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DEFORMATIONS FROM A GIVEN KÄHLER METRIC TO A TWISTED CSCK METRIC*

YU ZENG†

Abstract. In [3], X. Chen proposed a continuity path aiming to attack the existence problem of the constant scalar curvature Kähler(cscK) metric. He also proved the openness of the path at $t \in (0,1)$ by the standard implicit function theorem on solutions of fourth order PDE. However, the openness at $t = 0$ is quite different in nature and it is in fact a deformation result from the solution of a second order PDE to the solution of a fourth order PDE. In this paper, we give a proof of the openness at $t = 0$, which asserts the existence of twisted cscK metrics for $t > 0$ sufficient small.

Key words. CscK metric, twisted CscK metric, CscK metric deformation.

Mathematics Subject Classification. 53 Differential geometry.

1. Introduction. In the 1980's, E. Calabi([2]) initiated a program to find "the best" canonical metric in each Kähler class. To this end, he considered the L^2-norm of the curvature tensor as a functional on metrics and sought its critical points, called the extremal metric. The Kähler-Eintein(KE) metric and more generally the constant scalar curvature Kähler(cscK) metric are both special cases of the extremal metric. The existence problem of KE metrics has been completely settled thanks to the fundamental contributions of Yau([16]) and Aubin([1]) as well as the recent breakthrough on the remaining Fano case([4], [5] and [6]). After this major achievement, it becomes more plausible to study the Calabi's original problem on cscK/extremal metrics in its full generality. In [7], Donaldson presented a precise algebro-geometric condition, called the K-stability(Tian first gave an equivalent definition in the particular case of Fano varieties) and then he formulated the following conjecture on the existence of cscK metrics.

Conjecture 1.1 (Yau-Tian-Donaldson, [7]). *A smooth polarized manifold (V, L) admits a cscK metric in the class $c_1(L)$ if and only if it is K-stable.*

Donaldson himself proved this conjecture on toric surfaces([7], [9], [10] and [11]). However, in general, the existence problem of cscK metrics is very difficult as explained in an expository article by Donaldson([8]). Recently, Chen proposed a continuity path in [3] aiming to attack the existence problem of cscK metrics via a direct PDE approach. We describe it briefly below.

Given an n-dimensional closed Kähler manifold (M, ω), we denote the space of smooth Kähler potentials by $\mathcal{H}^\infty = \{\varphi \in C^\infty(M) | \omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0\}$. For any Kähler potential $\varphi \in \mathcal{H}^\infty$, we denote by $R_\varphi := -g_\varphi^{i\bar{j}} \frac{\partial^2}{\partial z_i \partial \bar{z}_j}\big(\log\det(g_\varphi)\big)$ the scalar curvature of the corresponding Kähler metric g_φ and by $\underline{R}$ its average, which is in fact a topological constant. Then, for a given closed positive $(1,1)$-form χ on M, the continuity path in [3] can be defined more precisely as a smooth family of solutions to the following equations parametrized by $t \in [0,1]$

$$t(R_\varphi - \underline{R}) - (1-t)(\mathrm{tr}_\varphi \chi - n) = 0, \tag{1}$$

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†Department of Mathematics, Stony Brook University, Stony Brook, NY 11794, USA (yu.zeng@stonybrook.edu).

where $\mathrm{tr}_\varphi\chi := g_\varphi^{i\bar{j}}\chi_{i\bar{j}}$ is the trace of χ taken with respect to the Kähler metric g_φ. Following the terminology in J. Stoppa([15]), a Kähler metric g_φ satisfying (1) is called twisted cscK metric.

For a smooth closed positive $(1,1)$-form χ, let I_χ be the set of $t \in [0,1]$ such that equation (1) with parameter t has a smooth solution. It was proved in [3] that I_χ was open at any $t \in (0,1) \cap I_\chi$ by directly applying the implicit function theorem. Note that if we simply choose $\chi = \omega$, then we have an obvious "starting point" $\varphi = 0$ at $t = 0$. Therefore, hereafter, we choose $\chi = \omega$. However, the openness at $t = 0$ is quite different from the case at $t \in (0,1)$, because equation (1) is a fourth order PDE for all positive t while it reduces to a second order PDE at $t = 0$. Thus, in this paper, we will prove the openness of I_ω at $t = 0$.

Theorem 1.2 (Main Theorem). *Given an n-dimensional closed Kähler manifold (M, ω), there exists a constant $\delta > 0$ only depending on the initial Kähler metric ω and dimension n such that $[0, \delta) \subset I_\omega$.*

We should mention here that the closedness of I_ω is related to K-stability as well as a priori estimates of fourth order elliptic PDE, and it is still out of reach at this moment. Soon after we posted our paper on arxiv, Y. Hashimoto([14]) also announced results similar to Theorem 1.2 about the openness of I_χ at $t = 0$ for arbitrary closed (1,1)-form $\chi > 0$. Besides essential difference in computations and PDE techniques, the fundamental idea of both our proof and Hashimoto's proof is to work on path (1) directly for fixed $t > 0$ and to show that equation (1) can be solved automatically provided that $t > 0$ is sufficiently small.

This paper is organized as follows. In Section 2, we introduce basic notions and reduce (1) from a 4th order PDE to a second order PDE by introducing the second order pseudo differential operator θ_φ. Then our main Theorem(Theorem 1.2) is equivalent to the solvability of equation $r\theta_\varphi + \varphi = 0$ for $r > 0$ sufficiently small(Theorem 2.1). Denote for fixed $r > 0$, $F_r(\varphi) = r\theta_\varphi + \varphi$. Then in order to find solution of $F_r(\varphi) = 0$, one could probably search around $\varphi = 0$ since

$$\|F_r(0)\|_{C^\alpha(M)} = r\|\theta_0\|_{C^\alpha(M)} \le Cr \to 0, \text{ as } r \to 0,$$

and by later computations in Section 4, for sufficiently small $r > 0$, $F_r : \mathcal{H}_\omega^{2,\alpha} \to C_\omega^\alpha(M)$ will be a local homeomorphism from 0 to $F_r(0)$. However the invertible neighborhood of $F_r(0)$ might have radius shrinking faster than r in C^α space in which case the invertible neighborhood wouldn't be large enough to include $0 \in C^\alpha(M)$. Thus, we have to search for a better approximation φ_1 with little modifications on $\varphi = 0$ such that it preserves the invertibility of F_r near φ_1 and moreover the distance $\|F_r(\varphi_1) - 0\|_{C^\alpha}$ reduces significantly in a way that the invertible neighborhood around $F_r(\varphi_1)$ includes $0 \in C_\omega^\alpha(M)$. In Section 3, we introduce the first ingredient of our proof, approximation of twisted cscK metrics for small $t > 0$, which is a possible candidate of the approximations described above. In Section 4, we show the second ingredient of our proof, the quantitative version of inverse function theorem for F_r, which justifies that the approximation φ_1 constructed in Section 3 meets both our expectations.

Without further notice, the "C" in each estimate means a constant depending on the complex dimension n, the background metric ω, the topological constant $\underline{R}$ and $0 < \alpha < 1$ unless specified.

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2. Preliminary. Suppose (M,ω) is a closed Kähler manifold. Denote the space of normalized smooth Kähler potentials as

$$\mathcal{H}_\omega = \{\varphi \in C^\infty(M) | \omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0, \int_M \varphi\omega^n = 0\}. \tag{2}$$

For $\varphi \in \mathcal{H}_\omega$, we denote R_φ the scalar curvature of ω_φ and $\underline{R} = \frac{[c_1(M)][\omega]^{[n-1]}}{[\omega]^{[n]}}$.

