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QUENCHED WEIGHTED MOMENTS OF A SUPERCRITICAL BRANCHING PROCESS IN A RANDOM ENVIRONMENT*

YUEJIAO WANG†, YINGQIU LI‡, QUANSHENG LIU§, AND ZAIMING LIU¶

Abstract. We consider a supercritical branching process (Z_n) in an independent and identically distributed random environment $\xi = (\xi_n)$. Let W be the limit of the natural martingale $W_n = Z_n/\mathbb{E}_\xi Z_n, n \geq 0$, where $\mathbb{E}_\xi$ denotes the conditional expectation given the environment ξ. We find a necessary and sufficient condition for the existence of quenched weighted moments of W of the form $\mathbb{E}_\xi W^\alpha l(W)$, where $\alpha > 1$ and l is a positive function slowly varying at ∞. The same conclusion is also proved for the maximum of the martingale $W^* = \sup_{n\geq 1} W_n$ instead of the limit variable W. In the proof we first show an extended version of Doob's inequality about weighted moments for nonnegative submartingales, which is of independent interest.

Key words. Branching process, random environment, weighted moments, Doob's inequality, slowly varying function.

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1. Introduction and main result. The Galton - Watson process is a famous population process where the particles behave independently; each particle gives birth to new particles of the next generation according to a fixed offspring distribution. A branching process in a random environment is a natural and important extension of the Galton-Watson process, where the offspring distributions vary from generation to generation according to a random environment. This model was first introduced by Wilkinson and Smith [29] in 1969. Basic limit theorems were established by Athreya and Karlin [4, 5] in 1971. Since then this model has attracted the attention of many authors, see for example the recent works by Bansaye and Berestycki [7], Bôinghoff, Dyakonova, Kersting and Vatutin [14], Huang and Liu [20], Bansaye and Vatutin [8], and the references therein.

We are interested in asymptotic properties of a supercritical branching process in a random environment. For a Galton-Watson process (Z_n) with one initial particle $(Z_0 = 1)$ and $m = \mathbb{E}Z_1 > 1$, the moments of the limit variable $W = \lim_{n\to\infty} \frac{Z_n}{m^n}$ has been studied by many authors, see for example [1, 6, 12, 18, 21]. Of particular interest is the existence of the weighted moments of W of the form $\mathbb{E}W^\alpha l(W)$, where $\alpha > 1$ and l is a positive function slowly varying at ∞. Bingham and Doney [12] showed that when $\alpha > 1$ is not an integer, $\mathbb{E}W^\alpha l(W) < \infty$ if and only if $\mathbb{E}Z_1^\alpha l(Z_1) < \infty$. Alsmeyer and Rösler [1] proved that the same result remains true for all non-dyadic integer $\alpha > 1$ (not of the form 2^k for some integer $k \geq 0$). Liang and Liu [27] proved that the result holds true for *all* $\alpha > 1$. In [28], this result was further extended to a branching process in a random environment for the annealed weighted moments $\mathbb{E}W^\alpha l(W)$. In this paper, we consider the extension to the quenched moments $\mathbb{E}_\xi W^\alpha l(W)$ (when

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†Central South University, School of Mathematics and Statistic, 410083, Changsha, China (wangyuejiaohujing@163.com); and College of Mathematics and Computational Science, Hunan First Normal University, 410205 Changsha, China.

‡Changsha University of Science and Technology, College of Mathematics and Computing Sciences, 410004, Changsha, China (liyq-2001@163.com).

§Corresponding author. Univ. Bretagne-Sud, UMR 6205, Laboratoire de Mathématiques de Bretagne Atlantique, F-56000 Vannes, France (quansheng.liu@univ-ubs.fr).

¶Central South University, School of Mathematics and Statistic, 410083, Changsha, China (math_lzm@csu.edu.cn).

the environment ξ is given), for which we will show that the existence condition is quite different to the annealed case. Meanwhile, we also consider the same problem for the maximum variable W^* instead of the limit variable W. We mention that the techniques of this paper can be used to study moments of more general models, such as Mandelbrot's cascades and branching random walks. In such models, the study of moments is an important topic, and is closely related to multifractal analysis, see for example Kahane and Peyrière [22], Barral and Mandelbrot [9], Barral and Jin [10], and Falconer [16].

Let us now describe precisely the model. Let $\xi = (\xi_n)_{n\geq 0}$ be a sequence of independent and identically distribution random variables taking values in some space Θ, whose realization corresponds to a sequence of probability distribution on $\mathbb{N} = \{0, 1, 2, \cdots\}$:

$$p(\xi_n) = \{p_i(\xi_n); i \geq 0\}, \quad \text{where} \;\; p_i(\xi_n) \geq 0 \quad \text{and} \quad \sum_{i=1}^{\infty} p_i(\xi_n) = 1.$$

A branching process $(Z_n)_{n\geq 0}$ in the random environment ξ can be defined as follows:

$$Z_0 = 1 \quad \text{and} \quad Z_{n+1} = \sum_{i=1}^{Z_n} X_{n,i} \;\; \text{for} \;\; n \geq 0,$$

where given the environment ξ, $X_{n,i}$ $(n \geq 0, i \geq 1)$ is a sequence of (conditionally) independent random variables each $X_{n,i}$ has distribution $p(\xi_n)$.

The total probability space on which all the random variables ξ_n and $X_{n,i}$ (consequently all the Z_n) are defined will be denoted by $(\Omega, \mathcal{F}, \mathbb{P})$; the conditional probability given the environment ξ will be denoted by $\mathbb{P}_\xi$. Therefore, by definition, for each realization of the environment sequence ξ, the random variables $X_{n,i}$ $(n \geq 0, i \geq 1)$ are independent of each other and independent of Z_n under $\mathbb{P}_\xi$. The probability $\mathbb{P}$ is usually called *annealed law*, while $\mathbb{P}_\xi$ is called *quenched law*. The expectation with respect to $\mathbb{P}$ and $\mathbb{P}_\xi$ will be denoted respectively by $\mathbb{E}$ and $\mathbb{E}_\xi$.

For $n \geq 0$, write

$$m_n = \sum_{i=0}^{\infty} i p_i(\xi_n), \qquad \Pi_0 = 1 \qquad \text{and} \qquad \Pi_n = \prod_{i=0}^{n-1} m_i \;\; \text{if} \;\; n \geq 1.$$

Then $m_n = \mathbb{E}_\xi X_{n,i}$ and $\Pi_n = \mathbb{E}_\xi Z_n$. We consider the supercritical case where $\mathbb{E} \log m_0 > 0$. It is well known that the normalized population size $W_n = \frac{Z_n}{\Pi_n}$, $n \geq 0$, is a nonnegative martingale under $\mathbb{P}_\xi$ with respect to the filtration

$$\mathcal{F}_0^0 = \{\emptyset, \Omega\} \quad \text{and} \quad \mathcal{F}_n^0 = \sigma\{\xi, X_{k,i} : k < n, i = 1, 2, \cdots\} \;\; \text{for} \;\; n \geq 1,$$

so that the limit $W = \lim_{n\to\infty} W_n$ exists almost surely (a.s.) with $\mathbb{E}_\xi W \leq 1$ by Fatou's Lemma. It is also known that W is non-degenerate (which is also equivalent to the convergence in L^1 of (W_n)) if and only if

$$\mathbb{E}\left(\frac{Z_1 \log^+ Z_1}{m_0}\right) < \infty \tag{1.1}$$

(see Athreya and Karlin [5] for the sufficiency and Tanny [30] for the necessity) and that $\mathbb{E}_\xi W = 1$ a.s. when the condition is satisfied, where $\log^+ x$ denotes the positive part of $\log x$: $\log^+ x = \log x$ if $x > 1$, and $\log^+ x = 0$ if $x \in [0, 1]$. It can be checked that under the supercritical condition $\mathbb{E} \log m_0 > 0$, the hypothesis (1.1) is equivalent to $\mathbb{E}W_1 \log^+ W_1 < \infty$. We will consider the existence of weighted moments of W of the form $\mathbb{E}_\xi W^\alpha l(W)$ with $\alpha > 1$ and l a positive function slowly varying at ∞, and the same problem for the maximum variable

