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F-MANIFOLDS, MULTI-FLAT STRUCTURES AND PAINLEVÉ TRANSCENDENTS*

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Abstract. In this paper we study F-manifolds equipped with multiple flat connections and multiple F-products, that are required to be compatible in a suitable sense. Multi-flat F-manifolds are the analogue for F-manifolds of Frobenius manifolds with multi-Hamiltonian structures.

In the semisimple case, we show that a necessary condition for the existence of such multiple flat connections can be expressed in terms of the integrability of a distribution of vector fields that are related to the eventual identities for the multiple products involved. These vector fields satisfy the commutation relations of the centerless Virasoro algebra. We prove that the distributions associated to bi-flat and tri-flat F-manifolds are integrable, while in other cases they are maximally non-integrable. Using this fact we show that there can not be non-trivial semisimple multi-flat structures with more than three flat connections. When the relevant distributions are integrable, coupling the invariants of the foliations they determine with Tsarev's conditions, we construct bi-flat and tri-flat semisimple F-manifolds in dimension 3. In particular we obtain a parameterization of three-dimensional bi-flat F-manifolds in terms of a system of six first order ODEs that can be reduced to the full family of P_{VI} equations.

In the second part of the paper we study the non-semisimple case. We show that three-dimensional regular non-semisimple bi-flat F-manifolds are locally parameterized by solutions of the full P_{IV} and P_V equations, according to the Jordan normal form of the endomorphism $L = E\circ$. As a consequence, combining this result with the result of the first part on the semisimple case we have that confluences of P_{IV}, P_V and P_{VI} correspond to collisions of eigenvalues of L preserving the regularity. Furthermore, we show that, contrary to the semisimple situation, it is possible to construct regular non-semisimple multi-flat F-manifolds, with any number of compatible flat connections.

Key words. Multi-flat F-manifolds, Painlevé transcendents.

Mathematics Subject Classification. 53D45, 34M55.

1. Introduction.

F-manifolds have been introduced in [21] as a unifying geometric scheme that encompasses several areas of modern Mathematics, ranging from the theory of Frobenius manifolds to special solutions of the oriented associativity equations ([29]), from quantum K-theory ([24]) to differential-graded deformation theory ([32]).

An F-manifold M is a smooth (or analytic) manifold equipped with a commutative and associative product $\circ : TM \times TM \to TM$ on sections of the tangent bundle TM, such that $\circ$ is $C(M)$-bilinear ($C(M)$ is the ring of smooth or analytic functions on M) and such that

$$P_{X\circ Y}(Z,W) = X \circ P_Y(Z,W) + Y \circ P_X(Z,W), \tag{1.1}$$

where $P_X(Z,W) := [X, Z\circ W] - [X,Z]\circ W - Z\circ[X,W]$. The condition (1.1) is usually called the Hertling-Manin condition and it implies that the deviation of the structure $(TM, \circ, [\cdot,\cdot])$ from that of a Poisson algebra on $(TM, \circ)$ is not arbitrary. Usually M is also required to be equipped with a distinguished vector field e, called unity or identity, such that for every vector field X, $X \circ e = X$.

Since the operation $\circ$ is $C(M)$-bilinear and commutative, it can be identified with a tensor field $c : S^2(TM) \to TM$. Once c is locally written in a coordinate

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system as $c^i_{jk} :=< c(\partial_j, \partial_k), dx^i >$, then the commutativity, the associativity and the Hertling-Manin condition (1.1) translate respectively as

$$c^i_{jk} = c^i_{kj},$$
$$c^i_{jl}c^l_{km} = c^i_{kl}c^l_{jm},$$
$$c^s_{im}\partial_s c^k_{jl} + c^k_{sl}\partial_j c^s_{im} - c^s_{jl}\partial_s c^k_{im} - c^k_{sm}\partial_i c^s_{jl} - c^k_{si}\partial_l c^s_{jm} - c^k_{js}\partial_m c^s_{li} = 0.$$

An F-manifold $(M, \circ, e)$ is called *semisimple* if locally $(TM, \circ)$ is isomorphic to $C(M)^n$ (where n is the dimension of the manifold M) with componentwise multiplication. This means that locally there exists a distinguished coordinate system such that, if X and Y are vector fields then $(X \circ Y)^i = X^iY^i$. This is equivalent to say that $c^i_{jk} = \delta^i_j\delta^i_k$ in this distinguished coordinate system (these are called *canonical coordinates* for $\circ$ whenever they exist). We will denote canonical coordinates with $\{u^1, \ldots, u^n\}$. If $\circ$ is semisimple, then the identity vector field e is given by $e = \sum_i \frac{\partial}{\partial u^i}$.

A few years later, Manin introduced F-manifolds with compatible flat structure ([30]), which we call flat F-manifold for simplicity. In particular, he proved that many constructions related to Frobenius manifolds [14, 31], such as Dubrovin's duality [15], do not require the presence of a (pseudo)-metric satisfying the condition $g(X\circ Y, Z) = g(X, Y \circ Z)$ for all vector fields X, Y, Z (such a metric is said to be *invariant*).

Definition 1.1 ([30]). *A flat F-manifold $(M, \circ, \nabla, e)$ is a manifold M equipped with the following data:*

1. *a commutative associative product $\circ : TM \times TM \to TM$ on sections of the tangent bundle TM,*
2. *a distinguished vector field e such that $X \circ e = X$ for every vector field X,*
3. *a flat torsionless affine connection ∇, such that $(\nabla_X c)(Y, Z) = (\nabla_Y c)(X, Z)$ for all vector fields X, Y, and Z.*
4. *$\nabla e = 0$ (flat identity).*

A semisimple flat F-manifold is defined analogously, with the requirement that the operation $\circ$ is semisimple.

The manifold M in the Definition 1.1 is a real or complex n-manifold. In the first case all the geometric data are supposed to be smooth. In the latter case TM is intended as the holomorphic tangent bundle and all the geometric data are supposed to be holomorphic.

Observe that in the Definition 1.1 there is no mention of the Hertling-Manin condition (1.1) since the symmetry condition on ∇c forces (1.1) to be automatically satisfied (see [20] for a proof).

The role played by flat F-manifolds in the study of integrable systems has been investigated in [26, 27]. Further generalizations of these structures that are tailored to the study of integrable dispersionless PDEs have been proposed in [1, 2, 28]). In this paper, following similar ideas, we introduce and study what we call *multi-flat F-manifolds*. They are a natural generalization of bi-flat F-manifolds (see [2, 28]) and they are deeply related to the notion of eventual identities and duality introduced in [30].

In order to define multi-flat F manifolds we need to recall a few facts about eventual identites:

Definition 1.2 ([30]). *A vector field E on an F-manifold is called an* eventual identity, *if it is invertible with respect to the product $\circ$, and if the bilinear product $*$*

defined via

$$X * Y := X \circ Y \circ E^{-1}, \qquad \text{for all } X, Y \text{ vector fields} \tag{1.2}$$

defines a new F-manifold structure on M. If E satisfies the additional condition $[e, E] = e$ then it is called Euler vector field*.*

By definition, an eventual identity is the unity of the associated product $*$. A useful criterion to detect eventual identities is the following:

THEOREM 1.3 ([11]). *An invertible vector field E is an eventual identity for the F-manifold $(M, \circ, e)$ if and only if*

$$\mathrm{Lie}_E(\circ)(X, Y) = [e, E] \circ X \circ Y, \qquad \forall X, Y \text{ vector fields.} \tag{1.3}$$

In the semisimple case, it is actually easier to characterize eventual identities. We have indeed the following theorem.

THEOREM 1.4 ([1]). *Let $(M, \circ, e)$ be a semisimple F-manifold and let E be an invertible vector field and assume that the eigenvalues of the endomorphism of the tangent bundle $V = E\circ$ are distinct. Then condition (1.3) is equivalent to the vanishing of the Nijenhuis torsion of V.*

In other words, in canonical coordinates for $\circ$ eventual identities are vector fields of the form

$$E = \sum_{i=1}^{n} E^i(u^i) \frac{\partial}{\partial u^i},$$

and the product $*$ has associated structure constants c^{*i}_{jk} given by :

$$c^{*i}_{jk} = \frac{1}{E^i(u^i)} \delta^i_j \delta^i_k. \tag{1.4}$$

Observe that powers of the Euler vector fields are eventual identities. This follows from the fact that eventual identities form a subgroup of the group of invertible vector fields on an F-manifold [11].

We have now all the ingredients to define multi-flat F-manifolds.

DEFINITION 1.5. *Let $(M, \nabla, \circ, e)$ be a flat F-manifold with unity e. A* multi-flat *F-manifold $(M, \nabla^{(l)}, \circ, e, E, l = 0...N-1)$ anchored at $(M, \nabla, \circ, e)$ is a manifold M endowed with N flat torsionless affine connections $\nabla^{(0)} := \nabla$, $\nabla^{(1)}, ..., \nabla^{(N-1)}$, a commutative associative product $\circ$ on sections of the tangent bundle TM, an invertible vector field E satisfying the following conditions:*

1. *E is an Euler vector field.*
2. *Given $E_{(l)} := E^{\circ l} = E \circ E \circ \cdots \circ E$ l-times, $l = 0, \ldots, N-1$, (by definition, $E_{(0)} = e$, $E_{(1)} = E$), we require*

$$\nabla^{(l)} E_{(l)} = 0. \tag{1.5}$$

3. *Given $E_{(l)}$ and the related commutative, associative product $\circ_{(l)}$ (defined as $X \circ_{(l)} Y := X \circ Y \circ E^{-1}_{(l)}$, so that $\circ_{(0)} = \circ$ and $\circ_{(1)} = *$), we require that the connection $\nabla^{(l)}$ is compatible with $\circ_{(l)}$. In other words we require that*

$$\left(\nabla^{(l)}_X c_{(l)}\right)(Y, Z) = \left(\nabla^{(l)}_Y c_{(l)}\right)(X, Z), \tag{1.6}$$

for all vector fields X, Y, and Z for all $l = 0, \ldots N-1$.

4. *The connections $\nabla^{(l)}$, $l = 0, \dots, N-1$ are almost hydrodynamically equivalent (see [2]) i.e.*

$$(d_{\nabla^{(l)}} - d_{\nabla^{(l')}})(X \circ_{(l)}) = 0, \tag{1.7}$$

for every vector fields X and for every pair l, l'; here $d_{\nabla^{(l)}}$ is the exterior covariant derivative constructed from the connection $\nabla^{(l)}$.

The conditions (1.5) and (1.7) fix uniquely the Christoffel symbols of the connection $\nabla^{(l)}$ in terms of the Christoffel symbols of the connection ∇, the structure constants of the product $\circ_{(l)}$ and the Euler vector field E. Indeed, in local coordinates, the condition (1.7) with $l' = 0$ reads

$$\Gamma^{(l)k}_{sj} c^{(l)s}_{im} - \Gamma^{(l)k}_{si} c^{(l)s}_{jm} = \Gamma^k_{sj} c^{(l)s}_{im} - \Gamma^k_{si} c^{(l)s}_{jm}$$

Multiplying both sides by $E^i_{(l)}$, taking the sum over i and using (1.5) one obtains

$$\Gamma^{(l)k}_{ij} = \Gamma^k_{ij} - c^{(l)s}_{ij} \nabla_s E^k_{(l)}. \tag{1.8}$$

Due to the invertibility of the operator $E^{-1}_{(l)}\circ$ the condition (1.7) must be checked only for $l = 0$. Remarkably, using formula (1.8) it is possible to check that also the condition (1.6) must be checked only for $l = 0$. In order to prove this fact, in the semisimple case it is sufficient to compare the explicit formulas for the Christoffel symbols in canonical coordinates obtained imposing the conditions (1.5) and (1.6) with the explicit formulas obtained writing (1.8) in the same coordinates. In the general case, the proof is based on the definition of eventual identity and on the formula

$$(E_{(l)}\circ)^m_h (\nabla^{(l)}_j c^{(l)i}_{rm} - \nabla^{(l)}_r c^{(l)i}_{jm}) = (E_{(l)}\circ)^i_s [c^s_{jm} \mathrm{Lie}_{E_{(l)}} c^m_{hr} - c^s_{rm} \mathrm{Lie}_{E_{(l)}} c^m_{hj}]$$

that can be obtained using (1.5), (1.6) with $l = 0$ and (1.7).

This means that multiflat F-manifolds can be defined as flat F-manifolds equipped with an Euler vector field E and $N-1$ additional almost hydrodynamically equivalent flat torsionless connections $\nabla_{(1)}, \dots, \nabla_{(N-1)}$ satisfying the conditions $\nabla_{(l)} E_{(l)} = 0$.