As in defining the Futaki invariant in [12], we could solve the Laplacian equation for any $\varphi \in \mathcal{H}_\omega$

$$\Delta_\varphi f = R_\varphi - \underline{R}. \tag{3}$$

We denote the unique solution of (3) as θ_φ with the normalization $\int_M \theta_\varphi \omega^n = 0$. Since

$$R_\varphi - \underline{R} = \mathrm{tr}_\varphi(\mathrm{Ric}_\varphi - \mathrm{Ric}(\omega) + \mathrm{Ric}(\omega) - \underline{R}\omega - \underline{R}\sqrt{-1}\partial\bar{\partial}\varphi), \tag{4}$$

$$= \Delta_\varphi(-\log\frac{\omega_\varphi^n}{\omega^n} - \underline{R}\varphi) + \mathrm{tr}_\varphi(\mathrm{Ric}(\omega) - \underline{R}\omega), \tag{5}$$

we have

$$\theta_\varphi = -\log\frac{\omega_\varphi^n}{\omega^n} - \underline{R}\varphi + \int_M \log\frac{\omega_\varphi^n}{\omega^n}\omega^n + P_\varphi, \tag{6}$$

where P_φ is determined by

$$\Delta_\varphi P_\varphi = \mathrm{tr}_\varphi(\mathrm{Ric}(\omega) - \underline{R}\omega), \int_M P_\varphi\omega^n = 0. \tag{7}$$

Given the notion of θ_φ above, we could reduce the continuity path equation (1) with $\chi = \omega$ from a 4th order PDE to a Monge-Ampère type of equation as

$$t\theta_\varphi + (1-t)\varphi = 0. \tag{8}$$

Following the discussion above, Theorem 1.2 will be an easy corollary of the following theorem:

THEOREM 2.1. *Suppose (M,ω) is a closed Kähler manifold. Then, for any $r > 0$ sufficiently small, there exists a unique $\varphi_r \in \mathcal{H}_\omega$ such that*

$$r\theta_{\varphi_r} + \varphi_r = 0. \tag{9}$$

In our paper, we'll repeatedly use Schauder estimate of Laplacian equation. Thus, let's introduce it here as the following Lemma:

LEMMA 2.2. *If $\varphi \in C^{2,\alpha}(M)$ with $\|\varphi\|_{C^{2,\alpha}(M)} \le \frac{1}{2}$ and $u \in C^{2,\alpha}(M)$ statisfies*

$$\Delta_\varphi u = f, \int_M u\omega^n = 0 \tag{10}$$

for some $f \in C^\alpha(M)$. Then

$$\|u\|_{C^{2,\alpha}(M)} \leq C\|f\|_{C^\alpha(M)}. \tag{11}$$

Proof. Proof of Lemma 2.2. By Schauder estimate [13, Theorem 6.2], we can get

$$\|u\|_{C^{2,\alpha}(M)} \leq C\big(\|u\|_{L^\infty(M)} + \|f\|_{C^\alpha(M)}\big). \tag{12}$$

To bound $\|u\|_{L^\infty(M)}$, we first multiply u on both hand sides of (10), integrate against ω_φ^n, and we get that

$$\int_M |\nabla u|_\varphi^2 \omega_\varphi^n = \int_M -fu\omega_\varphi^n \leq C\|f\|_{L^\infty(M)}\|u\|_{L^2(M,\omega)}. \tag{13}$$

On the other hand,

$$\int_M |\nabla u|_\varphi^2 \omega_\varphi^n \geq \frac{1}{C}\int_M |\nabla u|^2\omega^n \geq \frac{1}{C}\|u\|^2_{L^2(M,\omega)}. \tag{14}$$

Thus, combining the above two inequalities, we can get

$$\|u\|_{L^2(M,\omega)} \leq C\|f\|_{L^\infty(M)}. \tag{15}$$

Then, by Moser iteration [13, Theorem 8.15], we can get that

$$\|u\|_{L^\infty(M)} \leq C\big(\|u\|_{L^2(M)} + \|f\|_{L^\infty(M)}\big) \leq C\|f\|_{C^\alpha(M)}. \tag{16}$$

This ends the proof. □

3. Approximation of twisted cscK metrics for small $t > 0$. Let's first introduce the space we're going to work on. Define for $0 < \alpha < 1$ and $k \in \mathbb{N}$

$$\mathcal{H}_\omega^{2,\alpha} = \{\varphi \in C^{2,\alpha}(M) | \omega_\varphi = \omega + \sqrt{-1}\partial\bar\partial\varphi > 0, \int_M \varphi\omega^n = 0\}, \tag{17}$$

$$C_\omega^{k,\alpha}(M) = \{f \in C^{k,\alpha}(M) | \int_M f\omega^n = 0\}. \tag{18}$$

More generally, θ_φ could be defined on the space $\mathcal{H}_\omega^{2,\alpha}$ if we took the definition as in (6). Therefore, we define, still denoted by θ_φ,

$$\begin{aligned} \theta : \mathcal{H}_\omega^{2,\alpha} &\to C_\omega^\alpha(M) \\ \varphi &\mapsto \theta_\varphi = -\log\frac{\omega_\varphi^n}{\omega^n} - \underline{R}\varphi + \int_M \log\frac{\omega_\varphi^n}{\omega^n}\omega^n + P_\varphi, \end{aligned}$$

where $P_\varphi \in C_\omega^{2,\alpha}(M) \subset C_\omega^\alpha(M)$ is determined by

$$\Delta_\varphi P_\varphi = \mathrm{tr}_\varphi(\mathrm{Ric}(\omega) - \underline{R}\omega), \int_M P_\varphi\omega^n = 0. \tag{19}$$

Define

$$\begin{aligned} F_r : \mathcal{H}_\omega^{2,\alpha} &\to C_\omega^\alpha(M) \\ \varphi &\mapsto r\theta_\varphi + \varphi. \end{aligned}$$

For fixed $r > 0$, in Theorem 2.1, we are just looking for $\varphi_r \in \mathcal{H}^{2,\alpha}_\omega$ such that $F_r(\varphi_r) = 0$. Denote $\varphi_0 = 0 \in \mathcal{H}^{2,\alpha}_\omega$. Then in order to find φ_r described above, one could probably search around φ_0 since

$$\|F_r(\varphi_0)\|_{C^\alpha(M)} = r\|\theta_{\varphi_0}\|_{C^\alpha(M)} \leq Cr \to 0, \text{ as } r \to 0, \tag{20}$$

and by later computations in Section 4, for sufficiently small $r > 0$, $F_r : \mathcal{H}^{2,\alpha}_\omega \to C^\alpha_\omega(M)$ will be a local homeomorphism from φ_0 to $F_r(\varphi_0)$. However, the invertible neighborhood around $F_r(\varphi_0)$ has radius comparable to $r^{3+\epsilon}$ which is not large enough to include "0" in it. Thus, we have to search for better approximation φ_1 such that it preserves the invertibility of F_r around φ_1 and moreover the invertible neighborhood around $F_r(\varphi_1)$ includes $0 \in C^\alpha_\omega(M)$. More precisely, by later Theorem 4.1, we need to look for $\varphi \in \mathcal{H}_\omega$ such that $\|\varphi\|_{C^4(M)} \leq \frac{1}{2}$ and $\|F_r(\varphi)\|_{C^\alpha(M)} = o(r^{\frac{3-\alpha}{1-\alpha}})$. For this purpose, we fix $\alpha \in (0, \frac{1}{3})$ and it is enough to consider the correction to φ_0 below