$$W^* = \sup_{n\geq 1} W_n.$$

Recall that a positive and measurable function l is defined on $[0, \infty)$ is called *slowly varying* at ∞ if $\lim_{x\to\infty} \frac{l(\lambda x)}{l(x)} = 1$ for all $\lambda > 0$. (Throughout this paper, the term "positive" is used in the wide sense.) By the representation theorem (see [11],Theorem 1.3.1), any function l slowly varying at ∞ is of the form

$$l(x) = c(x) \exp\left(\int_{a_0}^{x} \frac{\varepsilon(t)}{t} dt\right), \quad x > a_0, \tag{1.2}$$

where $a_0 > 0$, $c(\cdot)$ and $\varepsilon(\cdot)$ are measurable with $c(x) \to c$ for some constant $c \in (0, \infty)$ and $\varepsilon(x) \to 0$ as $x \to \infty$. Moreover, it is known that any slowly varying function l posses a smoothed version l_1 in the sense that $l(x) \sim l_1(x)$ as $x \to \infty$, with l_1 of the form

$$l_1(x) = c\ \exp\left(\int_{a_0}^{x} \frac{\varepsilon_1(t)}{t} dt\right), \quad x > a_0, \tag{1.3}$$

with ε_1 infinitely differentiable on (a_0, ∞) and $\lim_{x\to\infty} \varepsilon_1(x) = 0$ (see [11], Theorem 1.3.3). The value of a_0 and those of $l(x)$ on $[0, a_0]$ will not be important. For convenience, we often take $a_0 = 1$. Notice also that the function $c(\cdot)$ in the representation of $l(\cdot)$ has no influence on the finiteness of moments of W of the form $\mathbb{E}_\xi W^\alpha l(W)$, so that we can suppose without loss of generality that $c(x) = 1$. Moreover, by choosing a smoothed version if necessary, we can suppose that the function ε in the representation form (1.2) is infinitely differentiable.

We can now formulate our main result.

THEOREM 1.1. *Let l be a function slowly varying at ∞ and $\phi(x) = x^\alpha l(x)$ with $\alpha > 1$. Assume (1.1) and $\mathbb{E} \log m_0 < \infty$. Then the following assertions are equivalent: (i) $\mathbb{E} \log^+ \mathbb{E}_\xi \phi(W_1) < \infty$; (ii) $\mathbb{E}_\xi \phi(W) < \infty$ a.s.; (iii) $\mathbb{E}_\xi \phi(W^*) < \infty$ a.s..*

REMARK 1.1. *For the equivalence between (ii) and (iii), we do not need the condition $\mathbb{E} \log m_0 < \infty$. Actually, this equivalence is a general result for martingales; we will prove it by establishing an extended version of Doob's inequality about weighted moments for nonnegative submartingales, which is of independent interest: see Theorem 2.1 below.*

When l is a constant, the equivalence between (i) and (ii) has been established in [19]. The general case where l is not necessarily a constant makes the proof much more delicate. Similar sufficient conditions were given in [25], where a completely different method was used.

The rest of the paper is organized as follows. In Section 2, we prove an extension of Doob's inequality about the ϕ-moments of the limit variable and the maximum of a non-negative submartimgale. In Section 3 we establish a key inequality, with which we will prove the main result in Section 4.

2. Extended Doob's inequality for ϕ-moments on submartingales. For a convergent nonnegative submartingale sequence $\{(f_n, \mathcal{G}_n) : n \geq 0\}$ with $\mathcal{G}_0 = \{\emptyset, \Omega\}$, defined on some probability space $(\Omega, \mathcal{G}, \mathbb{P})$, we set

$$f = \lim_{n\to\infty} f_n \quad \text{and} \quad f^* = \sup_{n\geq 0} f_n.$$

In this subsection, we will prove a relation between the weighted moments for the limit variable f and the maximum variable f^*, which implies that they are finite or infinite simultaneously. It can be considered as an extended version of Doob's inequality on submartingales.

THEOREM 2.1. *Let $(f_n, \mathcal{G}_n)$ be a nonnegative submartingale convergent a.s. and in L^1. Define f and f^* as above. Let $\phi(x) = x^\alpha l(x)$, where $\alpha > 1$, l is a positive function slowly varying at ∞ and locally bounded on $[0,\infty)$. Then there exist two constants $C_0 > 0$ and $C_1 > 0$ depending only on ϕ, such that*

$$\mathbb{E}\phi(f^*) \leq C_0 + C_1 \mathbb{E}\phi(f).$$

Two lemmas will be used for the proof of Theorem 2.1. The first is an extension of Doob's inequality on a submartingale, while the second is about a smoothed version of a regularly varying function.

LEMMA 2.1 ([2], Proposition 1.1). *Let $\phi : [0,\infty) \to [0,\infty)$ be an unbounded, nondecreasing convex function, with $\phi(0) = 0$,*

$$p_\phi := \inf_{0<x<\infty} \frac{x\phi'(x)}{\phi(x)} > 1 \quad \text{and} \quad p_\phi^* := \sup_{0<x<\infty} \frac{x\phi'(x)}{\phi(x)} < \infty,$$

where $\phi'(x)$ denotes the right derivative of ϕ at x. Then for each $n \geq 0$, the maximum variable $f_n^* = \sup_{0\leq k\leq n} f_k$ satisfies

$$\mathbb{E}\phi(f_n^*) \leq \left(\frac{p_\phi}{p_\phi - 1}\right)^{p_\phi^*} \mathbb{E}\phi(f_n).$$

As usual, we write $f(x) \sim g(x)$ if $\lim_{x\to\infty} \frac{f(x)}{g(x)} = 1$.

LEMMA 2.2. *([28],Lemma 3.1) Let $\phi(x) = x^\alpha l(x)$, with $\alpha > 1$ and l a positive function slowly varying at ∞. Then, for each $\beta \in (1,2]$ with $\beta < \alpha$, there is a function $\phi_1 \geq 0$ such that*

(i) $\phi_1(x) \sim \phi(x)$, *as $x \to \infty$;*
(ii) $x \mapsto \phi_1(x)$ *and* $x \mapsto \phi_1(x^{1/\beta})$ *are convex on* $[0,\infty)$;
(iii) $\phi_1(x) = x^\alpha l_1(x)$, *where l_1 is slowly varying at ∞ and $l_1(x) > 0$ for all $x \geq 0$.*
(iv) *If l has the representation form (1.2) with $c(x) = c$ for a constant $c \in (0,\infty)$ and $\varepsilon(x)$ continuous for $x > a_0$, then the function ϕ_1 satisfying* (i), (ii) *and* (iii) *can be taken as*

$$\phi_1(x) = \alpha \int_0^x u^{\alpha-1} l(u)du \text{ for } x \in [0,\infty).$$

We mention that Part (iv) does not appear explicitely in Lemma 3.1 of [28], but is shown in its proof.

Proof of Theorem 2.1. By choosing a smoothed version (see (1.3)) if necessary, we can suppose that l has the representation form (1.2) with $c(x) = c$ and ε infinitely differentiable. Let ϕ_1 be defined as in Lemma 2.2(iv) and $\alpha = \alpha_0 + b$ where $b > 0$ and $\alpha_0 > 1$. Let $\delta > 0$ be small enough such that $\alpha - \delta > \alpha_0$ and a_1 be large enough such that $\alpha - \delta < \frac{\alpha\phi(x)}{\phi_1(x)} < \alpha + \delta$ for all $x \geq a_1$.

