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HIGH ORDER LINEAR EXTENDED STATE OBSERVER AND ERROR ANALYSIS OF ACTIVE DISTURBANCE REJECTION CONTROL*

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Abstract. In this paper, we aim to give a delicate analysis of the error dynamics of linear active disturbance rejection control(LADRC) for nonlinear time-varying plants with unknown dynamics. We first give precise upper bounds for the estimation error of linear extended state observer(LESO) and tracing error and then generalize the results to a higher order LADRC. The settling time is studied numerically. The results can be very useful for determining the parameters of LADRC. The results are very effective as verified by various simulations.

Key words. Active disturbance rejection control, feedback linearization, output feedback and observers, stability of nonlinear systems, uncertain systems.

Mathematics Subject Classification. 65L20, 65L70, 93C10, 93D15, 93D20.

1. Introduction.

One of the fundamental problem of control theory is how to deal with uncertainties, since most of the control system in real world are not only nonlinear, time varying but also with uncertainty. The well known control theoretician Roger Brockett once said: “If there is no uncertainty in the system, the control, or the environment, feedback control is largely unnecessary”[1]. Scientists and engineers take a lot of effort to tackle this problem.

In 1930s, the Soviet Union scholar Schipanov came up with the principle of invariance. B. N. Petrov later gave the dual-channel design principle where the external disturbance signal is measured and used for its cancellation[2]. In 1970s, W. M. Wonham built the internal model principle (IMP) which can eliminate unmeasurable external disturbance if the mathematical model is exact for linear time invariant system[3]. The adaptive control method was emerged gradually after 1970s. This method is based on mathematical model whose characteristics can be identified online and the controller parameters is modified adaptively[4]. The widely used robust control method is designed to minimize the estimation error under the worst case situation to increase the robustness against uncertainty. The sliding model control (SMC) is a kind of variable structure control method, however, the chattering problem caused by discontinuous control hinders its application. There are many other advanced control methods, e.g. fuzzy logic control (FLC), artificial neural networks (ANN) control method and so on. These methods are either based on mathematical model or pursing to make the established model more accurate. However, do we have to have an accurate mathematical model for control design? Actually, the PID control method which is model free have dominated the vast majority of industrial control since 1930s.

The active disturbance rejection control (ADRC) method was firstly proposed by Han at Chinese Academy of Sciences in 1990s [27, 28, 29, 30] and was first introduced

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to English literature in[7]. The creative idea of Han is that we do not have to know the exact mathematical model of the plant, instead the external disturbance and the internal disturbance including the unknown dynamics of the plant can be considered as the "total disturbance" which can be estimated in real time via the extended state observer (ESO) and then compensated by control input. In this way, ADRC is almost model-free and naturally decoupling. It has been an emerging technology in industry control. The industrial giant Texas Instruments (TI) has adopted this method in the design of its new motor control chip. For a comprehensive introduction of ADRC, we recommend the readers to the excellent paper[12].

The key part of ADRC is ESO which provides estimations of the states of the plant as well as the "total disturbance". Actually, there are several classes of disturbance observer, including the unknown input observer (UIO)[13, 14], the disturbance observer (DOB)[15, 16], the perturbation observer (POB)[18, 19] and the extended state observer (ESO)[27, 8, 9]. UIO proposed in 1969 is the earliest disturbance estimator, where the external disturbance is treated as an augmented state of the plant. The state observer is designed to estimate both the original states and the augmented one. The disturbance can be compensated by using the estimated value by the observer. For DOB the disturbance is estimated by using the inverse of the nominal transfer function of the plant. POB is another class of disturbance observer which is formulated in state space in discrete time domain.

All the estimators mentioned above are effectively proven in practice. It was shown that the DOB and UIO is equivalent for some specific design choices[16]. In most cases, the disturbance estimated by originally proposed UIO, DOB or POB is assumed to be external to the plant, that is exactly the difference between ESO and UIO, DOB or POB. ESO can estimate not only the original states but also the "total disturbance" which contains the external disturbance and the un-modeled dynamics of the plant. There are also different variations of UIO and DOB and other approaches seen in scattered reports[20, 21]. Some robust stability analysis for Linear time invariant systems was performed for UIO and DOB [16, 17] based on small gain theorem.

The original ESO proposed by Han uses nonlinear gains which is difficult to tune the parameters and analyze the stability. Gao[8, 9] simplified it to linear ESO (LESO) and the corresponding linear active disturbance rejection control (LADRC) which facilitated the industrial applications of ADRC greatly. The convergence of nonlinear ADRC for a class of nonlinear system is analyzed in [22, 23]. The convergence and the bounds of the estimation error and tracking error of LADRC are presented in[10, 11]. The tracking performance of ADRC for a class of uncertain LTI systems was analyzed both in time domain and frequency domain in[24].

In practice, although the higher LESO gain is, the more accurate the estimation effect would be, it will render the estimation more sensitive to the noise, and the cost maybe expensive. Therefore, it is of great significance for real applications to provide as accurate as possible the upper bounds for estimation and tracking errors, which can be used to determine the lower bounds of LADRC gains given the demanded precision. However, the results in [10, 11] are quite conservative in that they did not give precise upper bounds for the the LADRC errors. In this paper, accurate upper bounds for the estimation and tracking error of LADRC with unknown dynamics are established. The numerical experiments verify that the established results are very effective.

The paper is organized as follows. The analysis of the estimation and tracking errors is given in Section 2. The numerical simulations are shown in Section 3. Finally, the conclusion including some remarks is given in Section 4.

2. Analysis of error dynamics about LADRC. In this paper we consider the following nonlinear SISO dynamic system:

$$y^{(n)}(t) = f(t, y(t), \cdots, y^{(n-1)}(t), d(t), u(t)) + bu(t). \tag{1}$$

Here $u(t)$ is the input, $y(t)$ is the output, b is a given constant, $d(t)$ represents the external disturbance and $f(t, y(t), \cdots, y^{(n-1)}(t), d(t), u(t))$ which can be simply denoted as f is the unknown nonlinear time varying dynamics of the plant with large uncertainty. The central idea of ADRC is that it is unnecessary to know exactly the form of f, instead the real time value of f matters. If a good real time estimation of f can be obtained, then f can be compensated by the control $u(t)$ which transfers the original system to a cascade integrators system. For this, assume f is differentiable and define $x_{n+1} = f$, $h = \dot{f}$. Then (1) can be written in the following state space form

$$\begin{aligned}
&\dot{x}_1 = x_2\\
&\dot{x}_2 = x_3\\
&\vdots\\
&\dot{x}_{n-1} = x_n\\
&\dot{x}_n = x_{n+1} + bu\\
&\dot{x}_{n+1} = h(t, x, d, u)\\
&y = x_1
\end{aligned} \tag{2}$$

where $x = [x_1, x_2, \cdots, x_{n+1}]^T \in \mathbb{R}^{n+1}$, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are the state, input, output of the system, respectively.

The following explicit formula for the matrix exponential plays the key role in our error analysis.

