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STABILITY INEQUALITIES FOR LAWSON CONES*

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Dedicated to Xunjing Wei

Abstract. In [1], Guido De Philippis and Francesco Maggi proved global quadratic stability inequalities and derived explicit lower bounds for the first eigenvalues of the stability operators for all area-minimizing Lawson cones M_{kh}, except for those with

$$(k,h),(h,k) \in S = \{(3,5),(2,7),(2,8),(2,9),(2,10),(2,11)\}.$$

We proved the corresponding inequalities and lower bounds for these Lawson cones M_{kh} with $(k,h),(h,k) \in S$ by using different sub-calibrations from theirs, thus extending their results to all area-minimizing Lawson cones.

Key words. Simons Cones, Lawson Cones.

Mathematics Subject Classification. 53Bxx Local differential geometry.

1. Introduction.

Suppose $h,k \geq 2$ are positive integers. The Lawson cone M_{kh} is the level set

$$M_{kh} = \left\{ z = (x,y) \in \mathbb{R}^k \times \mathbb{R}^h : \frac{|x|}{\sqrt{k-1}} = \frac{|y|}{\sqrt{h-1}} \right\}.$$

It is known to be area-minimizing (see [2], [3], [4], and [5]) provided

$$h + k \geq 9, \text{ or } (h,k) = (3,5),(4,4),(5,3). \tag{1}$$

In their paper [1], G. De. Philippis and F. Maggi proved global quadratic stability inequalities and derived explicit lower bounds for the first eigenvalues of the stability operators for all area-minimizing Lawson cones M_{kh}, except for

$$(h,k),(k,h) \in S = \{(3,5),(2,7),(2,8),(2,9),(2,10),(2,11)\}.$$

They achieved this by exploiting sub-calibrations for Lawson cones. Unfortunately, the sub-calibrations that they used did not work for the cones M_{kh} with $(h,k),(k,h) \in S$. Our main results, Theorem 1 and Theorem 2 in Section 1.1, extend these inequalities to the cones M_{kh} with $(h,k),(k,h) \in S$. We achieve this by carefully choosing sub-calibrations for these Lawson cones in Lemma 2 of Section 2.1. However, our sub-calibrations do not work for other cases in general.

We first review their results and explain their methods, which we mostly follow. Consider a variation with compact support of the Lawson cone M_{kh}. Suppose the variation can be realized as the boundary of a set F of finite perimeter. Roughly speaking, their first result controls the volume bounded between the Lawson cone and the variation ∂F by the difference between the area of the variation ∂F and that of the cone M_{kh} up to scaling. Their second result provides lower bounds for the first eigenvalues of the stability operators. For a more detailed discussion of the significance of these results, please refer to Section 1 of [1].

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The Lawson cone M_{kh} can be realized as the boundary ∂K_{kh} of the region

$$K_{kh} = \left\{ (x,y) \in \mathbb{R}^k \times \mathbb{R}^h : \frac{|x|}{\sqrt{k-1}} < \frac{|y|}{\sqrt{h-1}} \right\}.$$

Let $\mathcal{L}^m$ denote the m-dimensional Lebesgue measure, ω_n denote the volume of unit n-ball, and $P(A;B)$ denote the perimeter of A in B. Their results are as follows.

RESULT 1 (Theorem 5 in [1]). *If $R > 0, m = h+k, (h,k) \notin S$ satisfy all the conditions in (1), then*

$$\left(\frac{\mathcal{L}^m(K_{kh} \Delta F)}{R^m} \right)^2 \leq C \frac{P(F;H_R) - P(K_{kh};H_R)}{R^{m-1}},$$

whenever F is a set of locally finite perimeter with symmetric difference $K_{kh}\Delta F \subset\subset H_R = B_R^k \times B_R^h$. Possible values of C are

$$\begin{aligned} C =& \frac{2^{12}\sqrt{\omega_k \omega_h}}{(k-1)^{1/8}} \sqrt{\frac{hk}{m-1}} (\frac{h-1}{k-1})^{3/2}, \ \textit{if } 2 \leq k \leq h, (k,h) \neq (4,4), \\ & \textit{Interchange } k,h \textit{ if } 2 \leq h \leq k. \\ C =& 128\omega_4, \ \textit{if } (k,h) = (4,4). \end{aligned}$$