In the first part of the paper we will study semisimple F-manifolds endowed with N flat structures. In principle N might be arbitrary, however we will see that the coexistence of more than 3 flat structures is in general impossible. The case of two structures has been studied in details in [2, 28]. It turns out that three-dimensional bi-flat F-manifolds are parameterized by solutions of Painlevé VI equation. In this paper we will find an alternative proof of this fact. We will also study the case of tri-flat F-manifolds in the three-dimensional case.

In the second part of the paper we will consider the non-semisimple case. First we will study regular non-semisimple bi-flat F-manifolds, leveraging on the recent results obtained in [10] unveiling a deep relation between regular bi-flat F-manifolds in dimension three on one side, and the full Painlevé equations P_{VI}, P_V and P_{IV} on the other. More precisely, regular bi-flat F-manifolds are characterized by the Jordan normal form of the operator $L = E\circ$. For three-dimensional manifolds, this gives rise to three cases, corresponding to L_1, L_2 and L_3 given by:

$$L_1 := \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad L_2 := \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad L_3 := \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 1 \\ 0 & 0 & \lambda_1 \end{pmatrix}, \tag{1.9}$$

(here λ_i with different indices are assumed to be distinct). Regular bi-flat F-manifolds in dimension three whose endomophism L has the form L_1 are actually semisimple and, as recalled above, are locally parameterized by solutions of the full Painlevé VI.

We will focus our attention on three-dimensional regular bi-flat F-manifolds whose operator L has the form L_2 or L_3 and we will show that in the former case they are locally parameterized by solutions of the full P_V, while in the latter case they are locally parameterized by solutions of the full P_{IV}. This highlights a striking parallelism between confluences of Painlevé equations and collision of eigenvalues of the endomorphism L (preserving regularity), a fact which in our opinion deserves further investigation. It would be definitely interesting to extend this correspondence beyond the regular case. Unfortunately for the non-regular case there are no structural results similar to those developed in [10] at the moment.

We point out that the approach championed in [2] and [28] is based on the study of a generalized Darboux-Egorov system and cannot be applied to the non-semisimple case, while the methodology developed here, in which the key role is played by a geometric version of Tsarev's conditions of integrability paired with a commutativity condition between the Lie derivative with respect to a set of eventual identities defining a subalgebra of the centerless Virasoro algebra and the covariant derivative of the associated connections, does not require the semisimplicity of the product.

Finally, in the last part of the paper we show the remarkable phenomenon that, while in the semisimple case there are in general obstructions to the existence of multi-flat F-manifolds, in the regular non-semisimple case it is possible to construct multi-flat F-manifolds with an *arbitrary* (countable) number of compatible flat connections (all the powers of the Euler vector field).

2. Background material.

2.1. Flat F-manifolds and Integrable dispersionless PDEs. In this section, we survey the relationships between F-manifolds, flat F-manifolds and other geometric structures on one hand, and the theory of integrable dispersionless PDEs on the other. We also introduce Tsarev's conditions, which play a key role in determining multi-flat F-structures.

According to Tsarev's theory [40, 41], integrable quasilinear systems of PDEs of the form

$$u^i_t = v^i(u)u^i_x, \qquad i = 1, ..., n \tag{2.1}$$

are defined by a set of functions Γ^i_{ij} $(i \neq j)$ satisfying the conditions (called *Tsarev's conditions*)

$$\partial_j\Gamma^i_{ik} + \Gamma^i_{ij}\Gamma^i_{ik} - \Gamma^i_{ik}\Gamma^k_{kj} - \Gamma^i_{ij}\Gamma^j_{jk} = 0, \quad \text{if } i \neq k \neq j \neq i. \tag{2.2}$$

Once the conditions (2.2) are satisfied the solutions of the system

$$\partial_j v^i = \Gamma^i_{ij}(v^j - v^i) \tag{2.3}$$

define a set (depending on functional parameters) of commuting flows of the form (2.1). From (2.2) it follows that the solutions of (2.3) satisfy the conditions

$$\partial_j\left(\frac{\partial_k v^i}{v^i - v^k}\right) = \partial_k\left(\frac{\partial_j v^i}{v^i - v^j}\right) \qquad \forall i \neq j \neq k \neq i, \tag{2.4}$$

Conversely, given v^i satisfying (2.4) and using (2.3) as definition of Γ^i_{ij}, the compatibility conditions (2.2) are automatically satisfied. Quasilinear systems satisfying conditions (2.4) are called *semi-Hamiltonian* [40, 41] or *rich* [37, 38]. Sévennec [39] later found a nice characterization of semi-Hamiltonian systems. He showed they coincide with diagonalizable systems of conservation laws. As the notation suggests, the functions Γ^i_{ij} can be identified with (part of) the coefficients of a torsionless connection ∇. The reconstruction of ∇ can be done in essentially two non-equivalent ways. In the first case, we call the connection ∇ a *Hamiltonian connection*. In this case, ∇ is the Levi-Civita connection of a diagonal metric g:

$$\partial_j \ln \sqrt{g_{ii}} = \Gamma^i_{ij}, \qquad j \neq i. \tag{2.5}$$

Given a diagonal metric g for which the functions Γ^i_{ij} satisfy the above conditions, all the remaining Christoffel symbols are uniquely defined through the classical Levi-Civita's formula. However, as it is easy to check, the general solution of (2.5) depends on n arbitary functions of a single variable: if g_{ii} is a solution then $\varphi_i(u^i)g_{ii}$ is still a solution. The connections defined by (2.5) have been introduced by Dubrovin and Novikov in [16]. We call them Hamiltonian connections since they are related to the Hamiltonian formalism. For instance, in the flat case the differential operator

$$P^{ij} := g^{ii}\delta^i_j\partial_x - g^{il}\Gamma^j_{lk}u^k_x \tag{2.6}$$

defines a local Hamiltonian operator for the flows (2.1) defined by the solutions of (2.5). The non-flat case is more involved: the Hamiltonian operators are non-local and the non-local tail is related to the quadratic expansion of the Riemann tensor in terms of solutions of the system (2.3):

$$R^{ij}_{ij} = \sum_\alpha \epsilon_\alpha w^i_\alpha w^j_\alpha.$$

The existence of this quadratic expansion is a non-trivial property. It was conjectured by Ferapontov [17] that all solutions of the system (2.5) possess such a property. Ferapontov's conjecture has been checked for reductions of dKP and 2d Toda in [19] and [7] respectively.

The other way to reconstruct a torsionless affine connection ∇ having Γ^i_{ij} as a subset of its Christoffel symbols in a distinguished coordinate system was devised in [27]. This leads to the notion of natural connections and F-manifold with compatible connection and flat unity [26].

Definition 2.1. *An* F-manifold with compatible connection and flat unity *is an* F*-manifold* $(M, \circ, e)$ *equipped with a torsionless connection* ∇ *(not necessarily flat) such that the following requirements hold*

$$\begin{aligned}
&\nabla e = 0,\\
&(\nabla_X c)(Y,Z) = (\nabla_Y c)(X,Z),\\
&Z \circ R(W,Y)(X) + W \circ R(Y,Z)(X) + Y \circ R(Z,W)(X) = 0,
\end{aligned}$$

where c *is the tensor field associated to* $\circ$*,* R *is the Riemann tensor and* X, Y, Z, W *are arbitrary vector fields.*

Connections satisfying these conditions are called *natural connections.* In the semisimple case, in the distinguished coordinate system given by the canonical coordinates of $\circ$, the first and second requirements specify completely the Christoffel symbols in terms of the subset given by Γ^i_{ij}:

$$\begin{aligned} \Gamma^i_{jk} &:= 0, \qquad \forall i \neq j \neq k \neq i,\\ \Gamma^i_{jj} &:= -\Gamma^i_{ij}, \qquad i \neq j,\\ \Gamma^i_{ii} &:= -\sum_{l\neq i}\Gamma^i_{li}. \end{aligned} \tag{2.7}$$

It was proved in [26] that the second and the third conditions are equivalent to Tsarev's condition (2.2) for Γ^i_{ij}. In this framework, the quasilinear system (2.1) can be written as

$$u_t = X \circ u_x. \tag{2.8}$$

and the system (2.3) reads

$$c^i_{jl}\nabla_k X^l = c^i_{kl}\nabla_j X^l. \tag{2.9}$$

In this setting the characteristic velocities v^i are thought as the components of the vector fields X in canonical coordinates. Since the Riemann invariants are identified with the canonical coordinates, given a semi-Hamiltonian system, the associated natural connection is defined up to a reparameterization of the Riemann invariants, that is up to the choice of n arbitrary functions of a single variable.

Like in the case of Hamiltonian connections, the most interesting case is when the connection ∇ is flat. In this case, a countable set of solutions of the system (2.3) can be obtained from a frame of flat vector fields $(X_{(1,0)}, ..., X_{(n,0)})$ via the following recursive relations

$$\nabla X_{(p,\alpha+1)} = X_{(p,\alpha)} \circ . \tag{2.10}$$

In general, the two ways we just described to reconstruct a torsionelss connection ∇ starting from the functions Γ^i_{ij} satisfying (2.2) are inequivalent: Hamiltonian connections are not always natural connections and natural connections are not always Hamiltonian. Indeed, combining the conditions $\nabla g = 0$ and $\nabla_i c^k_{lj} = \nabla_l c^k_{ij}$, one obtains $\partial_j g_{ii} = \partial_i g_{jj}$, which implies that in order to have a connection which is both Hamiltonian and natural, the metric must be potential in canonical coordinates (Egorov case). This is for instance the case of semisimple Frobenius manifolds.

In many examples (including Frobenius manifolds) besides the recursive relation (2.10) there exists an additional one, which we called twisted Lenard-Magri chain [1]

$$d_{\nabla^{(1)}}(e \circ X_{(p,\alpha+1)}) = d_{\nabla^{(2)}}(E \circ X_{(p,\alpha)}). \tag{2.11}$$

It is based on the existence of an additional flat structure and on an eventual identity E. This leads naturally to define the class of bi-flat F-manifolds that was extensively studied in [2, 28].

Remark 2.2. *In the semisimple case removing the condition $\nabla e = 0$ in the definition of natural connections one has the freedom to choose the Christoffel symbols Γ^i_{ii} [27]. The same freedom can be also described in terms of the special family of connections [12]*

$$\tilde{\nabla}_X Y = \nabla_X Y + V \circ X \circ Y.$$

This is a family of connections satisfying the symmetry condition (1.5) (the product is not assumed to be semisimple). Like in the semisimple case the condition $\tilde{\nabla} e = 0$ fixes uniquely the vector field V.

2.2. Examples of bi-flat F-manifolds. In this section we present some examples of bi-flat F-manifolds.

2.2.1. Semisimple Frobenius manifolds. Any semisimple Frobenius manifold $(M, \eta, \circ, e, E)$ is endowed with a bi-flat structure $(\nabla, \nabla^*, \circ, *, e, E)$. More precisely:

- the connection ∇ is the Levi-Civita connection of the invariant metric η.
- $(\circ, e, E)$ are the same data from the Frobenius manifold, while $*$ is given by Dubrovin's almost dual product $X * Y = X \circ Y \circ E^{-1}$ [15].
- the connection ∇^* is defined by (1.8) with $l = 1$. The flatness of this connection is a consequence of the condition $\nabla\nabla E = 0$ (see theorem 4.4 of [3]).

Any Frobenius manifold is equipped with a contravariant flat metric called intersection form [14]. The connection ∇^* in general does not coincide with the Levi-Civita connection $\tilde{\nabla}$ of the intersection form. However, they satisfy the condition $(d_{\tilde{\nabla}} - d_{\nabla^*})(X\,\circ) = 0$ and they are compatible with the same product $*$. This implies that $\tilde{\nabla} = \nabla^* + \lambda *$ for some function λ. Using the properties of the intersection form it is not difficult to see that λ is indeed a constant.

2.2.2. Bi-flat F-manifolds and complex reflection groups. One of the main examples of Frobenius manifold is the orbit space of a Coxeter group W [13]. In this case, in the flat coordinates of $\tilde{\nabla}$ the dual product has the form [15]

$$* = \lambda \sum_{H \in \mathcal{H}} \frac{d\alpha_H}{\alpha_H} \otimes \pi_H, \tag{2.12}$$

where α_H is a linear form defining the mirror H, $\mathcal{H}$ is the collection of mirrors H associated to the reflections in W, π_H denotes the orthogonal projection onto the orthogonal complement of the hyperplane H and λ is a suitable normalizing factor. Products of this form appear in the theory of $\vee$-system [42, 43]. This example can be generalized to the case of well-generated complex reflection groups. A first generalization was given by Kato, Mano and Sekiguchi [22] that proved the existence of a flat structure (with *linear* Euler vector field) on the orbit space of well-generated complex reflection groups and computed some examples of vector potentials related to the agebraic solutions of PVI found in [5, 6] (see also [4]).