$$\varphi_1 = \varphi_0 + ru_1 + \frac{r^2}{2}u_2 + \frac{r^3}{6}u_3, \tag{21}$$

where $u_i's$ are fixed smooth functions on M with $\int_M u_i\omega^n = 0$ that we'll specify later. To determine u_i's, first we need to expand $F_r(\varphi_1)$ in terms of r at $r = 0$. Denote $u_r = ru_1 + \frac{r^2}{2}u_2 + \frac{r^3}{6}u_3$. Compute

$$\frac{\partial\theta_{\varphi_1}}{\partial r} = -\Delta_{\varphi_1}\dot{u}_r - \underline{R}\dot{u}_r + \int_M \Delta_{\varphi_1}\dot{u}_r\omega^n + \mathcal{D}P|_{\varphi_1}(\dot{u}_r), \tag{22}$$

where $\mathcal{D}P|_{\varphi_1} : C^{2,\alpha}_\omega(M) \to C^{2,\alpha}_\omega(M)$ is the linearization of P_φ at $\varphi = \varphi_1$ and it satisfies

$$\Delta_{\varphi_1}\big(\mathcal{D}P|_{\varphi_1}(u)\big) = \langle\partial\bar{\partial}u, \partial\bar{\partial}P_{\varphi_1} - (\mathrm{Ric}(\omega) - \underline{R}\omega)\rangle_{\varphi_1}, \int_M \big(\mathcal{D}P|_{\varphi_1}(u)\big)\omega^n = 0. \tag{23}$$

Take one more derivative of θ_{φ_1}, we get

$$\begin{aligned}\frac{\partial^2\theta_{\varphi_1}}{\partial^2 r} &= -(\Delta_{\varphi_1}\ddot{u}_r - |\partial\bar{\partial}\dot{u}_r|^2_{\varphi_1}) - \underline{R}\ddot{u}_r + \int_M (\Delta_{\varphi_1}\ddot{u}_r - |\partial\bar{\partial}\dot{u}_r|^2_{\varphi_1})\omega^n + \mathcal{D}P|_{\varphi_1}(\ddot{u}_r)\\ &\quad + (\frac{\partial}{\partial\varphi}\mathcal{D}P|_\varphi)|_{\varphi_1}(\dot{u}_r, \dot{u}_r),\end{aligned}$$

where the last term is given by the unique solution of the following elliptic equation

$$\begin{aligned}\Delta_{\varphi_1} f = 2\langle\partial\bar{\partial}\dot{u}_r, \partial\bar{\partial}\big(\mathcal{D}P|_{\varphi_1}(\dot{u}_r)\big)\rangle_{\varphi_1} - \dot{u}_{r,i\bar{p}}\dot{u}_{r,p\bar{j}}\big(P_{\varphi_1,j\bar{i}} - (\mathrm{Ric}(\omega) - \underline{R}\omega)_{j\bar{i}}\big)\\ - \dot{u}_{r,i\bar{p}}\dot{u}_{r,j\bar{i}}\big(P_{\varphi_1,p\bar{j}} - (\mathrm{Ric}(\omega) - \underline{R}\omega)_{p\bar{j}}\big),\end{aligned}$$

with $\int_M f\omega^n = 0$. Thus, we get the expansion of $F_r(\varphi_1)$ of r at $r = 0$,

$$F_r(\varphi_1) = r\theta_{\varphi_1} + \varphi_1 \tag{24}$$

$$= \varphi_0 + r(u_1 + \theta_{\varphi_0}) + \frac{r^2}{2}(u_2 + 2\frac{\partial\theta_{\varphi_1}}{\partial r}|_{r=0}) + \frac{r^3}{6}(u_3 + 3\frac{\partial^2\theta_{\varphi_1}}{\partial^2 r}|_{r=0}) + O(r^4). \tag{25}$$

It suggests that we should define

$$u_1 = -\theta_{\varphi_0},$$
$$u_2 = -2\frac{\partial\theta_{\varphi_1}}{\partial r}|_{r=0} = -2\big(- \Delta_{\varphi_0} u_1 - \underline{R}u_1 + \mathcal{D}P|_{\varphi_0}(u_1)\big),$$
$$u_3 = -3\frac{\partial^2\theta_{\varphi_1}}{\partial^2 r}|_{r=0} = -3\big(- \Delta_{\varphi_0} u_2 - \underline{R}u_2 + \mathcal{D}P|_{\varphi_0}(u_2) + |\partial\bar{\partial}u_1|^2_{\varphi_0} - \int_M |\partial\bar{\partial}u_1|^2_{\varphi_0}\omega^n$$
$$+ (\frac{\partial}{\partial\varphi}\mathcal{D}P|_{\varphi})|_{\varphi_0}(u_1, u_1)\big).$$

It's clear from definitions that $u_i's$ are fixed smooth functions with C^k norm bounds only depend on φ_0. Therefore, we could choose $r > 0$ sufficiently small such that $\varphi_1 \in \mathcal{H}^{2,\alpha}_{\omega}$ with $\|\varphi_1\|_{C^{2,\alpha}(M)} \le \frac{1}{2}$. And we expect that $F_r(\varphi_1)$ is r^4 close to "0" in C^α norm sense. This observation can be made more precise as the following lemma:

LEMMA 3.1. *Notations as described above, for $r > 0$ sufficiently small, we have*

$$\|F_r(\varphi_1)\|_{C^\alpha(M)} \le Cr^4.$$

Proof. Proof of Lemma 3.1. It suffices to show that

$$\|\theta_{\varphi_1} - (\theta_{\varphi_0} + r\frac{\partial\theta_{\varphi_1}}{\partial r}|_{r=0} + \frac{r^2}{2}\frac{\partial^2\theta_{\varphi_1}}{\partial^2 r}|_{r=0})\|_{C^\alpha(M)} \le Cr^3. \tag{26}$$

By Taylor expansion theorem, we could write the difference of the function θ_{φ_1} and its second order taylor expansion at $r = 0$ as an integral,

$$\text{Remainder}(x) = \frac{1}{2!}\int_0^r (r-s)^2\big(\frac{\partial^3\theta_{\varphi_1}}{\partial^3 r}|_{r=s}(x)\big)ds. \tag{27}$$

So it suffices to show that for any $s \in [0, r]$ with $r > 0$ sufficiently small

$$\|\frac{\partial^3\theta_{\varphi_1}}{\partial^3 r}|_{r=s}\|_{C^\alpha(M)} \le C. \tag{28}$$

Denote $u_s = su_1 + \frac{s^2}{2}u_2 + \frac{s^3}{6}u_3$ and $\varphi_s = \varphi_0 + u_s$. Compute

$$\frac{\partial^3\theta_{\varphi_1}}{\partial^3 r}|_{r=s} = -(\Delta_{\varphi_s}u_s^{(3)} - 3\langle\partial\bar{\partial}u_s^{(1)}, \partial\bar{\partial}u_s^{(2)}\rangle_{\varphi_s} + 2(\partial\bar{\partial}u_s^{(1)})^{*3})$$
$$- \int_M(\Delta_{\varphi_s}u_s^{(3)} - 3\langle\partial\bar{\partial}u_s^{(1)}, \partial\bar{\partial}u_s^{(2)}\rangle_{\varphi_s} + 2(\partial\bar{\partial}u_s^{(1)})^{*3})\omega^n - \underline{R}u_s^{(3)}$$
$$+ \mathcal{D}P|_{\varphi_s}(u_s^{(3)}) + 2(\frac{\partial}{\partial\varphi}\mathcal{D}P|_{\varphi})|_{\varphi=\varphi_s}(u_s^{(2)}, u_s^{(1)}) + (\frac{\partial}{\partial\varphi}\mathcal{D}P|_{\varphi})|_{\varphi=\varphi_s}(u_s^{(1)}, u_s^{(2)})$$
$$+ (\frac{\partial^2}{\partial^2\varphi}\mathcal{D}P|_{\varphi})|_{\varphi=\varphi_s}(u_s^{(1)}, u_s^{(1)}, u_s^{(1)}).$$