RESULT 2 (Theorem 2 in [1]). *If R, m, h, k are as in Result 1, and*

$$\lambda_{k,h}(R) = \inf \left\{ \int_{M_{kh}} |\nabla^{M_{kh}} \varphi|^2 - |\mathrm{II}_{M_{kh}}|^2 \varphi^2 d\mathcal{H}^{m-1} : \int_{M_{kh}} \varphi^2 = 1, \mathrm{spt}\varphi \subset\subset B_R^m \right\},$$

then

$$\lambda_{k,h}(R) \geq \frac{c_{k,h}}{R^2}.$$

Possible values of $c_{k,h}$ are

$$\begin{aligned} c_{k,h} =& \frac{1}{2^9} \left(\frac{k-1}{h-1} \right)^{9/4} \frac{(m-2)^{1/2}}{(h-1)^{1/4}}, \ \textit{if } 2 \leq k \leq h, (k,h) \neq (4,4). \\ & \textit{Interchange } k,h \textit{ if } 2 \leq h \leq k. \\ c_{k,h} =& \frac{\sqrt{2}}{16}, \ \textit{if } (k,h) = (4,4). \end{aligned}$$

As illustrated in Figure 1, their method is based on sub-calibrating the Lawson cones with a unit-length vector field g. In other words, the vector field g restricts to the unit normal on M_{kh}, and the divergence $\operatorname{div} g$ does not change sign in K_{kh} and $K_{kh}^{\complement}$, respectively.

After cleverly choosing g, they proved that

$$\operatorname{div} g(z) \geq c_{k,h} \frac{\operatorname{dist}(z, M_{kh})}{|z|^2}, \tag{2}$$

where dist is the Euclidean distance. Then they exploit inequality (2) to deduce the desired results. For a beautiful discussion of sub-calibrations (also called quantitative calibrations), please refer to their paper [1].

Unfortunately, the sub-calibrations they used did not work for $(h,k), (k,h) \in S$. The main results of this paper extend their stability inequalities to include those (k,h). We achieve this by using sub-calibrations inspired by [5].

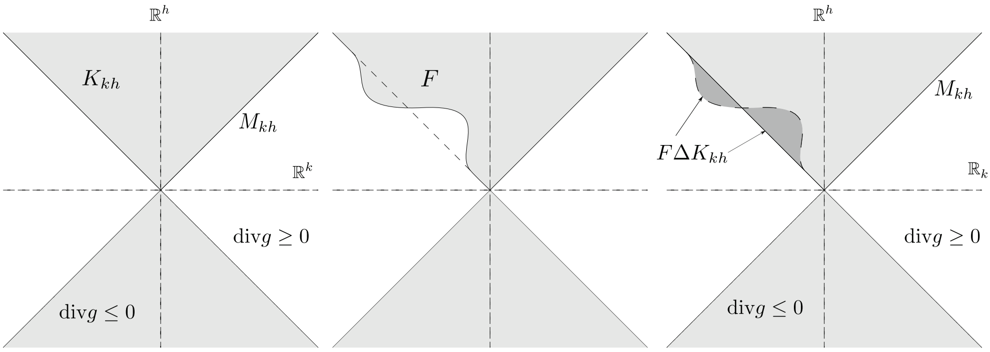

FIG. 1. *A sub-calibration g of the Lawson cone M_{kh} and a variation.*

1.1. Stability Inequalities Extended to $(h,k),(k,h) \in S$.

THEOREM 1. *If $R>0, m=h+k, (h,k),(k,h)\in S$, then*

$$\left(\frac{\mathcal{L}^m(K_{kh}\Delta F)}{R^m}\right)^2 \leq C\frac{P(F;H_R)-P(K_{kh};H_R)}{R^{m-1}},$$

whenever F is a set of locally finite perimeter with $K_{kh}\Delta F \subset\subset B^k_R \times B^h_R$. A possible value of C is $7^2 \times 12^2 \times 10^{20}$.

THEOREM 2. *If R,m,h,k are as in Theorem 1, and*

$$\lambda_{k,h}(R) = \inf\left\{\int_{M_{kh}} |\nabla^{M_{kh}}\varphi|^2 - |\mathrm{II}_{M_{kh}}|^2\varphi^2 d\mathcal{H}^{m-1} : \int_{M_{kh}}\varphi^2 = 1, \mathrm{spt}\varphi \subset\subset B^m_R\right\},$$

then

$$\lambda_{k,h}(R) \geq \frac{c_{k,h}}{R^2},$$

Possible values of $c_{k,h}$ are

$$c_{3,5} = c_{5,3} = \frac{\sqrt{3}}{21^3},$$
$$c_{k,2} = c_{2,h} = \frac{\sqrt{11}}{11^6},$$

for $k,h = 7,8,9,10,11$.