Lemma 1 (Corollary 5, [26]). *Suppose H is a $n \times n$ matrix with complex entries, m is the degree of the minimal polynomial of H. If H has a single eigenvalue λ, then*

$$e^{tH} = e^{\lambda t} \sum_{k=0}^{m-1} \left(\frac{t^k}{k!} \sum_{j=0}^{m-k-1} \frac{(-\lambda t)^j}{j!} \right) H^k.$$

2.1. LESO and its error dynamics. The extended state observer is designed to estimate system states and unknown dynamics in real time. In practice, it is convenient to consider linear ESO. With u and y as input for the LESO, we use the LESO as designed in [8, 9]

$$\begin{aligned}
&\dot{\hat{x}}_1 = \hat{x}_2 + l_1(x_1 - \hat{x}_1)\\
&\dot{\hat{x}}_2 = \hat{x}_3 + l_2(x_1 - \hat{x}_1)\\
&\vdots\\
&\dot{\hat{x}}_{n-1} = \hat{x}_n + l_{n-1}(x_1 - \hat{x}_1)\\
&\dot{\hat{x}}_n = \hat{x}_{n+1} + l_n(x_1 - \hat{x}_1) + bu\\
&\dot{\hat{x}}_{n+1} = l_{n+1}(x_1 - \hat{x}_1)\\
&y = x_1
\end{aligned} \tag{3}$$

where $\hat{x} = [\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_{n+1}]^T \in \mathbb{R}^{n+1}$, and $l_i, i = 1, 2, \cdots, n+1$ are the observer gain parameters to be chosen.

Define $\tilde{x}_i = x_i - \hat{x}_i, i = 1, 2, \cdots, n+1$, then from (2) and (3) the estimation error of the above LESO is given by

$$\begin{array}{l} \dot{\tilde{x}}_1 = \tilde{x}_2 - l_1\tilde{x}_1 \\ \dot{\tilde{x}}_2 = \tilde{x}_3 - l_2\tilde{x}_1 \\ \vdots \\ \dot{\tilde{x}}_{n-1} = \tilde{x}_n - l_{n-1}\tilde{x}_1 \\ \dot{\tilde{x}}_n = \tilde{x}_{n+1} - l_n\tilde{x}_1 \\ \dot{\tilde{x}}_{n+1} = h(t, x, d, u) - l_{n+1}\tilde{x}_1 \end{array} \tag{4}$$

Now define

$$\varepsilon_i = \frac{\tilde{x}_i}{w_o^{i-1}}, \quad \varepsilon = [\varepsilon_1, \cdots, \varepsilon_n + 1]^T, \tag{5}$$

$i = 1, 2, \cdots, n+1$, and also choose the gain parameters as

$$[l_1, l_2, \cdots, l_{n+1}] = [\alpha_1 w_o, \alpha_2 w_o^2, \cdots, \alpha_{n+1} w_o^{n+1}],$$

where $w_o > 0$ is the so called bandwidth of the LESO and $\alpha_i = \binom{n+1}{i}, i = 1, 2, \cdots, n+1$. Then (4) can be rewritten in matrix form as

$$\dot{\varepsilon} = w_o A\varepsilon + B\frac{h}{w_o^n}, \tag{6}$$

where

$$A = \begin{bmatrix} -\alpha_1 & 1 & 0 & \cdots & 0 \\ -\alpha_2 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\alpha_n & 0 & 0 & \cdots & 1 \\ -\alpha_{n+1} & 0 & 0 & \cdots & 0 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

We should mention that the formula (6) is well-known in literature. The reason why l_i are chosen as above is that by pole assignment, the matrix A is Hurwitz and the gain parameters which need to be tuned is only one. Now the matrix A has only one eigenvalue -1. From linear algebras, we know that the matix A is companion matrix, hence the minimal polynomial of the matrix A equals the characteristic polynomial of A.

THEOREM 1. *Assuming that* $\sup_{t \geq t_0} |h(t)| = \delta$, *then for any* $t \geq t_0$,

$$|\tilde{x}_i(t)| \leq w_o^{i-1} \|e^{w_o A(t-t_0)}\|_\infty \cdot \|\varepsilon(t_0)\|_\infty + \frac{\delta \cdot M_i}{w_o^{n+2-i}}, \tag{7}$$

where $M_i = \sum_{j=0}^{i-1} \binom{n+1-i+j}{n+1-i}$, $1 \leq i \leq n+1$, *and*

$$\|\varepsilon(t)\|_\infty \leq \|e^{w_o A(t-t_0)}\|_\infty \cdot \|\varepsilon(t_0)\|_\infty + \frac{\delta \cdot M}{w_o^{n+1}}, \tag{8}$$

where $M = \max_{1 \leq i \leq n+1} M_i$.

Proof. The explicit solution of (6) is given by

$$\varepsilon(t) = e^{w_o A(t-t_0)}\varepsilon(t_0) + P(t), \tag{9}$$

where $P(t) = \int_{t_0}^{t} e^{w_o A(t-s)} B \frac{h(s)}{w_o^n} ds$.

Since the minimal polynomial of A equals to the characteristic polynomial of A which implies the degree of the minimal polynomial of A is $n+1$. By Lemma 1

$$e^{w_o A t} = e^{-w_o t} \sum_{k=0}^{n} \left(\frac{t^k}{k!} \sum_{j=0}^{n-k} \frac{(w_o t)^j}{j!} \right) A^k. \tag{10}$$

Substitute (10) into $P(t)$, then

$$\begin{aligned} P(t) =& \frac{1}{w_o^n} \int_{t_0}^{t} e^{-w_o(t-s)} \sum_{k=0}^{n} \frac{(w_o(t-s))^k}{k!} \sum_{j=0}^{n-k} \frac{(w_o(t-s))^j}{j!} A^k B \cdot h(s) ds \\ =& \frac{1}{w_o^n} \sum_{k=0}^{n} N(k,t) A^k B, \end{aligned} \tag{11}$$

where

$$\begin{aligned} N(k,t) &= \sum_{j=0}^{n-k} \frac{w_o^{k+j}}{k! \cdot j!} \cdot \psi(k,j,t), \\ \psi(k,j,t) &= \int_{t_0}^{t} e^{-w_o(t-s)} (t-s)^{k+j} \cdot h(s) ds. \end{aligned} \tag{12}$$

Note $A^k B = E_{n+1-k}$, where E_i is the i-th basis of $n+1$ dimension Euclidean space. It follows that

$$P(t) = \frac{1}{w_o^n} N(t), \tag{13}$$

where $N(t) = [N(n,t), N(n-1,t), \cdots, N(0,t)]^T$.