Due to the linearity of the Euler vector field, Theorem 4.4 of [3] ensures the existence of a second flat structure compatible with Kato-Mano-Sekiguchy structure. In all the examples the dual connection and the dual product have the form (see [3] for details)

$$\nabla^* = \tilde{\nabla} - \sum_{H \in \mathcal{H}} \frac{d\alpha_H}{\alpha_H} \otimes \kappa_H \pi_H, \quad * = \sum_{H \in \mathcal{H}} \frac{d\alpha_H}{\alpha_H} \otimes \kappa_H \pi_H. \tag{2.13}$$

In this case, $\tilde{\nabla}$ is the standard flat connection on $\mathbb{C}^n$, π_H denotes the unitary projection onto the unitary complement of the hyperplane H and the weight κ_H is proportional to the order of the corresponding reflection. A further generalization was given in [3], where it was proved that the bi-flat structures associated with complex reflection groups might depend on parameters (related to a different choice of the weights in (2.13)), even in the case of Coxeter groups. In this case it was conjectured that, under suitable assumptions, the number of parameters coincides with the number of orbits for the action of the group on the collection of reflecting hyperplanes minus one. The conjecture was proved for Weyl groups of rank $2, 3, 4$ and for the groups $I_2(m)$.

2.2.3. Generalized ϵ-system and Lauricella bi-flat structure. The system of quasilinear PDEs

$$u^i_t = \left(u^i - \sum_{k=1}^{n} \epsilon_k u^k\right) u^i_x, \qquad i = 1, ..., n,$$

is called *the generalized ϵ-system* [34]. The integrable hierarchy associated with the (generalized) ϵ-system was studied in [27] and [28] where it was also proved that the associated natural and dual connections determined by the Tsarev's Christoffel symbols

$$\Gamma^i_{ij} = \frac{\epsilon_j}{u^i - u^j} \qquad i \neq j, \tag{2.14}$$

the structure constants $c^i_{jk} = \delta^i_j \delta^i_k$ and $c^{*i}_{jk} = \frac{1}{u^i}\delta^i_j\delta^i_k$, the vector fields $e = \sum \partial_i$ and $E = \sum u^i \partial_i$ define a bi-flat structure.

This example is related to the theory of Lauricella functions [23], Lauricella connections and Lauricella manifolds [9, 25]. In particular Lauricella functions provide $n-1$ flat homogenous coordinates for the natural connection of the generalized ϵ-system described above. For this reason we call this example *Lauricella bi-flat structure.*

3. Eventual identities and flatness conditions. Given an F- manifold with an eventual identity E we want to characterize *flat* symmetric connections ∇ compatible with the eventual identity E, i.e satisfying the following requirements

$$\begin{aligned} &(\nabla_X c^*)(Y, Z) = (\nabla_Y c^*)(X, Z) \\ &\nabla E = 0, \end{aligned}$$

where c^* is the $(1, 2)$-tensor field associated to the dual product $*$.

It is possible to provide an intrinsic characterization of the flatness condition, which is given in Theorem 3.2 below. Before stating this result and proving it, we elucidate a general fact:

LEMMA 3.1. *Let M be a manifold equipped with a bilinear product $\circ$ on sections of its tangent bundle $\circ : TM \times TM \to TM$ and with a torsionless affine connection ∇. Suppose $\circ$ is equipped with an unit vector field e and that ∇ and $\circ$ satisfy the following condition:*

$$Z \circ R(W, Y)(X) + W \circ R(Y, Z)(X) + Y \circ R(Z, W)(X) = 0, \tag{3.1}$$

where $R(X, Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$ for all vector fields X, Y, Z, W. Then ∇ is flat if and only if $R(e, W) = 0$ for all vector fields W.

Proof. If ∇ is flat, certainly $R(e, W) = 0$ for all vector fields W. Conversely, suppose $R(e, W) = 0$ for all vector fields. Then substituting $Z := e$ in (3.1) we get immediately $e \circ R(W, Y)(X) = 0$, i.e. $R(W, Y)(X) = 0$ for all vector fields W, Y, X, and we are done. ◻

Observe that the condition (3.1) appearing in the previous Lemma is exactly one of the conditions that define an F-manifold with compatible connection [26]. Using the above lemma we can prove the following useful flatness criterion.

THEOREM 3.2. *An F-manifold with compatible connection ∇ and flat unity e is flat if and only if the operator* Lie_e *and the covariant derivative ∇ satisfy the following condition:*

$$\mathrm{Lie}_e(\nabla_X T) - \nabla_X(\mathrm{Lie}_e T) - \nabla_{[e,X]}T = 0, \tag{3.2}$$

for any vector field X and for any tensor field T.

Proof. Since the unity e is assumed to be a flat vector field, we have that $\mathrm{Lie}_e = \nabla_e$ and therefore the condition (3.2) is equivalent to $R(e, X)(T) = 0$. Now by Lemma 3.1 we know that ∇ is flat if and only if $R(e, X) = 0$ for all vector fields X. □

Using the fact that X is an arbitrary vector field, we have the following Lemma

LEMMA 3.3. *Condition* (3.2) *is equivalent to*

$$\mathrm{Lie}_e(\nabla T) - \nabla(\mathrm{Lie}_e T) = 0, \tag{3.3}$$

for any tensor fied T.

Proof. Observe that we can write $\nabla_X T = (\nabla T)(X) = C(\nabla T \otimes X)$ for any vector field X, where C is the contraction. Therefore using the property that Lie_e commutes with contractions and it satisfies Leibniz rule with respect to the tensor product we have

$$\begin{aligned}[\mathrm{Lie}_e(\nabla T) - \nabla(\mathrm{Lie}_e T)](X) &= \mathrm{Lie}_e((\nabla T)(X)) - \nabla T(\mathrm{Lie}_e X) - \nabla_X(\mathrm{Lie}_e T)\\ &= \mathrm{Lie}_e(\nabla_X T) - \nabla T([e, X]) - \nabla_X(\mathrm{Lie}_e T)\\ &= \mathrm{Lie}_e(\nabla_X T) - \nabla_X(\mathrm{Lie}_e T) - \nabla_{[e,X]}T.\end{aligned}$$

□

Observe that in the proof of Theorem 3.2 the symmetry of ∇c does not play any role. The only hypotheses that were used are the presence of a flat identity e for the product $\circ$ and condition (3.1) for the torsionless connection ∇.

4. The semisimple case.

4.1. Multi-flatness conditions in the semisimple case. We apply now the flatness criterion discussed in the previous Section to study multi-flat structures in the semisimple case. As a consequence of the previous results we have the following

THEOREM 4.1. *A semisimple F-manifold with compatible connection ∇ and flat unity e is flat if and only if*

$$e(\Gamma^i_{ij}) = \sum_{i=1}^{n} \frac{\partial \Gamma^i_{ij}}{\partial u^i} = 0, \quad \forall i \neq j,$$

where Γ^i_{ij} are the Christoffel symbols of ∇ in the canonical coordinates of $\circ$.

Proof. Under the current hypotheses, ∇ is flat if and only if (3.3) holds for an arbitrary tensor field T. However, notice that (3.3) is automatically satisfied when T is a function since covariant and Lie derivatives coincide on functions. Moreover, the operators Lie_e and ∇_i commute with contractions and satisfy Leibniz rule with respect to tensor products. This easily implies that (3.3) holds for an arbitrary tensor fields T if and only if it holds for an arbitrary vector field T. Writing the right hand side of (3.3) in canonical coordinates of $\circ$, for T an arbitrary vector field, we get

$$e\left(\partial_j T^i + \Gamma^i_{jk}T^k\right) - \partial_j(e(T^i)) - \Gamma^i_{jk}e(T^k) = e(\Gamma^i_{jk})T^k,$$

since e commutes with ∂_j in canonical coordinates. Therefore (3.3) is fulfilled if and only if $e(\Gamma^i_{jk}) = 0$, due to the arbitrariness of T. On the other hand, for a natural connection in canonical coordinates one already has $\Gamma^i_{jk} = 0$ $i \neq j \neq k \neq i$, while all the other non-vanishing components are expressed as linear combinations with constant coefficients of Γ^i_{ij}, $i \neq j$ (see formula (2.7)). □

Obviously, the flatness criterion provided by relation (3.3) and its equivalent forms can be applied to the case of connections associated to general eventual identities. It is easy to check that a torsionless connection ∇ compatible with the dual product defined by E and satisfying the condition $\nabla E = 0$ has Christoffel symbols (in canonical coordinates for $\circ$) of the form:

$$\begin{aligned}
\Gamma^i_{jk} &:= 0, \qquad \forall i \neq j \neq k \neq i,\\
\Gamma^i_{jj} &:= -\frac{E^i}{E^j}\Gamma^i_{ij}, \qquad i \neq j,\\
\Gamma^i_{ii} &:= -\sum_{l\neq i}\frac{E^l}{E^i}\Gamma^i_{li} - \frac{\partial_i E^i}{E^i}.
\end{aligned} \tag{4.1}$$

Given an eventual identity E with associated dual product $*$, it is useful to have relations characterizing the flatness of the connection given by (4.1) in the canonical coordinate for $\circ$. This characterization is provided by the following:

Theorem 4.2. *Suppose that the functions Γ^i_{ij} satisfy Tsarev's conditions* (2.2), *then in canonical coordinates for $\circ$, the torsionless connection* (4.1) *is flat if and only if*

$$E(\Gamma^i_{ij}) = -(\partial_j E^j)\Gamma^i_{ij}, \qquad i \neq j.$$

Proof. Writing the invariant condition (3.3) in canonical coordinates for $\circ$ we get the following conditions

$$\begin{aligned}
E(\Gamma^i_{ij}) &= -\Gamma^i_{ij}\partial_j E^j,\\
E(\Gamma^i_{jj}) &= \Gamma^i_{jj}\partial_i E^i - 2\Gamma^i_{jj}\partial_j E^j,\\
E(\Gamma^i_{ii}) &= -\partial_i^2 E^i - \Gamma^i_{ii}\partial_i E^i.
\end{aligned}$$

The first condition is the statement of the Theorem. The second and third one follow using the first one, the defining relations of the natural connection and the obvious identities $E(E^i) = E^i\partial_i E^i$, $E(\partial_i E^i) = E^i\partial_i^2 E^i$. □

4.2. Non existence of semisimple F-manifolds with more than 3 compatible connections. We are going to apply Theorem 4.2 to study the existence of multi-flat structures on semisimple F-manifolds. Recall that, by definition, given an N-multi flat semisimple manifold, the N connections $\nabla^{(l)}$, $l = 0, \dots, N-1$ share the same Christoffel symbols Γ^i_{ij}, $i \neq j$ in the canonical coordinates for $\circ$, while the remaining ones are determined according to the formulas (4.1), where E is the corresponding eventual identity $E_{(l)}$. Therefore, given the $E_{(l)}$, $l = 0, \dots N-1$, it is possible to reconstruct N-multi-flat connections only if the system for Γ^i_{ij} (j is fixed):

$$E_{(l)}(\Gamma^i_{ij}) + (\partial_j E^j_{(l)})\Gamma^i_{ij} = 0, \quad l = 0, \dots N-1 \tag{4.2}$$

admits non-trivial solutions Γ^i_{ij} for all $i \neq j$. Indeed, (4.2) is just the flatness condition of Theorem 4.2. It is possible to reduce the non-homogenous system (4.2) to a homogenous one. To do this we introduce a fictitious additional variable u^{n+1} and assume that Γ^i_{ij} is defined implicitly via $\phi(u^1, \dots, u^n, u^{n+1}) = c$ where c is a constant. In this case the system (4.2) becomes

$$\hat{E}_{(l)}(\phi) := E_{(l)}(\phi) - (\partial_j E^j_{(l)})u^{n+1}\partial_{n+1}\phi = 0, \quad l = 0, \dots, N-1.$$

Notice that the above condition is required to hold only at the points of the hypersurface $\phi = c$. In this way, determining ϕ can be interpreted as the problem of finding hypersurfaces, locally represantable as

$$u^{n+1} = \Gamma^i_{ij}(u^1, \dots, u^n), \tag{4.3}$$

which are integral leaves of the distribution Δ generated by the vector fields $\{\hat{E}_{(l)}\}$.

Therefore we are interested in characterizing the integrable distributions generated by the extended vector fields $\hat{E}_{(l)}, l =, 0, \dots N-1$, where by definition of multi-flat F-manifold $E_{(l)} = (u^1)^l\partial_1 + ... + (u^n)^l\partial_n$, $l = 0, \dots, N-1$, in canonical coordinate for $\circ = \circ_{(0)}$.