2. Proof of the Theorems. We now prove, in order, Theorem 2 and Theorem 1. By the symmetry of Lawson cones, it suffices to prove the cases with $(h,k)\in S$. The following lemma is the basic tool to extract information from the sub-calibrations g.

LEMMA 3. *If $m \geq 2$, E is of locally finite perimeter in $\mathbb{R}^m$, and $g \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^m,\mathbb{R}^m)$,*

$$|g| \leq 1 \text{ on } \mathbb{R}^m,$$
$$\operatorname{div} g \geq 0, \text{ a.e. on } E^c,$$
$$\operatorname{div} g \leq 0, \text{ a.e. on } E,$$
$$g = \nu_E, \ \mathcal{H}^{m-1} - \text{a.e. on } \partial_{1/2}E,$$

then E is a local minimizer of the perimeter in $\mathbb{R}^m$, with

$$P(F;A)-P(E;A)=\int_{E\Delta F}|\operatorname{div} g|+\int_{A\cap\partial_{1/2}F}1-(g\cdot\nu_F)d\mathcal{H}^{m-1}. \tag{3}$$

Here $\mathcal{H}^{m-1}$ is the $m-1$-dimensional Hausdorff measure, ν_E is the out-pointing unit normal. If $|E|$ denote the $\mathcal{L}^m$-volume of a set E, then

$$\partial_{1/2}E=\{x\in\mathbb{R}^m:\lim_{r\to0^+}\frac{|E\cap B(x,r)|}{\omega_n r^n}=\frac{1}{2}\},$$

is defined as the set of points of density 1/2 in E. For proof of Lemma 1 and details about $\partial_{1/2}E$, please refer to the proof of Proposition 4.1 in [1] and the relevant discussions on page 416 in [1]. Roughly speaking, Lemma 1 can be proved by breaking down the integration definition of perimeter and then using the divergence theorem.

The left hand-side of (3) can be seen as variation of area, so it can provide information for second variation by Taylor expansion and choosing suitable variation F. The key to using this information is to find vector fields g that satisfy inequality (1) in Section 1.

2.1. Sub-calibrations for M_{kh} with $(h,k)\in S$.

LEMMA 1. *For $E=K_{kh}$, the vector field*

$$g=\frac{\nabla f}{|\nabla f|}$$

satisfies all the hypothesis in Lemma 1. The function f for $(h,k)=(3,5)$ is

$$f(x,y)=\begin{cases}\frac{(h-1)|x|^2-(k-1)|y|^2}{4}((h-1)|x|)^{3/2}, & \text{if } z\in K_{kh},\\ \frac{(h-1)|x|^2-(k-1)|y|^2}{4}((k-1)|y|)^{3/2}, & \text{if } z\in K_{kh}^{\complement},\end{cases}$$

and the functions f for $(h,k)=(2,k)$ with $k=7,8,9,10,11$ are

$$f(x,y)=\begin{cases}\frac{(h-1)|x|^2-(k-1)|y|^2}{4}((h-1)|x|)^{3}, & \text{if } z\in K_{kh},\\ \frac{(h-1)|x|^2-(k-1)|y|^2}{4}((k-1)|y|)^{2}, & \text{if } z\in K_{kh}^{\complement}.\end{cases}$$

Moreover, g also satisfy

$$|\operatorname{div} g|\geq\frac{c_{k,h}}{|z|^2}\operatorname{dist}(z,M_{kh}),$$

with values of $c_{k,h}$ the same as in Theorem 2.

The proof of Lemma 2 is left to Section 3. The sub-calibrations we choose work well for $(h,k)\in S$, but do not work for some other Lawson cones. In some sense, these are specifically chosen to cover the cases $(h,k)\in S$.

2.2. Proof of Theorem 2. By Lemma 1, we have

$$
\begin{aligned}
P(F;H_R)-P(K_{kh};H_R) \geq& \int_{K_{kh}\Delta f} |\operatorname{div} g| \\
\geq& c_{k,h}\int_{K_{kh}\Delta F} \frac{\operatorname{dist}(z,M_{kh})}{|z|^2} dz \\
\geq& \frac{c_{k,h}}{R^2}\int_{K_{kh}\Delta F} \operatorname{dist}(z,M_{kh}) dz.
\end{aligned}
\tag{4}
$$