Since $\forall t_0 \le \tau \le t,\ |h(\tau)| \le \delta(t,t_0)$, then

$$\begin{aligned} |\psi(k,j,t)| &= |\int_{t_0}^{t} e^{-w_o(t-s)} (t-s)^{k+j} \cdot h(s) ds| \\ &\le \delta \cdot \int_{t_0}^{t} e^{-w_o(t-s)} (t-s)^{k+j} ds \\ &= \delta \cdot \frac{(-1)^{k+j+1}}{w_o^{k+j+1}} \int_{0}^{-w_o(t-t_0)} e^{\tau} \tau^{k+j} d\tau. \end{aligned} \tag{14}$$

It can be easily verified that

$$\phi(k,j,t) := (-1)^{k+j+1} \int_{0}^{-w_o(t-t_0)} e^{\tau} \tau^{k+j} d\tau = (k+j)! - \zeta(k,j,t)$$

where

$$\zeta(k,j,t) := e^{-w_o(t-t_0)} \left(\sum_{l=0}^{k+j} A_{k+j}^{l} \cdot (w_o(t-t_0))^{k+j-l} \right).$$

It is easy to see that $\zeta(k,j,t)$ is monotonically decreasing to 0 on $[t_0,\infty)$, we have $\sup_{t\geq t_0}|\phi(k,j,t)| = (k+j)!$. From (14)

$$|\psi(k,j,t)| \leq \frac{\delta\cdot|\phi(k,j,t)|}{w_o^{k+j+1}} \leq \frac{\delta\cdot(k+j)!}{w_o^{k+j+1}}.$$

By (12)

$$|N(k,t)| \leq \sum_{j=0}^{n-k}\frac{w_o^{k+j}}{k!\cdot j!}\cdot|\psi(k,j,t)| \leq \frac{\delta}{w_o}\sum_{j=0}^{n-k}\binom{k+j}{k}. \tag{15}$$

From (9) and (15), we have

$$\begin{aligned}|\varepsilon_i(t)| &= |(e^{w_oA(t-t_0)}\varepsilon(t_0))_i + P_i(t)|\\ &\leq \|e^{w_oA(t-t_0)}\varepsilon(t_0)\|_\infty + \frac{N(n+1-i,t)}{w_o^n}\\ &\leq \|e^{w_oA(t-t_0)}\|_\infty\cdot\|\varepsilon(t_0)\|_\infty + \frac{\delta\cdot M_i}{w_o^{n+1}},\end{aligned} \tag{16}$$

where $M_i = \sum_{j=0}^{i-1}\binom{n+1-i+j}{n+1-i}$, $1\leq i\leq n+1$ and $P_i(t)$ means the i-th entry of $P(t)$.

Note $\tilde{x}_i = \varepsilon_i\cdot w_o^{i-1}$, $i=1,2,\cdots,n+1$, then

$$|\tilde{x}_i(t)| \leq w_o^{i-1}\cdot|\varepsilon_i| \leq w_o^{i-1}\|e^{w_oA(t-t_0)}\|_\infty\cdot\|\varepsilon(t_0)\|_\infty + \frac{\delta\cdot M_i}{w_o^{n+2-i}}.$$

This is (7).

From (16), we have

$$\|\varepsilon(t)\|_\infty \leq \|e^{w_oA(t-t_0)}\|_\infty\cdot\|\varepsilon(t_0)\|_\infty + \frac{\delta\cdot M}{w_o^{n+1}}, \tag{17}$$

where $M = \max_{1\leq i\leq n+1} M_i$. This is (8). ☐

COROLLARY 1. *The steady estimation error $\tilde{x}_i(\infty)$, $i=1,2,\cdots,n+1$ satisfies*

$$|\tilde{x}_i(\infty)| \leq \frac{\delta\cdot M_i}{w_o^{n-i+2}}, \tag{18}$$

where $M_i = \sum_{j=0}^{i-1}\binom{n+1-i+j}{n+1-i}$. Moreover, suppose the demanded steady estimation precision about x_i is E_i, then $w_o \geq (\frac{\delta\cdot M_i}{E_i})^{\frac{1}{n-i+2}}$ is sufficient to guarantee the precision.

2.2. LADRC and its error dynamics. Assuming the control design objective is to make the output $y(t)$ of (1) track a given, bounded, reference signal r, whose derivatives, $\dot{r},\ddot{r},\cdots,r^{(n)}$ are also bounded. Denote $[r_1,r_2,\cdots,r_{n+1}]^T = [r,\dot{r}_1,\cdots,\dot{r}_n]^T$. By employing the estimation values of the above LESO, the control law of LADRC is taken as

$$u = \frac{1}{b}(k_1(r_1-\hat{x}_1) + k_2(r_2-\hat{x}_2) + \cdots + k_n(r_n-\hat{x}_n) - \hat{x}_{n+1} + r_{n+1}), \tag{19}$$

where $k_i, i=1,2,\cdots,n$ are the controller gain parameters. We should also mention that the above feedback law (19) is a standard control meant to force a high order stable linear differential equation. Substitute u in (1), then the closed-loop system becomes

$$y^{(n)}(t) = (f - \hat{x}_{n+1}) + k_1(r_1 - \hat{x}_1) + k_2(r_2 - \hat{x}_2) + \cdots + k_n(r_n - \hat{x}_n) + r_{n+1} \quad (20)$$

If the estimation value $\hat{x}_{n+1}$ of f by LESO is good enough, then the first term on the right hand side of (20) is sufficiently small. When the first term on the RHS of (20) is dropped, the rest constitutes a generalized PID controller with a feedforward term, which generally works very well by appropriately choosing the controller gain parameters. However, it is important to give a precise analysis about the tracking error of LADRC.

Define $e_i = r_i - x_i, i = 1, 2, \cdots, n$, then

$$\begin{array}{lcl} \dot{e}_1 & = & \dot{r}_1 - \dot{x}_1 = r_2 - x_2 = e_2 \\ & \vdots & \\ \dot{e}_{n-1} & = & \dot{r}_{n-1} - \dot{x}_{n-1} = r_n - x_n = e_n \\ \dot{e}_n & = & \dot{r}_n - \dot{x}_n = r_{n+1} - (x_{n+1} + bu) \\ & = & -k_1(e_1 + \tilde{x}_1) - k_2(e_2 + \tilde{x}_2) \\ & & - \cdots - k_n(e_n + \tilde{x}_n) - \tilde{x}_{n+1}, \end{array} \quad (21)$$

where $\tilde{x}_i, i = 1, 2, \cdots, n+1$ is the estimation error of LESO. Let $e = [e_1, e_2, \cdots, e_n]^T \in \mathbb{R}^n$, $\tilde{x} = [\tilde{x}_1, \tilde{x}_2, \cdots, \tilde{x}_{n+1}]^T \in \mathbb{R}^{n+1}$, then (21) can be rewritten as

$$\dot{e}(t) = A_1 e(t) + B_1 g(\tilde{x}(t)), \quad (22)$$

where

$$A_1 = \begin{bmatrix} 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -k_1 & -k_2 & \cdots & -k_n \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

and $g(\tilde{x}(t)) = -k_1\tilde{x}_1 - k_2\tilde{x}_2 - \cdots - k_n\tilde{x}_n - \tilde{x}_{n+1}$.

Since the matrix A_1 is companion matrix, then the minimal polynomial of A_1 equals A_1's characteristic polynomial which is $s^n + k_1 s^{n-1} + \cdots + k_n$. For the purpose of parameters tuning simplicity and that the matrix A_1 is Hurwitz, choose the controller parameters $k_i = \binom{n}{i-1} w_c^{n+1-i} = \frac{n!}{(i-1)!\cdot(n+1-i)!} w_c^{n+1-i}$, $i = 1, 2, \cdots, n$, $w_c > 0$ is the controller bandwidth. Then the characteristic polynomial of A_1 is $(\lambda + w_c)^n$ and A_1's eigenvalue is $-w_c < 0$.