We set $\phi_2(x) = \phi_1(x)$ if $x \geq a_1$; $\phi_2(x) = x^{\alpha_0} a_1^b l_1(a_1)$ if $x \in [0, a_1)$. Then ϕ_2 is convex on $[0, a_1)$ and (a_1, ∞). At a_1, the right derivative $\phi_2'(a_1+)$ and left derivative $\phi_2'(a_1-)$ satisfy

$$\frac{\phi_2'(a_1+)}{\phi_2'(a_1-)} = \frac{\phi_1'(a_1)}{a_1^b l_1(a_1)\alpha_0 a_1^{\alpha_0-1}} = \frac{a_1\phi_1'(a_1)}{\alpha_0\phi_1(a_1)} = \frac{\alpha\phi(a_1)}{\alpha_0\phi_1(a_1)} > \frac{\alpha-\delta}{\alpha_0} > 1,$$

where the last equality holds by Lemma 2.2(iv). Therefore ϕ_2 is convex on $[0, \infty)$. Since

$$\inf_{0<x<a_1} \frac{x\phi_2'(x)}{\phi_2(x)} = \inf_{0<x<a_1} \frac{\alpha_0 x^{\alpha_0} a_1^b l_1(a_1)}{x^{\alpha_0} a_1^b l_1(a_1)} = \alpha_0 > 1,$$

and

$$\begin{aligned}\inf_{a_1\leq x<\infty} \frac{x\phi_2'(x)}{\phi_2(x)} &= \inf_{a_1\leq x<\infty} \frac{x\phi_1'(x)}{\phi_1(x)} \\ &= \inf_{a_1\leq x<\infty} \frac{x\alpha x^{\alpha-1} l(x)}{\phi_1(x)} = \inf_{a_1\leq x<\infty} \frac{\alpha\phi(x)}{\phi_1(x)} > \alpha-\delta > 1,\end{aligned}$$

we have $p_{\phi_2} > 1$. Similarly,

$$\sup_{0<x<a_1} \frac{x\phi_2'(x)}{\phi_2(x)} = \sup_{0<x<a_1} \frac{\alpha_0 x^{\alpha_0} a_1^b l_1(a_1)}{x^{\alpha_0} a_1^b l_1(a_1)} = \alpha_0 < \infty,$$

$$\begin{aligned}\sup_{a_1\leq x<\infty} \frac{x\phi_2'(x)}{\phi_2(x)} &= \sup_{a_1\leq x<\infty} \frac{x\phi_1'(x)}{\phi_1(x)} = \sup_{a_1\leq x<\infty} \frac{x\alpha x^{\alpha-1} l(x)}{\phi_1(x)} \\ &= \sup_{a_1\leq x<\infty} \frac{\alpha\phi(x)}{\phi_1(x)} < \alpha+\delta < \infty,\end{aligned}$$

so that $p_{\phi_2}^* < \infty$. Then, by Lemma 2.1, we have

$$\mathbb{E}\phi_2(f_n^*) \leq (\frac{p_{\phi_2}}{p_{\phi_2}-1})^{p_{\phi_2}^*}\mathbb{E}\phi_2(f_n), \tag{2.1}$$

where $f_n^* = \sup_{0\leq k\leq n} f_k$. Since (f_m) converges a.s. and in L^1, we know that for all n

$$\mathbb{E}\big(f|\mathcal{G}_n\big) \geq f_n.$$

Therefore, by Jensen's inequality and the monotonicity of ϕ_2, we get

$$\mathbb{E}\big(\phi_2(f)|\mathcal{G}_n\big) \geq \phi_2\Big(\mathbb{E}\big(f|\mathcal{G}_n\big)\Big) \geq \phi_2(f_n). \tag{2.2}$$

Taking expectation at both sides of (2.2), we have

$$\mathbb{E}\phi_2(f) \geq \mathbb{E}\phi_2(f_n). \tag{2.3}$$

Passing to the limit in (2.1) as $n \to \infty$, using the monotone convergence theorem for the left side and the bound (2.3) for the right side, we obtain

$$\mathbb{E}\phi_2(f^*) \leq (\frac{p_{\phi_2}}{p_{\phi_2}-1})^{p^*_{\phi_2}} \mathbb{E}\phi_2(f). \tag{2.4}$$

Because $\phi_2(x) \sim \phi(x)$ as $x \to \infty$, for $a > 0$ large enough, we have

$$\frac{1}{2}\phi(x) \leq \phi_2(x) \leq 2\phi(x) \quad \text{for all} \quad x \geq a.$$

Hence

$$\frac{1}{2}\mathbb{E}\phi(f)I_{\{f \geq a\}} \leq \mathbb{E}\phi_2(f)I_{\{f\geq a\}} \leq 2\mathbb{E}\phi(f)I_{\{f\geq a\}}.$$

As $\mathbb{E}\phi_2(f)I_{\{f<a\}} \leq \phi_2(a)$, it follows that

$$\frac{1}{2}\left(\mathbb{E}\phi(f) - \phi(a)\right) \leq \mathbb{E}\phi_2(f) \leq 2\left(\phi(a) + \mathbb{E}\phi(f)\right).$$

Similarly, the same result holds with f replaced by f^*. Therefore the desired conclusion follows from (2.4).

3. A key inequality. In this section, we show an inequality (see Lemma 3.1 below) which will play a key role in the proof of the main result.

For $n \geq 0$ and $i \geq 1$, write

$$\tilde{X}_{n,i} = \frac{X_{n,i}}{m_n} - 1.$$

For simplicity, set $\tilde{X}_n = \tilde{X}_{n,1}$ which has the same distribution as $\tilde{X}_{n,i}$ under $\mathbb{P}_\xi$, for each $i \geq 1$. For $n \geq 1$, write

$$D_n = W_n - W_{n-1} = \frac{1}{\Pi_{n-1}}\left(\sum_{i=1}^{Z_{n-1}} \tilde{X}_{n-1,i}\right).$$

Then $W_n = 1 + D_1 + \cdots + D_n$, so that $W^* = \sup_{n\geq 1} W_n$ can be written as $W^* = 1 + \sup_{n\geq 1}(D_1 + \cdots + D_n)$, so that $|W^* - 1| \leq \sup_{n\geq 1}|D_1 + \cdots + D_n|$. Define

$$\mathcal{F}_0 = \{\emptyset, \Omega\} \quad \text{and} \quad \mathcal{F}_n = \sigma\{\xi_k, X_{l,i}, k < n, l < n, i = 1, 2, \cdots\} \quad \text{for } n \geq 1.$$

Then, as observed in Liang and Liu [28], $(W_n, \mathcal{F}_n)_n \geq 0$ also forms a non negative martingale under $\mathbb{P}_\xi$ since

$$\mathbb{E}_\xi(W_n|\mathcal{F}_{n-1}) = \mathbb{E}_\xi\Big(\mathbb{E}_\xi\big((W_n|\mathcal{F}^0_{n-1})|\mathcal{F}_{n-1}\big)\Big) = \mathbb{E}_\xi(W_{n-1}|\mathcal{F}_{n-1}) = W_{n-1}.$$

Following [28], for technical reasons we will use the martingale $(W_n, \mathcal{F}_n)$ rather than the more frequently used one $(W_n, \mathcal{F}^0_n)$.

LEMMA 3.1. *Let $\phi : [0,\infty) \to [0,\infty)$ be a convex and increasing function with $\phi(0) = 0$ and $\phi(2x) \leq c\phi(x)$ for some constant $c \in (0,\infty)$ and all $x > 0$. Let $\beta \in (1, 2]$.*

If the function $x \mapsto \phi(x^{1/\beta})$ is convex, then writing $A = \sum_{n=1}^{\infty} 1/\Pi_{n-1}^{\beta-1}$ which is a.s. finite, we have

$$\mathbb{E}_\xi \phi(|W^* - 1|) \le C \sum_{n=1}^{\infty} \left[\mathbb{E}_\xi \left(\frac{1}{A\Pi_{n-1}^{\beta-1}} \phi\Big(A^{1/\beta} W_{n-1}^{1/\beta} \left(\mathbb{E}_\xi |\tilde{X}_{n-1}|^\beta\right)^{1/\beta} \Big) \right) + \mathbb{E}_\xi \phi \left(\frac{|\tilde{X}_{n-1}|}{\Pi_{n-1}^{(\beta-1)/\beta}} \, W_{n-1}^{1/\beta} \right) \right],$$

where $C > 0$ is a constant depending only c and β.