Now, suppose $\varphi \in C^1(M_{kh})$, with $0 \notin \operatorname{spt}\varphi \subset\subset B^m_R$. For $t_0 > 0$ small enough, there exists an open set $F \subset \mathbb{R}^m$ with $\partial F - \{0\}$ a C^1 hypersurface and $K_{kh}\Delta F \subset\subset H_R$, such that

$$\partial F - \{0\} = \{z + t\varphi(z)\nu_{K_{kh}}(z) : z \in M_{kh} - \{0\}\}.$$

By second variation and Taylor expansion, we have

$$P(F;H_R) - P(K_{kh};H_R) = \frac{t^2}{2}\int_{M_{kh}} |\nabla^{M_{kh}}\varphi|^2 - |\mathrm{II}_{M_{kh}}|^2\varphi^2 d\mathcal{H}^{m-1} + O(t^3).$$

Calculating the integral directly by pulling back the volume form on $\mathbb{R}^m$, we have

$$
\begin{aligned}
\int_{K_{kh}\Delta F} \operatorname{dist}(z,M_{kh})dz =& (1+O(t))\int_{M_{kh}} d\mathcal{H}^{m-1}(z)\int_0^{t|\varphi(z)|} sds \\
=& \frac{t^2}{2}\int_{M_{kh}} \varphi^2 d\mathcal{H}^{m-1} + O(t^3).
\end{aligned}
\tag{5}
$$

For details, please refer to Lemma 3.1 in [1]. Combining (4) and (5), and letting $t \to 0$, we deduce that

$$\int_{M_{kh}} |\nabla^{M_{kh}}\varphi|^2 - |\mathrm{II}_{M_{kh}}|^2\varphi^2 d\mathcal{H}^{m-1} \geq \frac{c_{k,h}}{R^2}\int_{M_{kh}} \varphi^2 d\mathcal{H}^{m-1}. \tag{6}$$

To extend (6) to all $\phi \in C^1(M_{kh})$, let ψ_j be a sequence of cut-off functions so that $\operatorname{spt}\psi_j \subset B^m_{2/j}$ and $\psi_j = 1$ on $B^m_{1/j}$ with $|D\psi_j| \leq C_m j$ everywhere, where C_m is a positive constant depending only on m. We know that $\mathcal{H}^{m-1}(M_{kh} \cap B^m_r) \leq c(m)r^{m-1}$ for some constant $c(m)$ depending only on m and $|\mathrm{II}_{M_{kh}}| \leq \frac{C}{|z|}$ for some constant C depending only on k, h. Combining these estimates, we can see that the integrand on the left hand side of (6) is dominated by $O(\frac{1}{|z|^2})$, and thus the integral on the left hand side converges as $j \to \infty$. Let $j \to \infty$ and use dominated convergence. We deduce that (6) is true for all $\varphi \in C^1(M_{kh})$. □

2.3. Proof of Theorem 1. Define

$$p(z) = \left|\frac{|x|}{\sqrt{k-1}} - \frac{|y|}{\sqrt{h-1}}\right|.$$

By Lemma 1 and Lemma 2, we have

$$\begin{aligned}|K_{kh}\Delta F| &\leq |(K_{kh}\Delta F)\cap\{p>\epsilon\}| + |H_R\cap\{p<\epsilon\}| \\ &\leq \int_{(K_{kh}\Delta F)\cap\{p>\epsilon\}} \frac{p(z)}{\epsilon}\frac{R^2}{|z|^2}dz + |H_R\cap\{p<\epsilon\}| \\ &= \frac{lR^2}{\epsilon}\int_{(K_{kh}\Delta F)\cap\{p>\epsilon\}} \frac{\operatorname{dist}(z, M_{kh})}{|z|^2}dz + |H_R\cap\{p<\epsilon\}| \\ &\leq \frac{lR^2}{c_{k,h}\epsilon}\int_{(K_{kh}\Delta F)\cap\{p>\epsilon\}} |\operatorname{div} g|dz + |H_R\cap\{p<\epsilon\}| \\ &\leq \frac{lR^2}{c_{k,h}\epsilon}\Big(P(F;H_R) - P(K_{kh};H_R)\Big) + |H_R\cap\{p<\epsilon\}|,\end{aligned}$$

where $l = \sqrt{\frac{1}{h-1}+\frac{1}{k-1}}$ by elementary geometry. Now, we need to get a suitable upper bound for $|H_R\cap\{p<\epsilon\}|$. We have

$$\begin{aligned}|H_R\cap\{p<\epsilon\}| &= \int_{B_R^h}\mathcal{H}^k\left(\left\{x\in B_R^k : \frac{|y|}{\sqrt{h-1}}-\epsilon < \frac{|x|}{\sqrt{k-1}} < \frac{|y|}{\sqrt{h-1}}+\epsilon\right\}\right)dy \\ &\leq \omega_k(k-1)^{k/2}\int_{B_R^h}\left(\frac{|y|}{\sqrt{h-1}}+\epsilon\right)^k - \left(\frac{|y|}{\sqrt{h-1}}-\epsilon\right)_+^k dy.\end{aligned}$$