THEOREM 4.3. *Let $\Delta_{(i_1,\dots,i_k)}$ be the distribution spanned by the vector fields $\hat{E}_{(i_1)}, \dots, \hat{E}_{(i_k)}$ in the $n+1$-dimensional space with coordinates $(u^1, \dots, u^n, u^{n+1})$. Then:*

1. *The distributions $\Delta_{(1,m)}$ with $m \in \mathbb{Z} \setminus \{1\}$ are integrable and these are the only integrable distributions of rank 2 among $\Delta_{(i_1,i_2)}$.*
2. *$\Delta_{(0,1,2)}$ is integrable.*
3. *$\Delta_{(0,1,2,3)}$ is not integrable. Furthermore, at the points where $u^i \neq u^k$ ($i \neq k, i, k = 1, ..., n$) and $u^{n+1} \neq 0$ it is totally non-holonomic, that is the minimal integrable distribution $\bar{\Delta}$ containing $\Delta_{(0,1,2,3)}$ has dimension $n+1$.*
4. *More in general $\Delta_{(i_1,\dots,i_k)}$, with $i_1 < i_2 < \dots < i_k$ is not integrable for $4 \leq k \leq n$.*

Proof. We have

$$[\hat{E}_{(l)}, \hat{E}_{(m)}]^i = \begin{cases} (m-l)(u^i)^{l+m-1} & \text{if } i = 1, ..., n \\ -(m-l)(m+l-1)(u^j)^{m+l-2}u^{n+1} & \text{if } i = n+1 \end{cases} \tag{4.4}$$

that is

$$[\hat{E}_{(l)}, \hat{E}_{(m)}] = (m-l)\hat{E}_{(m+l-1)}, \quad l \neq m,$$

$$[\hat{E}_{(l)}, \hat{E}_{(m)}] = 0, \quad l = m.$$

Since $[\hat{E}_{(m)}, \hat{E}_{(1)}] = \hat{E}_{(m)}$, the distribution $\Delta_{(m,1)}$ is integrable. Moreover, any other distribution of rank 2, $\Delta_{(i_1,i_2)}$ is not integrable since $[\hat{E}_{i_1}, \hat{E}_{i_2}] = (i_2 - i_1)\hat{E}_{(i_1+i_2-1)}$, and $i_1 + i_2 - 1 = i_1$ or $i_1 + i_2 - 1 = i_2$ implies either $i_2 = 1$ or $i_1 = 1$.

Since $\hat{E}_{(0)}$, $\hat{E}_{(1)}$ and $\hat{E}_{(2)}$ satisfy the commutation relations of $sl(2,\mathbb{C})$: $[\hat{E}_{(0)}, \hat{E}_{(1)}] = \hat{E}_{(0)}$, $[\hat{E}_{(0)}, \hat{E}_{(2)}] = 2\hat{E}_{(1)}$ and $[\hat{E}_{(1)}, \hat{E}_{(2)}] = \hat{E}_{(2)}$, we have that also the distribution $\Delta_{(0,1,2)}$ is integrable.

With regard to the fourth point, consider $\Delta_{(i_1,\dots,i_k)}$, with $i_1 < \dots < i_k$ and $4 \leq k \leq n$. If the two indices i_1, i_2 are both strictly negative or if $i_1 < 0$ and $i_2 = 0$, then $[\hat{E}_{(i_1)}, \hat{E}_{(i_2)}] \notin \Delta_{(i_1,\dots,i_k)}$, due to the commutation relations. Thus we can assume $i_1 \geq 0$ and the indices i_{k-1}, i_k strictly greater than 1. Therefore again we have $[\hat{E}_{(i_{k-1})}, \hat{E}_{(i_k)}] = (i_k - i_{k-1})\hat{E}_{(i_k+i_{k-1}-1)} \notin \Delta_{(i_1,\dots,i_k)}$, since $i_k + i_{k-1} - 1 > i_k$.

Finally, it remains to prove the third point. By the fourth point, the distribution $\Delta_{(0,1,2,3)}$ is not integrable. Given a collection of vector fields $\{\hat{E}_{(l)}\}_{l\in L}$ their Lie hull is the collection of all vector fields of the form $\{\hat{E}_{(l)}, [\hat{E}_{(l)}, \hat{E}_{(m)}], [\hat{E}_{(n)}, [\hat{E}_{(l)}, \hat{E}_{(m)}]], \dots\}$ generated by the iterated Lie brackets. The minimal integrable distribution containing $\Delta_{(i_1,\dots,i_k)}$ is the minimal integrable distribution containing the Lie hull of the vector fields $\{\hat{E}_{(i_1)}, \dots \hat{E}_{(i_k)}\}$.

In order to compute the minimal integrable distribution containing $\Delta_{(0,1,2,3)}$, we consider the sub-bundle of the tangent bundle spanned by $\hat{E}_{(0)}, \hat{E}_{(1)}, \hat{E}_{(2)}, \hat{E}_{(3)}, \hat{E}_{(m)} = \frac{1}{m-2}[\hat{E}_{(2)}, \hat{E}_{(m-1)}]$, $m = 4, 5, \dots, n$. To show that its rank is $n+1$, it is sufficient to show that the determinant of the A matrix does not vanish on an open set, where

$$A := \begin{pmatrix} 1 & \dots & 1 & 0 \\ u^1 & \dots & u^n & -u^{n+1} \\ \vdots & \ddots & \vdots & \vdots \\ (u^1)^n & \dots & (u^n)^n & -n(u^j)^{n-1}u^{n+1} \end{pmatrix}.$$

The matrix A can be written as

$$A = \begin{pmatrix} 1 & \dots & 1 & -u^{n+1}\frac{\partial}{\partial u^{n+1}}_{|u^{n+1}=u^j} 1 \\ u^1 & \dots & u^n & -u^{n+1}\frac{\partial}{\partial u^{n+1}}_{|u^{n+1}=u^j}(u^{n+1}) \\ \vdots & \ddots & \vdots & \vdots \\ (u^1)^n & \dots & (u^n)^n & -u^{n+1}\frac{\partial}{\partial u^{n+1}}_{|u^{n+1}=u^j}(u^{n+1})^n \end{pmatrix}.$$

Expanding the determinant of A along the last column, we get

$$\det(A) = \sum_{k=0}^{n} A_k \left(-u^{n+1}\frac{\partial}{\partial u^{n+1}}_{|u^{n+1}=u^j}(u^{n+1})^k\right),$$

where A_k are the corresponding minors. Since the A_k's do not depend on u^{n+1} we can factor the derivative operator in front of the expansion and get:

$$\det(A) = -u^{n+1}\frac{\partial}{\partial u^{n+1}}_{|u^{n+1}=u^j}\left(\sum_{k=0}^{n} A_k (u^{n+1})^k\right) = -u^{n+1}\frac{\partial}{\partial u^{n+1}}_{|u^{n+1}=u^j}\det(V_{0,\dots,n}),$$

where $V_{0,\dots,n}$ is the Vandermonde matrix. By the form of the Vandermonde determinant, it is clear that $\det(A) \neq 0$ in the open subset $\Omega := \{(u^1, \dots, u^n, u^{n+1}) \mid u^i \neq u^k (i \neq k, i, k = 1, \dots, n), u^{n+1} \neq 0\}$. $\square$

Remark 4.4. *Notice that the extended vector fields $Z_{(l)} := \hat{E}_{(l+1)}$ satisfy the commutation relation*

$$[Z_{(l)}, Z_{(m)}] = [\hat{E}_{(l+1)}, \hat{E}_{(m+1)}] = (m-l)\hat{E}_{(m+l+1)} = (m-l)Z_{(m+l)},$$

of the centerless Virasoro algebra.

The above theorem has this important consequence

Corollary 4.5. *In the semisimple case non-trivial multi-flat F-manifolds with more than* 3 *compatible flat structure do not exist.*

Proof. Since the distribution $\Delta_{(0,1,2,\dots,m)}$ with $m \geq 3$ is totally non-holonomic in Ω, the Chow-Rashevsky Theorem [8, 35] implies that there are no codimension one integral leaves in each connected component of this subset. On the other hand, the complement of Ω in $\mathbb{R}^{n+1}$ is the union of the hyperplanes $u^i = u^k$ ($i \neq k$ and $i, k < n+1$) which are integral leaves of the distribution $\Delta_{(0,1,2\dots,m)}$ but are not related to solutions of the system (4.2) (since they cannot written in the form (4.3)) and the hyperplane $u^{n+1} = 0$ which is also an integral leave of the distribution $\Delta_{(0,1,2\dots,m)}$. The corresponding solution of the system (4.2) ($\Gamma^i_{ij} = 0$) defines the trivial multi-flat structure. □

Remark 4.6. *Unlike the multi-flat case we have been analyzing, it is possible to have multi-Hamiltonian structures encompassing more than three structures. An example is given by the following $n+1$ metrics introduced in [18]:*

$$g^{(\alpha)}_{ii} = \frac{\prod_{k\neq i}(u^k - u^i)}{(u^i)^\alpha}, \qquad \alpha = 0, \dots, n. \tag{4.5}$$

They are flat and thus their inverses define $n+1$ Hamiltonian structures of hydrodynamic type that turn out to be compatible among each other.

4.3. Bi-flat and tri-flat semisimple F-manifolds. In this Section we present the classification of three-dimensional semisimple bi-flat and tri-flat F-manifolds. Due to the results of the previous section semisimple bi-flat F-manifolds are parameterized by the solutions of the system

$$\partial_k \Gamma^i_{ij} = -\Gamma^i_{ij}\Gamma^i_{ik} + \Gamma^i_{ij}\Gamma^j_{jk} + \Gamma^i_{ik}\Gamma^k_{kj}, \quad i \neq k \neq j \neq i, \tag{4.6}$$

$$E_{(0)}(\Gamma^i_{ij}) = 0, \qquad i \neq j \tag{4.7}$$

$$E_{(1)}(\Gamma^i_{ij}) = -\Gamma^i_{ij}, \qquad i \neq j \tag{4.8}$$

where $E_{(0)} = \sum_{i=1}^n \partial_i$ and $E_{(1)} = \sum_{i=1}^n u^i \partial_i$. It is easy to prove that the above system is compatible and thus its general solution depends on $n(n-1)$ arbitrary constants.

Remark 4.7. *For $n = 2$ Tsarev's conditions* (4.6) *are empty. The general solution of the remaining conditions* (4.7) *and* (4.8) *depends on two arbitrary constants ϵ_1 and ϵ_2. It coincides with the natural connection associated with the two-component generalized ϵ-system.*

4.3.1. Three-dimensional bi-flat F-manifolds . Three-dimensional bi-flat F-manifolds are parameterized by solutions of Painlevé VI equation [2, 28]. This result has been obtained reducing a generalized version of the Darboux-Egorov system for the rotation coefficients β_{ij} to a system of ODEs equivalent to the sigma form of Painlevé VI (see [2, 28] for details).

In this Section, we follow a different approach, based on the study of the sytem (4.6,4.7,4.8). In particular we show that this system is equivalent to a system of six first order ODEs admitting 4 independent first integrals. Moreover, we provide an explicit relation between the solutions of this system and the solutions of the generic Painlevé VI equation. The values of the 4 parameters of the Painlevé VI equation are related to the values of the first integrals of the system.