It's obvious that the first two lines has uniform C^α norm as we expected. Therefore, we next focus on estimating the last four terms of the above equation. Let's first consider P_{φ_s}. It satisfies the Laplacian equation as described in Lemma 2.2, so we get that

$$\|P_{\varphi_s}\|_{C^{2,\alpha}(M)} \le C. \tag{29}$$

Then we can estimate $\mathcal{D}P|_{\varphi_s}(u)$ using (29) and Lemma 2.2 since it satisfies the similar Laplacian equation with right hand side depending on second order derivatives of P_{φ_s}, we can conclude that

$$\|\mathcal{D}P|_{\varphi_s}(u)\|_{C^{2,\alpha}(M)} \leq C\|u\|_{C^{2,\alpha}(M)}. \tag{30}$$

Thus, we could further estimate the term using the same argument in Lemma 2.2

$$\|(\frac{\partial}{\partial\varphi}\mathcal{D}P|_{\varphi})|_{\varphi_s}(u,v)\|_{C^{2,\alpha}(M)} \leq C\|u\|_{C^{2,\alpha}(M)}\|v\|_{C^{2,\alpha}(M)}. \tag{31}$$

Finally, we could estimate the term $(\frac{\partial^2}{\partial^2\varphi}\mathcal{D}P|_{\varphi})|_{\varphi_s}(u,v,w)$ which satisfies the equation

$$\Delta_{\varphi_s} f = \langle\partial\bar{\partial}w, \partial\bar{\partial}\big((\frac{\partial}{\partial\varphi}\mathcal{D}P|_{\varphi})|_{\varphi_s}(u,v)\big)\rangle_{\varphi_s} + \langle\partial\bar{\partial}u, \partial\bar{\partial}\big((\frac{\partial}{\partial\varphi}\mathcal{D}P|_{\varphi})|_{\varphi_s}(v,w)\big)\rangle_{\varphi_s} \tag{32}$$

$$+ \langle\partial\bar{\partial}v, \partial\bar{\partial}\big((\frac{\partial}{\partial\varphi}\mathcal{D}P|_{\varphi})|_{\varphi_s}(u,w)\big)\rangle_{\varphi_s} + \partial\bar{\partial}v * \partial\bar{\partial}w * \partial\bar{\partial}\big(\mathcal{D}P|_{\varphi_s}(u)\big) \tag{33}$$

$$+ \partial\bar{\partial}u * \partial\bar{\partial}w * \partial\bar{\partial}\big(\mathcal{D}P|_{\varphi_s}(v)\big) + \partial\bar{\partial}v * \partial\bar{\partial}u * \partial\bar{\partial}\big(\mathcal{D}P|_{\varphi_s}(w)\big) \tag{34}$$

$$+ \partial\bar{\partial}u * \partial\bar{\partial}v * \partial\bar{\partial}w * \big(\partial\bar{\partial}P_{\varphi_s} - \mathrm{Ric}(\omega) - \underline{R}\omega\big), \int_M f\omega^n = 0. \tag{35}$$

Thus by the Lemma 2.2, we can conclude that

$$\|(\frac{\partial^2}{\partial^2\varphi}\mathcal{D}P|_{\varphi})|_{\varphi_s}(u,v,w)\|_{C^{2,\alpha}(M)} \leq C\|u\|_{C^{2,\alpha}(M)}\|v\|_{C^{2,\alpha}(M)}\|w\|_{C^{2,\alpha}(M)}. \tag{36}$$

Since $\|u_s^{(i)}\|_{C^{2,\alpha}(M)} \leq C$ for $1 \leq i \leq 3$,

$$\|\frac{\partial^3\theta_{\varphi_1}}{\partial^3 r}|_{r=s}\|_{C^{\alpha}(M)} \leq C. \tag{37}$$

Thus it ends the proof of the lemma. □

4. Quantitative inverse function theorem. In this section, we will prove the following quantitative version of inverse function theorem of F_r.

THEOREM 4.1. *Suppose that $\varphi \in \mathcal{H}_\omega$ and $\|\varphi\|_{C^4(M)} \leq \frac{1}{2}$. Then there exists a constant $\epsilon = \epsilon(\omega, n) > 0$ such that for any $r \in (0,\epsilon)$, $F_r : \mathcal{H}_\omega^{2,\alpha} \to C_\omega^\alpha(M)$ is a local homeomorphism from φ to $F_r(\varphi)$. Moreover, there exists a constant $\eta = \eta(\omega, n) > 0$ such that for any $r \in (0,\epsilon)$, if $y \in C_\omega^\alpha(M)$ satisfies $\|y - F_r(\varphi)\|_{C^\alpha(M)} \leq \eta r^{\frac{3-\alpha}{1-\alpha}}$, we can find an $x \in \mathcal{H}_\omega^{2,\alpha}$ with $F_r(x) = y$.*

In the previous section, for fixed $r > 0$, we defined an approximation of the twisted cscK metric as

$$\varphi_1 = \varphi_0 + ru_1 + \frac{r^2}{2}u_1 + \frac{r^3}{6}u_3,$$

where u_i's are smooth functions only depending on initial Kähler metric ω. Then Theorem 2.1 will be a direct corollary of Theorem 4.1 and Lemma 3.1. We explain here briefly the proof of Theorem 2.1.

Proof. Proof of Theorem 2.1. Let $\alpha = \frac{1}{4}$. We consider the approximation φ_1 as described above. One can choose $r_0 > 0$ small such that for any $r \in (0, r_0)$,

$\|\varphi_1\|_{C^4(M)} \le \frac{1}{2}$ since u_i's are fixed smooth functions on M. Thus, by Theorem 4.1, there exist constants $0 < \epsilon \le r_0$ and $\eta > 0$ such that for any $r \in (0, \epsilon)$, if $y \in C^{\alpha}_{\omega}(M)$ satisfies $\|y - F_r(\varphi)\|_{C^{\alpha}(M)} \le \eta r^{\frac{3-\alpha}{1-\alpha}}$, we can find an $x \in \mathcal{H}^{2,\alpha}_{\omega}$ with $F_r(x) = y$. In particular, when $y = 0$, by Lemma 3.1,

$$\|0 - F_r(\varphi_1)\|_{C^{\alpha}(M)} \le Cr^4 \le (C_2 r^{\frac{1-3\alpha}{1-\alpha}}) r^{\frac{3-\alpha}{1-\alpha}} \le \eta r^{\frac{3-\alpha}{1-\alpha}},$$

where we choose $\epsilon > 0$ even smaller such that $C_2 \epsilon^{\frac{1-3\alpha}{1-\alpha}} < \eta$. Thus we have that for any $r \in (0, \epsilon)$ there exists an $x \in \mathcal{H}^{2,\alpha}_{\omega}$ with $F_r(x) = 0$. □