For the proof of Lemma 3.1, we will use the Burkholder - Davis- Gundy (BDG) inequality that we are going to state in the following lemma. For a martingale sequence $\{(f_n, \mathcal{G}_n) : n \ge 1\}$ defined on some probability space $(\Omega, \mathcal{G}, \mathbb{P})$, set $f_0 = 0$, $\mathcal{G}_0 = \{\emptyset, \Omega\}$, $d_n = f_n - f_{n-1}$ for $n \ge 1$,

$$f^* = \sup_{n \ge 1} |f_n| \quad \text{and} \quad d^* = \sup_{n \ge 1} |d_n|.$$

Lemma 3.2 ([15], Theorem 2). *Let $\Phi : [0,\infty) \to [0,\infty)$ be an increasing and continuous function with $\Phi(0) = 0$ and $\Phi(2\lambda) \le c\Phi(\lambda)$ for some $c \in (0,\infty)$ and all $\lambda > 0$.*

(i) *For every $\beta \in (1,2]$, there exists a constant $B = B_{c,\beta} \in (0,\infty)$ depending only on c and β such that for any martingale $\{(f_n, \mathcal{G}_n) : n \ge 1\}$, we have*

$$\mathbb{E}\Phi(f^*) \le B\mathbb{E}\Phi(s(\beta)) + B\mathbb{E}\Phi(d^*) \quad \text{with} \quad s(\beta) = \left(\sum_{n=1}^{\infty} \mathbb{E}\big(|d_n|^\beta | \mathcal{G}_{n-1}\big) \right)^{1/\beta} \tag{3.1}$$

and

$$\mathbb{E}\Phi(f^*) \le B\mathbb{E}\Phi(s(\beta)) + B \sum_{n=1}^{\infty} \mathbb{E}\Phi(|d_n|). \tag{3.2}$$

(ii) *If Φ is convex on $[0,\infty)$, then there exist constants $A = A_c \in (0,\infty)$ and $B = B_c \in (0,\infty)$, depending only on c, such that for any martingale $\{(f_n, \mathcal{G}_n) : n \ge 1\}$, we have*

$$A\mathbb{E}\Phi(S) \le \mathbb{E}\Phi(f^*) \le B\mathbb{E}\Phi(S), \quad \text{where } S = \left(\sum_{n=1}^{\infty} d_n^2 \right)^{1/2};$$

moreover, for any $\beta \in (0,2]$,

$$\mathbb{E}\Phi(f^*) \le B\mathbb{E}\Phi(S(\beta)), \quad \text{where } S(\beta) = \left(\sum_{n=1}^{\infty} |d_n|^\beta \right)^{1/\beta}.$$

If, additionally, for some $\beta \in (0,2]$ the function $\Phi_{1/\beta}(x) = \Phi(x^{1/\beta})$ is subadditive on $[0,\infty)$, then

$$\mathbb{E}\Phi(f^*) \le B \sum_{n=1}^{\infty} \mathbb{E}\Phi(|d_n|).$$

Proof of Lemma 3.1. By the law of large numbers, $\lim_{n\to\infty}(1/\Pi_n)^{1/n} = e^{-E\log m_0} < 1$ a.s., so that $A < \infty$. By (3.2), we have

$$\mathbb{E}_\xi\phi(|W^*-1|) \le B\left(\mathbb{E}_\xi\phi\left(\left(\sum_{n=1}^{\infty}\mathbb{E}_\xi(|D_n|^\beta|\mathcal{F}_{n-1})\right)^{1/\beta}\right) + \sum_{n=1}^{\infty}\mathbb{E}_\xi\phi(|D_n|)\right), \quad (3.3)$$

where $B > 0$ is a constant depending only on c and β.

For fixed $n \ge 1$, let $\tilde{X}(j) = \tilde{X}_{n-1,j}$ for $j \ge 1$. By the fact that $\mathbb{E}_\xi\tilde{X}(i) = 0$ and that $\{\tilde{X}(i)\}$ are independent of each other under $\mathbb{P}_\xi$, we know that under $\mathbb{P}_\xi(\cdot|\mathcal{F}_n)$, $\{\tilde{X}(i), i = 1, 2, \cdots, Z_{n-1}\}$ is a sequence of martingale differences with respect to the natural filtration

$$\tilde{\mathcal{F}}_k := \sigma\{\xi_l, X_{h,i}, \tilde{X}(j) :\ 0 \le l < n-1, 0 \le h < n-1, i \ge 1, 1 \le j \le k\}, \quad k \ge 1.$$

For this martingale difference sequence, using Lemma 3.2 (ii) and the subadditivity of $x \mapsto x^{\beta/2}$, we get

$$\begin{aligned}
\mathbb{E}_\xi(|D_n|^\beta|\mathcal{F}_{n-1}) &= \mathbb{E}_\xi\left(\left(\frac{|\sum_{i=1}^{Z_{n-1}}\tilde{X}_{n-1,i}|}{\Pi_{n-1}}\right)^\beta \middle| \mathcal{F}_{n-1}\right)\\
&\le B\mathbb{E}_\xi\left(\left(\sum_{i=1}^{Z_{n-1}}\frac{|\tilde{X}_{n-1,i}|^2}{\Pi_{n-1}^2}\right)^{\beta/2} \middle| \mathcal{F}_{n-1}\right)\\
&\le B\mathbb{E}_\xi\left(\left(\sum_{i=1}^{Z_{n-1}}\frac{|\tilde{X}_{n-1,i}|^\beta}{\Pi_{n-1}^\beta}\right) \middle| \mathcal{F}_{n-1}\right)\\
&= B\frac{Z_{n-1}}{\Pi_{n-1}^\beta}\mathbb{E}_\xi|\tilde{X}_{n-1}|^\beta = B\frac{W_{n-1}}{\Pi_{n-1}^{\beta-1}}\mathbb{E}_\xi|\tilde{X}_{n-1}|^\beta. \qquad (3.4)
\end{aligned}$$

Since $\phi_{1/\beta}(x) := \phi(x^{1/\beta})$ is increasing and convex, and $\sum_{n=1}^\infty \frac{1}{A\Pi_{n-1}^{\beta-1}} = 1$, from (3.4) we get

$$\begin{aligned}
&\mathbb{E}_\xi\phi\left(\left(\sum_{n=1}^{\infty}E_\xi(|D_n|^\beta|\mathcal{F}_{n-1})\right)^{1/\beta}\right) \le \mathbb{E}_\xi\phi_{1/\beta}\left(\sum_{n=1}^{\infty}\frac{1}{A\Pi_{n-1}^{\beta-1}}ABW_{n-1}\mathbb{E}_\xi|\tilde{X}_{n-1}|^\beta\right)\\
&\le \mathbb{E}_\xi\sum_{n=1}^{\infty}\frac{1}{A\Pi_{n-1}^{\beta-1}}\phi_{1/\beta}\Big(ABW_{n-1}\mathbb{E}_\xi|\tilde{X}_{n-1}|^\beta\Big)\\
&\le C\sum_{n=1}^{\infty}\mathbb{E}_\xi\frac{1}{A\Pi_{n-1}^{\beta-1}}\phi\Big(A^{1/\beta}W_{n-1}^{1/\beta}\big(\mathbb{E}_\xi|\tilde{X}_{n-1}|^\beta\big)^{1/\beta}\Big), \qquad (3.5)
\end{aligned}$$

where $C > 0$ is a constant depending only on B and c. The last inequality holds since for any constant $c_1 > 0$, we have $\phi(c_1x) \le c_2\phi(x)$ for $c_2 = c^{n_1}$ and any $x > 0$, where n_1 is an integer such that $c_1 \le 2^{n_1}$; the last inequality follows from the condition that $\phi(2x) \le c\phi(x)$ for all $x > 0$ and that $\phi(x)$ is increasing, which imply that $\phi(c_1x) \le \phi(2^{n_1}x) \le c^{n_1}\phi(x)$. This gives the desired bound for the first part of (3.3). For the second part of (3.3), by Lemma 3.2 (ii) and the convexity of $\phi_{1/\beta}(x) = \phi(x^{1/\beta})$, we obtain

$$\begin{aligned}
\mathbb{E}_\xi \phi(|D_n|) &= \mathbb{E}_\xi\left(\phi\left(\left|\frac{1}{\Pi_{n-1}}\sum_{i=1}^{Z_{n-1}}\tilde{X}_{n-1,i}\right|\right)\right) \le B\mathbb{E}_\xi\phi\left(\left(\sum_{i=1}^{Z_{n-1}}\frac{|\tilde{X}_{n-1,i}|^\beta}{\Pi_{n-1}^\beta}\right)^{1/\beta}\right)\\
&= B\mathbb{E}_\xi\phi_{1/\beta}\left(\sum_{i=1}^{Z_{n-1}}\frac{|\tilde{X}_{n-1,i}|^\beta}{\Pi_{n-1}^\beta}\right) = B\mathbb{E}_\xi\phi_{1/\beta}\left(\sum_{i=1}^{Z_{n-1}}\frac{1}{Z_{n-1}}Z_{n-1}\frac{|\tilde{X}_{n-1,i}|^\beta}{\Pi_{n-1}^\beta}\right)\\
&\le B\mathbb{E}_\xi\sum_{i=1}^{Z_{n-1}}\frac{1}{Z_{n-1}}\phi_{1/\beta}\left(Z_{n-1}\frac{|\tilde{X}_{n-1,i}|^\beta}{\Pi_{n-1}^\beta}\right)\\
&= B\mathbb{E}_\xi\phi\left(\frac{|\tilde{X}_{n-1}|}{\Pi_{n-1}^{(\beta-1)/\beta}}W_{n-1}^{1/\beta}\right). \qquad (3.6)
\end{aligned}$$

The conclusion of Lemma 3.1 then follows from (3.3), (3.5) and (3.6).

