omega(\nabla f, F_*e_i) - \omega_H(\nabla_{e_i}e_i)\\ =& -\lambda \operatorname{Hess} f(F_*e_i, JF_*e_i) + \lambda\omega(\nabla f, \nabla_{F_*e_i}F_*e_i) - \lambda\omega(\nabla f, F_*(\nabla_{e_i}e_i))\\ =& -\lambda \operatorname{Hess} f(F_*e_i, JF_*e_i) + \lambda\omega(\nabla f, H(F)).\end{aligned}$$

Since the ambient is a gradient shrinking Kähler Ricci soliton, we have

$$\operatorname{Hess} f(F_*e_i, JF_*e_i) = -\operatorname{Ric}(F_*e_i, JF_*e_i) + \frac{1}{2}g(F_*e_i, JF_*e_i) = 0.$$

Furthermore, we have

$$\omega(\nabla f, H(F)) = \omega(\nabla f^{\top}, H(F)) = \omega(F_*\nabla(F^*f), H(F)) = -(F^*g)(\nabla(F^*f), \nabla\theta).$$

Hence, θ satisfies the following linear elliptic equation:

$$\Delta\theta + \lambda(F^*g)(\nabla(F^*f), \nabla\theta) = 0.$$

Since L is compact, by the maximum principle, we obtain that θ is a constant, and this implies that $H(F) = 0$. □

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