THEOREM 2. *Under the assumption of Theorem 1, the tracking error satisfies, for any* $t \geq t_1$,

$$\begin{aligned} |e_1(t)| &\leq \|e^{A_1(t-t_1)}\|_\infty \|e(t_1)\|_\infty + \frac{\gamma(t, t_1)}{w_c^n}, \\ |e_2(t)| &\leq \|e^{A_1(t-t_1)}\|_\infty \|e(t_1)\|_\infty + \frac{2\gamma(t, t_1)}{w_c^{n-1}}, \end{aligned} \quad (23)$$

where $\gamma(t, t_1) = (w_c + w_o)^n \cdot \sup_{t_1 \leq s \leq t} \|\varepsilon(s)\|_\infty$, *and* $\varepsilon(t)$ *is as defined in (5). Moreover, the steady tracking error satisfies*

$$\begin{aligned} |e_1(\infty)| &\leq \left(\frac{1}{w_o} + \frac{1}{w_c}\right)^n \cdot \frac{\delta \cdot M}{w_o}, \\ |e_2(\infty)| &\leq \left(\frac{1}{w_o} + \frac{1}{w_c}\right)^n \cdot \frac{2\delta w_c \cdot M}{w_o}. \end{aligned} \quad (24)$$

Proof. The explicit solution of (22) is

$$e(t) = e^{A_1(t-t_1)}e(t_1) + \int_{t_1}^{t} e^{A_1(t-\tau)}B_1 g(\tilde{x}(\tau))d\tau, \tag{25}$$

where $g(\tilde{x}(\tau)) = -k_1\tilde{x}_1 - k_2\tilde{x}_2 - \cdots - k_n\tilde{x}_n - \tilde{x}_{n+1}$.

$$\begin{aligned} |g(\tilde{x}(\tau))| &\leq k_1 \cdot |\tilde{x}_1| + k_2 \cdot |\tilde{x}_2| + \cdots k_n \cdot |\tilde{x}_n| + |\tilde{x}_{n+1}| \\ &= k_1 \cdot |\varepsilon_1| + k_2 w_o \cdot |\varepsilon_2| + \cdots k_n w_o^{n-1} \cdot |\varepsilon_n| + w_o^n \cdot |\varepsilon_{n+1}| \\ &\leq (k_1 + k_2 \cdot w_o + + \cdots + k_n \cdot w_o^{n-1} + w_o^n) \cdot \|\varepsilon(\tau)\|_\infty \\ &= \left(\sum_{i=1}^{n} k_i \cdot w_o^{i-1} + w_o^n \right) \cdot \|\varepsilon(\tau)\|_\infty. \end{aligned} \tag{26}$$

Note $k_i = \binom{n}{i-1} w_c^{n+1-i}, i = 1, 2, \cdots, n$, then

$$\sum_{i=1}^{n} k_i \cdot w_o^{i-1} + w_o^n = \sum_{i=1}^{n} \binom{n}{i-1} w_c^{n+1-i} \cdot w_o^{i-1} + w_o^n = (w_c + w_o)^n. \tag{27}$$

From (26) and (27) , we have

$$|g(\tilde{x}(\tau))| \leq (w_c + w_o)^n \cdot \|\varepsilon(\tau)\|_\infty. \tag{28}$$

Define

$$\gamma(t, t_1) = (w_c + w_o)^n \cdot \sup_{t_1 \leq s \leq t} \|\varepsilon(s)\|_\infty,$$

then $\sup_{t_1 \leq s \leq t} |g(\tilde{x}(s))| \leq \gamma(t, t_1)$.

Let $W(t) = \int_{t_1}^{t} e^{A_1(t-\tau)}B_1 g(\tilde{x}(\tau))d\tau$. Since the minimal polynomial of A_1 equals to the characteristic polynomial of A_1 which implies the degree of minimal polynomial of A_1 is n. By Lemma 1

$$e^{A_1 t} = e^{-w_c t} \sum_{k=0}^{n-1} \left(\frac{t^k}{k!} \sum_{j=0}^{n-k-1} \frac{(w_c t)^j}{j!} \right) A_1^k. \tag{29}$$

Substitute (29) into $W(t)$, then

$$\begin{aligned} W(t) &= \int_{t_1}^{t} e^{-w_c(t-s)} \sum_{k=0}^{n-1} \frac{(t-s)^k}{k!} \sum_{j=0}^{n-k-1} \frac{(w_c(t-s))^j}{j!} A_1^k B_1 \cdot g(s)ds \\ &= \sum_{k=0}^{n-1} T(k,t) N_1(k), \end{aligned} \tag{30}$$

where $N_1(k) := A_1^k B_1$ and

$$\begin{aligned} T(k,t) &= \sum_{j=0}^{n-k-1} \int_{t_1}^{t} e^{-w_c(t-s)} \frac{(t-s)^k}{k!} \frac{(w_c(t-s))^j}{j!} \cdot g(s)ds \\ &= \sum_{j=0}^{n-k-1} \frac{w_c^j}{k! \cdot j!} \cdot \psi_1(k,j,t), \end{aligned} \tag{31}$$

$$\psi_1(k,j,t) = \int_{t_1}^{t} e^{-w_c(t-s)} (t-s)^{k+j} \cdot g(s)ds.$$

By direct calculation, $N_1(k)$ is

$$\begin{array}{l} N_1(0) = B_1 = (0, 0, \cdots, 0, 1)^T, \\ N_1(1) = A_1 B_1 = (0, 0, \cdots, 1, -k_n)^T, \\ \vdots \\ N_1(n-1) = A_1^{n-1} B_1 = (1, -k_n, \cdots, *, *)^T, \end{array} \tag{32}$$

where "*" in $N_1(n-1)$ are terms with unknown form which we also do not care. Now

$$W(t) = \begin{bmatrix} T(n-1,t) \\ T(n-2,t) - k_n T(n-1,t) \\ * \\ \vdots \\ * \end{bmatrix}. \tag{33}$$

Suppose $e^{A_1(t-t_1)} e(t_1) = [q_1(t), q_2(t), \cdots, q_n(t)]^T$.
From (25) and (33), we have

$$\begin{array}{l} e_1(t) = q_1(t) + T(n-1,t) \\ e_2(t) = q_2(t) + T(n-2,t) - k_n T(n-1,t). \end{array} \tag{34}$$

From (31), we have

$$\begin{aligned} |T(n-1,t)| &= \frac{1}{(n-1)!} |\psi_1(n-1,0,t)| \\ &\leq \gamma(t,t_1) \cdot \frac{(-1)^n}{(n-1)! \cdot w_c^n} \int_1^{-w_c(t-t_1)} e^\tau \tau^{n-1} d\tau. \end{aligned} \tag{35}$$

It is easy to see that

$$\begin{aligned} \eta(n-1,t) &:= (-1)^n \int_1^{-w_c(t-t_1)} e^\tau \tau^{n-1} d\tau \\ &= (n-1)! - e^{-w_c(t-t_1)} \left(\sum_{l=0}^{n-1} \frac{(n-1)!}{(n-1-l)!} \cdot (w_c(t-t_1))^{n-l-1} \right). \end{aligned}$$

and $\eta(n-1,t)$ is monotonically increasing to $(n-1)!$ on $[t_1, \infty)$. Then from (35) $|T(n-1,t)| \leq \frac{\gamma(t,t_1)}{w_c^n}$, from (34)

$$\begin{aligned} |e_1(t)| &= |q_1(t) + T(n-1,t)| \\ &\leq |q_1(t)| + |T(n-1,t)| \\ &\leq \|e^{A_1(t-t_1)}\|_\infty \|e(t_1)\|_\infty + \frac{\gamma(t,t_1)}{w_c^n}. \end{aligned} \tag{36}$$

This is the first inequality in (23).