4. Proof of main result. For the proof of our main result Theorem 1.1, we will use the following Lemmas.

LEMMA 4.1 ([19], Theorem 1.1). *Let (Z_n) be a branching process in an i.i.d. random environment with $0 < \mathbb{E}\log m_0 < \infty$. Then for each fixed $p > 1$, $0 < \mathbb{E}_\xi W^p < \infty$ a.s. if and only if $\mathbb{E}\log^+\mathbb{E}_\xi W_1^p < \infty$.*

LEMMA 4.2. *Let X be a non negative random variable, l be a function slowly varying at ∞ and $\phi(x) = x^\alpha l(x)$ with $\alpha > 1$. The following assertions are equivalent:*
(i) $\mathbb{E}\log^+\mathbb{E}_\xi\phi(X) < \infty$;
(ii) $\mathbb{E}\log^+\mathbb{E}_\xi\phi(|X-c|) < \infty$, *where $c > 0$ is a constant.*

Proof. Recall that we can take l as the form (1.2) with $a_0 = 1$ and $c(x) = 1$ for all $x \ge 1$. Consequently, we have for $x > c+1$,

$$\frac{l(x-c)}{l(x)} = \exp\left(-\int_{x-c}^{x}\frac{\varepsilon(t)}{t}dt\right),$$

where $c > 0$ is a constant. Because $\varepsilon(x) \to 0$ as $x \to \infty$, so for $\delta > 0$, there exists $x_0 > c+1$ such that $-\delta < \varepsilon(x) < \delta$ for all $x > x_0$. Therefore for $x > x_0$,

$$-\delta\log\frac{x}{x-c} < \int_{x-c}^{x}\frac{\varepsilon(t)}{t}dt < \delta\log\frac{x}{x-c},$$

so that $\lim_{x\to\infty}\int_{x-c}^{x}\frac{\varepsilon(t)}{t}dt = 0$. Therefore

$$\lim_{x\to\infty}\frac{l(x-c)}{l(x)} = 1. \qquad (4.1)$$

The conclusion of the lemma then follows easily from (4.1) together with the following elementary inequalities: for all $a, b > 0$,

$$(a+b)^\alpha \le 2^{\alpha-1}(a^\alpha + b^\alpha) \qquad (4.2)$$

and

$$\log^+(a+b) \le 1 + \log^+ a + \log^+ b. \qquad (4.3)$$

(To see the last inequality (4.3), it suffices to notice that if $a \geq 1$ and $b \geq 1$, then $a+b \leq 2ab$; if $a < 1$ and $b < 1$, then $a+b \leq 2$; if $a < 1$ and $b \geq 1$, then $a+b \leq 2b$; if $a \geq 1$ and $b < 1$, then $a+b \leq 2a$.)

LEMMA 4.3 ([19], Lemma 3.1). *Let $(\alpha_n, \beta_n)_{n\geq 0}$ be a stationary and ergodic sequence of non-negative random variables. If $\mathbb{E}\log\alpha_0 < 0$ and $\mathbb{E}\log^+\beta_0 < \infty$, then*

$$\sum_{n=0}^{\infty} \alpha_0 \cdots \alpha_{n-1}\beta_n < \infty \quad a.s.. \tag{4.4}$$

Conversely, when $(\alpha_n, \beta_n)_{n\geq 0}$ are i.i.d. and $\mathbb{E}\log\alpha_0 \in (-\infty, 0)$, then (4.4) implies that $\mathbb{E}\log^+\beta_0 < \infty$.

See also [17] and [24] for a discussion about the convergence of the series (4.4). We are now ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $\beta \in (1, 2]$ and $\beta < \alpha$. Write $\phi(x) = x^\alpha l(x)$, by Lemma 2.2, we can assume that the functions ϕ and $x \mapsto \phi(x^{1/\beta})$ are convex on $[0, \infty)$, and $l(x) > 0$ for all $x \geq 0$. Moreover, by choosing a smoothed version if necessary, we can suppose that l is differentiable.

The equivalence between (ii) and (iii) follows from Theorem 2.1. The rest of the proof is composed of the following two parts.

Part 1: prove that (i) implies (iii). Suppose that $\mathbb{E}\log^+\mathbb{E}_\xi\phi(W_1) < \infty$. By Lemma 3.1, we get

$$\mathbb{E}_\xi\phi(|W^* - 1|) \leq C\sum_{n=1}^{\infty}\big(I_1(n) + I_2(n)\big), \tag{4.5}$$

where

$$I_1(n) = \mathbb{E}_\xi\bigg(\frac{1}{A\Pi_{n-1}^{\beta-1}}\phi\Big(A^{1/\beta}W_{n-1}^{1/\beta}\big(\mathbb{E}_\xi|\widetilde{X}_{n-1}|^\beta\big)^{1/\beta}\Big)\bigg),$$
$$I_2(n) = \mathbb{E}_\xi\phi\bigg(\frac{|\widetilde{X}_{n-1}|}{\Pi_{n-1}^{(\beta-1)/\beta}}W_{n-1}^{1/\beta}\bigg).$$

Hence, in order to prove $\mathbb{E}_\xi\phi(|W^* - 1|) < \infty$ a.s., we only need to prove that $\sum_{n=1}^{\infty} I_1(n) < \infty$ and $\sum_{n=1}^{\infty} I_2(n) < \infty$ a.s..

We first prove that $\sum_{n=1}^{\infty} I_1(n) < \infty$ a.s.. Since l is bounded away from 0 and ∞ on any compact subset of $[0, \infty)$, by Potter's theorem (see [11]), for $\delta > 0$, there exists $C = C(l, \delta) > 1$ such that $l(x) \leq C\max(x^\delta, x^{-\delta}) \leq C(x^\delta + x^{-\delta})$ for all $x > 0$. Hence

$$\begin{aligned} I_1(n) &= \Pi_{n-1}^{1-\beta}A^{\alpha/\beta-1}\mathbb{E}_\xi\bigg(W_{n-1}^{\alpha/\beta}\Big(\mathbb{E}_\xi\big(|\tilde{X}_{n-1}|^\beta\big)\Big)^{\alpha/\beta} l\Big(A^{1/\beta}W_{n-1}^{1/\beta}\Big(\mathbb{E}_\xi|\tilde{X}_{n-1}|^\beta\Big)^{1/\beta}\Big)\bigg) \\ &\leq C(I_1^+(n) + I_1^-(n)), \end{aligned}$$

where

$$I_1^+(n) = \Pi_{n-1}^{1-\beta}A^{(\alpha+\delta)/\beta-1}\mathbb{E}_\xi W_{n-1}^{(\alpha+\delta)/\beta}\big(\mathbb{E}_\xi|\tilde{X}_{n-1}|^\beta\big)^{(\alpha+\delta)/\beta},$$
$$I_1^-(n) = \Pi_{n-1}^{1-\beta}A^{(\alpha-\delta)/\beta-1}\mathbb{E}_\xi W_{n-1}^{(\alpha-\delta)/\beta}\big(\mathbb{E}_\xi|\tilde{X}_{n-1}|^\beta\big)^{(\alpha-\delta)/\beta}.$$