We can break down the estimate into two parts, namely

$$\begin{aligned}&\int_{B^h_{\epsilon\sqrt{h-1}}}\left(\frac{|y|}{\sqrt{h-1}}+\epsilon\right)^k - \left(\frac{|y|}{\sqrt{h-1}}-\epsilon\right)_+^k dy \\ =&\int_{B^h_{\epsilon\sqrt{h-1}}}\left(\frac{|y|}{\sqrt{h-1}}+\epsilon\right)^k dy \\ \leq& 2^k\epsilon^{h+k}\omega_h(h-1)^{k/2},\end{aligned}$$

and

$$\begin{aligned}&\int_{B_R^h\setminus B^h_{\epsilon\sqrt{h-1}}}\left(\frac{|y|}{\sqrt{h-1}}+\epsilon\right)^k - \left(\frac{|y|}{\sqrt{h-1}}-\epsilon\right)_+^k dy \\ =&\int_{B_R^h\setminus B^h_{\epsilon\sqrt{h-1}}}\left(\frac{|y|}{\sqrt{h-1}}+\epsilon\right)^k - \left(\frac{|y|}{\sqrt{h-1}}-\epsilon\right)^k dy \\ \leq&\frac{1}{(h-1)^{k/2}}\int_{B_R^h\setminus B^h_{\epsilon\sqrt{h-1}}}|y|^k\left(\left(1+\frac{\epsilon\sqrt{h-1}}{|y|}\right)^k - \left(1-\frac{\epsilon\sqrt{h-1}}{|y|}\right)^k\right)dy \\ \leq&\frac{2^k}{(h-1)^{k/2}}\int_{B_R^h\setminus B^h_{\epsilon\sqrt{h-1}}}|y|^k\frac{\epsilon\sqrt{h-1}}{|y|}dy \\ \leq&\frac{2^k\epsilon}{(h-1)^{(k-1)/2}}\int_{\epsilon\sqrt{h-1}}^R r^{k-1}\mathcal{H}^{m-1}(S_r^{h-1})dr \\ \leq&\frac{2^kh\omega_h\epsilon}{(h-1)^{(k-1)/2}}\int_{\epsilon\sqrt{h-1}}^R r^{k+h-2}dr \\ \leq&\frac{2^kh\omega_h\epsilon}{(h-1)^{(k-1)/2}(m-1)}R^{m-1},\end{aligned}$$

where we use $(1+t)^k - (1-t)^k \le 2^k t$ for $t \in (0,1), k \in \mathbb{N}$. Combining the two parts, we have

$$|H_R \cap \{p < \epsilon\}| \le 2^k \omega_k \omega_h (k-1)^{k/2}(h-1)^{h/2}\epsilon\left(\epsilon^{m-1} + \frac{hR^{m-1}}{(h-1)^{(m-1)/2}(m-1)}\right).$$

Now, note that $\omega_j < 6$ for all $2 \le j \le 11$, so by substituting the explicit values for $c_{k,h}$, we have

$$\begin{aligned}|K_{kh}\Delta F| \le & \frac{2\times 11^5\sqrt{11}R^2}{\epsilon}\Big(P(F;H_R) - P(K_{kh};H_R)\Big) \\ & + 2^{11}6^2 10^{11/2}2^{3/2}\epsilon(\epsilon^{m-1} + \frac{3}{6}R^{m-1}) \qquad (7)\\ \le & 7\times 10^{10}\left(\frac{R^2}{\epsilon}\big(P(F;H_R) - P(K_{kh};H_R)\big) + \epsilon(\epsilon^{m-1} + R^{m-1})\right). \qquad (8)\end{aligned}$$

Let

$$\alpha = \frac{\mathcal{L}^m(K_{kh}\Delta F)}{R^m},$$
$$\delta = \frac{P(F;H_R) - P(K_{kh};H_R)}{R^{m-1}}.$$