From (31),

$$
\begin{aligned}
&|T(n-2,t)-k_nT(n-1,t)| \\
=& \left|\frac{1}{(n-1)!}\int_{t_1}^{t} e^{-w_c(t-s)}(t-s)^{n-2}[(n-1-w_c(t-s)]g(s)ds\right| \\
\le& \frac{\gamma(t,t_1)}{(n-2)!}\int_{t_1}^{t} e^{-w_c(t-s)}(t-s)^{n-2}ds+\frac{\gamma(t,t_1)\cdot w_c}{(n-1)!}\int_{t_1}^{t} e^{-w_c(t-s)}(t-s)^{n-1}ds \\
=& \frac{\gamma(t,t_1)}{w_c^{n-1}}\{2-\varphi(t)\},
\end{aligned} \tag{37}
$$

where

$$
\varphi(t)=\frac{(w_c(t-t_1))^{n-1}+2\sum_{l=1}^{n-1}\frac{(n-1)!}{(n-1-l)!}(w_c(t-t_1))^{n-l-1}}{(n-1)!\cdot e^{w_c(t-t_1)}} \tag{38}
$$

which is monotonically decreasing to 0 on $[t_1,\infty)$. From (34) and (37), we have

$$
\begin{aligned}
|e_2(t)| &\le |q_2(t)|+|T(n-2,t)-k_nT(n-1,t)| \\
&\le \|e^{A_1(t-t_1)}\|_\infty\|e(t_1)\|_\infty+\frac{2\gamma(t,t_1)}{w_c^{n-1}},
\end{aligned} \tag{39}
$$

which is the second one of (23).

Take the limit on both sides of (36) and (39) as $t\to\infty$, then

$$
\begin{aligned}
|e_1(\infty)| &\le \frac{\gamma(\infty,t_1)}{w_c^{n}}, \\
|e_2(\infty)| &\le \frac{2\gamma(\infty,t_1)}{w_c^{n-1}},
\end{aligned} \tag{40}
$$

where $\gamma(\infty,t_1)=(w_c+w_o)^n\cdot\sup_{t\ge t_1}\|\varepsilon(t)\|_\infty$. By (8), we have

$$
\lim_{t_1\to\infty}\sup_{t\ge t_1}\|\varepsilon(t)\|_\infty=\|\varepsilon(\infty)\|_\infty\le\frac{\delta\cdot M}{w_o^{n+1}}.
$$

Again let $t_1\to\infty$ in (40), we have (24). ∎

REMARK 1. *We should emphasis that if the initial tracking error $e(t_1)$ is not zero, then $e^{A_1(t-t_1)}$ will enlarge the tracking error at the beginning stage considerably which is the so called "Peaking value" phenomenon in engineering practice. However, we can either use the Tracking differentiator (TD) [29] to smooth the transient profile where the "Peaking value" phenomenon occurs or adopt time-varying gain method as proposed in [25] to reduce the "Peaking value".*

REMARK 2. *From Theorem 1 and Theorem 2, we define the saturated estimation error (SEE) as*

$$
SEE:=\frac{\delta M}{w_o^{n+1}},
$$

and the saturated tracking error (STE) as

$$
STE:=(\frac{1}{w_o}+\frac{1}{w_c})^n\cdot\frac{\delta M}{w_o}.
$$

In practice, the initial estimation error $\tilde{x}(t_0)$ *and initial tracking error* $e(t_0)$ *are not always equal to 0. However, since the matrix* A *has negative eigenvalues, for every* w_o *and* w_c*, there exists a* $t_{w_o} \geq t_0$ *such that in (8), for* $t \geq t_{w_o}$*,* $\|e^{w_0 A(t-t_0)}\|_\infty \cdot \|\varepsilon(t_0)\|_\infty$ *is negligible compared to the saturation estimation error (SEE). If we choose* $t_1 = t_{w_o}$ *in (23), then the first term on the RHS of (8) is negligible and then we have* $\gamma(t, t_1) = (w_c + w_o)^n \cdot \frac{\delta M}{w_o^{n+1}}$*. Since the matrix* A_1 *has negative eigenvalues, there exists a* $t_{w_c} > t_1$ *such that in (23), for* $t \geq t_{w_c}$*,* $\|e^{A_1(t-t_1)}\|_\infty \cdot \|e(t_1)\|_\infty$ *is negligible compared to the the saturation tracking error (STE). Define the following notations*

$$\begin{aligned} t_{w_o} &= \min\{t \| \|e^{w_o A t}\|_\infty \cdot \|\varepsilon(t_0)\|_\infty \leq \theta \cdot SEE\}, \\ t_{w_c} &= \min\{t \| \|e^{A_1 t}\|_\infty \cdot \|e(t_1)\|_\infty \leq \theta \cdot STE\}. \end{aligned}$$

where $0 \leq \theta \leq 1$ *is a given threshold.*

For a given reference signal $r(t)$ *and tracking decision* E*, the settling time of a controlled system is defined as*

$$T_s = \min\{t | \sup_{s \geq t} |r(s) - y(s)| \leq E\}.$$

If we choose w_o*,* w_c *in this way, then*

$$\hat{T}_s = t_0 + t_{w_o} + t_{w_c}$$

can be considered as an estimate for the real settling time T_s*.*

Remark 3. *In real applications the explicit form of* f *usually is very complicated or we even do not know it. Hence it is difficult to determine* δ *which is the crux of the established real time upper bounds. However, if the estimated value* $\hat{f}$ *of* f *by LESO is good enough, then we can use the derivative* $\hat{h}$ *of* $\hat{f}$ *as an estimated value of* h *and then an estimated value of* δ *is obtained.*

2.3. More general case. We now consider a more general LESO which is more suitable for application in industry. We allow that the high order derivative of unknown "total disturbance" f, i.e. $f^{(m)}$, $m > 1$ can be bounded.

Denote $x_{n+1} = f,\ x_{n+2} = \dot{f},\ \cdots,\ x_{n+m} = f^{(m-1)},\ h = f^{(m)}$, then (1) can be written in state space form as

$$\begin{aligned} &\dot{x}_1 = x_2 \\ &\vdots \\ &\dot{x}_n = x_{n+1} + bu \\ &\dot{x}_{n+1} = x_{n+2} \\ &\vdots \\ &\dot{x}_{n+m} = h \\ &y = x_1 \end{aligned} \tag{41}$$

where $x = [x_1, x_2, \cdots, x_{n+m}]^T \in \mathbb{R}^{n+m}$, $u \in \mathbb{R}$ and $y \in \mathbb{R}$ are the state, input, output of the system, respectively. In this case the LESO is designed as

$$
\begin{aligned}
&\dot{\hat{x}}_1 = \hat{x}_2 + l_1(x_1 - \hat{x}_1)\\
&\vdots\\
&\dot{\hat{x}}_n = \hat{x}_{n+1} + l_n(x_1 - \hat{x}_1) + bu\\
&\dot{\hat{x}}_{n+1} = \hat{x}_{n+2} + l_{n+1}(x_1 - \hat{x}_1)\\
&\vdots\\
&\dot{\hat{x}}_{n+m} = l_{n+m}(x_1 - \hat{x}_1)
\end{aligned}
\tag{42}
$$

and the corresponding matrix A and vector B in (6) is

$$
A = \begin{bmatrix} -\alpha_1 & 1 & 0 & \cdots & 0\\ -\alpha_2 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ -\alpha_{n+m-1} & 0 & 0 & \cdots & 1\\ -\alpha_{n+m} & 0 & 0 & \cdots & 0 \end{bmatrix},\quad B = \begin{bmatrix} 0\\ 0\\ \vdots\\ 0\\ 1 \end{bmatrix}.
$$

Here $l_i = \alpha_i w_o^i,\ \alpha_i = \binom{n+m}{i}, i = 1, 2, \cdots, n+m.$

The analysis of estimation error and tracking error for this generalized case is given by the following theorems and the proofs are basically the same as those in section 2.1 and 2.2.