As a first step we have to solve the system

$$\begin{aligned}E_{(0)}(\Gamma^i_{ij}) &= [\partial_1+\partial_2+\partial_3]\Gamma^i_{ij}=0,\\ E_{(1)}(\Gamma^i_{ij}) &= [u^1\partial_1+u^2\partial_2+u^3\partial_3]\Gamma^i_{ij}=-\Gamma^i_{ij},\end{aligned}$$

the solutions of which are given by $\Gamma^i_{ij}=\frac{F_{ij}\left(\frac{u^2-u^3}{u^1-u^2}\right)}{u^i-u^j}$ where F_{ij} are arbitrary smooth functions. Imposing Tsarev's conditions and introducing the auxiliary variable $z=\frac{u^2-u^3}{u^1-u^2}$, we obtain the system

$$\begin{aligned}\frac{dF_{12}}{dz} &= \frac{(F_{12}F_{13}-F_{12}F_{23})z+F_{12}F_{23}-F_{13}F_{32}}{z(z-1)},\\ \frac{dF_{21}}{dz} &= \frac{(F_{21}F_{23}-F_{21}F_{13})z+F_{23}F_{31}-F_{21}F_{23}}{z(z-1)},\\ \frac{dF_{13}}{dz} &= \frac{(F_{12}F_{23}-F_{12}F_{13})z-F_{12}F_{23}+F_{13}F_{32}}{z(z-1)},\\ \frac{dF_{31}}{dz} &= \frac{(F_{31}F_{12}-F_{32}F_{21})z-F_{31}F_{32}+F_{32}F_{21}}{z(z-1)},\\ \frac{dF_{23}}{dz} &= \frac{(F_{21}F_{13}-F_{21}F_{23})z-F_{23}F_{31}+F_{21}F_{23}}{z(z-1)},\\ \frac{dF_{32}}{dz} &= \frac{(F_{32}F_{21}-F_{31}F_{12})z+F_{31}F_{32}-F_{32}F_{21}}{z(z-1)}.\end{aligned} \tag{4.9}$$

It is straightforward to check that the above system admits three linear first integrals

$$I_1=F_{12}+F_{13}, \tag{4.10}$$

$$I_2=F_{23}+F_{21}, \tag{4.11}$$

$$I_3=F_{31}+F_{32}, \tag{4.12}$$

and one quadratic first integral

$$I_4=F_{31}F_{13}+F_{12}F_{21}+F_{23}F_{32}. \tag{4.13}$$

We consider also the cubic first integral

$$\begin{aligned}I_5 &= -I_3I_4+I_1I_2I_3\\ &= F_{21}F_{13}F_{32}+F_{12}F_{23}F_{31}+(I_2-I_3)F_{13}F_{31}+(I_1-I_3)F_{23}F_{32}.\end{aligned} \tag{4.14}$$

where I_1, I_2, I_3 are given by (4.10), (4.11) and (4.12) respectively. The correspondence between solutions of the system (4.9) and solutions of the Painlevé VI equation is given in terms of purely *algebraic* operations, as it is highlighted by the following Theorem:

THEOREM 4.8. *Let $(F_{12}(z), F_{21}(z), F_{13}(z), F_{31}(z), F_{23}(z), F_{32}(z))$ be a solution of the system* (4.9)*, then the function $f(z)=F_{23}F_{32}+zF_{12}F_{21}-\frac{q_1}{2}$ is a solution of the equation*

$$[z(z-1)f'']^2=[q_2-(d_2-d_3)g_2-(d_1-d_3)g_1]^2-4f'g_1g_2, \tag{4.15}$$

where $g_1 = f - zf' + \frac{q_1}{2}$ *and* $g_2 = (z-1)f' - f + \frac{q_1}{2}$ *and the parameters* d_1, d_2, d_3, q_1, q_2 *coincide with the values of the first integrals* I_1, I_2, I_3, I_4, I_5 *on the given solution of* (4.9). *Furthermore, equation* (4.15) *can be reduced to the sigma form of the generic Painlevé VI equation.*

Proof. Let $(F_{12}(z), F_{21}(z), F_{13}(z), F_{31}(z), F_{23}(z), F_{32}(z))$ be a solution of the system (4.9) and d_1, d_2, d_3, q_1, q_2 the corresponding values of the first integrals I_1, I_2, I_3, I_4, I_5. In analogy with [2, 28] we introduce the function $f(z) = F_{23}F_{32} + zF_{12}F_{21} - \frac{q_1}{2}$ satisfying $f' := F_{12}F_{21}$. Indeed

$$\begin{aligned}
\frac{d}{dz}(F_{12}F_{21}) &= \frac{F_{23}F_{31}F_{12} - F_{13}F_{32}F_{21}}{z(z-1)},\\
\frac{d}{dz}(F_{23}F_{32}) &= -\frac{F_{23}F_{31}F_{12} - F_{13}F_{32}F_{21}}{z-1},\\
\frac{d}{dz}(F_{13}F_{31}) &= \frac{F_{23}F_{31}F_{12} - F_{13}F_{32}F_{21}}{z}.
\end{aligned}$$

Summarizing we have

$$F_{12}F_{21} = f', \quad F_{23}F_{32} = f - zf' + \frac{q_1}{2} = g_1.$$

Taking into account that $F_{31}F_{13} + F_{12}F_{21} + F_{23}F_{32} = q_1$, we obtain

$$F_{31}F_{13} = (z-1)f' - f + \frac{q_1}{2} = g_2.$$

Using these relations we get (4.15). Up to an inessential sign the above equation coincides with the equation (4.3) appearing in [28] and, as a consequence, it is equivalent to the sigma form of the generic Painlevé VI equation (see [28] for details). □

This proves that each solution of (4.9) determines a specific Painlevé VI equation and it identifies a unique solution of the corresponding Painlevé VI equation itself. To get solutions of the system (4.9) starting from solutions of the equation (4.15) one can proceed in the following way.

Given a specific instance of equation (4.15) and a solution $f(z)$, define d_1 as a root of the cubic polynomial

$$\lambda^3 - (2d_{13} - d_{23})\lambda^2 + (d_{13}^2 - d_{13}d_{23} - q_1)\lambda + q_1 d_{13} - q_2$$

and d_2 and d_3 as $d_2 = d_1 - d_{13} + d_{23}$, $d_3 = d_1 - d_{13}$. In this way the parameters d_1, d_2, d_3, q_1, q_2 satisfy the identity $q_2 = -d_3 q_1 + d_1 d_2 d_3$. Notice that the constants d_1, d_2, d_3, q_2 are determined up to a sign, since the equation (4.15) is invariant under the simultaneous substitutions

$$d_1 \to -d_1, \quad d_2 \to -d_2, \quad d_3 \to -d_3, \quad q_2 \to -q_2.$$

Given d_1, d_2, d_3, q_1 and $f(z)$ one can reconstruct the solution of the system (4.9) solving the algebraic system

$$\begin{aligned}
&F_{12} + F_{13} = \pm d_1, \quad F_{23} + F_{21} = \pm d_2, \quad F_{31} + F_{32} = \pm d_3, \quad F_{12}F_{21} = f',\\
&F_{23}F_{32} = g_1 = f - zf' + \frac{q_1}{2}, \qquad F_{13}F_{31} = g_2 = (z-1)f' - f + \frac{q_1}{2}.
\end{aligned}$$

The solution is

$$\begin{aligned} &F_{12} = \pm\frac{\mu f'}{\mu d_2 - g_1}, \qquad F_{21} = \pm\left(d_2 - \frac{g_1}{\mu}\right), \qquad F_{13} = \pm\left(d_1 - \frac{\mu f'}{\mu d_2 - g_1}\right),\\ &F_{31} = \pm\left(-\mu + d_3\right), \qquad F_{23} = \pm\frac{g_1}{\mu}, \qquad F_{32} = \pm\mu, \end{aligned} \tag{4.16}$$

where μ satisfies

$$(f' - d_1 d_2)\mu^2 + (d_1 d_2 d_3 + d_1 g_1 - d_2 g_2 - d_3 f')\mu - d_1 d_3 g_1 + g_1 g_2 = 0.$$

Indeed, defining the constants d_1, d_2, d_3 and the functions F_{ij} as above it is not difficult to prove that

$$[z(z-1)f'']^2 = [F_{23}F_{31}F_{12} - F_{13}F_{32}F_{21}]^2.$$

Since the functions F_{ij} are defined up to a sign, in a neighborhood of a point $z_0 \neq 0, 1$ such that $f''(z_0) \neq 0$ we can always choose the simultaneous sign of F_{ij} in such a way that the following relation holds:

$$z(z-1)f'' = F_{23}F_{31}F_{12} - F_{13}F_{32}F_{21}.$$

Taking into account the definition of the functions g_1 and g_2 we obtain the system

$$\begin{aligned} &(F_{12}F_{21})' = f'', \quad (F_{13}F_{31})' = (z-1)f'', \quad (F_{23}F_{32})' = -zf'',\\ &(F_{12} + F_{13})' = 0, \quad (F_{21} + F_{23})' = 0, \quad (F_{31} + F_{32})' = 0. \end{aligned}$$

It is easy to check that it is equivalent to the system (4.9) provided that $f'' \neq 0$.

4.3.2. Three-dimensional tri-flat F-manifolds. In this brief Section we provide a complete classification of tri-flat F-manifolds in dimension 3.

First of all we have to solve the system

$$\begin{aligned} &E_{(0)}(\Gamma^i_{ij}) = [\partial_1 + \partial_2 + \partial_3]\Gamma^i_{ij} = 0,\\ &E_{(1)}(\Gamma^i_{ij}) = [u^1\partial_1 + u^2\partial_2 + u^3\partial_3]\Gamma^i_{ij} = -\Gamma^i_{ij},\\ &E_{(2)}(\Gamma^i_{ij}) = [(u^1)^2\partial_1 + (u^2)^2\partial_2 + (u^3)^2\partial_3]\Gamma^i_{ij} = -2u^j\Gamma^i_{ij}. \end{aligned}$$

The general solution is given by

$$\begin{aligned} &\Gamma^1_{12} = \frac{C_{12}(u^3 - u^1)}{(u^2 - u^1)(u^2 - u^3)}, \; \Gamma^1_{13} = \frac{C_{13}(u^1 - u^2)}{(u^3 - u^1)(u^3 - u^2)}, \; \Gamma^2_{21} = \frac{C_{21}(u^2 - u^3)}{(u^1 - u^3)(u^1 - u^2)},\\ &\Gamma^2_{23} = \frac{C_{23}(u^1 - u^2)}{(u^3 - u^1)(u^3 - u^2)}, \; \Gamma^3_{31} = \frac{C_{31}(u^2 - u^3)}{(u^1 - u^3)(u^1 - u^2)}, \; \Gamma^3_{32} = \frac{C_{32}(u^3 - u^1)}{(u^2 - u^1)(u^2 - u^3)}, \end{aligned}$$

where $C_{12}, C_{21}, C_{13}, C_{31}, C_{23}, C_{32}$ are arbitrary constants. Imposing Tsarev's condition we obtain immediately the following constraints

$$C_{13} = -C_{12}, \qquad C_{23} = -C_{21}, \qquad C_{32} = -C_{31}, \qquad C_{12} + C_{23} + C_{31} = 1.$$

REMARK 4.9. *Semisimple Frobenius manifolds are special examples of bi-flat F-manifolds, but it is not difficult to prove that Frobenius manifolds with tri-Hamiltonian structure studied in [36], in general, do not constitute a special subclass of tri-flat F-manifolds.*

5. The non-semisimple regular case. In this Section we are interested in F-manifolds that are not semisimple, but that satisfy still a regularity condition. In order to deal with the non-semisimple regular case we will use a result of David and Hertling [10] about the existence of local "canonical coordinates" for non-semisimple regular F-manifolds with an Euler vector field. Let us summarize the main results of their work which are relevant for our situation.

DEFINITION 5.1 ([10]). *An F-manifold $(M, \circ, e, E)$ where E is an Euler vector field is called* regular *if for each $p \in M$ the endomorphism $L_p := E_p \circ : T_pM \to T_pM$ has exactly one Jordan block for each distinct eigenvalue.*

THEOREM 5.2 ([10]). *Let $(M, \circ, e, E)$ be a regular F-manifold of dimension greater or equal to 2 with an Euler vector field E of weight one. Furthermore assume that locally around a point $p \in M$, the operator L has only one eigenvalue. Then there exists locally around p a distinguished system of coordinates $\{u^1, \dots, u^n\}$ (a sort of "generalized canonical coordinates" for $\circ$) such that $u^2(p) \neq 0$ and*

$$e = \partial_{u^1}, \quad c^k_{ij} = \delta^k_{i+j-1}, \quad E = u^1\partial_{u^1} + \cdots + u^n\partial_{u^n}. \tag{5.1}$$

Notice that here we have performed a shift of the variables u^1 and u^2 compared to the coordinate system identified in [10] to obtain simpler formulas. □

Let us point out that if the endomorphisms $L_p := E_p\circ$ consist of different Jordan blocks with distinct eigenvalues, then the results of [10] can be readily extended using Hertling's Decomposition Lemma [20]. However, in the case in which there are multiple Jordan blocks with the same eigenvalues no results are available to the best of our knowledge.

5.1. Three-dimensional non-semisimple bi-flat F-manifolds. In the three dimensional case, assuming regularity, one has only three possibilities (see the matrices L_1, L_2, L_3 in formula (1.9)). One is the semisimple case (in which L has the form L_1), one is the case with one Jordan block and all eigenvalues equal (this corresponds to $L = L_3$) and this is the situation we analyze in detail in the first part of the next section. In the third case (corresponding to $L = L_2$) there is a non-trivial 2×2 Jordan block with one eigenvalue and a second distinct eigenvalue. This last case is analyzed in detail in the second part of the next section

5.1.1. The case of one single eigenvalue and one Jordan block. In this section, we use canonical coordinates for a regular non-semisimple bi-flat F-manifold in dimension three to show that locally these structures are parameterized by solutions of the full Painlevé IV equations.