Note that $\alpha \le R^{-m}\mathcal{L}^m(H_R) = \omega_k\omega_h \le 6^2$. If $\delta \ge 6^2$, then $\alpha \le \omega_k\omega_h \le 6\sqrt{\delta}$. Thus we assume $\delta \le 6^2$. Inequality (8) implies

$$\alpha \le 7\times 10^{10}\left(\frac{R}{\epsilon}\delta + \frac{\epsilon}{R}\big((\epsilon/R)^{m-1}) + 1\big)\right). \qquad (9)$$

If $\epsilon < \sqrt[13]{35}R$, then inequality (9) implies

$$\alpha \le 7\times 10^{10}(\frac{R}{\epsilon}\delta + 36\frac{\epsilon}{R}).$$

Note that

$$\frac{R}{\epsilon}\delta + 36\frac{\epsilon}{R} \ge 12\sqrt{\delta}$$

with equality if and only if $\epsilon = R\sqrt{\frac{\delta}{36}}$. Since $\frac{\delta}{36} \le 1$, we can let $\epsilon = R\sqrt{\frac{\delta}{36}}$, and deduce that

$$\alpha \le 7\times 12\times 10^{10}\sqrt{\delta}.$$

□

3. Proof of Lemma 2.

3.1. Calculating $\operatorname{div} g$ **on** K_{kh}**.** To make calculations simpler, let $u = (h-1)|x|^2, v = (k-1)|y|^2$. First, consider the function

$$f(z) = \frac{1}{4}(u-v)u^d.$$

We have

$$
\begin{aligned}
\partial_{x_i} f =& \frac{h-1}{2} x_i\big((d+1)u^d - dvu^{d-1}\big),\\
\partial_{x_i}\partial_{x_j} f =& \frac{h-1}{2}\delta_{ij}\big((d+1)u^d - dvu^{d-1}\big) + (h-1)^2 x_i x_j\big((d+1)du^{d-1} - d(d-1)vu^{d-2}\big),\\
\partial_{y_i} f =& -\frac{k-1}{2} y_i u^d,\\
\partial_{y_j}\partial_{y_i} f =& -\frac{k-1}{2}\delta_{ij} u^d,\\
\partial_{y_j}\partial_{x_i} f =& -d(h-1)(k-1)u^{d-1} x_i y_j.
\end{aligned}
$$

This gives

$$
\begin{aligned}
|\nabla f|^2 =& \frac{h-1}{4} u\big((d+1)u^d - dvu^{d-1}\big)^2 + \frac{k-1}{4} vu^{2d},\\
\Delta f =& \frac{(h-1)k}{2}\big((d+1)u^d - dvu^{d-1}\big) - \frac{(k-1)h}{2} u^d\\
&+ (h-1)u\big((d+1)du^{d-1} - d(d-1)vu^{d-2}\big),\\
(\partial_{x_i} f)(\partial_{x_j} f)(\partial_{x_i}\partial_{x_j} f) =& \frac{(h-1)^2}{8} u\big((d+1)u^d - dvu^{d-1}\big)^3\\
&+ \frac{(h-1)^2}{4} u^2\big((d+1)u^d - dvu^{d-1}\big)^2\\
&\times \big((d+1)du^{d-1} - d(d-1)vu^{d-2}\big),\\
(\partial_{y_i} f)(\partial_{y_j} f)(\partial_{y_i}\partial_{y_j} f) =& -\frac{(k-1)^2}{8} vu^{3d},\\
(\partial_{x_i} f)(\partial_{y_j} f)(\partial_{x_i}\partial_{y_j} f) =& \frac{(h-1)(k-1)}{4} du^{2d} v\big((d+1)u^d - dvu^{d-1}\big).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
|\nabla f|^3 \operatorname{div} g =& |\nabla f|^3 \operatorname{div} \frac{\nabla f}{|\nabla f|}\\
=& |\nabla f|^2 \Delta f - (\partial_{x_i} f)(\partial_{x_j} f)(\partial_{x_i}\partial_{x_j} f) - (\partial_{y_i} f)(\partial_{y_j} f)(\partial_{y_i}\partial_{y_j} f)\\
&- 2(\partial_{x_i} f)(\partial_{y_j} f)(\partial_{x_i}\partial_{y_j} f)\\
=& \frac{(h-1)(k-1)}{8} u^{3d-2}(u-v)\Big((1+d)^2\big(-1+d(-1+h)\big)u^2\\
&+ d(-2+d(1+2d-2(1+d)h)+k)uv + d^3(-1+h)v^2\Big).
\end{aligned}
$$