Choose $\delta_1 > 0$ and $\delta_2 > 0$ small enough such that $\alpha - \delta_1 > \beta$, $\beta - 1 - 2\delta_2 > 0$, and let $\delta = \min\{\delta_1, \delta_2\}$. Then $\alpha - \delta > \beta$ and $\beta - 1 - 2\delta > 0$. Writing $Z_{n-1}^{(\alpha+\delta)/\beta} = XY$ with $X = Z_{n-1}^{(\alpha+\delta-\beta+1)/\beta}$, $Y = Z_{n-1}^{(\beta-1)/\beta}$, and using Hölder's inequality $\mathbb{E}_\xi(XY) \le (\mathbb{E}_\xi X^\beta)^{1/\beta}(\mathbb{E}_\xi Y^{\beta^*})^{1/\beta^*}$ with $\frac{1}{\beta} + \frac{1}{\beta^*} = 1$, together with the fact that $X^\beta = Z_{n-1}^{\alpha+\delta-\beta+1} \le Z_{n-1}^{\alpha-\delta}$ (since $\alpha+\delta-\beta+1 < \alpha-\delta$) and $\mathbb{E}_\xi(Y^{\beta^*}) = \mathbb{E}_\xi Z_{n-1} = \Pi_{n-1}$, we obtain (see also (4.3) of [28]),

$$\mathbb{E}_\xi Z_{n-1}^{(\alpha+\delta)/\beta} \le \Pi_{n-1}^{(\alpha+\beta-1-\delta)/\beta}(\mathbb{E}_\xi W_{n-1}^{\alpha-\delta})^{1/\beta},$$

which also reads

$$\mathbb{E}_\xi W_{n-1}^{(\alpha+\delta)/\beta} \le \Pi_{n-1}^{(\beta-1-2\delta)/\beta}(\mathbb{E}_\xi W_{n-1}^{\alpha-\delta})^{1/\beta}. \tag{4.6}$$

By Jensen's inequality, we have

$$\mathbb{E}_\xi|\tilde{X}_{n-1}|^\beta = \mathbb{E}_\xi|\tilde{X}_{n-1}|^{(\alpha-\delta)\times\frac{\beta}{\alpha-\delta}} \le (\mathbb{E}_\xi|\tilde{X}_{n-1}|^{\alpha-\delta})^{\frac{\beta}{\alpha-\delta}}. \tag{4.7}$$

By (4.6) and (4.7), we obtain

$$\begin{aligned} I_1^+(n) &\le \Pi_{n-1}^{-((\beta-1)^2+2\delta)/\beta} A^{(\alpha+\delta)/\beta-1}(\mathbb{E}_\xi W_{n-1}^{\alpha-\delta})^{1/\beta}(\mathbb{E}_\xi|\tilde{X}_{n-1}|^\beta)^{(\alpha+\delta)/\beta} \\ &\le \Pi_{n-1}^{-((\beta-1)^2+2\delta)/\beta} A^{(\alpha+\delta)/\beta-1}(\mathbb{E}_\xi W_{n-1}^{\alpha-\delta})^{1/\beta}(\mathbb{E}_\xi|\tilde{X}_{n-1}|^{\alpha-\delta})^{(\alpha+\delta)/(\alpha-\delta)}. \end{aligned} \tag{4.8}$$

By Potter's theorem, for $\delta > 0$, there exists $C = C(l, \delta) > 0$ such that $l(x) \ge Cx^{-\delta}$ for all $x \ge 1$, so that $x^{\alpha-\delta} \le 1 + \frac{1}{C}x^\alpha l(x) = 1 + \frac{1}{C}\phi(x)$ for all $x \ge 0$. Therefore

$$\mathbb{E}_\xi W_1^{\alpha-\delta} \le 1 + \frac{1}{C}\mathbb{E}_\xi\phi(W_1|). \tag{4.9}$$

Together with the elementary inequalities $\log^+(1+xy) \le 1 + \log^+(xy) \le 1 + \log^+ x + \log^+ y$, this implies

$$\begin{aligned} \mathbb{E}\log^+\mathbb{E}_\xi W_1^{\alpha-\delta} &\le \mathbb{E}\log^+\left(1 + \frac{1}{C}\mathbb{E}_\xi\phi(W_1)\right) \\ &\le 1 + \log^+\frac{1}{C} + \mathbb{E}\log^+\mathbb{E}_\xi W_1^\alpha l(W_1) < \infty. \end{aligned}$$

Since $\mathbb{E}\log^+\mathbb{E}_\xi W_1^{\alpha-\delta} < \infty$ and $\mathbb{E}\log m_0 < \infty$, by Lemma 4.1 we see that $\mathbb{E}_\xi W^{\alpha-\delta} < \infty$, which implies that

$$\sup_{n\ge1}\mathbb{E}_\xi W_{n-1}^{\alpha-\delta} < \infty. \tag{4.10}$$

By (4.8), we get

$$\begin{aligned} &\sum_{n=1}^\infty I_1^+(n) \le \sum_{n=1}^\infty \Pi_{n-1}^{-((\beta-1)^2+2\delta)/\beta} A^{(\alpha+\delta)/\beta-1}(\mathbb{E}_\xi W_{n-1}^{\alpha-\delta})^{1/\beta}\left(\mathbb{E}_\xi|\tilde{X}_{n-1}|^{\alpha-\delta}\right)^{(\alpha+\delta)/(\alpha-\delta)} \\ &\le \sup_{n\ge1}(\mathbb{E}_\xi W_{n-1}^{\alpha-\delta})^{1/\beta} A^{(\alpha+\delta)/\beta-1}\sum_{n=1}^\infty \Pi_{n-1}^{-((\beta-1)^2+2\delta)/\beta}\left(\mathbb{E}_\xi|\tilde{X}_{n-1}|^{\alpha-\delta}\right)^{(\alpha+\delta)/(\alpha-\delta)}. \end{aligned}$$

By notation, $\tilde{X}_0 = W_1 - 1$. So by Lemma 4.2, $\mathbb{E}\log^+\mathbb{E}_\xi|X_0|^{\alpha-\delta} < \infty$ if and only if $\mathbb{E}\log^+\mathbb{E}_\xi W_1^{\alpha-\delta} < \infty$. Therefore, since $\mathbb{E}\log 1/m_0 < 0$ and $\mathbb{E}\log^+\mathbb{E}_\xi W_1^{\alpha-\delta} < \infty$, by Lemma 4.3 we get

$$\sum_{n=1}^\infty I_1^+(n) < \infty \quad a.s.. \tag{4.11}$$

We now use a similar argument to estimate $I_1^-(n)$. By Jensen's inequality and (4.7), we have

$$\begin{aligned} I_1^-(n) &\le \Pi_{n-1}^{1-\beta}\, A^{(\alpha-\delta)/\beta}(\mathbb{E}_\xi W_{n-1}^{\alpha-\delta})^{1/\beta}\big(\mathbb{E}_\xi|\tilde{X}_{n-1}|^\beta\big)^{(\alpha-\delta)/\beta} \\ &\le \Pi_{n-1}^{1-\beta}\, A^{(\alpha-\delta)/\beta}(\mathbb{E}_\xi W_{n-1}^{\alpha-\delta})^{1/\beta}\mathbb{E}_\xi|\tilde{X}_{n-1}|^{\alpha-\delta}. \end{aligned}$$

It follows that

$$\begin{aligned} \sum_{n=1}^\infty I_1^-(n) &\le \sum_{n=1}^\infty \Pi_{n-1}^{1-\beta} A^{(\alpha-\delta)/\beta}(\mathbb{E}_\xi W_{n-1}^{\alpha-\delta})^{1/\beta}\mathbb{E}_\xi|\tilde{X}_{n-1}|^{\alpha-\delta} \\ &\le A^{(\alpha-\delta)/\beta}\sup_{n\ge1}(\mathbb{E}_\xi W_{n-1}^{\alpha-\delta})^{1/\beta}\sum_{n=1}^\infty \Pi_{n-1}^{1-\beta}\mathbb{E}_\xi|\tilde{X}_{n-1}|^{\alpha-\delta}. \end{aligned}$$