Since we have proved Lemma 3.1 in Section 3, in the rest of this section, we will focus on proving Theorem 4.1. First, we have to understand the linearization of $F_r : \mathcal{H}^{2,\alpha}_{\omega} \to C^{\alpha}_{\omega}(M)$ at φ. Compute

$$\begin{aligned} \mathcal{D}F_r|_{\varphi} : C^{2,\alpha}_{\omega}(M) &\to C^{\alpha}_{\omega}(M) \\ u &\mapsto -r\Delta_{\varphi} u + (1 - r\underline{R})u + r\Big(\int_M (\Delta_{\varphi} u)\omega^n + \mathcal{D}P|_{\varphi}(u)\Big), \end{aligned}$$

where $\mathcal{D}P|_{\varphi}(u)$ satisfies

$$\Delta_{\varphi}\big(\mathcal{D}P|_{\varphi}(u)\big) = \langle \partial\bar{\partial} u, \big(\partial\bar{\partial} P_{\varphi} - (\mathrm{Ric}(\omega) - \underline{R}\omega)\big)\rangle_{\varphi}, \int_M \big(\mathcal{D}P|_{\varphi}(u)\big)\omega^n = 0. \tag{38}$$

We summarize the properties of $\mathcal{D}F_r|_{\varphi}$ as the following lemma:

Lemma 4.2. *Suppose $0 < \alpha < 1$ and $\varphi \in \mathcal{H}_{\omega}$ with $\|\varphi\|_{C^4(M)} \le \frac{1}{2}$. Then there exists an $\epsilon = \epsilon(\omega, n) > 0$ such that for any $r \in (0, \epsilon)$, the linearizaiton of $F_r : \mathcal{H}^{2,\alpha}_{\omega} \to C^{\alpha}_{\omega}(M)$ at φ, $\mathcal{D}F_r|_{\varphi} : C^{2,\alpha}_{\omega}(M) \to C^{\alpha}_{\omega}(M)$, is injective and also surjective. Moreover, the operator norm of the inverse of $\big(\mathcal{D}F_r|_{\varphi}\big)$ has the upper bound*

$$\|\big(\mathcal{D}F_r|_{\varphi}\big)^{-1}\| \le Cr^{-\frac{2-\alpha}{1-\alpha}}.$$

Before proving Lemma 4.2, we'll need the uniform estimate of $\mathcal{D}P|_{\varphi}(u)$ for $\|\varphi\|_{C^{2,\alpha}(M)} \le \frac{1}{2}$. We summarize it as the following lemma:

Lemma 4.3. *Suppose $\varphi \in \mathcal{H}_{\omega}$ and $\|\varphi\|_{C^{2,\alpha}(M)} \le \frac{1}{2}$, then we have the estimate for any $1 < p < \infty$,*

$$\|\big(\mathcal{D}P|_{\varphi}(u)\big)\|_{L^p(M)} \le C_p \|u\|_{L^p(M)}. \tag{39}$$

Remark 4.1. Since ω and ω_{φ} are equivalent metrics if $\|\varphi\|_{C^{2,\alpha}(M)} \le \frac{1}{2}$, we make no efforts to distinguish between L^p spaces with respect to the two metrics hereafter.

Proof. We first introduce the Green function $G_{\varphi}(x, y)$ of the metric ω_{φ}. Then we define

$$T(u)(x) = \int_M G_{\varphi}(x, y)\big(u(P_{\varphi, i\bar{j}} - (\mathrm{Ric}(\omega) - \underline{R}\omega)_{i\bar{j}})\big)_{,\bar{i}j}(y)\omega^n_{\varphi} \tag{40}$$

$$= \int_M (G_{\varphi}(x, y))_{,\bar{i}j}\big(u(P_{\varphi, i\bar{j}} - (\mathrm{Ric}(\omega) - \underline{R}\omega)_{i\bar{j}})\big)(y)\omega^n_{\varphi}. \tag{41}$$

Since

$$\Delta_\varphi\big(\mathcal{D}P|_\varphi(u)\big) = \big(u(P_{\varphi,i\bar{j}} - (\mathrm{Ric}(\omega) - \underline{R}\omega)_{i\bar{j}})\big)_{,\bar{i}j}, \int_M \big(\mathcal{D}P|_\varphi(u)\big)\omega^n = 0, \tag{42}$$

we have

$$\mathcal{D}P|_\varphi(u) = T(u) - \int_M T(u)\omega^n. \tag{43}$$

For $i, j \in \mathbb{N}$, we define the operator

$$T_{\bar{i}j} f = \int_M (G_\varphi(x,y))_{,\bar{i}j} f(y)\omega_\varphi^n.$$

$T_{\bar{i}j}$ is a Calderon-Zygmund([13, Theorem 9.9]) operator which maps L^p functions to L^p functions for any $1 < p < \infty$. Moreover we can show $T_{\bar{i}j}$ has uniform norms. To see this, we consider the Laplacian equation

$$\Delta_\varphi u = f, \int_M u\omega_\varphi^n = 0. \tag{44}$$

Thus, we see that the solution satisfies

$$\frac{\partial^2}{\partial z_{\bar{i}} \partial z_j} u(x) = \big(T_{\bar{i}j} f\big)(x). \tag{45}$$

So it suffices to show the uniform $W^{2,2}$ estimates of (44), which follows from the fact that $\|\varphi\|_{C^{2,\alpha}(M)} \leq \frac{1}{2}$ and the standard L^p theory of elliptic equation([13, Theorem 9.9]). We have the estimate for any $p \in (1, +\infty)$

$$\|T_{\bar{i}j} f\|_{L^p(M)} \leq C_p \|f\|_{L^p(M)}. \tag{46}$$

Thus, taking advantages of the above estimate, we can get

$$\|T(u)\|_{L^p(M)} \leq \sum_{k,l} C_p \|u\big(g_\varphi^{i\bar{l}} g_\varphi^{k\bar{j}} (P_{\varphi,i\bar{j}} - (\mathrm{Ric}(\omega) - \underline{R}\omega)_{i\bar{j}})\big)\|_{L^p(M)} \tag{47}$$

$$\leq C_p \|u\|_{L^p(M)}. \tag{48}$$

Thus, we have for any $1 < p < \infty$

$$\|\big(\mathcal{D}P|_\varphi(u)\big)\|_{L^p(M)} \leq C_p \|u\|_{L^p(M)}. \tag{49}$$

This ends the proof of Lemma 4.3. □

Now we can prove Lemma 4.2.