3.2. Calculating $\operatorname{div} g$ **on** K^{C}_{kh}**.** Now, define

$$
f(z) = \frac{1}{4}(u-v)v^d,
$$

which can be obtained by interchanging u, v and h, k and adding an additional minus sign to f in the previous subsection. Thus, by symmetry or by direct computations, we must have

$$
\begin{aligned}
|\nabla f|^2 =& \frac{h-1}{4} u v^{2d} + \frac{k-1}{4} v (d u v^{d-1} - (d+1) v^d)^2, \\
|\nabla f|^3 \operatorname{div} g =& |\nabla f|^3 \operatorname{div} \frac{\nabla f}{|\nabla f|} \\
=& \frac{(h-1)(k-1)}{8} (u-v) v^{3d-2} \Big(d^3 (k-1) u^2 \\
& + d\big(-2 + d + 2d^2 + h - 2d(1+d)k\big) uv \\
& + (d+1)^2 (-1 + d(k-1)) v^2 \Big).
\end{aligned}
$$

Note that if we set $g = \frac{\nabla f}{|\nabla f|}$, then g is clearly continuous, and smooth except on M_{kh}. Calculations can show that the derivative of g is of order $O(|z|^{-1})$ near origin, so $g \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^m, \mathbb{R}^m)$.

3.3. The Cases $(2, k)$. We use the basic inequalities $\max\{|x|, |y|\} \le z \le \sqrt{2} \max\{|x|, |y|\}$ and $(a^q + b^q)^{1/q} \le (a^2 + b^2)^{1/2}$ for $a, b > 0, q \ge 2$. Also note that

$$
d(z, M_{kh}) = \frac{|\sqrt{u} - \sqrt{v}|}{\sqrt{h+k-2}}
$$

by elementary geometry.

If $u > v$, then choosing $d = 3/2$, we have

$$
\operatorname{div} g = \frac{\frac{1}{64}(-1+k) u^{5/2} (u-v)(25u^2 + 12(-11+k)uv + 27v^2)}{\Big(\frac{1}{16} u^2 (25u^2 + 2(-17+2k)uv + 9v^2)\Big)^{3/2}}.
$$

Let $p_2(t) = 27t^2 - 48t + 25$. We have $\min_{[0,1]} p_2 = p_2(8/9) = 11/3$.

This gives

$$
\begin{aligned}
\operatorname{div} g \ge& (k-1)(\sqrt{u} - \sqrt{v}) \frac{\sqrt{u} + \sqrt{v}}{\sqrt{u}} \frac{25u^2 + 12(-11+7)uv + 27v^2}{(25u^2 + 2(-17+2k)uv + 9v^2)^{3/2}} \\
\ge& (k-1)(\sqrt{u} - \sqrt{v}) \frac{u^2 p_3(v/u)}{(25u^2 + 10uv + 9v^2)^{3/2}} \\
\ge& (k-1)(\sqrt{u} - \sqrt{v}) \frac{\frac{11}{3}(|z|/\sqrt{2})^4}{(44u^2)^{3/2}} \\
\ge& (k-1)(\sqrt{u} - \sqrt{v}) \frac{\frac{11}{3}(|z|/\sqrt{2})^4}{(44|z|^4)^{3/2}} \\
\ge& \frac{(k-1)}{2^5 3\sqrt{11}} \frac{\sqrt{u} - \sqrt{v}}{|z|^2} \\
\ge& \frac{\sqrt{11}}{2^5 3} \frac{\sqrt{u} - \sqrt{v}}{|z|^2}.
\end{aligned}
$$

If $u < v$, choosing $d = 1$, we have

$$
\operatorname{div} g = \frac{\frac{1}{8}(k-1)(u-v) v ((k-1)u^2 + (3-4k)uv + 4(-2+k)v^2)}{\Big(\frac{1}{4} v ((k-1)(u-2v)^2 + uv)\Big)^{3/2}}.
$$

Let $q_2(t) = (k-1)t^2 + (3-4k)t + 4(k-2)$. We know that $\min_{[0,1]} q_2 = q_2(1) = k-6$. This gives