Therefore, as in the preceding, since $\mathbb{E}\log m_0 > 0$, $\mathbb{E}\log^+\mathbb{E}_\xi W_1^{\alpha-\delta} < \infty$ and $\tilde{X}_0 = W_1 - 1$, using Lemmas 4.2 and 4.3 we get

$$\sum_{n=1}^\infty I_1^-(n) < \infty \quad a.s.. \tag{4.12}$$

By (4.11) and (4.12), we obtain

$$\sum_{n=1}^\infty I_1(n) < \infty \quad a.s.. \tag{4.13}$$

We next prove that $\sum_{n=1}^\infty I_2(n) < \infty$. By Potter's theorem, for $\delta > 0$, there exists $C = C(l,\delta) > 0$ such that $l(xy) \le Cl(x)\max\{y^\delta, y^{-\delta}\} \le Cl(x)(y^\delta + y^{-\delta})$ for all $x > 0, y > 0$. Using this and the independence between W_{n-1} and $\tilde{X}_{n-1}$ under $\mathbb{P}_\xi$, we obtain

$$\begin{aligned} I_2(n) &= \Pi_{n-1}^{-\alpha(\beta-1)/\beta}\mathbb{E}_\xi\left(W_{n-1}^{\alpha/\beta}|\tilde{X}_{n-1}|^\alpha l\left(\left(\frac{W_{n-1}}{\Pi_{n-1}^{\beta-1}}\right)^{1/\beta}|\tilde{X}_{n-1}|\right)\right) \\ &\le C(I_2^+(n) + I_2^-(n)), \end{aligned}$$

where $C = C(l,\delta,\beta)$ is a constant depending only on l, δ and β,

$$\begin{aligned} I_2^+(n) &= \Pi_{n-1}^{-(\alpha+\delta)(\beta-1)/\beta}\mathbb{E}_\xi W_{n-1}^{(\alpha+\delta)/\beta}\mathbb{E}_\xi\phi(|\tilde{X}_{n-1}|), \\ I_2^-(n) &= \Pi_{n-1}^{-(\alpha-\delta)(\beta-1)/\beta}\mathbb{E}_\xi W_{n-1}^{(\alpha-\delta)/\beta}\mathbb{E}_\xi\phi(|\tilde{X}_{n-1}|). \end{aligned}$$

In order to prove that $\sum_{n=1}^\infty I_2(n) < \infty$, we only need to prove $\sum_{n=1}^\infty I_2^+(n) < \infty$ and $\sum_{n=1}^\infty I_2^-(n) < \infty$. We now consider $\sum_{n=1}^\infty I_2^+(n)$. By (4.6),

$$\sum_{n=1}^\infty I_2^+(n) \le \sup_{n\ge1}\big(\mathbb{E}_\xi W_{n-1}^{\alpha-\delta}\big)^{1/\beta}\sum_{n=1}^\infty \Pi_{n-1}^{-[(\alpha+\delta-1)(\beta-1)+2\delta]/\beta}\mathbb{E}_\xi\phi(|\tilde{X}_{n-1}|).$$

Since $\mathbb{E}\log^+\mathbb{E}_\xi\phi(W_1) < \infty$, from Lemma 4.2 and the fact that $\tilde{X}_0 = W_1 - 1$, we see that $\mathbb{E}\log^+\mathbb{E}_\xi\phi(|\tilde{X}_0|) < \infty$. Using this together with the condition $\mathbb{E}\log m_0 > 0$, by lemma 4.3 and the above display we get

$$\sum_{n=1}^\infty I_2^+(n) < \infty \quad a.s.. \tag{4.14}$$

We now consider $\sum_{n=1}^{\infty} I_2^-(n)$. By Jensen's inequality, we get

$$I_2^-(n) \leq \Pi_{n-1}^{-(\alpha-\delta)(\beta-1)/\beta}(\mathbb{E}_\xi W_{n-1}^{\alpha-\delta})^{1/\beta}\mathbb{E}_\xi\phi(|\tilde{X}_{n-1}|).$$

Since $\mathbb{E}\log^+\mathbb{E}_\xi W_1^{\alpha-\delta} < \infty$, by Lemma 4.1 we get $\sup_{n\geq 1}(\mathbb{E}_\xi W_{n-1}^{\alpha-\delta})^{1/\beta} < \infty$; by Lemma 4.2 and the condition $\mathbb{E}\log^+\mathbb{E}_\xi\phi(W_1) < \infty$, we get $\mathbb{E}\log^+\mathbb{E}_\xi\phi(|\tilde{X}_0|) < \infty$. Using this and the condition $\mathbb{E}\log m_0 > 0$, by lemma 4.3 and the above inequality on $I_2^-(n)$ we obtain

$$\sum_{n=1}^{\infty} I_2^-(n) \leq \sup_{n\geq 1}(\mathbb{E}_\xi W_{n-1}^{\alpha-\delta})^{1/\beta}\sum_{n=1}^{\infty}\Pi_{n-1}^{-(\alpha-\delta)(\beta-1)/\beta}\mathbb{E}_\xi\phi(|\tilde{X}_{n-1}|) < \infty \quad a.s.. \tag{4.15}$$

By (4.14) and (4.15), we finally get

$$\sum_{n=1}^{\infty} I_2(n) < \infty \quad a.s.. \tag{4.16}$$

Combining (4.5), (4.13) and (4.16), we obtain $\mathbb{E}_\xi\phi(|W^* - 1|) < \infty$ a.s., which is equivalent to $\mathbb{E}_\xi\phi(W^*) < \infty$ a.s. by Lemma 4.2. Thus we have proved that (i) implies (iii).

Part 2: prove that (iii) implies (i). Assume that $\mathbb{E}_\xi\phi(W^*) < \infty$ a.s. We only need to prove that $\mathbb{E}\log\mathbb{E}_\xi\phi(|\tilde{X}_0|) < \infty$ (recall that $\tilde{X}_0 = W_1 - 1$). We divide the proof into two cases, according to $\alpha > 2$ and $1 < \alpha \leq 2$.

(a) We first consider the case $\alpha > 2$. By Lemma 3.2(ii) and the convexity of $\phi_{1/2}(x)$, we obtain

$$\begin{aligned}\mathbb{E}_\xi\phi(|W^* - 1|) &\geq A\mathbb{E}_\xi\phi\left(\left(\sum_{n=1}^{\infty}|D_n|^2\right)^{1/2}\right)\\ &\geq A\sum_{n=1}^{\infty}\mathbb{E}_\xi\phi\big((|D_n|^2)^{1/2}\big) = A\sum_{n=1}^{\infty}\mathbb{E}_\xi\phi(|D_n|\big).\end{aligned}$$

For the same reason, we obtain

$$\begin{aligned}\mathbb{E}_\xi(\phi(|D_n|)|\mathcal{F}_{n-1}) &= \mathbb{E}_\xi\phi\big(|\frac{1}{\Pi_{n-1}}\sum_{i=1}^{Z_{n-1}}\tilde{X}_{n-1,i}||\mathcal{F}_{n-1}\big)\\ &\geq A\mathbb{E}_\xi\phi\big((\sum_{i=1}^{Z_{n-1}}\frac{|\tilde{X}_{n-1,i}|^2}{\Pi_{n-1}^2})^{1/2}|\mathcal{F}_{n-1}\big)\\ &\geq A\mathbb{E}_\xi\big(\sum_{i=1}^{Z_{n-1}}\phi(\frac{|\tilde{X}_{n-1,i}|}{\Pi_{n-1}})|\mathcal{F}_{n-1}\big) = AZ_{n-1}\mathbb{E}_\xi\big(\phi(\frac{|\tilde{X}_{n-1}|}{\Pi_{n-1}})|\mathcal{F}_{n-1}\big),\end{aligned}$$

so that

$$\mathbb{E}_\xi\phi(|D_n|) \geq A\mathbb{E}_\xi Z_{n-1}\mathbb{E}_\xi\left(\phi\left(\frac{|\tilde{X}_{n-1}|}{\Pi_{n-1}}\right)\right) = A\Pi_{n-1}\mathbb{E}_\xi\left(\phi\left(\frac{|\tilde{X}_{n-1}|}{\Pi_{n-1}}\right)\right).$$