Proof. Proof of Lemma 4.2. First we show that $\mathcal{D}F_r|_\varphi$ is injective. Suppose there exists $u \in C^{2,\alpha}_\omega(M)$ such that

$$-r\Delta_\varphi u + (1 - r\underline{R})u + r\Big(\int_M (\Delta_\varphi u)\omega^n + \mathcal{D}P|_\varphi(u)\Big) = 0. \tag{50}$$

It suffices to show that $u = 0$. Multiply u on both hand sides of (50) and integrate against ω_φ^n.

$$\begin{aligned} 0 &= r\int_M |\nabla u|_\varphi^2\omega_\varphi^n + (1-r\underline{R})\int_M u^2\omega_\varphi^n + r\Big(\int_M (\Delta_\varphi u)\omega^n\Big)\Big(\int_M u\omega_\varphi^n\Big) \\ &\quad + r\int_M \big(\mathcal{D}P|_\varphi(u)\big)u\omega_\varphi^n \qquad (51)\\ &\geq (1-r\underline{R})\int_M u^2\omega_\varphi^n + r\{\Big(\int_M (\Delta_\varphi u)\omega^n\Big)\Big(\int_M u\omega_\varphi^n\Big) + \int_M \big(\mathcal{D}P|_\varphi(u)\big)u\omega_\varphi^n\}. \qquad (52)\end{aligned}$$

We focus on estimates of the later two terms in (52). Consider

$$\begin{aligned} \int_M (\Delta_\varphi u)\omega^n &= \int_M (\Delta_\varphi u)\Big(\frac{\omega^n}{\omega_\varphi^n}\Big)\omega_\varphi^n = \int_M u\Big(\Delta_\varphi \frac{\omega^n}{\omega_\varphi^n}\Big)\omega_\varphi^n \\ &= \int_M u\frac{\omega^n}{\omega_\varphi^n} g_\varphi^{i\bar{j}}\big(-g_\varphi^{k\bar{l}}\varphi_{,k\bar{l}i\bar{j}} + g_\varphi^{k\bar{p}}g_\varphi^{q\bar{l}}\varphi_{,\bar{p}q\bar{j}}\varphi_{,k\bar{l}i} + g_\varphi^{\bar{p}q}g_\varphi^{k\bar{l}}\varphi_{,\bar{p}q\bar{j}}\varphi_{,k\bar{l}i}\big)\omega_\varphi^n \\ &\geq -C\Big(\int_M |u|\omega_\varphi^n\Big)\end{aligned}$$

where the derivatives are covariant derivatives of ω. Thus, we have that

$$r\Big(\int_M (\Delta_\varphi u)\omega^n\Big)\Big(\int_M u\omega_\varphi^n\Big) \geq -Cr\int_M u^2\omega_\varphi^n. \qquad (53)$$

To estimate the last term of (52), we need the estimate of $(\mathcal{D}P|_\varphi)$ in Lemma 4.3. Thus we get

$$\|\big(\mathcal{D}P|_\varphi(u)\big)\|_{L^2(M)} \leq C\|u\|_{L^2(M)}. \qquad (54)$$

Thus for the last term in (52) we have the estimate

$$\int_M \big(\mathcal{D}P|_\varphi(u)\big)u\omega_\varphi^n \geq -C\int_M u^2\omega_\varphi^n. \qquad (55)$$

Therefore, combining the estimates above, we have that

$$0 \geq (1-Cr)\int_M u^2\omega_\varphi^n. \qquad (56)$$

It implies that when $r > 0$ sufficiently small, we have that $u = 0$. So we have proved the injectivity of $\big(\mathcal{D}F_r|_\varphi\big)$.

Next, we show the surjectivity of $\big(\mathcal{D}F_r|_\varphi\big)$ and the upper bound of $\|\big(\mathcal{D}F_r|_\varphi\big)^{-1}\|$ together. For $f \in C_\omega^\alpha(M)$, we'll use continuity method to solve the equation

$$\mathcal{D}F_r|_\varphi(u) = f. \qquad (57)$$

Define for $s \in [0,1]$,

$$L_s : C^{2,\alpha}(M) \to C^\alpha(M) \qquad (58)$$

$$u \mapsto -r\Delta_\varphi u + (1-r\underline{R})u + sr\Big(\int_M (\Delta_\varphi u)\omega^n + \mathcal{D}P|_\varphi(u)\Big). \qquad (59)$$

First, we show that for any $s \in [0,1]$

$$\|u\|_{C^{2,\alpha}(M)} \leq C_r \|L_s u\|_{C^\alpha(M)}. \tag{60}$$

From the definition of L_s, we get that,

$$\Delta_\varphi u = -\frac{1}{r} L_s u + \frac{1 - r\underline{R}}{r} u + s\big(\int_M (\Delta_\varphi u)\omega^n + \mathcal{D}P|_\varphi(u)\big). \tag{61}$$

Since we choose $r > 0$ sufficiently small s.t. $\|\varphi\|_{C^{2,\alpha}(M)} \leq \frac{1}{2}$, we can get from Schauder estimate,

$$\begin{aligned}
\|u\|_{C^{2,\alpha}(M)} &\leq C\big(\|\Delta_\varphi u\|_{C^\alpha(M)} + \|u\|_{L^\infty(M)}\big) \\
&\leq C\big(\frac{1}{r}\|L_s u\|_{C^\alpha(M)} + \frac{1}{r}\|u\|_{C^\alpha(M)} + |\int_M (\Delta_\varphi u)\omega^n| \\
&+ \|\big(\mathcal{D}P|_\varphi(u)\big)\|_{C^\alpha(M)} + \|u\|_{L^\infty(M)}\big) \\
&\leq C_0\big(\frac{1}{r}\|L_s u\|_{C^\alpha(M)} + \frac{1}{r}\|u\|_{C^\alpha(M)} + \|\big(\mathcal{D}P|_\varphi(u)\big)\|_{C^\alpha(M)} + \|u\|_{L^\infty(M)}\big).
\end{aligned}$$

By interpolations on Holder spaces in [13, Lemma 6.32], we have

$$\|u\|_{C^\alpha(M)} \leq \frac{r}{4C_0}\|u\|_{C^{2,\alpha}(M)} + Cr^{-\frac{\alpha}{1-\alpha}}\|u\|_{L^\infty(M)}. \tag{62}$$

Also, for term $\|\mathcal{D}P|_\varphi(u)\|_{C^\alpha(M)}$, since it satisfies equation (38), we have estimate

$$\|\mathcal{D}P|_\varphi(u)\|_{C^\alpha(M)} \leq C\|\langle \partial\bar\partial u, \big(\partial\bar\partial P_\varphi - (\mathrm{Ric}(\omega) - \underline{R}\omega)\big)\rangle_\varphi\|_{L^\infty(M)} \tag{63}$$

$$\leq C\|\partial\bar\partial u\|_{L^\infty(M)} \leq \frac{1}{4C_0}\|u\|_{C^{2,\alpha}(M)} + C\|u\|_{L^\infty(M)}. \tag{64}$$

Combining estimates of (62) and (63), we have that

$$\|u\|_{C^{2,\alpha}(M)} \leq C\big(\frac{1}{r}\|L_s u\|_{C^\alpha(M)} + r^{-\frac{1}{1-\alpha}}\|u\|_{L^\infty(M)}\big). \tag{65}$$

Now we focus on estimates of $\|u\|_{L^\infty(M)}$. For $p > 1$, We could first multiply $|u|^p$ on both hand sides of (61) and integrate against ω_φ^n on the region $\{u > 0\}$. Then we'll get by a similar argument which we use to prove the injectivity,

$$\begin{aligned}
\frac{1}{r}\int_{u>0}(L_s u)u^p\omega_\varphi^n &\geq \int_{u>0} p u^{p-1}|\nabla u|^2_\varphi \omega_\varphi + \frac{1 - r\underline{R}}{r}\int_{u>0} u^{p+1}\omega_\varphi^n \\
&- Cr\big(\int_M |u|\omega_\varphi^n\big)\big(\int_{u>0} u^p\omega_\varphi^n\big) - C_p\|u\|_{L^{p+1}(M)}\big(\int_{u>0} u^{p+1}\omega_\varphi\big)^{\frac{p}{p+1}} \\
&\geq \frac{1 - r\underline{R}}{r}\int_{u>0} u^{p+1}\omega_\varphi^n - C_p\int_M u^{p+1}\omega_\varphi^n.
\end{aligned}$$