$$\begin{aligned}
|\operatorname{div} g| \geq& (k-1)|\sqrt{u}-\sqrt{v}|\frac{\sqrt{u}+\sqrt{v}}{\sqrt{v}}\frac{v^2 q_2(u/v)}{((k-1)4v^2+v^2)^{3/2}}\\
\geq& (k-1)|\sqrt{u}-\sqrt{v}|\frac{(k-6)(k-1)^2(|y|)^4}{(4(k-1)^3+(k-1)^2)^{3/2}|y|^6}\\
\geq& (k-1)|\sqrt{u}-\sqrt{v}|\frac{(k-6)(k-1)^2(|z|/\sqrt{k})^4}{(4(k-1)^3+(k-1)^2)^{3/2}|z|^6}\\
\geq& \frac{(k-6)(k-1)^3}{k^2(4(k-1)^3+(k-1)^2)^{3/2}}\frac{|\sqrt{u}-\sqrt{v}|}{|z|^2}\\
\geq& \frac{6^3}{11^2(4\times 10^3+10^2)^{3/2}}\frac{|\sqrt{u}-\sqrt{v}|}{|z|^2}\\
\geq& \frac{1}{11^5}\frac{|\sqrt{u}-\sqrt{v}|}{|z|^2},
\end{aligned}$$

where we use $v > u$ if and only if $|x| < \sqrt{k-1}|y|$ and thus $|z| < \sqrt{k}|y|$

This gives

$$|\operatorname{div} g| \geq \frac{1}{11^5}\frac{|\sqrt{u}-\sqrt{v}|}{|z|^2} = \frac{1}{11^5\sqrt{11}}\frac{\operatorname{dist}(z, M_{2,k})}{|z|^2}.$$

3.4. The Case $(h,k) = (3,5)$**.** Choose $d = 3/4$. If $u > v$, we have

$$\operatorname{div} g = \frac{\frac{1}{32}u^{1/4}(u-v)(49u^2-72uv+27v^2)}{\left(\frac{1}{32}\sqrt{u}(49u^2-10uv+9v^2)\right)^{3/2}}.$$

Let

$$p_3(t) = 27t^2 - 72t + 49.$$

We know that $\min_{[0,1]} p_3 = p_3(1) = 4$. This yields

$$\begin{aligned}
\operatorname{div} g \geq& 4\sqrt{2}(\sqrt{u}-\sqrt{v})\frac{\sqrt{u}+\sqrt{v}}{\sqrt{u}}\frac{u^2 p_3(v/u)}{\left(49\times 4|x|^4+9\times 16|y|^4\right)^{3/2}}\\
\geq& 4\sqrt{2}(\sqrt{u}-\sqrt{v})\frac{4u^2}{\left((49\times 4)(|x|^4+|y|^4)\right)^{3/2}}\\
\geq& 4\sqrt{2}(\sqrt{u}-\sqrt{v})\frac{4\times 4(|z|/\sqrt{2})^4}{14^3|z|^6}\\
\geq& \frac{2\sqrt{2}}{7^3}\frac{\sqrt{u}-\sqrt{v}}{|z|^2}.
\end{aligned}$$

If $u < v$, we have

$$\operatorname{div} g = \frac{\frac{1}{16}(u-v)v^{1/4}(27u^2-123uv+98v^2)}{\left(\frac{1}{16}\sqrt{v}(9u^2-34uv+49v^2)\right)^{3/2}}.$$

Let $q_3(t) = 27t^2 - 123t + 98$. We have $\min_{[0,1]} q_3(t) = q_3(1) = 2$. This gives

$$
\begin{aligned}
|\operatorname{div} g| \geq & 4|\sqrt{u} - \sqrt{v}| \frac{\sqrt{u}+\sqrt{v}}{\sqrt{v}} \frac{2v^2 q_3(u/v)}{\left(9 \times 4|x|^4 + 49 \times 16|y|^4\right)^{3/2}} \\
\geq & 4|\sqrt{u} - \sqrt{v}| \frac{2v^2}{\left(49 \times 16(|x|^4 + |y|^4)\right)^{3/2}} \\
\geq & 4|\sqrt{u} - \sqrt{v}| \frac{2 \times 4^2 |y|^4}{28^3 |z|^6} \\
\geq & 4|\sqrt{u} - \sqrt{v}| \frac{2 \times 4^2 (|z|/\sqrt{3})^4}{28^3 |z|^6} \\
\geq & \frac{2}{3^2 7^3} \frac{|\sqrt{u} - \sqrt{v}|}{|z|^2},
\end{aligned}
$$

where we use $u < v \Leftrightarrow 2|y|^2 > |x|$ and thus $|z|^2 < 3|y|^2$. This yields

$$
|\operatorname{div} g| \geq \frac{2}{3^2 7^3} \frac{|\sqrt{u} - \sqrt{v}|}{|z|^2} = \frac{\sqrt{3}}{21^3} \frac{\operatorname{dist}(z, M_{5,3})}{|z|^2}.
$$

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