THEOREM 3. *Assuming that* $\sup_{t\ge t_0}|h(t)| = \delta$, *then for any* $t \ge t_0,\ 1 \le i \le n+m$,

$$
|\tilde{x}_i(t)| \le w_o^{i-1}\|e^{w_o A(t-t_0)}\|_\infty \cdot \|\varepsilon(t_0)\|_\infty + \frac{\delta \cdot M_i}{w_o^{n+m-i+1}},
\tag{43}
$$

where $M_i = \sum_{j=0}^{i-1}\binom{n+m-i+j}{n+m-i},\ 1 \le i \le n+m$, *and*

$$
\|\varepsilon(t)\|_\infty \le \|e^{w_o A(t-t_0)} \cdot \|_\infty \|\varepsilon(t_0)\|_\infty + \frac{\delta \cdot M}{w_o^{n+m}},
\tag{44}
$$

where $M = \max_{1\le i\le n+m} M_i$.

COROLLARY 2. *Under the assumption of Theorem 3, the steady estimation error* $\tilde{x}_i(\infty), i = 1, 2, \cdots, n+m$ *satisfies*

$$
|\tilde{x}_i(\infty)| \le \frac{\delta \cdot M_i}{w_o^{n+m-i+1}}.
\tag{45}
$$

and $\|\varepsilon(\infty)\|_\infty$ *satisfies,*

$$
\|\varepsilon(\infty)\|_\infty \le \frac{\delta \cdot M}{w_o^{n+m}},
\tag{46}
$$

where $M_i,\ 1 \le i \le n+m$ *and* M *are as defined in Theorem 3.*

THEOREM 4. *Under the assumption of Theorem 3, the tracking error satisfies, for any* $t \ge t_1 \ge t_0$,

$$
\begin{aligned}
|e_1(t)| &\le \|e^{A_1(t-t_1)}\|_\infty \|e(t_1)\|_\infty + \frac{\gamma(t,t_1)}{w_c^n},\\
|e_2(t)| &\le \|e^{A_1(t-t_1)}\|_\infty \|e(t_1)\|_\infty + \frac{2\gamma(t,t_1)}{w_c^{n-1}},
\end{aligned}
\tag{47}
$$

where $\gamma(t,t_1) = (w_c + w_o)^n \cdot \sup_{t_1 \le s \le t} \|\varepsilon(s)\|_\infty$. *Moreover, the steady tracking error satisfies*

$$\begin{aligned} |e_1(\infty)| &\le \left(\frac{1}{w_o} + \frac{1}{w_c}\right)^n \cdot \frac{\delta \cdot M}{w_o^m}, \\ |e_2(\infty)| &\le \left(\frac{1}{w_o} + \frac{1}{w_c}\right)^n \cdot \frac{2w_c\delta \cdot M}{w_o^m}. \end{aligned} \tag{48}$$

We can see from the above theorems that the estimation precision and the tracking precision can be improved if we use a higher order LESO.

3. Simulation. In this section, the following second order system is considered

$$\ddot{y} = f(y(t), \dot{y}(t), d(t)) + u. \tag{49}$$

The reference signal is taken as a step input at $t = 1$ second, i.e.

$$r(t) = \begin{cases} 0, & t < 1; \\ 1, & t \ge 1. \end{cases}$$

The total simulation time is $T = 6$ seconds. Since h is usually unknown, there is no easy way to give a priori bound δ of h. Hence to determine the bound δ of h and verify the effectiveness of our upper bound for the estimation and tracking error, we make the following assumption in this section.

Assumption: When the system (49) enter steady state, the derivatives of output $y(t)$ tracks the derivatives of reference signal $r(t)$, i.e. $y(t) \to r(t)$, $\dot{y}(t) \to \dot{r}(t), \cdots, y^{(n)}(t) \to r^{(n)}(t)$.

We should mention that this assumption is always realizable as long as w_o and w_c are chosen large enough.

3.1. Example I. In this example, we take $f = -2\dot{y} - y + a \cdot t$ in (49). We use 3rd order LESO as designed in (3) and the control law is taken as (19), which is $u = k_p(y - \hat{x}_1) - k_d\hat{x}_2$ in this case. Since in this case $n = 2$, we have

$$M_1 = 1,\ M_2 = 3,\ M_3 = 3,\ M = \max_{1 \le i \le 3} M_i = 3.$$

We choose $w_o = 10,\ w_c = 5,\ a = 1$ in this experiment. In the following two cases, the upper bounds for the steady error of LADRC under different initial conditions are studied. The upper bounds for the steady estimation and tracking error are given by (18) and (24), respectively.

(a). There is no initial estimation error and tracking error, i.e. $\tilde{x}(0) = [0,0,0]^T$ and $e(0) = [0,0]^T$. The LADRC performance is given by Fig. 1. Note that $h = \dot{f} = -2\ddot{y} - \dot{y} + a$, thus by assumption we choose $\delta = \sup_{t \ge t_0} |h| = |a| = 1$. The real steady estimation and tracking error as well as their upper bounds are given in Table. 1.

(b). The initial estimation error $\tilde{x}(0) = [1, -0.5, -2]^T$ and the initial tracking error $e(0) = [-1.5, 0]^T$. The LADRC performance in this case is given by Fig. 2. We can see that the estimation error and tracking error enlarged at the beginning stage which was cause by $e^{w_0 A t}$ and $e^{A_1 t}$. The comparison of real steady error and their theoretical upper bounds is given in Table 1.

From Table 1, we can see that the established upper bounds for the steady error of LADRC are very effective.