THEOREM 5.3. *Let $(M, \nabla, \nabla^*, \circ, *, e, E)$ be a regular bi-flat F-manifold in dimension three such that L_p has three equal eigenvalues. Then there exist local coordinates $\{u^1, u^2, u^3\}$ such that*

1. *$e, E, \circ$ are given by* (5.1).
2. *The Christoffel symbols Γ^i_{jk} for ∇ are given by:*

$$\Gamma^1_{23} = \Gamma^1_{32} = \Gamma^2_{33} = \frac{F_1\left(\frac{u^3}{u^2}\right)}{u^2}, \ \Gamma^3_{23} = \Gamma^3_{32} = \frac{F_2\left(\frac{u^3}{u^2}\right)}{u^2}, \ \Gamma^2_{23} = \Gamma^2_{32} = \frac{F_3\left(\frac{u^3}{u^2}\right)}{u^2},$$

$$\Gamma^1_{22} = \frac{F_4\left(\frac{u^3}{u^2}\right)}{u^2}, \ \Gamma^2_{22} = \frac{F_5\left(\frac{u^3}{u^2}\right)}{u^2}, \ \Gamma^3_{22} = \frac{F_6\left(\frac{u^3}{u^2}\right)}{u^2}, \ \Gamma^3_{33} = \frac{F_3\left(\frac{u^3}{u^2}\right) - F_4\left(\frac{u^3}{u^2}\right)}{u^2},$$

where the functions $F_1, \dots, F_6$ satisfy the system

$$
\begin{aligned}
\frac{dF_1}{dz} &= 0, \\
\frac{dF_2}{dz} &= 2F_4F_3z + 2F_2F_1z - 2F_5F_1z + F_6F_1 - F_2F_3 + F_4 - F_3, \\
\frac{dF_3}{dz} &= -F_4F_3 - F_2F_1 + F_5F_1 - F_1, \\
\frac{dF_4}{dz} &= F_4F_3 + F_2F_1 - F_5F_1 - F_1, \\
\frac{dF_5}{dz} &= F_4F_3z + F_2F_1z - F_5F_1z - F_6F_1 + F_2F_3 + F_1z - F_3, \\
\frac{dF_6}{dz} &= -2F_4F_3z^2 - 2F_2F_1z^2 + 2F_5F_1z^2 - F_6F_1z + F_2F_3z \\
&\qquad + F_4F_6 - F_4z + F_2^2 - F_2F_5 + F_3z - F_2.
\end{aligned}
\tag{5.2}
$$

in the variable $z = \frac{u^3}{u^2}$ while the other symbols are identically zero.

3. *The dual product $*$ is obtained via formula* (1.2) *using $\circ$ and E.*
4. *The Christoffel symbols Γ^{*i}_{jk} for ∇^* are obtained via formula* (1.8).

Proof. Due to David-Hertling results, there exist local coordinates such that $e, E, \circ$ are given by (5.1). To determine the Christoffel symbols Γ^k_{ij} for the torsionless connection ∇ in these coordinates we start imposing the following conditions:

- compatibility with $\circ$: $\Gamma^i_{ml}c^m_{jk} - \Gamma^m_{lk}c^i_{jm} - \Gamma^i_{mj}c^m_{lk} + \Gamma^m_{jk}c^i_{lm} = 0,\ 1 \le l, j, k \le 3$,
- torsion freeness of the connection: $\Gamma^k_{ij} = \Gamma^k_{ji}$,
- flatness of unity: $\nabla e = 0 \iff \Gamma^i_{1j} = 0$.

This provides a system of algebraic equations for Γ^k_{ij}. These symbols are in general functions of u^1, u^2, u^3. Imposing the commutativity of ∇ and $\mathrm{Lie_e}$, coming from the flatness of ∇ we obtain that the symbols $\Gamma^k_{i,j}$ do not depend on u^1.

Now we use the expression of the Euler vector field in the canonical coordinates and impose the commutativity of ∇^* with $\mathrm{Lie_E}$, coming from the flatness of ∇^*. We obtain:

$$E^m\partial_m\Gamma^{*i}_{jk} - \Gamma^{*m}_{jk}\partial_m E^i + \Gamma^{*i}_{mk}\partial_j E^m + \Gamma^{*i}_{jm}\partial_k E^m + \partial_j\partial_k E^i = 0.$$

Since Γ^{*i}_{jk} are expressed uniquely in terms of Γ^i_{jk}, the previous system of PDEs reduces to a system for the unknown Γ^i_{jk}. In particular, we observe that for $[j,k,i] = [3,2,2]$ we get the PDE:

$$u^2(\partial_{u^2}\Gamma^2_{32}) + \partial_{u^3}\Gamma^2_{32}u^3 + \Gamma^2_{32} = 0,$$

and for $[j,k,i] = [2,3,3]$ and $[j,k,i] = [3,1,1]$ we get an identical PDE for Γ^3_{32} and Γ^1_{32} respectively. The general solutions of these PDEs can be obtained directly with the method of characteristics yielding

$$\Gamma^1_{32} = F_1\left(\frac{u^3}{u^2}\right)\frac{1}{u^2},\ \Gamma^3_{32} = F_2\left(\frac{u^3}{u^2}\right)\frac{1}{u^2},\ \Gamma^3_{32} = F_3\left(\frac{u^3}{u^2}\right)\frac{1}{u^2}.$$

Substituting these solutions in the remaining equations, we obtain similar conditions for Γ^2_{22}, Γ^1_{22} and Γ^3_{22}:

$$\Gamma^1_{22} = F_4\left(\frac{u^3}{u^2}\right)\frac{1}{u^2},\ \Gamma^2_{22} = F_5\left(\frac{u^3}{u^2}\right)\frac{1}{u^2},\ \Gamma^3_{22} = F_6\left(\frac{u^3}{u^2}\right)\frac{1}{u^2}.$$

Imposing the zero curvature conditions for ∇, we obtain the system (5.2) for the unknown functions F_i in the variable $z = \frac{u^3}{u^2}$. □

The system (5.2) reduces to the full Painlevé IV family of equations.

THEOREM 5.4. *Regular bi-flat F-manifolds in dimension three such that L_p has three equal eigenvalues and one Jordan block are locally parameterized by solutions of the full Painlevé IV equation.*

Proof. It is straightforward to check that the system of ordinary differential equations given by (5.2) admits the following integrals of motion:

$$I_1 = F_1,\ I_2 = 2F_1 z + F_3 + F_4,\ I_3 = -2F_3 z + F_4 z - F_2 - F_5.$$

Using these first integrals, the system can be reduced to a system of three ODEs given by:

$$\begin{aligned}
\frac{dF_4}{dz} &= 4I_1^2 z^2 - 2I_1 I_2 z + F_4 I_1 z + I_2 F_4 - I_3 I_1 - F_4^2 - 2I_1 F_5 - I_1,\\
\frac{dF_5}{dz} &= -4I_1^2 z^3 + 6I_2 I_1 z^2 - 9F_4 I_1 z^2 - 2I_2^2 z + 6I_2 F_4 z + I_3 I_1 z - 4F_4^2 z - I_2 I_3\\
&\quad -I_2 F_5 + I_3 F_4 + F_4 F_5 - I_1 F_6 + 3I_1 z - I_2 + F_4,\\
\frac{dF_6}{dz} &= -4I_2 I_1 z^3 + 12F_4 I_1 z^3 + 2I_2^2 z^2 - 9I_2 F_4 z^2 - 4I_3 I_1 z^2 + 8F_4^2 z^2\\
&\quad -6I_1 F_5 z^2 + 3I_2 I_3 z + 5I_2 F_5 z - 5I_3 F_4 z - 8F_4 F_5 z - I_1 F_6 z - 6I_1 z^2\\
&\quad +3I_2 z + I_3^2 + 3I_3 F_5 + F_6 F_4 - 5F_4 z + 2F_5^2 + I_3 + F_5.
\end{aligned}$$

We further reduce this system to a second order ODE in the following way. First we express F_5 in terms of F_4 and its first derivative using the first equation, obtaining (here and thereafter we assume $I_1 \neq 0$):

$$F_5 = \frac{1}{2I_1}\left(-\frac{dF_4}{dz} + 4I_1^2 z^2 - 2I_2 I_1 z - I_3 I_1 - I_1 + F_4 I_1 z + I_2 F_4 - F_4^2\right).$$

We substitute this in the second equation and solve for F_6:

$$\begin{aligned}
F_6 = \frac{1}{2I_1^2}\Bigg[&\frac{d^2F_4}{dz^2} - (I_1 z - F_4)\frac{dF_4}{dz} + I_1 I_2 - I_1 I_2 I_3 - 2I_1^2 z - 2I_1 I_2^2 z + 2I_1^2 I_3 z +\\
&+8I_1^2 I_2 z^2 - 8I_1^3 z^3 + (I_1 I_3 - I_2^2 + 9I_1 I_2 z - 14I_1^2 z^2)F_4 + (2I_2 - 7I_1 z)F_4^2 - F_4^3\Bigg].
\end{aligned}$$

Substituting these expressions for F_5 and F_6 in terms of F_4 and its derivatives in the last ODE of the system above, we obtain a third order nonlinear ODE for F_4. Multiplying it by $2I_1^2(-I_1 z + I_2 - F_4)$, it is possible to recognize that it is a total derivative with respect to z of an expression involving the second derivative of F_4. Integrating this expression one obtains the nonlinear second order ODE for the function $F_4 = F$:

$$\begin{aligned}
&+(I_2 - I_1 z - F)\frac{d^2F}{dz^2} + \frac{1}{2}\left(\frac{dF}{dz}\right)^2 + I_1\frac{dF}{dz} + C + 2I_1 I_2(I_1 I_3 - I_2^2 - I_1)z +\\
&+I_1^2(8I_2^2 - I_1 I_3 + I_1)z^2 - 10I_1^3 I_2 z^3 + 4I_1^4 z^4 + (2I_1 I_3 - 2I_1 - I_2^2)I_2 F +\\
&+(2I_1 + 11I_2^2 - 2I_1 I_3 - 23I_1 I_2 z + 13I_1^2 z^2)I_1 z F + \left(I_1 - I_1 I_3 + \frac{7}{2}I_2^2\right)F^2 +\\
&-\left(17I_2 - \frac{31}{2}I_1 z\right)I_1 z F^2 + 8I_1 z F^3 - 4I_2 F^3 + \frac{3}{2}F^4 = 0,
\end{aligned}$$

where C is the constant of integration. Now we show that this ODE can be reduced to the full Painlevé IV equation.

First we do a change of variables of the form $F(z) = f(z) - I_1 z + I_2$ in order to obtain a term of the form $f(z)\frac{d^2 f}{dz^2}$ which is the term that appears in Painlevé IV. Doing this we obtain the following ODE:

$$f\frac{d^2 f}{dz^2} = \frac{1}{2}\left(\frac{d^2 f}{dz^2}\right) + \frac{3}{2}f^4 + (2zI_1 + 2I_2)\, f^3 + \left(\frac{1}{2}I_1^2 z^2 + I_1 I_2 z - I_1 b + \frac{1}{2}I_2^2\right) f^2 + c,$$

where $b = I_3 - 1$ and $c = C + I_1 I_2^2 I_3 - \frac{1}{2}I_1^2 - I_2^2 I_1$. Introducing the affine transformation $z = \sqrt{\frac{2}{I_1}}t - \frac{I_2}{I_1}$ and the function $y(t) = \sqrt{\frac{2}{I_1}} f\left(\sqrt{\frac{2}{I_1}}t - \frac{I_2}{I_1}\right)$ the previous ODE becomes:

$$y\frac{d^2 y}{dt^2} = \frac{1}{2}\left(\frac{dy}{dt}\right)^2 + \frac{3}{2}y^4 + 4ty^3 + 2\left(t^2 - b\right) y^2 + c,$$

which is indeed the full Painlevé IV family. □

REMARK 5.5. *In the proof of the previous Theorem we have assumed that $I_1 \neq 0$, hence the genericity statement. If $I_1 = 0$ then the system* (5.2) *reduces to a system of ODEs that can be integrated explicitly. In particular, using the integrals of motion $I_1 = 0$, I_2 and I_3, the system obtained by reduction and involving only F_4, F_5 and F_6 is lower triangular.*

5.1.2. The case of two distinct eigenvalues and two Jordan blocks. In this subsection we analyze the case in which the operator L_p has two distinct eigevanlues, one eigenvalue with algebraic multiplicity two (and nontrivial 2×2 Jordan block), while the other eigenvalue is simple.