Multiply $|u|^p$ on both hand sides of (61) and integrate against ω_φ^n on the region $\{u < 0\}$. Similarly we get

$$-\frac{1}{r}\int_{u<0}(L_s u)|u|^p\omega_\varphi^n \geq \frac{1 - r\underline{R}}{r}\int_{u<0} |u|^{p+1}\omega_\varphi^n - C_p\int_M |u|^{p+1}\omega_\varphi^n. \tag{66}$$

Thus, choose $p = n$, and then we could choose our $r > 0$ small such that

$$\frac{1}{r}\int_M |u|^{n+1}\omega^n \le \frac{C}{r}\|u\|^{\frac{n}{n+1}}_{L^{n+1}(M)}\|L_s u\|_{L^{n+1}(M)}. \tag{67}$$

And then

$$\|u\|_{L^{n+1}(M)} \le C\|L_s u\|_{L^{n+1}(M)} \le C\|L_s u\|_{L^\infty(M)}. \tag{68}$$

By L^p theory of elliptic equation for (61), we get

$$\|u\|_{W^{2,n+1}(M)} \le C\big(\frac{1}{r}\|L_s u\|_{L^{n+1}(M)} + \frac{1}{r}\|u\|_{L^{n+1}(M)}\big) \tag{69}$$

$$\le \frac{C}{r}\|L_s u\|_{L^\infty(M)}. \tag{70}$$

By sobolev embedding, we can get that

$$\|u\|_{L^\infty(M)} \le C\|u\|_{W^{2,n+1}(M)} \le \frac{C}{r}\|L_s u\|_{L^\infty(M)}. \tag{71}$$

Therefore, we conclude that

$$\|u\|_{C^{2,\alpha}(M)} \le Cr^{-\frac{2-\alpha}{1-\alpha}}\|L_s u\|_{C^\alpha(M)}. \tag{72}$$

Since the norm is independent of $s \in [0,1]$ and obviously $L_0 : C^{2,\alpha}(M) \to C^\alpha(M)$ is onto, thus by continuity method in [13, Theorem 5.2], we conclude that $L_1 : C^{2,\alpha}(M) \to C^\alpha(M)$ is also onto. Thus we have shown that $\mathcal{D}F_r|_\varphi = L_1$ is surjective. Moreover,

$$\|\big(\mathcal{D}F_r|_\varphi\big)^{-1}(f)\|_{C^{2,\alpha}(M)} \le Cr^{-\frac{2-\alpha}{1-\alpha}}\|f\|_{C^\alpha(M)}. \tag{73}$$

This ends the proof of Lemma 4.2. □

Given $\varphi \in \mathcal{H}_\omega$ and $\|\varphi\|_{C^4(M)} \le \frac{1}{2}$, for any $y \in C^\alpha_\omega(M)$ we define the functional Ψ_y in a $C^{2,\alpha}$-neighborhood of φ as

$$\Psi_y : \mathcal{H}^{2,\alpha}_\omega \to C^{2,\alpha}_\omega(M)$$
$$x \mapsto x + \big(\mathcal{D}F_r|_\varphi\big)^{-1}\big(y - F_r(x)\big).$$

Our goal is to find $x \in \mathcal{H}^{2,\alpha}_\omega$ such that $F_r(x) = y$. Given the definition of Ψ_y, our problem comes down to find the fixed point of Ψ_y. So it suffices to show that Ψ_y is a contraction in a small neighborhood of $\varphi \in \mathcal{H}^{2,\alpha}_\omega$.

LEMMA 4.4. *Suppose that $\varphi \in \mathcal{H}_\omega$ and $\|\varphi\|_{C^4(M)} \le \frac{1}{2}$. Then there exists some $\delta = \delta(\omega, n) > 0$, such that if $x_1, x_2 \in \mathcal{H}^{2,\alpha}_\omega$ with $\|x_1 - \varphi\|_{C^{2,\alpha}(M)} < r^{\frac{1}{1-\alpha}}\delta$ and $\|x_2 - \varphi\|_{C^{2,\alpha}(M)} < r^{\frac{1}{1-\alpha}}\delta$, then*

$$\|\Psi_y(x_1) - \Psi_y(x_2)\|_{C^{2,\alpha}(M)} \le \frac{1}{2}\|x_1 - x_2\|_{C^{2,\alpha}(M)}. \tag{74}$$

Proof. Denote $x_s = sx_1 + (1-s)x_2$. Suppose $\|\varphi - x_1\|_{C^{2,\alpha}(M)} < r^{\frac{1}{1-\alpha}}\delta$ and $\|\varphi - x_2\|_{C^{2,\alpha}(M)} < r^{\frac{1}{1-\alpha}}\delta$. We'll specify $\delta > 0$ later.

We have

$$\begin{aligned}\Psi_y(x_1)-\Psi_y(x_2) &= \int_0^1 \frac{\partial}{\partial s}\Psi_y(x_s)ds\\ &= (x_1-x_2) - \int_0^1 \big(\mathcal{D}F_r|_\varphi\big)^{-1}\big(\mathcal{D}F_r|_{x_s}(x_1-x_2)\big)ds\\ &= -\int_0^1 \big(\mathcal{D}F_r|_\varphi\big)^{-1}\{\big(\mathcal{D}F_r|_{x_s}-\mathcal{D}F_r|_\varphi\big)(x_1-x_2)\}ds.\end{aligned}$$

We consider the term

$$\begin{aligned}\big(\mathcal{D}F_r|_{x_s}-\mathcal{D}F_r|_\varphi\big)(x_1-x_2) = r\Big(&-(\Delta_{x_s}-\Delta_\varphi)(x_1-x_2)+\int_M \big((\Delta_{x_s}-\Delta_\varphi)(x_1-x_2)\big)\omega^n\\ &+(\mathcal{D}P|_{x_s}-\mathcal{D}P|_\varphi)(x_1-x_2)\Big).\end{aligned}$$

Thus, we know that

$$\|\big(\mathcal{D}F_r|_{x_s}-\mathcal{D}F_r|_\varphi\big)(x_1-x_2)\|_{C^\alpha(M)} \tag{75}$$
$$\leq Cr\big(r^{\frac{1}{1-\alpha}}\delta\|x_1-x_2\|_{C^{2,\alpha}(M)}+\|(\mathcal{D}P|_{x_s}-\mathcal{D}P|_\varphi)(x_1-x_2)\|_{C^\alpha(M)}\big). \tag{76}$$

By definitons of $\mathcal{D}P|_\varphi$ in (38), we have

$$\Delta_\varphi\big(\mathcal{D}P|_\varphi(u)\big) = \langle\partial\bar\partial u,\big(\partial\bar\partial P_\varphi-(\mathrm{Ric}(\omega)-\underline{R}\omega)\big)\rangle_\varphi,\ \int_M\big(\mathcal{D}P|_\varphi(u)\big)\omega^n=0, \tag{77}$$
$$\Delta_{x_s}\big(\mathcal{D}P|_{x_s}(u)\big) = \langle\partial\bar\partial u,\big(\partial\bar\partial P_{x_s}-(\mathrm{Ric}(\omega)-\underline{R}\omega)\big)\rangle_{x_s},\ \int_M\big(\mathcal{D}P|_{x_s}(u)\big)\omega^n=0. \tag{78}$$