Choose $\delta > 0$ small enough such that $1-\alpha+\delta < 0$. By Potter's theorem, there exists $C = C(l,\delta)$ such that $l(xy) \geq Cl(x)\min\{y^\delta, y^{-\delta}\}$, for all $x > 0$ and $y > 0$. Hence, we have

$$\begin{aligned}\mathbb{E}_\xi\phi(|W^*-1|) &\geq \sum_{n=1}^{\infty} A\Pi_{n-1}\mathbb{E}_\xi\left(\phi\left(\frac{|\tilde{X}_{n-1}|}{\Pi_{n-1}}\right)\right)\\ &= \sum_{n=1}^{\infty} A\Pi_{n-1}\mathbb{E}_\xi\Pi_{n-1}^{-\alpha}|\tilde{X}_{n-1}|^{\alpha} l(|\tilde{X}_{n-1}|\Pi_{n-1})\\ &\geq \sum_{n=1}^{\infty} AC\Pi_{n-1}^{1-\alpha}\mathbb{E}_\xi|\tilde{X}_{n-1}|^{\alpha} l(|\tilde{X}_{n-1}|)\min\left(\Pi_{n-1}^{\delta},\Pi_{n-1}^{-\delta}\right)\\ &= AC\sum_{n=1}^{\infty}\min\left(\Pi_{n-1}^{1-\alpha-\delta},\Pi_{n-1}^{1-\alpha+\delta}\right)\mathbb{E}_\xi\phi(|\tilde{X}_{n-1}|).\end{aligned}$$

By Lemma 4.2, we know that if $\mathbb{E}_\xi\phi(W^*)<\infty$, then $\mathbb{E}_\xi\phi(|W^*-1|)<\infty$. Since $0<\mathbb{E}\log m_0<\infty$, by the law of large numbers there is $a\in(0,\infty)$ such that a.s. $\Pi_{n-1}\geq e^{(n-1)a}$ for $n\geq n(\xi)$ large enough. Therefore the preceding lower bound of $\mathbb{E}_\xi\phi(|W^*-1|)$ implies that

$$\sum_{n=1}^{\infty} e^{-(n-1)a(\alpha-1+\delta)}\,\mathbb{E}_\xi\phi(|\tilde{X}_{n-1}|)<\infty \quad a.s..$$

Hence by Lemma 4.3 we obtain $\mathbb{E}\log^+\mathbb{E}_\xi\phi(|\tilde{X}_0|)<\infty$, which implies $\mathbb{E}\log^+\mathbb{E}_\xi\phi(W_1)<\infty$ by Lemma 4.2.

(b) We next consider the case where $1<\alpha\leq 2$. Let $F=\sum_{n=1}^{\infty}\frac{1}{\Pi_{n-1}}$. By Lemma 3.2(ii) and the concavity of $\phi_{1/2}(x)$, we have

$$\begin{aligned}\mathbb{E}_\xi\phi(|W^*-1|) &\geq A\mathbb{E}_\xi\left(\phi\left(\left(\sum_{n=1}^{\infty}D_n^2\right)^{1/2}\right)\right) = A\mathbb{E}_\xi\phi_{1/2}\left(\sum_{n=1}^{\infty}\frac{1}{F\Pi_{n-1}}F\Pi_{n-1}D_n^2\right)\\ &\geq A\mathbb{E}_\xi\sum_{n=1}^{\infty}\frac{1}{F\Pi_{n-1}}\phi_{1/2}\left(F\Pi_{n-1}D_n^2\right)\\ &= A\sum_{n=1}^{\infty}\frac{1}{F\Pi_{n-1}}\mathbb{E}_\xi\phi\left(F^{1/2}\Pi_{n-1}^{1/2}|D_n|\right).\end{aligned}$$

Again by Lemma 3.2(ii) and the concavity of $\phi_{1/2}(x)$, we have

$$\begin{aligned}&\mathbb{E}_\xi\phi\left(F^{1/2}\Pi_{n-1}^{1/2}|D_n|\right) = \mathbb{E}_\xi\phi\left(F^{1/2}\Pi_{n-1}^{1/2}\left|\sum_{i=1}^{Z_{n-1}}\frac{\tilde{X}_{n-1,i}}{\Pi_{n-1}}\right|\right)\\ &\geq \mathbb{E}_\xi\phi\left(\left(\sum_{i=1}^{Z_{n-1}}F\Pi_{n-1}\frac{|\tilde{X}_{n-1,i}|^2}{\Pi_{n-1}^2}\right)^{1/2}\right) = \mathbb{E}_\xi\phi_{1/2}\left(\sum_{i=1}^{Z_{n-1}}F\Pi_{n-1}\frac{|\tilde{X}_{n-1,i}|^2}{\Pi_{n-1}^2}\right)\\ &= \mathbb{E}_\xi\phi_{1/2}\left(\sum_{i=1}^{Z_{n-1}}\frac{1}{Z_{n-1}}F\Pi_{n-1}\frac{Z_{n-1}|\tilde{X}_{n-1,i}|^2}{\Pi_{n-1}^2}\right)\\ &\geq \mathbb{E}_\xi\sum_{i=1}^{Z_{n-1}}\frac{1}{Z_{n-1}}\phi_{1/2}\left(F\Pi_{n-1}\frac{Z_{n-1}|\tilde{X}_{n-1}|^2}{\Pi_{n-1}^2}\right) = \mathbb{E}_\xi\phi\left(F^{1/2}W_{n-1}^{1/2}|\tilde{X}_{n-1}|\right).\end{aligned}$$

By Potter's theorem, for $\delta > 0$, there exists $C = C(l, \delta) > 1$ such that for all $x > 0, y > 0$, $l(xy) \geq C\, l(y) \min\{x^{\delta}, x^{-\delta}\}$. Therefore

$$\begin{aligned}
&\mathbb{E}_\xi \phi(|W^* - 1|) \\
&\geq A \sum_{n=1}^{\infty} \frac{1}{F\Pi_{n-1}} \mathbb{E}_\xi \min\left(F^{(\alpha+\delta)/2} W_{n-1}^{(\alpha+\delta)/2}, F^{(\alpha-\delta)/2} W_{n-1}^{(\alpha-\delta)/2}\right) \mathbb{E}_\xi \phi(|\tilde{X}_{n-1}|) \\
&\geq \frac{A}{F}\, \mathbb{E}_\xi(U) \sum_{n=1}^{\infty} \frac{1}{\Pi_{n-1}} \mathbb{E}_\xi \phi(|\tilde{X}_{n-1}|),
\end{aligned}$$

where $U = \min\left(F^{(\alpha+\delta)/2} \inf_{n\geq 1} W_{n-1}^{(\alpha+\delta)/2}, F^{(\alpha-\delta)/2} \inf_{n\geq 1} W_{n-1}^{(\alpha-\delta)/2}\right)$. Notice that when $W > 0$, then $W_n > 0$ for all $n \geq n_0$ with some $n_0 = n_0(\omega)$ large enough. Since $Z_k = 0$ implies $Z_n = 0$ for all $n > k$, it follows that $\inf_{n\geq 0} W_n > 0$ a.s. on $\{W > 0\}$. Hence $U > 0$ a.s. on $\{W > 0\}$. Since $P_\xi(W > 0) > 0$, this implies that $\mathbb{E}_\xi U > 0$. Therefore $\mathbb{E}_\xi \phi(|W^* - 1|) < \infty$ implies that

$$\sum_{n=1}^{\infty} \frac{1}{\Pi_{n-1}} \mathbb{E}_\xi \phi(|\tilde{X}_{n-1}|) < \infty \quad a.s..$$

Since $\mathbb{E}_\xi \phi(W^*) < \infty$, by Lemma 4.2 we have $\mathbb{E}_\xi \phi(|W^* - 1|) < \infty$. Hence, from the condition $0 < \mathbb{E} \log m_0 < \infty$ together with Lemma 4.3 and the above implication, we get $\mathbb{E} \log^+ \mathbb{E}_\xi \phi(|\tilde{X}_0|) < \infty$, which implies that $\mathbb{E} \log^+ \mathbb{E}_\xi \phi(W_1) < \infty$ again by Lemma 4.2. Thus we have proved that (iii) implies (i).

The proof of Theorem 1.1 is then finished.

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