THEOREM 5.6. *Let $(M, \nabla, \nabla^*, \circ, *, e, E)$ be a non-semisimple regular bi-flat F-manifold in dimension three such that L_p has exactly two distinct eigenvalues and two Jordan blocks. Then there exist local coordinates $\{u^1, u^2, u^3\}$ around p such that $u^2(p) \neq 0$, $u^3(p) \neq 0$ and*

1. $e, E, \circ$ are given by

$$e = \partial_{u^1} + \partial_{u^3} \tag{5.3}$$
$$E = u^1\partial_{u^1} + u^2\partial_{u^2} + u^3\partial_{u^3} \tag{5.4}$$
$$c^i_{jk} = \delta^k_{i+j-1} \quad \textit{if} \;\; 1 \leq i, j, k \leq 2 \tag{5.5}$$
$$c^3_{33} = 1 \tag{5.6}$$
$$c^i_{jk} = 0 \quad \textit{in all other cases} \tag{5.7}$$

2. The Christoffel symbols Γ^i_{jk} for ∇_1 are given by:

$$\Gamma^3_{13} = \frac{F_4\left(\frac{u^3-u^1}{u^2}\right)}{u^2}, \quad \Gamma^1_{22} = \frac{F_3\left(\frac{u^3-u^1}{u^2}\right)}{u^2}, \quad \Gamma^2_{22} = \frac{F_6\left(\frac{u^3-u^1}{u^2}\right)}{u^2},$$

$$\Gamma^3_{23} = \frac{F_1\left(\frac{u^3-u^1}{u^2}\right)}{u^2}, \quad \Gamma^1_{31} = \frac{F_2\left(\frac{u^3-u^1}{u^2}\right)}{u^2}, \quad \Gamma^2_{31} = \frac{F_5\left(\frac{u^3-u^1}{u^2}\right)}{u^2},$$

$$\Gamma^1_{11} = -\Gamma^1_{31}, \quad \Gamma^2_{11} = -\Gamma^2_{31}, \quad \Gamma^3_{11} = -\Gamma^3_{13}, \quad \Gamma^2_{12} = -\Gamma^1_{31}, \quad \Gamma^3_{12} = -\Gamma^3_{23},$$

$$\Gamma^2_{21} = -\Gamma^1_{31},\ \ \Gamma^3_{21} = -\Gamma^3_{23},\ \ \Gamma^2_{23} = \Gamma^1_{31},$$

$$\Gamma^1_{33} = -\Gamma^1_{31},\ \ \Gamma^2_{33} = -\Gamma^2_{31},\ \ \Gamma^3_{33} = -\Gamma^3_{13},$$

where the functions $F_1, \dots F_6$ satisfy the system

$$\begin{aligned}
\frac{dF_1}{dz} &= -\frac{F_3F_4 - F_1^2 + F_1F_6 + F_1}{z},\\
\frac{dF_2}{dz} &= \frac{F_3F_5 - F_2F_1 - F_2}{z},\\
\frac{dF_3}{dz} &= 0,\\
\frac{dF_4}{dz} &= -\frac{F_3F_4 - F_1^2 + F_1F_6 + F_4z + F_1}{z^2},\\
\frac{dF_5}{dz} &= \frac{-F_5F_1z + F_5F_6z + F_2F_4z + F_3F_5 - F_5z - F_2F_1 - F_2}{z^2},\\
\frac{dF_6}{dz} &= -2F_3F_5 + 2F_2F_1.
\end{aligned} \tag{5.8}$$

in the variable $z = \frac{u^3-u^1}{u^2}$ while the other symbols not obtainable from the above list using the symmetry of the connection are identically zero.

3. *The dual product $*$ is obtained via formula* (1.2) *using $\circ$ and E.*
4. *The Christoffel symbols Γ^{*i}_{jk} for ∇^* are obtained via formula* (1.8).

Proof. The first point of the Theorem is a direct consequence of the results of [10] and of Hertling's Decomposition Lemma (Thereom 2.11 from [20]) . Imposing that ∇ is torsionless, that it is compatible with $\circ$, and that it satisfies $\nabla e = 0$, we obtain the following constraints on Γ^k_{ij}:

$$\Gamma^1_{11} = -\Gamma^1_{31},\ \ \Gamma^2_{11} = -\Gamma^2_{31},\ \ \Gamma^3_{11} = -\Gamma^3_{13},\ \ \Gamma^1_{12} = 0,\ \ \Gamma^2_{12} = -\Gamma^1_{31},$$

$$\Gamma^1_{33} = -\Gamma^1_{31},\ \ \Gamma^2_{33} = -\Gamma^2_{31},\ \ \Gamma^3_{33} = -\Gamma^3_{13},$$

$$\Gamma^3_{12} = -\Gamma^3_{23},\ \ \Gamma^1_{21} = 0,\ \ \Gamma^3_{21} = -\Gamma^3_{23},$$

$$\Gamma^3_{22} = 0,\ \ \Gamma^1_{23} = 0,\ \ \Gamma^2_{23} = \Gamma^1_{31},$$

together with the trivial constraints $\Gamma^k_{ij} = \Gamma^k_{ji}$.

Once we have expressed the Christoffel symbols of ∇^* in terms of the Christoffel symbols of ∇ (using (1.8)), imposing the commutativity of ∇^* with Lie_E and the commutativity of Lie_e with ∇, we obtain a system of PDEs for the variables Γ^k_{ij}. This system in particular implies that $\Gamma^k_{ij}(u^1, u^2, u^3)$ can be expressed as functions of two variables, as $\Gamma^k_{ij}(u^2, u^3 - u^1)$. Following a procedure similar to the process described in the proof of Theorem 5.3, we can solve the two systems and we find that (here $z = \frac{u^3-u^1}{u^2}$):

$$\Gamma^3_{13} = \frac{F_4(z)}{u^2},\ \ \Gamma^1_{22} = \frac{F_3(z)}{u^2},\ \ \Gamma^2_{22} = \frac{F_6(z)}{u^2},$$

$$\Gamma^3_{23} = \frac{F_1(z)}{u^2},\ \Gamma^1_{31} = \frac{F_2(z)}{u^2},\ \Gamma^2_{31} = \frac{F_5(z)}{u^2},\ \Gamma^2_{32} = \frac{F_2(z)}{u^2},$$

for arbitrary smooth functions $F_i(z)$. At this point, we impose the remaining conditions. Imposing the zero curvature conditions for ∇, we obtain the system of equations (5.8). □

Now we prove that the system (5.8) can be reduced to Painlevé V equation.

THEOREM 5.7. *Regular bi-flat F-manifolds in dimension three such that L_p has two distinct eigenvalues and two Jordan blocks are locally parameterized by solutions of the full Painlevé V equation.*

Proof. It is straightforward to check that $I_1 = F_3$, $I_2 = F_1 - F_4 z$ and $I_3 = F_6 + 2F_2$ are constant along the solutions of the system (5.8). Using these three integrals of motion in the system above we reduce it to the following three ODEs:

$$\frac{dF_2}{dz} = -F_2F_4 + \frac{I_1F_5 - I_2F_2 - F_2}{z}, \tag{5.9}$$

$$\frac{dF_4}{dz} = 2F_2F_4 + F_4^2 + \frac{2I_2(F_2+F_4) - (I_3+2)F_4}{z} - \frac{I_1F_4 - I_2^2 + I_2I_3 + I_2}{z^2}, \tag{5.10}$$

$$\frac{dF_5}{dz} = -2F_5F_2 - F_5F_4 - \frac{I_2F_5 - I_3F_5 + F_5}{z} + \frac{I_1F_5 - I_2F_2 - F_2}{z^2}. \tag{5.11}$$

Solving for F_5 in (5.9) and for F_2 in (5.10) and substituting in (5.11) we obtain a third order ODE. Using the integrating factor $\mu = (-F_4(z)z^2 - I_2z + I_1)z$ we get the second order nonlinear ODE for the function $F_4 = F$:

$$\begin{aligned}
&2z^4(I_2+Fz)(I_1 - I_2z - Fz^2)\frac{d^2F}{dz^2} - z^5(I_1 - 2I_2z - 2Fz^2)\left(\frac{dF}{dz}\right)^2 + \\
&+2z^3(4I_1I_2 - 3I_2^2z + 3I_1Fz - 4I_2Fz^2 - F^2z^3)\frac{dF}{dz} + \\
&+I_2^2(2I_2^3z^2 - 2I_2^2I_3z^2 - 2I_2^2z^2 + 4CI_1z + 2I_1^2I_2 - 2I_1^2I_3 - 2I_1^2 - 3I_1z) + \\
&+2I_2\left(5I_2^3z^3 - 4I_2^2I_3z^3 - 5I_1I_2^2z^2 + 2I_1I_2I_3z^2 - 4I_2^2z^3 + 4CI_1z^2 + \right.\\
&\left.+5I_1^2I_2z - 2I_1^2I_3z + 2I_1I_2z^2 - I_2z^3 - I_1^3 - 2I_1^2z - I_1z^2\right)F + \\
&z\left(20I_2^3z^3 - 12I_2^2I_3z^3 - 25I_1I_2^2z^2 + 8I_1I_2I_3z^2 - 12I_2^2z^3 + 4CI_1z^2 + \right.\\
&\left.+12I_1^2I_2z - 2I_1^2I_3z + 8I_1I_2z^2 - 2I_2z^3 - I_1^3 - 2I_1^2z\right)F^2 + \\
&4z^3(5I_2^2z^2 - 2I_2I_3z^2 - 5I_1I_2z + I_1I_3z - 2I_2z^2 + I_1^2 + I_1z)F^3 + \\
&-z^5(-10I_2z + 2I_3z + 5I_1 + 2z)F^4 + 2F^5z^7 = 0,
\end{aligned}$$

where C is an integration constant. The above equation can be reduced to Painlevé V first using the nonlinear transformation:

$$F(z) = \frac{I_1}{z^2}\frac{H(z)}{H(z)-1} - \frac{I_2}{z}$$

and then setting $z = \frac{1}{s}$ and $G(s) = H(z) = H\left(\frac{1}{s}\right)$. The final result is the Painlevé V equation

$$\frac{d^2G}{ds^2} = \left(\frac{1}{2G} + \frac{1}{G-1}\right)\left(\frac{dG}{ds}\right)^2 - \frac{1}{s}\frac{dG}{ds} + \frac{1}{s^2}(G-1)^2\left(aG + \frac{b}{G}\right) + \frac{cG}{s} + \frac{dG(G+1)}{G-1}$$

where $a = -2I_2^2 + 2I_2I_3 - 2C + 2I_2 + 2$, $b = -\frac{1}{2}I_2^2$, $c = I_1(I_3+1)$, $d = -\frac{1}{2}I_1^2$.

□

Remark 5.8. *In the proof of the previous Theorem we have assumed that $I_1 \neq 0$, hence the genericity statement. If $I_1 = 0$ then the system* (5.8) *reduces to a system of ODEs that can be integrated explicitly.*

5.2. Regular case and confluences of Painlevé equations. In this Section, we have shown that there exists an intimate relationship between regular bi-flat F-manifolds in dimension three on one hand and Painlevé transcendents on the other. Our analysis leads us to conclude that regular bi-flat F-manifolds in dimension three are characterized by continuous and discrete moduli. The discrete moduli are provided by the Jordan normal form for the operator L, which in turns determines which of the Painlevé equations controls the continuous moduli.

Furthermore, the well-known confluence of the Painlevé equations is associated to a corresponding degeneration of the form of the operator L characterizing regular three-dimensional bi-flat F-manifold. In this way, confluences of the Painlevé equations are mirrored in the collision of eigenvalues and the creation of non-trivial Jordan blocks according to the following diagram:

$$\begin{tikzcd}
P_{VI} \arrow[r, "\text{confluence}"] \arrow[d, dashed, leftrightarrow] & P_V \arrow[r, "\text{confluence}"] \arrow[d, dashed, leftrightarrow] & P_{IV} \arrow[d, dashed, leftrightarrow] \\
L_1 \arrow[r, "\text{degeneration of distinct eigenvalues}", "\text{preserving regularity}"'] & L_2 \arrow[r, "\text{degeneration of distinct eigenvalues}", "\text{preserving regularity}"'] & L_3
\end{tikzcd}$$

As an open problem, let us mention the fact that it would be interesting to extend this correspondence to include the remaining Painlevé transcendents on one side and possibly non-regular bi-flat F-manifolds on the other.

5.3. Multi-flat F-manifolds in the regular non-semisimple case. In this Subsection we are going to study three-flat and multi-flat F-manifolds in the regular non-semisimple case. For simplicity we focus our attention on the case in which the Jordan normal form of the operator L contains only one Jordan block with the same eigenvalues.

5.3.1. Tri-flat F-manifolds. The next theorem shows that regular three-flat F-manifolds in dimension three such that L_p has three equal eigenvalues are locally represented as follows.