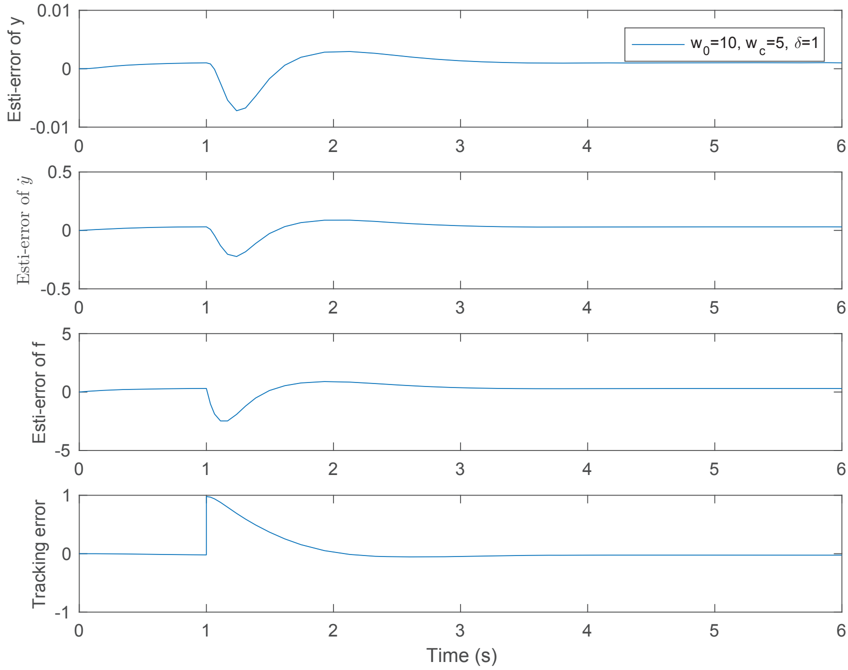

Fig. 1. *The error dynamics of LADRC with no initial error.*

Table 1
Comparison of LADRC errors and their upper bounds

Error	$\tilde{x}_1(t)$	$\tilde{x}_2(t)$	$\tilde{x}_3(t)$	$e_1(t)$
Esti-1[1]	0.0010	0.0299	0.2996	0.0247
Esti-2[2]	0.0010	0.0297	0.2970	0.0246
Theo[3]	0.0010	0.0300	0.3000	0.0270

[1] The data is the average of real estimation, tracking errors in experiment (a) after 4 seconds.
[2] The data is the average of real estimation, tracking errors in experiment (b) after 4 seconds.
[3] The theoretical upper bounds for 3rd order LADRC errors.

3.2. Example II. In this example, we consider the case $f = \dot{y}^2 + y + \sin t$ and there is no initial estimation and tracking error since they do not affect the steady errors of LADRC. We use 3rd order LADRC as in Example I, then $M_i,\ M$ are given in Example I. We choose $w_o = 10,\ w_c = 5$.

The performance of 3rd order LADRC is given in Fig. 3. Now in this case $h = \dot{f} = 2\dot{y}\ddot{y} + \dot{y} + \cos t$, by assumption we can choose $\delta = 1$. The comparison of real steady estimation and tracking error and their theoretical upper bounds is given in Table 2. We can see that the established upper bounds are very effective.

3.3. Example III. In this example, we consider $f = -2\dot{y} - y + t^2$ and there is no initial estimation and tracking error. Since $\dot{f} = -2\ddot{y} - \dot{y} + 2t$ is not bounded and

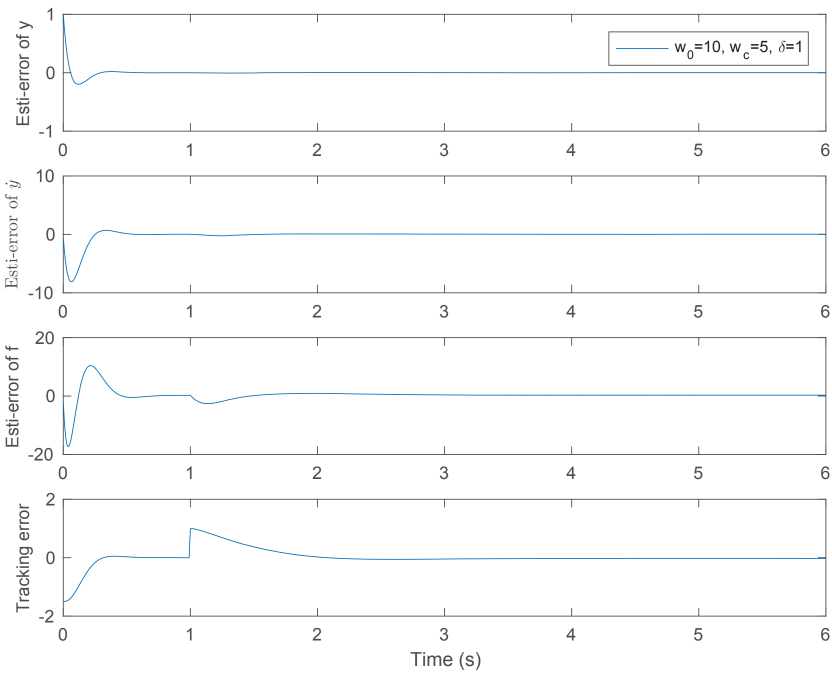

FIG. 2. *The error dynamics of LADRC with initial error.*

TABLE 2
Comparison of LADRC errors and their upper bounds

Error	$\tilde{x}_1(t)$	$\tilde{x}_2(t)$	$\tilde{x}_3(t)$	$e_1(t)$
Esti[1]	0.0005	0.0149	0.1507	0.0120
Theo[2]	0.0010	0.0300	0.3000	0.0270

[1] The data is the average of real estimation, tracking errors after 4 seconds.
[2] The theoretical upper bounds for 3rd order LADRC errors.

$h = \ddot{f} = -2y^{(3)} - \ddot{y} + 2$ is bounded, we use 4th order LESO designed in (42). Note in this case $n = 2,\ m = 2$, we have

$$M_1 = 1,\ M_2 = 4,\ M_3 = 6,\ M_4 = 4,$$
$$M = \max_{0\le k\le 3} \sum_{j=0}^{3-k} \binom{k+j}{k} = 6.$$

We choose $w_o = 10,\ w_c = 5$. By assumption, we can choose $\delta = 2$. The upper bounds for the steady estimation and tracking error are given by (45) and (48), respectively.

The LADRC performance in this case is shown in Fig. 4 and Table 3. We can see that the 4th order LADRC performs a lot better than the 3rd order LADRC and the upper bounds for 4th order LESO and LADRC are also very effective.

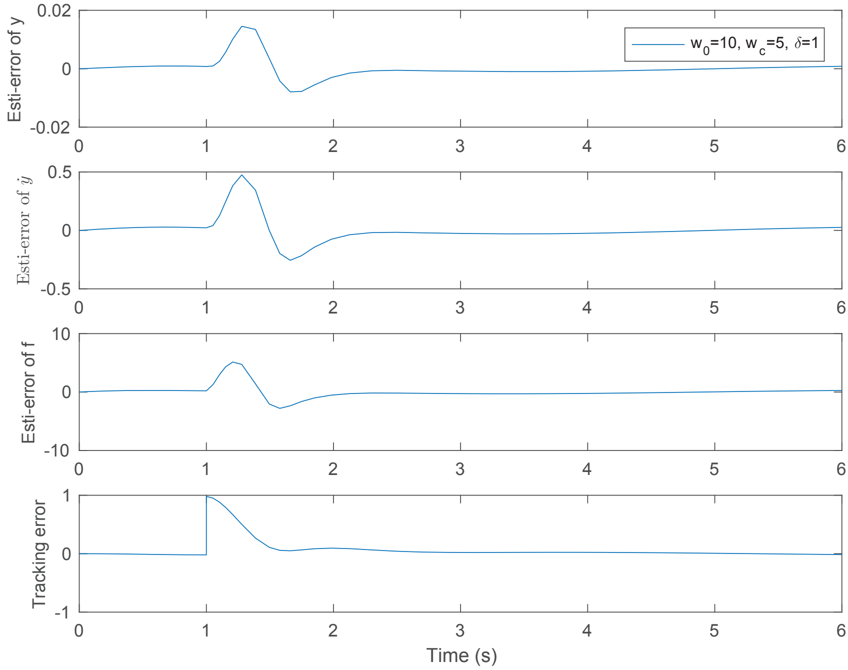

FIG. 3. *The error dynamics of LADRC with no initial error.*

TABLE 3
Comparison of different order LADRC errors and their upper bounds

Error	$\tilde{x}_1(t)$	$\tilde{x}_2(t)$	$\tilde{x}_3(t)$	$e_1(t)$
Esti-1[1]	0.0093	0.2824	2.8652	0.2170
Esti-2[2]	2.000×10^{-4}	0.0077	0.1169	0.0082
Theo[3]	2.000×10^{-4}	0.008	0.12	0.0108

[1] The data is the average of the real estimation error, tracking errors with 3rd order LADRC after 4 seconds.
[2] The data is the average of the real estimation error, tracking errors with 4th order LADRC after 4 seconds.
[3] The theoretical upper bounds for 4th order LADRC errors.