So

$$\begin{aligned}&\Delta_\varphi\big(\mathcal{D}P|_\varphi(u)-\mathcal{D}P|_{x_s}(u)\big)\\ &=\langle\partial\bar\partial u,\big(\partial\bar\partial P_\varphi-(\mathrm{Ric}(\omega)-\underline{R}\omega)\big)\rangle_\varphi-\langle\partial\bar\partial u,\big(\partial\bar\partial P_{x_s}-(\mathrm{Ric}(\omega)-\underline{R}\omega)\big)\rangle_{x_s}\\ &+\big(\Delta_{x_s}-\Delta_\varphi\big)\big(\mathcal{D}P|_{x_s}(u)\big),\\ &=u_{,i\bar j}(g_\varphi^{i\bar l}g_\varphi^{k\bar j}-g_{x_s}^{i\bar l}g_{x_s}^{k\bar j})P_{\varphi,k\bar l}+u_{,i\bar j}g_{x_s}^{i\bar l}g_{x_s}^{k\bar j}(P_\varphi-P_{x_s})_{,k\bar l}\\ &-u_{,i\bar j}(g_\varphi^{i\bar l}g_\varphi^{k\bar j}-g_{x_s}^{i\bar l}g_{x_s}^{k\bar j})(\mathrm{Ric}(\omega)-\underline{R}\omega)_{k\bar l}+(g_{x_s}^{k\bar l}-g_\varphi^{k\bar l})\big(\mathcal{D}P|_{x_s}(u)\big)_{,k\bar l}.\end{aligned}$$

Thus, by Lemma 2.2 and the previous estimates about P_φ(29) and $\mathcal{D}P|_\varphi(u)$(30) in Section 3,

$$\begin{aligned}\|\mathcal{D}P|_\varphi(u)-\mathcal{D}P|_{x_s}(u)\|_{C^{2,\alpha}(M)} &\leq Cr^{\frac{1}{1-\alpha}}\delta\big(\|u\|_{C^{2,\alpha}(M)}+\|\mathcal{D}P|_{x_s}(u)\|_{C^{2,\alpha}(M)}\big)\\ &+C\|P_\varphi-P_{x_s}\|_{C^{2,\alpha}(M)}\|u\|_{C^{2,\alpha}(M)}\\ &\leq Cr^{\frac{1}{1-\alpha}}\delta\|u\|_{C^{2,\alpha}(M)}+C\|P_\varphi-P_{x_s}\|_{C^{2,\alpha}(M)}\|u\|_{C^{2,\alpha}(M)}.\end{aligned}$$

Since we have

$$\Delta_\varphi(P_\varphi-P_{x_s})=(g_{x_s}^{k\bar l}-g_\varphi^{k\bar l})P_{x_s,k\bar l}+(g_\varphi^{k\bar l}-g_{x_s}^{k\bar l})(\mathrm{Ric}(\omega)-\underline{R}\omega)_{k\bar l},$$

then

$$\|P_\varphi-P_{x_s}\|_{C^{2,\alpha}(M)}\leq Cr^{\frac{1}{1-\alpha}}\delta. \tag{79}$$

Thus, we have

$$\|\big(\mathcal{D}P|_{\varphi}-\mathcal{D}P|_{x_s}\big)(x_1-x_2)\|_{C^{\alpha}(M)} \le \|\big(\mathcal{D}P|_{\varphi}-\mathcal{D}P|_{x_s}\big)(x_1-x_2)\|_{C^{2,\alpha}(M)} \tag{80}$$

$$\le Cr^{\frac{1}{1-\alpha}}\delta\|x_1-x_2\|_{C^{2,\alpha}(M)}. \tag{81}$$

By Lemma 4.2, we have that

$$\|\big(\mathcal{D}F_r|_{\varphi}\big)^{-1}\{\big(\mathcal{D}F_r|_{x_s}-\mathcal{D}F_r|_{\varphi}\big)(x_1-x_2)\}\|_{C^{2,\alpha}(M)}$$

$$\le Cr^{-\frac{2-\alpha}{1-\alpha}}rr^{\frac{1}{1-\alpha}}\delta\|x_1-x_2\|_{C^{2,\alpha}(M)} \tag{82}$$

$$\le C\delta\|x_1-x_2\|_{C^{2,\alpha}(M)}. \tag{83}$$

And then

$$\|\Psi_y(x_1)-\Psi_y(x_2)\|_{C^{2,\alpha}(M)} \le C\delta\|x_1-x_2\|_{C^{2,\alpha}(M)}. \tag{84}$$

We could choose $\delta>0$ sufficiently small such that $C\delta<\frac{1}{2}$, and thus it ends the proof of Lemma 4.4. □

Now we're ready to prove the Theorem 4.1.

Proof. From Lemma 4.2, we can conclude that F_r is a local homeomorphism from φ to $F_r(\varphi)$ if $\varphi\in\mathcal{H}_\omega$ with $\|\varphi\|_{C^4(M)}\le\frac{1}{2}$ and $r<\epsilon$ where the constant ϵ is the same constant in Lemma 4.2. Denote the constant $\delta>0$ in Lemma 4.4 as δ_0. Define for $k\in\mathbb{Z}$

$$\varphi_k=\Psi_y^{k-1}(\varphi). \tag{85}$$

Ultimately, we want to show that $\varphi_k\to\varphi_\infty$ in $C^{2,\alpha}(M)$ norm for some $x\in\mathcal{H}_\omega^{2,\alpha}$ as $k\to\infty$. We choose the start point to be φ, thus we need to show that $\varphi_2=\Psi_y(\varphi)$ stays in the neighborhood of φ of radius $\delta_0 r^{\frac{1}{1-\alpha}}$. Compute

$$\begin{aligned}\|\varphi_2-\varphi\|_{C^{2,\alpha}(M)} &= \|\big(\mathcal{D}F_r|_{\varphi}\big)^{-1}\big(y-F_r(\varphi)\big)\|_{C^{2,\alpha}(M)},\\ &\le Cr^{-\frac{2-\alpha}{1-\alpha}}\|y-F_r(\varphi)\|_{C^{\alpha}(M)},\\ &\le (C_1r^{-\frac{3-\alpha}{1-\alpha}}\|y-F_r(\varphi)\|_{C^{\alpha}(M)})r^{\frac{1}{1-\alpha}}.\end{aligned}$$

If we consider all $y\in C^{\alpha}_{\omega}(M)$ such that $\|y-F_r(\varphi)\|_{C^{\alpha}(M)}\le\frac{\delta_0}{2C_1}r^{\frac{3-\alpha}{1-\alpha}}$(it suggests that we could choose $\eta=\frac{\delta_0}{2C_1}$), then

$$\|\varphi_2-\varphi\|_{C^{2,\alpha}(M)}\le\frac{1}{2}r^{\frac{1}{1-\alpha}}\delta_0. \tag{86}$$

By induction, we could get that for any $k\in\mathbb{Z}$

$$\|\varphi_k-\varphi\|_{C^{2,\alpha}(M)}<r^{\frac{1}{1-\alpha}}\delta_0, \tag{87}$$

and for any $k,l\in\mathbb{Z}$

$$\|\varphi_{k+l}-\varphi_k\|_{C^{2,\alpha}(M)}\le(\frac{1}{2})^{k-1}r^{\frac{1}{1-\alpha}}\delta_0. \tag{88}$$

Thus, we conclude that $\{\varphi_k\}_{k=2}^{\infty}$ converges and the limit $x\in\mathcal{H}_\omega^{2,\alpha}$ is the unique fixed point of Ψ_y. Thus, we get

$$F_r(x)=y. \tag{89}$$

Then we finish the proof of Theorem 4.1. □

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