Theorem 5.9. *Let $(M, \nabla_0 = \nabla, \nabla_1, \nabla_2, \circ_0 = \circ, \circ_1, \circ_2, E_0 := e, E_1 := E, E_2 := E \circ E)$ be a regular three-flat F-manifold in dimension three such that L_p has three equal eigenvalues. Then there exist local coordinates $\{u^1, u^2, u^3\}$ such that*

1. *The vector fields e and E and the product $\circ$ are given by* (5.1).
2. *The Christoffel symbols Γ^i_{jk} for ∇ are given by:*

$$\Gamma^1_{23} = \Gamma^1_{32} = \Gamma^2_{33} = \frac{f_1}{u^2}, \; \Gamma^3_{32} = \Gamma^3_{23} = \frac{F_2\left(\frac{u^3}{u^2}\right)}{u^2}, \; \Gamma^2_{32} = \Gamma^2_{23} = \frac{f_3}{u^2},$$

$$\Gamma^1_{22} = \frac{F_4\left(\frac{u^3}{u^2}\right)}{u^2}, \; \Gamma^2_{22} = \frac{F_5\left(\frac{u^3}{u^2}\right)}{u^2}, \; \Gamma^3_{22} = \frac{F_6\left(\frac{u^3}{u^2}\right)}{u^2}, \; \Gamma^3_{33} = \frac{f_3 - F_4\left(\frac{u^3}{u^2}\right)}{u^2},$$

where f_1 and f_3 are constants and the functions F_2, F_4, F_5, F_6 are given by

$$F_2 = -f_1 z^2 - 1,\ F_4 = -2f_1 z,\ F_5 = -f_1 z^2 - 2f_3 z,\ F_6 = -f_3 z^2 + 2z \quad (5.12)$$

in the variable $z = \frac{u^3}{u^2}$ while the other Christoffel symbols are identically zero.

3. *The products $\circ_1$ and $\circ_2$ are obtained via formula* (1.2) *using as eventual identities E and E_2 respectively.*
4. *The Christoffel symbols $\Gamma^{(1)i}_{jk}$ and $\Gamma^{(2)i}_{jk}$ are obtained via formula* (1.8).

Proof. The first part of the proof is the same as the proof given for Theorem 5.3. Imposing the additional conditions coming from the flatness of $\nabla^{(2)}$ we get the formulas for Γ^i_{jk} appearing in the statement of the theorem. □

5.3.2. An example with infinitely many compatible flat structures. With similar computations it is possible to add further connections and try to construct F-manifolds with four or more compatible flat connections. A very remarkable phenomenon is the following: once a quadri-flat F-manifold has been constructed, no new conditions arise if one tries to equip it with further flat compatible connections. In other words, regular quadri-flat F-manifolds in dimension three with an operator L consisting of a single Jordan block are automatically "infinitely"-flat F-manifolds.

Theorem 5.10. *The data*

$$c^k_{ij} = \delta^k_{i+j-1},$$
$$E_{(l)} = E^l = (u^1)^l \partial_{u^1} + l u^2 (u^1)^{l-1} \partial_{u^2} + \left(l u^3 (u^1)^{l-1} + \frac{1}{2}(l^2 - l)(u^2)^2 (u^1)^{l-2} \right) \partial_{u^3},$$

and

$$\Gamma^{(l)1}_{11} = -\frac{l}{u^1},\ \Gamma^{(l)2}_{11} = \frac{l u^2 (l a^2 + l a + a + 2)}{(a+2)(u^1)^2}$$
$$\Gamma^{(l)3}_{11} = \frac{l((2la^2 + 2la + a + 2)u^1 u^3 - (la^2 + 2la + a + 2)(u^2)^2 + (lab + 2lb)u^1 u^2)}{(a+2)(u^1)^3}$$
$$\Gamma^{(l)1}_{12} = \Gamma^{(l)1}_{21} = 0,\ \Gamma^{(l)2}_{12} = \Gamma^{(l)2}_{21} = -\frac{l(a^2 + 2a + 2)}{(u^1)(a+2)},\ \Gamma^{(l)3}_{23} = \Gamma^{(l)3}_{32} = \frac{a}{u^2}$$
$$\Gamma^{(l)3}_{12} = \Gamma^{(l)3}_{21} = \frac{l((la^2 + a^2 + 2la + 4a + 4)(u^2)^2 - 2a^2 u^1 u^3 - (2ab + 4b)u^1 u^2)}{2u^2 (a+2)(u^1)^2},$$
$$\Gamma^{(l)1}_{13} = \Gamma^{(l)1}_{31} = \Gamma^{(l)2}_{13} = \Gamma^{(l)2}_{31} = \Gamma^{(l)1}_{22} = 0,\ \Gamma^{(l)3}_{13} = \Gamma^{(l)3}_{31} = -\frac{l(a+1)}{u^1},$$
$$\Gamma^{(l)3}_{22} = -\frac{((la^2 + 3la + 2l)(u^2)^2 - (ab - 2b)u^1 u^2 + 2a u^1 u^3)}{(a+2)u^1 (u^2)^2},\ \Gamma^{(l)2}_{22} = \frac{a(a+1)}{u^2(a+2)}$$
$$\Gamma^{(l)1}_{23} = \Gamma^{(l)1}_{32} = \Gamma^{(l)2}_{23} = \Gamma^{(l)2}_{32} = \Gamma^{(l)1}_{33} = \Gamma^{(l)2}_{33} = \Gamma^{(l)3}_{33} = 0,$$

locally define a regular three dimensional multi-flat F-manifold $(M, \nabla_{(l)}, \circ_l, E_{(l)},\ l = 0, 1, 2...)$ for any value of the constants a and b.

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REFERENCES

[1] A. Arsie and P. Lorenzoni, *F-manifolds with eventual identities, bidifferential calculus and twisted Lenard-Magri chains*, Int Math Res Notices, 17 (2013), pp. 3931–3976.

[2] A. Arsie and P. Lorenzoni, *From Darboux-Egorov system to bi-flat F-manifolds*, Journal of Geometry and Physics, 70 (2013), pp. 98–116.

[3] A. Arsie and P. Lorenzoni, *Complex reflection groups, logarithmic connections and bi-flat F-manifolds*, Lett Math Phys, 107 (2017), pp. 1919–1961.

[4] P. Boalch, *Painlevé equations and complex reflections*, Ann. Inst. Fourier, 53:4 (2003), pp. 1009–1022.

[5] P. Boalch, *From Klein to Painlevé via Fourier, Laplace and Jimbo*, Proc. London Math. Soc. (3), 90 (2005), pp. 167–208.

[6] P. Boalch, *The fifty-two icosahedral solutions to Painlevé VI*, J. Reine Angew. Math., 596 (2006), pp. 183–214.

[7] G. Carlet, P. Lorenzoni and A. Raimondo, *The reductions of the dispersionless 2D Toda hierarchy and their Hamiltonian structures*, J. Phys. A: Math. Theor., 4 (2009).

[8] W. L. Chow, *Über Systeme von linearen partiellen Differentialgleichungen erster Ordnung*, Math. Ann., 117 (1939), pp. 98–105.

[9] W. Couwenberg, G. Heckman and E. Looijenga, *Geometric Structures on the Complement of a Projective Arrangement*, Publications Mathématiques de l'IHÉS, 101 (2005), pp. 69-161.

[10] L. David and C. Hertling, *Regular F-manifolds: initial conditions and Frobenius metrics*, Annali della Scuola Normale di Pisa, Classe di Scienze Vol. XVII, issue, 3 (2017), pp. 1121–1152.

[11] L. David and I. A. B. Strachan, *Dubrovin's duality for F-manifolds with eventual identities*, Advances in Mathematics, 226:5 (2011), pp. 4031–4060.

[12] L. David and I. A. B. Strachan, *Symmetries of F-manifolds with eventual identities and special families of connections*, Annali della Scuola Normale di Pisa, Classe di Scienze Vol. XIII, issue 3 (2014), pp. 641–674.

[13] B. A. Dubrovin, *Differential geometry of the space of orbits of a Coxeter group*, Surv. Diff. Geom., IV (1999), pp. 181–211.

[14] B. A. Dubrovin, *Geometry of 2D topological field theories*, in: Integrable Systems and Quantum Groups, Montecatini Terme, 1993. Editors: M. Francaviglia, S. Greco. Springer Lecture Notes in Math., 1620 (1996), pp. 120–348.

[15] B. A. Dubrovin, *On almost duality for Frobenius manifolds*, Geometry, topology and mathematical physics, Amer. Math. Soc. Transl. Ser. 2, 212 (2004), pp. 75–132.

[16] B. A. Dubrovin and S. P. Novikov, *On Hamiltonian brackets of hydrodynamic type*, Soviet Math. Dokl., 279:2 (1984), pp. 294–297.

[17] E. V. Ferapontov, *Differential geometry of nonlocal Hamiltonian operators of hydrodynamic type*, Funkts. Anal. i Pril., 25 (3), 95 (1991), pp. 37–49.

[18] E. V. Ferapontov and M. V. Pavlov, *Quasiclassical limit of coupled KdV equations, Riemann invariants and multi-Hamiltonian structures*, Phys. D, 52 (1991), pp. 211–219.

[19] J. Gibbons, P. Lorenzoni and A. Raimondo, *Hamiltonian Structures of Reductions of the Benney System*, Commun. Math. Phys., 287 (2009,) pp. 291–322.

[20] C. Hertling, *Frobenius manifolds and moduli spaces for singularities*, Cambridge University Press, (2002).

[21] C. Hertling and Yu. Manin, *Weak Frobenius Manifolds*, Int Math Res Notices, 6 (1999), pp. 277–286.

[22] M. Kato, T. Mano, and J. Sekiguchi, *Flat structure on the space of isomonodromic deformations*, `arXiv:1511.01608v1`.

[23] G. Lauricella, *Sulle funzioni ipergeometriche a piu' variabili*, Rendiconti del Circolo Mat. Palermo, 7 (1893), pp. 111–158.

[24] Y. P. Lee, *Quantum K-theory: Foundations*, Duke Math. J., 121:3 (2004), pp. 389–424.

[25] E. Looijenga, *Uniformization by Lauricella functions: an overview of the theory of Deligne-Mostow*, in Arithmetic and Geometry Around Hypergeometric Functions, Prog. Math., 260

Birkhäuser, Basel (2007), pp. 207–244.

[26] P. Lorenzoni, M. Pedroni, and A. Raimondo, *F-manifolds and integrable systems of hydrodynamic type*, Archivum Mathematicum, 47 (2011), pp. 163–180.

[27] P. Lorenzoni and M. Pedroni, *Natural connections for semi-Hamiltonian systems: the case of the ϵ-system*, Lett. Math. Phys., 97:1 (2011), pp. 85–108.

[28] P. Lorenzoni, *Darboux-Egorov system, bi-flat F-manifolds and Painlev VI*, Int Math Res Notices, 12 (2014), pp. 3279–3302.

[29] A. Losev and Yu. Manin, *Extended modular operad*, in C. Hertling, M. Marcolli (Eds.) Frobenius Manifolds, Vieweg & Sohn Verlag, Wiesbaden, (2004), pp. 181–211.

[30] Y. Manin, *F-manifolds with flat structure and Dubrovin's duality*, Adv. Math., 198:1 (2005), pp. 5–26.

[31] Y. Manin, *Frobenius Manifolds, Quantum Cohomology and Moduli Spaces*, AMS, Colloquium Publications, 47 (1999).

[32] S. Merkulov, *Operads, deformation theory and F-manifolds* in C. Hertling, M. Marcolli (Eds.) Frobenius Manifolds, Vieweg & Sohn Verlag, Wiesbaden, (2004).

[33] M. V. Pavlov, S. I. Svinolupov, and R. A. Sharipov, *An invariant criterion for hydrodynamic integrability*, (Russian) Funktsional. Anal. i Prilozhen., 30:1 (1996), pp. 18–29, 96; translation in Funct. Anal. Appl., 30:1 (1996), pp. 15–22.

[34] M. V. Pavlov, *Integrable hydrodynamic chains*, Journal of Mathematical Physics, 44:9 (2003), pp. 4134–56.

[35] P. K. Rashevskiĭ, *On the connectability of two arbitrary points of a totally nonholonomic space by an admissible curve*, Uchen. Zap. Mosk. Ped. Inst. Ser. Fiz.-Mat. Nauk, 3:2 (1938), pp. 83–94.

[36] S. Romano, 4-*dimensional Frobenius manifolds and Painlevé VI*, Mathematische Annalen, 360 Issue 3-4 (2012), pp. 715–751.

[37] D. Serre, *Richness and the classification of quasilinear hyperbolic systems*, IMA vol. in Math and their appl. Vol.29, Springer Verlag, (1991).

[38] D. Serre, *Systèmes hyperboliques riches de lois de conservation*, Prepubl. Lyon-St.Etienne, 74 (1988).

[39] B. Sévennec, *Géométrie des systèmes hyperboliques riches de lois des conservation*, Mémoires de la S.M.F., deuxieme série tome 56 (1994).

[40] S. P. Tsarev, *Ph.D thesis*, Moscow State University (1986).

[41] S. P. Tsarev, *The geometry of Hamiltonian systems of hydrodynamic type. The generalised hodograph transform*, USSR Izv., 37 (1991), pp. 397–419.

[42] A. P. Veselov, *Deformations of the root systems and new solutions to generalised WDVV equations*, Physics Letters A, 261 (1999), pp. 297–302.

[43] A. P. Veselov, *On geometry of a special class of solutions to generalized WDVV equations*, Integrability: the Seiberg-Witten and Whitham equations (Edinburgh, 1998), pp. 125–135, Gordon and Breach.