3.4. Numerical study of the settling time of LADRC. In this section, we first study the properties of $e^{w_o At}$ and $e^{A_1 t}$, since they play a crucial role at the beginning stage of the error dynamics of LADRC with nonzero initial error. Then we give an numerical estimate of the settling time.

We consider the same example and initial errors as in Example I (b). We take the total simulation time $T = 6$ seconds, the tracking precision $E = 3\%$ and $w_0 = 10,\ w_c = 5,\ t_0 = 1.0$ seconds. From Remark 2, in this case the saturated estimation

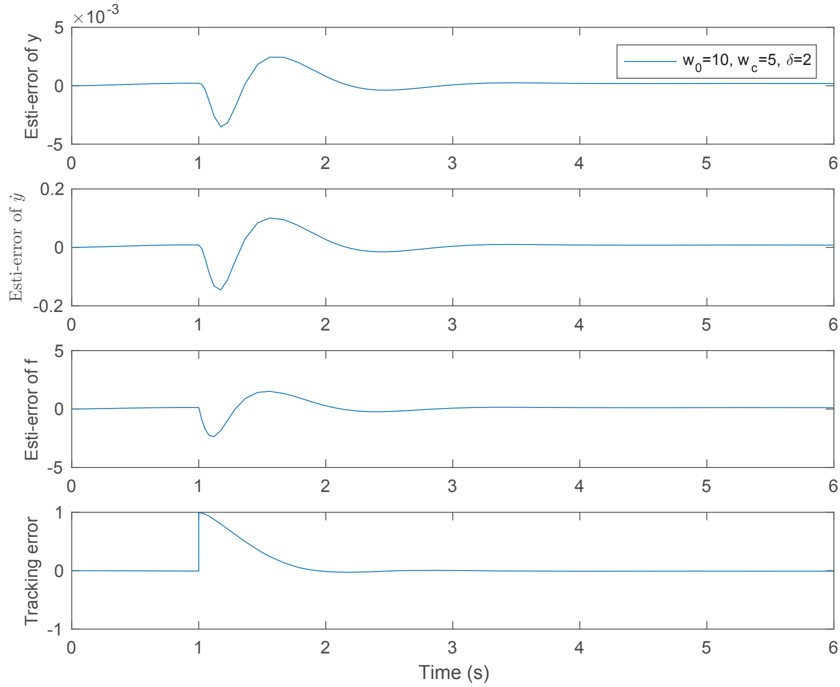

FIG. 4. *The error dynamics of 4th order LADRC with no initial error.*

error and the saturated tracking error are

$$SEE = \frac{\delta M}{w_o^{n+1}} = 0.003,$$
$$STE := \left(\frac{1}{w_o} + \frac{1}{w_c}\right)^n \cdot \frac{\delta M}{w_o} = 0.027.$$

By simulation, we have

$$\|\varepsilon(t_0)\|_\infty = 0.0024,$$
$$\|e(t_0)\|_\infty = 0.9968,$$

and the real settling time $T_s = 3.56$ seconds.

The evolutions of $\|e^{w_o At}\|_\infty$ and $\|e^{A_1 t}\|_\infty$ when $w_o = 10,\ w_c = 5$ are shown in Fig. 5, we can see that $\|e^{w_o At}\|_\infty$ is monotonically increasing first and then monotonically decreasing to 0. $\|e^{A_1 t}\|_\infty$ basically is the same except at the peak value point.

With the threshold $\theta = 0.1$ defined in Remark 2, from numerical experiments we easily have

$$t_{w_o} = 0.69, \quad t_{w_c} = 2.00$$

seconds, thus $\hat{T}_s = t_0 + t_{w_o} + t_{w_c} = 3.69$ seconds. We can see that $\hat{T}_s$ is very close to the real settling time T_s.

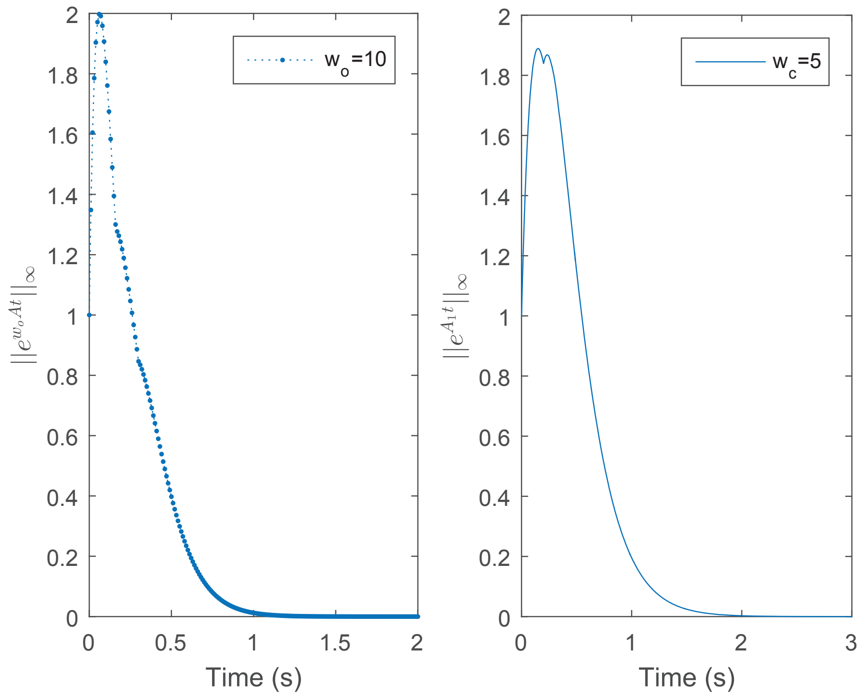

FIG. 5. *Numerical result about the infinite norm of* $e^{w_o At}$ *and* $e^{A_1 t}$

4. Conclusion. As a response to the eagerly demand from the engineering application of ADRC, we investigate the error dynamics of LADRC with unknown model in this paper. Compared to the previous results, we provide very effective upper bounds for the estimation error and tracking error of LADRC which can be useful for parameters tuning of LADRC. In addition, for a generalized system in which only high order derivative of total disturbance with bound is assumed, we have also established precise results. The settling time is analyzed numerically and we give a very effective estimate. Our simulations show that the established upper bounds for the estimation errors and tracking errors are indeed very effective.

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